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Vector Measures with Values in $\ell^\infty(\Gamma)$ and Interpolation of Banach Lattices

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An explicit construction for the representation of the Calderón interpolation of spaces of vector measure integrable functions is given as well as for the representation of the real interpolation of these spaces using the K-functional. In order to do this, we introduce a technique based on interpolation of function valued matrices. For the real interpolation, we develop a vector-valued version of the K-functional having values in ℓ^∞-spaces, providing in this way a new procedure for the study of the interpolation of general Banach lattices.

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1. Introduction

Order continuous Banach function lattices with weak order unit can always be represented as spaces of vector measure integrable functions. Indeed, a Banach lattice with these properties is always isometrically order isomorphic to such a space of integrable functions (see for example Proposition 3.9 in [23]). This is the reason why interpolation of spaces of vector measure integrable functions constructed by using classical interpolation methods have been recently studied by several authors; see [4, 10, 12, 13, 14, 17, 18] for the theoretical results and [9, 11] for applications to operator theory.

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In [15] an explicit construction of an interpolated vector measure starting with two positive vector measures is given. The ideas used there – together with some tools coming from [22] – are the starting point of the technique that we show here. However, the vector measures with values in order continuous Banach function lattices appearing in [15] are substituted by a new construction in the present paper. This new definition associates an $\ell^\infty(\Lambda, L^1(\mu))$-valued vector measure to any Banach-space-valued vector measure in such a way that the corresponding spaces of integrable functions coincide. This provides a procedure that allows to write a representation of real and complex interpolation spaces as spaces of integrable functions with respect to an *interpolated vector measure.* Concretely, we will develop the cases of the Calderón interpolation method and the K-method for real interpolation due to Peetre.

2. Definitions and preliminary results

In this section we sketch some background from the areas of vector measures and interpolation theory. We will also take the opportunity to establish some notation. For more details we refer to [2, 3, 16, 19, 20, 21, 23].

Throughout this section, let X, Y, Z be *complex Banach spaces*; it must be noted that we can also assume that they are real, but we write only the complex case for simplicity of presentation. By B_X we denote the *unit ball* of X. As usual X^* denotes the *dual Banach space* of X. If Y and Z are *isometric* then we write $Y \cong Z$. Let (Ω, Σ, μ) be a σ-finite measure space, and let $L^0(\mu)$ be the space of (equivalence classes of μ-a.e. equal) complex valued measurable functions on Ω equipped with the topology of convergence in measure on sets of finite measure. We write simΣ for the collection of all Σ-*simple functions.*

An *ideal* $(Y(\mu), \|\cdot\|_Y)$ of the vector lattice $L^0(\mu)$ (i.e., $Y := Y(\mu)$ is a vector subspace such that $f \in Y$ whenever $f \in L^0(\mu)$ satisfies $|f| \leqslant |g|$ μ-a.e. for some $g \in Y$) is called a *Banach function lattice* (for short a Banach lattice) on (Ω, Σ, μ) if sim$\Sigma \subset Y$ and $\|\cdot\|_Y$ is a lattice norm (i.e., whenever $f, g \in Y$ and $|f| \leqslant |g|$ μ-a.e., then $\|f\|_Y \leqslant \|g\|_Y$) for which it is complete.

The *positive cone* of $L^0(\mu)$ is denoted by $L^0(\mu)^+$; the positive cone of Y is then denoted by $Y^+ := Y \cap L^0(\mu)^+$. We say that a positive function $e \in Y$ is a weak order unit of Y if it is a *weak order unit* of the real part $Y_{\mathbb{R}}$ (i.e., if $f \wedge (ne) \uparrow f$ for every $f \in Y^+$ or equivalently $x \wedge e = 0$ implies that $x = 0$), which is a real Banach lattice (i.e., $Y_{\mathbb{R}} := Y \cap L^0_{\mathbb{R}}(\mu)$ where $L^0_{\mathbb{R}}(\mu) := \{f \in L^0(\mu) :\ f \text{ is } \mathbb{R}\text{-valued } \mu\text{-a.e.}\}$).

The constant function χ_Ω is a weak order unit of (the Banach lattice) Y. The Banach lattice Y is a complexification of the real Banach lattice $Y_{\mathbb{R}}$ equipped with its given norm $\|\cdot\|_{Y_{\mathbb{R}}}$ and that $\|\cdot\|_{Y_{\mathbb{R}}}$ coincides with the restriction of $\|\cdot\|_Y$ to $Y_{\mathbb{R}}$ (i.e., $\|f\|_Y = \|\,|f|\,\|_{X_{\mathbb{R}}}$ for every $f \in Y$).

Given a Banach lattice $(Y, \|\cdot\|_Y)$, the *order continuous (absolutely continuous)* part Y_a of Y is defined by

$$Y_a := \{ f \in Y \,:\, |f| \geqslant |f_n| \downarrow 0 \text{ with } f_n \in Y \text{ implies } \|f_n\|_Y \downarrow 0 \}.$$

Y_a is a closed order ideal of Y; it is the largest order ideal of Y to which the restriction of the norm $\|\cdot\|_Y$ is order continuous. Let us also note that $f \in Y_a$ if and only if $\lim_{n\to\infty} \|f\chi_{B_n}\|_Y = 0$ for any sequence $\{B_n\}_{n=1}^{\infty} \subset \Sigma$ such that $B_n \searrow \emptyset$ (i.e., $\{B_n\}$ is a decreasing sequence such that $\mu(B_n \cap B) \to 0$ for every set B of finite measure). The Banach lattice Y is said to be *order continuous* if $Y = Y_a$. Let us also recall that every ideal Z with the support Ω (or equivalently $\chi_\Omega \in Z$) of an order continuous Banach lattice Y is dense in Y.

Let (Ω, Σ) be a measurable space, and let $m\colon \Sigma \to X$ be a vector measure (i.e., m is a σ-additive set function). The *variation measure* of m is denoted by $|m|$ and corresponds to the set function $|m| : \Sigma \to [0, \infty]$ given by $|m|(A) = \sup_\pi \sum_{B\in\pi} \|m(B)\|_X$ for all $A \in \Sigma$, where the supremum is taken over all the finite (disjoint) partitions π of A; $|m|$ is the smallest $[0, \infty]$-valued measure dominating m (i.e., $\|m(A)\|_E \leqslant |m|\,(A)$ for every $A \in \Sigma$). For each $x^* \in X^*$, let $\langle m, x^*\rangle\colon \Sigma \mapsto \mathbb{C}$ denote the scalar measure

$$\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle, \quad A \in \Sigma.$$

The *semivariation* $\|m\|$ of m is the set function defined by

$$\|m\|(A) := \sup_{x^*\in B_{X^*}} |\langle m, x^*\rangle|(A), \quad A \in \Sigma,$$

where B_{X^*} is the unit ball of X^*. The Orlicz-Pettis Theorem says that a Banach-space-valued, finitely additive set function $m\colon \Sigma \to X$ is σ-additive if and only if it is scalarly σ-additive (i.e, that is, $\langle m, x^*\rangle$ is σ-additive for every $x^* \in X^*$).

Let $\mathcal{L}^0(\Sigma)$ denote the space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω. According to [20, Definition 2.1], a function $f \in \mathcal{L}^0(\Sigma)$ is called *m-integrable* if

(1) it is $\langle m, x^*\rangle$-integrable for every $x^* \in X^*$, and
(2) for every $A \in \Sigma$ there exists a unique element $m_f(A) \in X$ satisfying

$$\langle m_f(A), x^*\rangle = \int_A f\, d\langle m, x^*\rangle, \quad x^* \in X^*.$$

By the Orlicz-Pettis Theorem, [16, Ch. I, Corollary 2.4], the X-valued set function $m_f\colon A \mapsto m_f(A)$ on Σ is a vector measure; m_f is called the indefinite integral of f relative to m and we will use a classical notation

$$\int_A f\, dm := m_f(A), \quad A \in \Sigma.$$

Let $L^1(m)$ denote the space of all m-integrable functions on Ω, equipped with the seminorm $\|\cdot\|_{L^1(m)}$ defined by

$$\|f\|_{L^1(m)} := \sup_{x^*\in B_{X^*}} \int_\Omega |f|\, d|\langle m, x^*\rangle|, \quad f \in L^1(m);$$

it also satisfies the equality $\|f\|_{L^1(m)} = \|m_f\|(\Omega)$. Each Σ-simple function $s\colon \Omega \to \mathbb{C}$ (i.e., $s = \sum_{j=1}^n a_j \chi_{A_j}$, for some $a_j \in \mathbb{C}$ and $A_j \in \Sigma$ for $j = 1, \ldots, n$, and $n \in \mathbb{N}$) is m-integrable.

Every function $f \in L^1(m)$ satisfying $\|f\|_{L^1(m)} = 0$ is called *m-null*. Let $\mathcal{N}(m)$ denote the subspace of $L^1(m)$ consisting of all m-null functions. A set $A \in \Sigma$ is called m-null if $\chi_A \in \mathcal{N}(m)$ (i.e., $\|m\|(A) = 0$). We denote the family of all m-null sets by $\mathcal{N}_0(m)$; the m-null and $\|m\|$-null sets coincide (i.e., $\mathcal{N}_0(m) = \mathcal{N}_0(\|m\|)$).

We identify $L^1(m)$ with its quotient space $L^1(m)/\mathcal{N}(m)$ and the seminorm $\|\cdot\|_{L^1(m)}$ will be identified with its associated quotient norm; $L^1(m)$ is a vector lattice with respect to the m-a.e. pointwise order and $\|\cdot\|_{L^1(m)}$ is then a lattice norm on $L^1(m)$. As usual, we will write I_m for the *integration operator* $I_m : \Sigma \to X$, $I_m(f) := \int_\Omega f dm$, where $f \in L^1(m)$. If only (1) in the definition above is required, the same construction provides the Banach function lattice $L^1_w(m)$ of weakly integrable functions with respect to m with the same norm and order.

Let us recall the following result due to Lewis [20]; a function $f \in \mathcal{L}^0(\Sigma)$ is m-integrable if and only if there exists a sequence $\{s_n\}_{n=1}^\infty \subset \mathrm{sim}\Sigma$ (resp., $\{f_n\}_{n=1}^\infty \subset L^1(m)$) such that $s_n \to f$ (resp., $f_n \to f$) pointwise as $n \to \infty$ and such that the sequence $\{\int_A s_n\, dm\}_{n=1}^\infty$ (resp., $\{\int_A f_n\, dm\}_{n=1}^\infty$) converges in X for every $A \in \Sigma$ with

$$\int_A f\, dm = \lim_{n\to\infty} \int_A s_n\, dm \quad \Big(\text{resp., } \int_A f\, dm = \lim_{n\to\infty} \int_A f_n\, dm\Big).$$

A finite measure $\mu\colon \Sigma \to [0, \infty)$ is called a *control measure* for a vector measure $m\colon \Sigma \to X$ if μ and m are mutually absolutely continuous (i.e., $\mu(A) \to 0$ if and only if $m(A) \to 0$); by a theorem of B. J. Pettis, this is equivalent to $\mathcal{N}_0(\mu) = \mathcal{N}_0(m)$. By the Rybakov Theorem, [16, Ch. IX, Theorem 2.2], there exists a linear functional $x^* \in X^*$ such that the scalar measure $|\langle m, x^*\rangle|\colon \Sigma \to [0, \infty)$ is a control measure for m; such an x^* is called a Rybakov functional.

The Lebesgue Dominated Convergence Theorem holds for the space $L^1(m)$; given $g \in L^1(m)^+$ (the positive cone of $L^1(m)$), if $\{f_n\}_{n=1}^\infty \subset \mathcal{L}^0(\Sigma)$ converges m-a.e. to a function $f \in \mathcal{L}^0(\Sigma)$ and if $|f_n| \leqslant g$ m-a.e. for all $n \in \mathbb{N}$, then f is m-integrable and $\lim_{n\to\infty} f_n = f$ in the norm $\|\cdot\|_{L^1(m)}$. The space $L^1(m)$ is a weakly compactly generated Banach space in which $\mathrm{sim}\Sigma$ is dense. Given any control measure μ for m, the space $L^1(m) \subset L^0(\mu)$ is a Banach function space based on the measure space (Ω, Σ, μ), its norm $\|\cdot\|_{L^1(m)}$ is an order continuous lattice norm and the constant function χ_Ω is a weak order unit of $L^1(m)$. Since the space $L^1(m)$ is order continuous, we have a representation of its dual space $L^1(m)^*$ as $L^1(m)'$ – the so called Köthe dual –, that is defined by measurable functions that allow to write the elements of the dual as integrals. Given any Rybakov functional x^*, $L^1(m) \subset L^1(|\langle m, x^*\rangle|)$ as vector sublattices of $L^0(|\langle m, x^*\rangle|)$ and the natural inclusion map is continuous.

Let E be an order continuous Banach lattice with a weak order unit. Then there exists an E-valued, positive vector measure m such that E and $L^1(m)$ are lattice isometric (see [7, Theorem 8] or [23, Proposition 3.9]). If E is a Banach lattice and $m\colon \Sigma \to E$ is a positive vector measure, then the norm $\|\cdot\|_{L^1(m)}$ admits a simpler form

$$\|f\|_{L^1(m)} = \left\| \int_\Omega |f|\, dm \right\|_E .$$

Let $\vec{A} = (A_0, A_1)$ be a Banach couple. The K*-functional* of the couple $\vec{A}$ is the map from the sum $A_0 + A_1$ into the cone of the nonnegative concave functions defined on $(0, \infty)$, given by the formula

$$K(t, a) = K(t, a; \vec{A}) := \inf_{a = a_0 + a_1} \left\{ \|a_0\|_{A_0} + t\, \|a_1\|_{A_1} \right\}, \quad t > 0$$

For given $t > 0$, the K-functional is an equivalent norm on the sum $A_0 + A_1$. In particular, $K(1, a) = \|a\|_{A_0 + A_1}$. A Banach couple (A_0, A_1), is called *regular* if $A_j^\circ = A_j$, where A_j° denotes the closure of $A_0 \cap A_1$ in A_j for $j = 0, 1$.

Let us recall definitions of continuous variants of abstract real interpolation methods. Let E be a lattice over a measurable space $(\mathbb{R}_+, dt/t)$, satisfying the condition $1 \wedge t \in E$. The K*-method space* $\vec{A}_{E;K} := (A_0, A_1)_{E;K}$ consists of all elements $a \in A_0 + A_1$ such that $K(\,\cdot\,, a; \vec{A}) \in E$, and equipped with the norm

$$\|a\| := \|K(\,\cdot\,, a; \vec{A})\|_E .$$

The *closure* of $A_0 \cap A_1$ in $\vec{A}_{E;K}$ which will be denoted by $\vec{A}^\circ_{E;K}$. It is well-known that in case where $E = L^q(t^{-\theta}, dt/t)$ for $1 \leqslant q < \infty$ and $\theta \in (0, 1)$ or $q = \infty$ and $\theta \in [0, 1]$, we obtain the classical real interpolation spaces $\vec{A}_{\theta,q}$ (see, e.g., [2, 3])

$$(A_0, A_1)_{E;K} = (A_0, A_1)_{\theta,q} .$$

Let us also recall (see [2, Theorem 3.4.2]) that in the case of $1 \leqslant q < \infty$, $A_0 \cap A_1$ is dense in $\vec{A}_{\theta,q}$. The closure of $A_0 \cap A_1$ in $\vec{A}_{\theta,\infty}$ is the space $\vec{A}^\circ_{\theta,\infty}$ of all a such that

$$t^{-\theta} K(t, a; \vec{A}) \to 0 \quad \text{as} \quad t \to 0 \quad \text{or} \quad t \to \infty.$$

Let us remark here that in case where $\vec{A}$ is a non-trivial couple (i.e., $A_0 \cap A_1$ is not closed in the norm of $A_0 + A_1$) the intersection $A_0 \cap A_1$ in not dense in $\vec{A}_{\theta,\infty}$ (see [8]).

Let us briefly recall the definition of the (lower) *complex interpolation method* given by Calderón [5]. Let $S = \{z \in \mathbb{C} \colon 0 \leqslant \Re z \leqslant 1\}$. Then $\mathcal{F}(A_0, A_1)$ is the space of all bounded continuous functions $f \colon S \to A_0 + A_1$ which are analytic in the interior of S and such that, for $j = 0, 1$, the map $t \mapsto f(j + it)$ is continuous and bounded from $\mathbb{R}$ into A_j and satisfies $\lim_{|t|\to\infty} \|f(j + it)\|_{A_j} = 0$ for $j = 0, 1$. The space $\mathcal{F}(A_0, A_1)$ equipped with the norm

$$\|f\|_{\mathcal{F}(A_0,A_1)} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{A_1}\right\}$$

is a Banach space, and for each $\theta \in (0, 1)$ the lower complex interpolation space $\vec{A}_\theta = [A_0, A_1]_\theta$ consists of all $a \in A_0 + A_1$ of the form $a = f(\theta)$ for some $f \in \mathcal{F}(A_0, A_1)$. Furthermore

$$\|a\|_{[A_0,A_1]_\theta} = \inf\left\{\|f\|_{\mathcal{F}(A_0,A_1)} : f \in \mathcal{F}(A_0, A_1),\ f(\theta) = a\right\}.$$

Given any pair of Banach lattices E_0, E_1 on (Ω, Σ, μ), Calderón [5] constructs the Banach lattice $E_\theta = E_0^{1-\theta} E_1^\theta$ to consists of all $f \in L^0(\mu)$ satisfying $|f| \leqslant |f_0|^{1-\theta}|f_1|^\theta$ μ-a.e. for some $f_0 \in E_0$, $f_1 \in E_1$, and equipped with the norm

$$\|f\|_{E_\theta} := \inf_{|f|\leqslant|f_0|^{1-\theta}|f_1|^\theta} \|f_0\|_{E_0}^{1-\theta} \|f_1\|_{E_1}^\theta,$$

where the infimum runs over all possible representations of f. If either E_0 or E_1 is order continuous then E_θ is also order continuous and the equality $E_0^{1-\theta} E_1^\theta = [E_0, E_1]_\theta$ holds with equality of norms (see, e.g. [5] or [19, Theorem 4.1.14]) where $[E_0, E_1]_\theta$ is the lower complex interpolation space (see, e.g. [2, 5]).

Let us finish this section by presenting some advanced results on vector measures with values in $\ell^\infty(\Gamma, X(\mu))$. Let (Ω, Σ, μ) be a finite measure space. Let us show some properties of vector measures taking values in $\ell^\infty(\Gamma, X(\mu))$ and given by families indexed by Γ of functions belonging to $X(\mu)$. We analyze countable additivity in relation with Γ and the particular family of functions of $X(\mu)$ used in the definition. Some of the concepts that are explained and used here and in the next section have been already studied and partially developed in [22, Section 5]. Here we recall the two more relevant ones. Recall that a *positive norming set* in the dual L^* of a Banach lattice L is a subset B in this dual such that $\|x\|_L = \sup_{x^*\in B}\langle |x|, x^*\rangle$.

Lemma 2.1. ([22, Lemma 5.1]). *Let $N \subseteq L^1(\mu)$ be a bounded set and let $m_N \colon \Sigma \to \ell^\infty(N)$ be given by $m_N(A) = (\int_A g d\mu)_{g\in N}$. The following statements hold.*

(1) *If N is a relatively weakly compact subset of $L^1(\mu)$, then m_N is a vector measure.*

(2) *If N is so that m_N is σ-additive and the space $L_1(m_N)$ is included in $L^1(\mu)$, then N is relatively weakly compact in $L^1(\mu)$.*

Proposition 2.2. ([22, Proposition 5.2]). *Let $m\colon \Sigma \to E$ be a vector measure, let μ be a Rybakov measure for m and consider the positively norming set N for $L^1(m)$ given by*

$$N := \{d|\langle m, x^*\rangle|/d\mu : x^* \in B_{E^*}\}.$$

Take the vector measure $m_N\colon \Sigma \to \ell^\infty(N)$ defined as in Lemma 2.1. Then $L^1(m_N) = L^1(m)$ and $L^1_w(m_N) = L^1_w(m)$.

It must be noted that the set N appearing above is a bounded equi-integrable (uniformly integrable) subset of $L^1(\mu)$, and therefore a relatively weakly compact subset.

3. Calderón interpolation of $\ell^\infty(\Gamma, L^1(\mu))$-valued vector measures

Let us provide now some technical tools for the representation of $L^1(m)$ spaces as spaces of integrable functions with respect to $\ell^\infty(\Gamma, L^1(\mu))$-valued measures. Consider two vector measures $m_0\colon \Sigma \to X_0$ and $m_1\colon \Sigma \to X_1$ taking values on different Banach spaces. Let μ_0 and μ_1 be two Rybakov measures for m_0 and m_1, respectively, and take the measure μ defined as $\mu = (\mu_0 + \mu_1)/2$.

Define the sets

$$N_0 := \Big\{h_\phi := d|\langle m_0, \phi\rangle|/d\mu \ ; \ \phi \in \Lambda_0 := B_{X_0^*}\Big\},$$
$$N_1 := \Big\{g_\psi := d|\langle m_1, \psi\rangle|/d\mu \ ; \ \psi \in \Lambda_1 := B_{X_1^*}\Big\}.$$

The proof of the following result is straightforward.

Lemma 3.1. *The maps*

$$I_0\colon \ell^\infty(\Lambda_0, L^1(\mu)) \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

and

$$I_1\colon \ell^\infty(\Lambda_1, L^1(\mu)) \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

given by

$$I_0\big((h_\phi)_{\phi\in\Lambda_0}\big) := (h_{\phi,\psi})_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

and

$$I_1\big((g_\psi)_{\phi\in\Lambda_1}\big) := (g_{\phi,\psi})_{(\phi,\psi)\in\Lambda_0\times\Lambda_1},$$

where $h_{\phi,\psi} := h_\phi$ and $g_{\phi,\psi} := g_\psi$ for every $(\phi,\psi) \in \Lambda_0 \times \Lambda_1$, are both isometric embeddings.

Consider the set functions

$$\overline{m}_0\colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu)) \text{ and } \overline{m}_1\colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

given by

$$\overline{m}_0(A) := (h_{\phi,\psi}\chi_A)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \quad \text{and} \quad \overline{m}_1(A) := (g_{\phi,\psi}\chi_A)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1},$$

where $A \in \Sigma$, $h_{\phi,\psi} := h_\phi$ and $g_{\phi,\psi} := g_\psi$. Both of them define vector measures.

Proposition 3.2. *The spaces $L^1(m_0)$ and $L^1(\overline{m}_0)$ are isometrically order isomorphic, and the same happens with the corresponding spaces of weakly integrable functions.*

Proof. It is easy to prove that the spaces $L^1(\overline{m}_0)$ and $L^1(m_{N_0})$ of Proposition 2.2 are isometrically isomorphic, and the same happens with $L^1_w(\overline{m}_0)$ and $L^1_w(m_{N_0})$. Then the result is a direct consequence of Proposition 2.2. □

We use now the elements that have been introduced above. We show here how they can be used for obtaining a representation of the Calderón interpolation of L^1-spaces of vector measures.

Given two function families $(h_\phi)_{\phi\in\Lambda_0}$ and $(g_\psi)_{\psi\in\Lambda_1}$ in $L^1(\mu)$, we define the pointwise θ-interpolation product by

$$(h_\phi)^{1-\theta}_{\phi\in\Lambda_0} \cdot (g_\psi)^{\theta}_{\psi\in\Lambda_1} := (h_\phi^{1-\theta} g_\psi^\theta)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

Given two vector measures as $\overline{m}_0$ and $\overline{m}_1$, we define the θ-interpolated vector measure

$$\overline{m}_\theta\colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

by

$$\overline{m}_\theta(A) := (h_\phi^{1-\theta} g_\psi^\theta \chi_A)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}, \quad A \in \Sigma.$$

We can assume that N_0 and N_1 are subsets of $B_{(L^1(m_0))'}$ and $B_{(L^1(m_1))'}$, respectively.

Theorem 3.3. *Let $m_0\colon \Sigma \to X_0$ and $m_1\colon \Sigma \to X_1$ be Banach space valued measures. The following equalities hold isometrically and in the order.*

$$\big(L^1(m_0)\big)^{1-\theta}\big(L^1(m_1)\big)^\theta = \big(L^1(\overline{m}_0)\big)^{1-\theta}\big(L^1(\overline{m}_1)\big)^\theta = L^1(\overline{m}_\theta).$$

Proof. We write the proof for the case that the suprema and the infima in it are exactly attained; the "up to an ε" case follows in the usual way from this case. On one hand, consider the $\ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$ valued vector measure $\overline{m}_\theta$ defined above. Since it is a positive vector measure, if $f \in L^1(\overline{m}_\theta)$ and

$|f| = |f_0|^{1-\theta}|f_1|^{\theta}$ (where each f_j belongs to $L^1(m_j)$, $j = 0,1$) we have that there is a pair of indexes (ϕ_0, ψ_0) such that

$$\begin{aligned}\|f\|_{L^1(\overline{m}_\theta)} &= \Big\|\int |f|\, d\overline{m}_\theta\Big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} = \sup_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \int |f|\, h_\phi^{1-\theta} g_\psi^{\theta}\, d\mu \\ &= \int |f_0|^{1-\theta}|f_1|^{\theta}\, h_{\phi_0}^{1-\theta} g_{\psi_0}^{\theta}\, d\mu \leqslant \Big(\int |f_0| h_{\phi_0}\, d\mu\Big)^{1-\theta}\Big(\int |f_1| g_{\psi_0}\, d\mu\Big)^{\theta}.\end{aligned}$$

Consequently, we have

$$\|f\|_{L^1(\overline{m}_\theta)} \leqslant \|f_0\|_{L^1(m_0)}^{1-\theta}\|f_1\|_{L^1(m_1)}^{\theta}.$$

For the converse, take a norm one function $f \in (L^1(m_0))^{1-\theta}(L^1(m_1))^{\theta}$ and consider a pair of functions $f_0 \in L_1(m_0)$ and $f_1 \in L^1(m_1)$ such that $|f| = |f_0|^{1-\theta}|f_1|^{\theta}$ and $\|f_0\|^{1-\theta}\|f_1\|^{\theta} = 1$. Note that we can assume that f, f_0 and f_1 are positive, since the metric computations are made over the absolute value of the functions. Then there are functions h and g in the corresponding sets of Radon-Nikodým derivatives N_0 and N_1 that norm f_0 and f_1, respectively. Let us prove that $\|f\|_{L^1(\overline{m}_\theta)} \geq 1$.

Define first the functions $\widehat{f_0} := f_0^{1-\theta} f_1^{\theta}(g/h)^{\theta}$ and $\widehat{f_1} := f_0^{1-\theta} f_1^{\theta}(h/g)^{1-\theta}$. Clearly, $\widehat{f_0}^{\,1-\theta}\widehat{f_1}^{\,\theta} = f_0^{1-\theta} f_1^{\theta}$ and $\widehat{f_0}h = \widehat{f_1}g$. We can assume that $\|h\|_{(L^1(m_0))'} = 1 = \|g\|_{(L^1(m_1))'}$. Thus,

$$\|f_0\|_{L^1(m_0)}^{1-\theta}\|f_1\|_{L^1(m_1)}^{\theta} = \Big(\int f_0 h d\mu\Big)^{1-\theta}\Big(\int f_1 g d\mu\Big)^{\theta} = 1$$

and so

$$\|f\|_{L^1(\overline{m}_\theta)} \geq \int f_0^{1-\theta} f_1^{\theta} h^{1-\theta} g^{\theta} d\mu = \Big(\int \widehat{f_0} h d\mu\Big)^{1-\theta}\Big(\int \widehat{f_1} g d\mu\Big)^{\theta} = 1.$$

This shows the equality between both norms and proves the theorem. □

4. Direct interpolation of vector measures by the real method

Let $\vec{A} = (A_0, A_1)$ be a Banach couple. It is an easy exercise to show that

$$\begin{aligned} A_0^\circ &= \big\{a \in A_0 + A_1 \,:\, \lim_{t\to 0} K(t, a; \vec{A}) = 0\big\} \quad \text{and} \\ A_1^\circ &= \big\{a \in A_0 + A_1 \,:\, \lim_{t\to\infty} K(t, a; \vec{A})/t = 0\big\}.\end{aligned}$$

Consequently, we can check that

$$\overline{A_0 \cap A_1}^{A_0+A_1} = A_0^\circ + A_1^\circ.$$

In the case where Y_0 and Y_1 are Banach lattices based on (Ω, Σ, μ), they are continuously embedded into the corresponding linear metric space of measurable functions $L^0(\mu)$ and therefore $\vec{Y}$ form a Banach couple. If additionally Y_0 and Y_1 are order continuous, then $\vec{Y}$ is regular since the intersection $Y_0 \cap Y_1$ is dense in $Y_0 + Y_1$ (as an ideal of the order continuous Banach lattice $Y_0 + Y_1$); therefore, $Y_0 \cap Y_1$ is dense in both Y_0 and Y_1.

Let X_0 and X_1 be Banach function lattices. Let $m_0 \colon \Sigma \to X_0$ and $m_1 \colon \Sigma \to X_1$ be given vector measures, and write μ for the mean of two Rybakov measures as defined in Section 3. We define $\widehat{m}_0 \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1)$ and $\widehat{m}_1 \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1)$ by

$$\widehat{m}_0(A) := \Big(\int_A h_{\phi,\psi} d\mu \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}, \quad h_{\phi,\psi} = h_\phi, \textit{ for all } (\phi,\psi) \in \Lambda_0 \times \Lambda_1,$$

and

$$\widehat{m}_1(A) := \Big(\int_A g_{\phi,\psi} d\mu \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}, \quad g_{\phi,\psi} = g_\psi, \textit{ for all } (\phi,\psi) \in \Lambda_0 \times \Lambda_1,$$

for $A \in \Sigma$. Define a new set function $\Sigma(\widehat{m}_0, \widehat{m}_1) \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1)$ by

$$\Sigma(\widehat{m}_0, \widehat{m}_1)(A) := \Big(\int_A \min\{h_\phi, g_\psi\} d\mu \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}.$$

It is clearly well-defined, and it is easy to prove that it is countably additive.

Lemma 4.1. *For all functions $f \in L^1(m_0) + L^1(m_1)$ and $t > 0$, the following equality holds*

$$K(t, f; (L^1(m_0), L^1(m_1))) = \sup_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} K(t, f; (L^1(h_\phi), L^1(g_\psi))).$$

In particular, the space $L^1(\Sigma(\widehat{m}_0, \widehat{m}_1))$ is an ideal of $L^0(\mu)$, and $L^1(\Sigma(\widehat{m}_0, \widehat{m}_1))$ coincides up to equivalence of norms with $L^1(\widehat{m}_0) + L^1(\widehat{m}_1)$, and so with

$$L^1(m_0) + L^1(m_1).$$

Proof. Let us first remark that for $t > 0$ and a compatible couple $\vec{E} = (E_0, E_1)$ of Banach function lattices based on some (Ω, Σ, μ), for every $x \in E_0 + E_1$,

$$\begin{aligned} K(t, x; \vec{E}) &= \inf_{x=x_0+x_1} \big\{ \|x_0\|_{E_0} + t\,\|x_1\|_{E_1} \big\} = \inf_{|x|\leqslant|x_0|+|x_1|} \big\{ \|x_0\|_{E_0} + t\,\|x_1\|_{E_1} \big\} \\ &= \inf_{|x|\leqslant x_0+x_1} \big\{ \|x_0\|_{E_0} + t\,\|x_1\|_{E_1} \big\} = \inf_{|x|=|x_0|+|x_1|} \big\{ \|x_0\|_{E_0} + t\,\|x_1\|_{E_1} \big\} \\ &= \inf_{|x|=x_0+x_1} \big\{ \|x_0\|_{E_0} + t\,\|x_1\|_{E_1} \big\} = K(t, |x|\,; \vec{E}). \end{aligned}$$

We will give the proof in several steps.

(a) Observe first that for all $(\phi,\psi) \in \Lambda_0 \times \Lambda_1$, the couple $(L^1(h_\phi), L^1(g_\psi))$ is a Banach couple of weighted L^1 spaces. Note also that for each $f \in L^1(m_0) + L^1(m_1)$ we have (see, e.g. p.116 (3) for $p = 1$ in [2])

$$K(t, f; (L^1(h_\phi), L^1(g_\psi))) = \int |f| \min\{h_\phi, t\, g_\psi\} d\mu;$$

in particular, we get $\|f\|_{L^1(h_\phi)+L^1(g_\psi)} = \int |f| \min\{h_\phi, g_\psi\} d\mu$.

(b) Since $L^1(m_0) \subseteq L^1(h_\phi d\mu)$ and $L^1(m_1) \subseteq L^1(g_\psi d\mu)$ and the norms of the inclusions are less or equal to 1, we obtain

$$\sup_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} K(t, f; (L^1(h_\phi), L^1(g_\psi))) \leqslant K(t, f; (L^1(m_0), L^1(m_1))) =: K(t, f).$$

(c) Let us show now the converse inequality. We assume that the set of the Radon-Nikodým derivatives $\mathcal{R}(m_0) \cap B_{(L^1(m_0))'}$, (resp., $\mathcal{R}(m_1) \cap B_{(L^1(m_1))'}$) is weak* compact and convex; this simplifies the argument and causes no loss of generality. Indeed, if this is not the case, notice that both sets are norm bounded. Take the weak* closure of each one in a multiple of the corresponding unit ball. Then the weak* closed convex hull of the weak* closure of the resulting set of $L^1(m_0)'$ (resp., $L^1(m_1)'$), is weak* compact too. This can be proved using for example Theorem 2.2 in [6]; see also the references therein, taking into account that the spaces L^1 of a vector measure are weakly compactly generated (see Theorem 2.2 in [7]). Obviously these sets are also positively norming for $L^1(m_0)$ (resp., $L^1(m_1)$).

Under these assumptions, we can use a classical Ky Fan's Lemma argument. Fix f and take all the representation of $|f|$ as sums of f_0 and f_1, $0 \leq f_0 \leq |f|$; write $r = \operatorname{rep}|f|$ for a couple (f_0, f_1) and consider the family $\mathcal{R}ep$ of all such representations. Let $C_0 \subseteq B_{L^1(m_0)'}$ and $C_1 \subseteq B_{L^1(m_1)'}$ be the convex weak* closures of the set of Radon-Nikodým derivatives of the functions in these pairs with respect to the vector measures m_0 and m_1, respectively. Fix $\varepsilon > 0$. Define for each $r \in \mathcal{R}ep$ a function $\tau_r\colon C_0 \times C_1 \to \mathcal{R}ep$ by

$$\tau_r(h_\phi, g_\psi) = C - \varepsilon - \int f_0 h_\phi d\mu - t \int f_1 g_\psi d\mu$$

where $C = \inf\{\|f_0\|_0 + t\|f_1\|_1 : (f_0, f_1) \in \mathcal{R}ep\}$. It is a (convex and) concave family of convex (and concave) weak* continuous functions, since for each $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$, we have that $\sum \alpha_i f_0^i = \overline{f}_0$ and $\sum \alpha_i f_1^i = \overline{f}_1$ obviously satisfies that $(\overline{f}_0, \overline{f}_1) \in \mathcal{R}ep$. Therefore,

$$\sum_{i=1}^n \alpha_i C - \sum_{i=1}^n \alpha_i \varepsilon - \sum_{i=1}^n \alpha_i \int f_0^i h_\phi d\mu - t \sum_{i=1}^n \alpha_i \int f_1^i g_\psi d\mu =$$

$$= C - \varepsilon - \int \overline{f}_0 h_\phi d\mu - t \int \overline{f}_1 g_\psi d\mu.$$

Note also that for each such function τ_r given by $r = (f_0, f_1)$, we have that there are h_ϕ and g_ψ in C_0 and C_1 such that $\tau_r(h_\phi, g_\psi) \le 0$. Indeed, there are such functions that satisfy

$$C \le \|f_0\|_{L^1(m_0)} + t\|f_1\|_{L^1(m_1)} \le \varepsilon + \int f_0 h_\phi d\mu + t \int f_1 g_\psi d\mu,$$

since the sets C_0 and C_1 are normable for the positive elements of $L^1(m_0)$ and $L^1(m_1)$, respectively, and both of them are weak*-compact.

Therefore, as a consequence of Ky Fan's Lemma we have that there are $\hat{h}_\phi$ and $\hat{g}_\psi$ such that for all f_0 and f_1 with $f = f_0 + f_1$ and $0 \le f_0 \le |f|$,

$$C \le \varepsilon + \int f_0 \hat{h}_\phi d\mu + t \int f_1 \hat{g}_\psi d\mu.$$

Since this holds for all $\varepsilon > 0$ we get the result, taking into account that for each $r = (f_0, f_1)$,

$$C = \inf_{(f_0,f_1)\in\mathcal{R}ep} \big\{\|f_0\|_{L^1(m_0)} + t\|f_1\|_{L^1(m_1)}\big\} \le \varepsilon + \int f_0 \hat{h}_\phi d\mu + t \int f_1 \hat{g}_\psi d\mu$$

and so

$$\begin{aligned} \inf_{(f_0,f_1)\in\mathcal{R}ep} \big\{\|f_0\|_{L^1(m_0)} + t\|f_1\|_{L^1(m_1)}\big\} &\le \varepsilon + \inf_{(f_0,f_1)\in\mathcal{R}ep} \Big\{\int f_0 \hat{h}_\phi d\mu + t \int f_1 \hat{g}_\psi d\mu\Big\} \\ &\le \varepsilon + \sup_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \inf_{(f_0,f_1)\in\mathcal{R}ep} \Big\{\int f_0 h_\phi d\mu + t \int f_1 g_\psi d\mu\Big\}. \end{aligned}$$

(d) In particular, for $t = 1$ this gives

$$\begin{aligned} \|f\|_{L^1(m_0)+L^1(m_1)} &= \sup\big\{\|f\|_{L^1(h_\phi d\mu)+L^1(g_\psi d\mu)} : (\phi,\psi) \in \Lambda_0 \times \Lambda_1\big\} \\ &= \Big\| \int |f|\, d\Sigma(\widehat{m}_0, \widehat{m}_1)\Big\| = L^1(\Sigma(\widehat{m}_0, \widehat{m}_1)). \end{aligned}$$

This finishes the proof. $\square$

Let us define in what follows the key tool of our construction, that is an ℓ^∞-valued version of the K-functional. Let us write

$$\widehat{K}(t,f) \colon (0,\infty) \times L^1(\Sigma(\widehat{m}_0, \widehat{m}_1)) \to \ell^\infty(\Lambda_0 \times \Lambda_1)$$

for the function

$$\widehat{K}(t,f) := \Big(\int |f| \min\{h_\phi, t\, g_\psi\} d\mu\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1},$$

for $f \in L^1(\Sigma(\widehat{m}_0, \widehat{m}_1))$ and $t \in (0, \infty)$. By the the above lemma, we have

$$\begin{aligned}\widehat{K}(t,f) &= \Big(\inf_{f=f_0+f_1} \Big\{ \int |f_0| h_\phi d\mu + t \int |f_1| g_\psi d\mu \Big\} \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \\ &= \Big(K\big(t, f; (L^1(h_\phi), L^1(g_\psi))\big) \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}\end{aligned}$$

and

$$\big\| \widehat{K}(t,f) \big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} = K(t, f; (L^1(m_0), L^1(m_1)))$$

for $f \in L^1(m_0) + L^1(m_1)$ and $t > 0$.

Lemma 4.2. *For a fixed $t > 0$, the function $\widehat{K}(t, \chi_A)$ is a positive vector measure. Moreover, if $f = \sum_{i=1}^n \lambda_i \chi_{A_i}$ is a simple function ($\{A_i\}$ disjoint), we get that*

$$\widehat{K}(t,f) = \sum_{i=1}^n |\lambda_i| \, \widehat{K}(t, \chi_{A_i}).$$

Proof. Fix $t > 0$ and $(\phi, \psi) \in \Lambda_0 \times \Lambda_1$. Take two disjoint sets $A, B \in \Sigma$. Then

$$\int \chi_{A\cup B} \min\big\{h_\phi, t\, g_\psi\big\} d\mu = \int \chi_A \min\big\{h_\phi, t\, g_\psi\big\} d\mu + \int \chi_B \min\big\{h_\phi, t\, g_\psi\big\} d\mu$$

This proves that

$$\widehat{K}(t, \chi_A) + \widehat{K}(t, \chi_B) = \widehat{K}(t, \chi_{A\cup B}),$$

and then we have that $\widehat{K}(t, \chi_{\{\cdot\}})$ defines an additive set function. For proving that it is countably additive, we only need to take into account that for every sequence of disjoint measurable sets (A_n), we have that

$$\big\| \widehat{K}(t, \chi_{\bigcup_{n=k}^\infty A_n}) \big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \leqslant \|\chi_{\bigcup_{n=k}^\infty A_n}\|_{L^1(m_0)} \to_k 0.$$

Finally, let us prove that the equality in the statement is true for simple functions. Let f be a simple function $\sum_{i=1}^n \lambda_i \chi_{A_i}$, with $\{A_i\}$ disjoint. Then it is clear that

$$\int \sum_{i=1}^n \lambda_i \chi_{A_i} \min\big\{h_\phi, t\, g_\psi\big\} d\mu = \sum_{i=1}^n \lambda_i \int \chi_{A_i} \min\big\{h_\phi, t\, g_\psi\big\} d\mu,$$

which together with $|f| = \sum_{i=1}^n |\lambda_i| \chi_{A_i}$, gives the required statement. $\square$

Let us write $\Sigma_t(\widehat{m}_0, \widehat{m}_1)(\,\cdot\,)$ for the vector measure given by $\widehat{K}(t, \chi_{\{\,\cdot\,\}})$ for each $t > 0$. The definition of $\widehat{K}(t, \,\cdot\,)$ makes clear that

$$\Sigma_t(\widehat{m}_0, \widehat{m}_1)(A) = \Big(\int_A \min\big\{h_\phi, t\, g_\psi\big\} d\mu \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

for each $t>0$. We write $L^1\big(\Sigma_t(\widehat{m}_0,\widehat{m}_1)\big)$ for the corresponding space of integrable functions. Since this vector measure is positive, we have that

$$\Big\| \int |f|\, d\,\Sigma_t(\widehat{m}_0,\widehat{m}_1)\Big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} = \|f\|_{L^1(\Sigma_t(\widehat{m}_0,\widehat{m}_1))}, \quad f\in L^1(\Sigma(\widehat{m}_0,\widehat{m}_1)).$$

Further, we can define the following vector measure. Consider the vector space

$$\mathcal{F}\big((0,\infty);\ell^\infty(\Lambda_0\times\Lambda_1)\big)$$

of all the $\ell^\infty(\Lambda_0\times\Lambda_1)$-valued functions, and the function $\mathcal{I}_f$ defined on $(0,\infty)$ by

$$\mathcal{I}_f(t) = \int f\, d\,\Sigma_t(\widehat{m}_0,\widehat{m}_1)$$

for every $f\in L^1(\Sigma(\widehat{m}_0,\widehat{m}_1))$. Clearly, the function

$$(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}\colon \Sigma\to\mathcal{F}\big((0,\infty);\ell^\infty(\Lambda_0\times\Lambda_1)\big)$$

given by $(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}(A) := \mathcal{I}_{\chi_A}$ is a (pointwise) countably additive, positive set function with

$$(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}(A)(t) = \Sigma_t(\widehat{m}_0,\widehat{m}_1)(A) = \Big(\int_A \min\big\{h_\phi, t\, g_\psi\}d\mu\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1},\ t>0.$$

Moreover, if $f=\sum_{i=1}^n \lambda_i\chi_{A_i}$ is a simple function (with $\{A_i\}$ disjoint), we have

$$\mathcal{I}_f = \sum_{i=1}^n \lambda_i(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}(A_i)$$

because of

$$\begin{aligned}\mathcal{I}_f(t) &= \Big(\int\sum_{i=1}^n \lambda_i\chi_{A_i}\min\big\{h_\phi, t\, g_\psi\}d\mu\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}\\ &= \Big(\sum_{i=1}^n \lambda_i\int_{A_i}\min\big\{h_\phi, t\, g_\psi\}d\mu\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}\\ &= \sum_{i=1}^n \lambda_i(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}(A_i)(t), \quad t>0.\end{aligned}$$

Consider now a lattice of (classes of measurable) real functions E over a measurable space $(\mathbb{R}_+, dt/t)$ satisfying the condition $1\wedge t\in E$ and denote by

$$E\big(\ell^\infty(\Lambda_0\times\Lambda_1)\big) := \Big\{\mathcal{G}\in\mathcal{F}\big((0,\infty);\ell^\infty(\Lambda_0\times\Lambda_1)\big)\ :\ \big\|\|\mathcal{G}(\,\cdot\,)\|_{\ell^\infty(\Lambda_0\times\Lambda_1)}\big\|_E<\infty\Big\}$$

the Köthe-Bochner space equipped with the norm

$$\|\mathcal{G}\|_{E(\ell^\infty(\Lambda_0\times\Lambda_1))} := \big\| \|\mathcal{G}(\,\cdot\,)\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \big\|_E$$

for $\mathcal{G} \in E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$.

Let us write in this case $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ for the set function defined as above, and satisfying that $A \mapsto \mathcal{I}_{\chi_A}$; it is countably additive on $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$ as the following lemma shows.

Lemma 4.3. *Let E be a Banach lattice over a measurable space $(\mathbb{R}_+, dt/t)$ such that $1 \wedge t \in E$. Then the set function $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ is a positive vector measure on $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$.*

Proof. Fix any two disjoint sets $A, B \in \Sigma$. Since by Lemma 4.2

$$\Sigma_t(\widehat{m}_0, \widehat{m}_1)(A \cup B) = \Sigma_t(\widehat{m}_0, \widehat{m}_1)(A) + \Sigma_t(\widehat{m}_0, \widehat{m}_1)(B), \quad t > 0,$$

we have

$$(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}(A) + (\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}(B) = (\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}(A \cup B).$$

Therefore, $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ is an additive set function. To see that $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ is a vector measure, we only need to use the inequality

$$\big\|\widehat{K}(t, \chi_A)\big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \leqslant \min\{1, t\} \max\big\{\|\chi_A\|_{L^1(m_0)}, \|\chi_A\|_{L^1(m_1)}\big\}, \quad t > 0,$$

namely, for every sequence of disjoint measurable sets (A_n) we have

$$\begin{aligned}\big\|(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}(B_k)\big\|_{E(\ell^\infty(\Lambda_0\times\Lambda_1))} &= \Big\| \big\|\Sigma_{(\,\cdot\,)}(\widehat{m}_0, \widehat{m}_1)(B_k)\big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \Big\|_E = \\ &= \Big\| \big\|\widehat{K}(\,\cdot\,, \chi_{B_k})\big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \Big\|_E \leqslant c_k \|\min\{1, \cdot\,\}\|_E \to_k 0,\end{aligned}$$

where $B_k := \bigcup_{n=k}^\infty A_n$ and $c_k := \max\{\|\chi_{B_k}\|_{L^1(m_0)}, \|\chi_{B_k}\|_{L^1(m_1)}\}$. □

Strictly speaking, the parameter E in the symbol $(\widehat{m}_0, \widehat{m}_1)_{E,\widehat{K}}$ only reflects the target space $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$ of our new vector measure. In fact, $E = L^\infty \cap L^\infty(1/t)$ is the smallest Banach lattice E possible in this setting (by $1 \wedge t \in E$).

We are now ready to show our main result.

Theorem 4.4. *Suppose that both X_0 and X_1 are Banach lattices and let $m_0 \colon \Sigma \to X_0$ and $m_1 \colon \Sigma \to X_1$ be vector measures. If E is a Banach lattice of functions on $(\mathbb{R}_+, dt/t)$ satisfying the condition $1 \wedge t \in E$, then we have*

$$(L^1(m_0), L^1(m_1))^\circ_{E;K} = \big((L^1(m_0), L^1(m_1))_{E;K}\big)_a \cong L^1\big((\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}\big)\,.$$

Proof. Let us consider a simple function $f = \sum_{i=1}^{n} \lambda_i \chi_{A_i}$, with $\{A_i\}$ disjoint. Clearly, $|f| = \sum_{i=1}^{n} |\lambda_i| \chi_{A_i}$. Now, by definition, and taking into account that $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ is a positive vector measure in $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$, we have by Lemma 4.3 that

$$\begin{aligned}
\|f\|_{L^1\left((\widehat{m}_0,\widehat{m}_1)_{E;\widehat{K}}\right)} &= \Big\| \int |f| \, d(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}} \Big\|_{E(\ell^\infty(\Lambda_0\times\Lambda_1))} \\
&= \Big\| \sum_{i=1}^{n} |\lambda_i| \, (\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}(A_i) \Big\|_{E(\ell^\infty(\Lambda_0\times\Lambda_1))} \\
&= \Big\| \big\| \mathcal{I}_{|f|}(\,\cdot\,) \big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \Big\|_E = \Big\| \big\| \widehat{K}(\,\cdot\,, f) \big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \Big\|_E \\
&= \big\| K(\,\cdot\,, f) \big\|_E = \|f\|_{(L^1(m_0), L^1(m_1))_{E;K}} \,.
\end{aligned}$$

Since $\mathrm{sim}\Sigma$ is dense in $L^1(m_0) \cap L^1(m_1)$, it follows that $\mathrm{sim}\Sigma$ is also dense in

$$(L^1(m_0), L^1(m_1))^\circ_{E;K},$$

and therefore

$$\big((L^1(m_0), L^1(m_1))_{E;K}\big)_a = (L^1(m_0), L^1(m_1))^\circ_{E;K}. \qquad \square$$

Note that by choosing $E = L^q(t^{-\theta}, dt/t)$, with $1 \leqslant q < \infty$ and $\theta \in (0,1)$, we obtain the following result.

Corollary 4.5. *If $1 \leqslant q < \infty$ and $0 < \theta < 1$, then*

$$(L^1(m_0), L^1(m_1))_{\theta,q} \cong L^1\big((\widehat{m}_0, \widehat{m}_1)_{L^q(t^{-\theta}, dt/t);\widehat{K}}\big)\,.$$

In the case where $E = L^\infty(t^{-\theta}, dt/t)$ with $\theta \in [0,1]$ we have

Corollary 4.6. *If $0 \leqslant \theta \leqslant 1$, then*

$$(L^1(m_0), L^1(m_1))^\circ_{\theta,\infty} \cong L^1\big((\widehat{m}_0, \widehat{m}_1)_{F;\widehat{K}}\big)$$

for

$$F = \overline{L^\infty \cap L^\infty(t^{-1})}^{L^\infty(t^{-\theta},dt/t)}.$$

Proof. By [2, Theorem 3.4.2]) we directly get

$$(L^1(m_0), L^1(m_1))^\circ_{\theta,\infty} = (L^1(m_0), L^1(m_1))_{F;K}.$$

This gives the result. $\square$

Let us remark that in the case when $\vec{A}_{E,K}$ is described using discrete a variants of real interpolation method, for a lattice $E = \ell^\infty(2^{-n\theta})$ (modeled on $\mathbb{Z}$) we have

$$F = \overline{\ell^\infty \cap \ell^\infty(2^{-n})}^E = c_0(2^{-n\theta}).$$

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