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On Countable Tightness and the Lindelöf Property in Non-Archimedean Banach Spaces

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Let $\mathbb{K}$ be a non-archimedean valued field and let E be a non-archimedean Banach space over $\mathbb{K}$. By E_w we denote the space E equipped with its weak topology and by $E^*_{w^*}$ the dual space E^* equipped with its weak* topology. Several results about countable tightness and the Lindelöf property for E_w and $E^*_{w^*}$ are provided. A key point is to prove that for a large class of infinite-dimensional polar Banach spaces E, countable tightness of E_w or $E^*_{w^*}$ implies separability of $\mathbb{K}$. As a consequence we obtain the following two characterizations of $\mathbb{K}$:
(a) A non-archimedean valued field $\mathbb{K}$ is locally compact if and only if for every Banach space E over $\mathbb{K}$ the space E_w has countable tightness if and only if for every Banach space E over $\mathbb{K}$ the space $E^*_{w^*}$ has the Lindelöf property.
(b) A non-archimedean valued separable field $\mathbb{K}$ is spherically complete if and only if every Banach space E over $\mathbb{K}$ for which E_w has the Lindelöf property must be separable if and only if every Banach space E over $\mathbb{K}$ for which $E^*_{w^*}$ has countable tightness must be separable.
Both results show how essentially different are non-archimedean counterparts from the "classical" corresponding theorems for Banach spaces over the real or complex field.

Keywords: Non-archimedean Banach spaces, weak topology, Lindelöf property, countable tightness.

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1. Introduction

In [3] Corson asked if, in the context of real or complex Banach spaces E, weakly compactly generated Banach spaces are exactly those E that are weakly Lindelöf, i.e. endowed with the weak topology $\sigma(E, E^*)$ have the Lindelöf property.

Recall that a Banach space E is called *weakly compactly generated* if it admits a $\sigma(E, E^*)$-compact set whose linear hull is dense in E. It was proved in [17] that every weakly compactly generated Banach space is weakly Lindelöf, see also [11]. However, there are examples of weakly Lindelöf Banach spaces which are not weakly compactly generated, see [13, Section 3.3]. Notice that there are concrete non-separable weakly compactly generated (hence, weakly Lindelöf) Banach spaces, for example $c_0(I, \mathbb{R})$ if I is uncountable, see e.g. [7] also as a good source of references. Although E_w does not necessarily have the Lindelöf property, it has always the following useful property called countable tightness.

A topological space X is said to have *countable tightness* if for every $A \subset X$ and $x \in X$ with $x \in \overline{A}$ there is a countable set $T \subset A$ such that $x \in \overline{T}$. Recall also that X is said to have the *Lindelöf property* if it is regular and every open cover of X has a countable subcover.

By Kaplansky's theorem (see [5, Theorem 4.49], [6, Theorem 3.54]), for every real or complex Banach space E, the space E_w has countable tightness. The proof of this fact essentially uses the compactness of the dual unit ball equipped with the weak* topology. Indeed, by Arkhangell'ski-Pytkeev's theorem, see [1, II.1.1], the space $C_p(X, \mathbb{R})$ of all real-valued continuous maps on a completely regular space X, endowed with the pointwise topology, has countable tightness if and only if every finite product X^n of X has the Lindelöf property. This result applies for many concrete spaces X, for example if $X = E^*_{w^*}$, the weak*-dual of a metrizable real or complex locally convex space E. Then, as E_w embeds into $C_p(X, \mathbb{R})$ and X is σ-compact, we obtain Kaplansky's result. In [2] (see also [7, Theorem 12.2]) it was proved that in a large class of locally convex spaces E (which contains for example all metrizable locally convex spaces, (DF)-spaces, (LF)-spaces, etc.), the space E_w has countable tightness if and only if $E^*_{w^*}$ has the Lindelöf property. In particular, for every real or complex Banach space E, its weak*-dual $E^*_{w^*}$ has the Lindelöf property.

In this paper we follow this line of research when our main object is a non-archimedean Banach space E over a non-archimedean valued field $\mathbb{K}$.

Clearly, for every finite-dimensional E, E_w and $E^*_{w^*}$ have countable tightness since the weak and weak* topologies coincide with the norm topologies on E and E^*, respectively; in this case E_w (resp. $E^*_{w^*}$) has the Lindelöf property if and only if $\mathbb{K}$ is separable (see [4, Corollary 4.1.16]). Therefore, we will center our attention on infinite-dimensional Banach spaces.

Kąkol and Śliwa proved a non-archimedean counterpart of Kaplansky's theorem, which states that if $\mathbb{K}$ is locally compact then, for every E over $\mathbb{K}$, E_w has countable tightness ([8, Proposition 2]). Also, we prove here that, for every E over such $\mathbb{K}$, $E^*_{w^*}$ has the Lindelöf property (Corollary 4.2). In this context it is natural to ask if these two results are true without the assumption of the local compactness of $\mathbb{K}$. Then, the main question arises:

Problem 1.1. *Let E be a Banach space over $\mathbb{K}$. Describe conditions on E and $\mathbb{K}$ under which E_w has countable tightness (resp. $E^*_{w^*}$ has the Lindelöf property).*

We show that for a polar Banach space E, countable tightness of E_w implies separability of $\mathbb{K}$, see Proposition 3.5 (since $\mathbb{K}$ is homeomorphically embedded in $E^*_{w^*}$, the Lindelöf property of this weak*-dual also implies separability of $\mathbb{K}$, by [4, Corollary 4.1.16]). This result covers a large class of Banach spaces over $\mathbb{K}$ not being necessarily spherically complete. Nevertheless, for non-locally compact $\mathbb{K}$, we prove (Theorem 4.3) that if either E has a base or $\mathbb{K}$ is spherically complete, then E_w has countable tightness if and only if E is separable if and only if $E^*_{w^*}$ has the Lindelöf property. A direct application of our Theorem 4.3 yields the following purely non-archimedean corollary: assume that $\mathbb{K}$ is not locally compact. Then, the Banach space $C(X, \mathbb{K})$ of all $\mathbb{K}$-valued continuous maps on a zero-dimensional compact space X, has countable tightness in the weak topology if and only if X is ultrametrizable and $\mathbb{K}$ is separable, see Remark 4.4.5.

On the other hand, we show also that the previous situation differs if $\mathbb{K}$ is not spherically complete. For this case, we provide an example of a non-separable normpolar Banach space E such that E_w has countable tightness and $E^*_{w^*}$ has the Lindelöf property, see Remark 4.4.3.

These results together lead us to the following two interesting characterizations of the field $\mathbb{K}$.

Theorem 1.2. *A non-archimedean valued field $\mathbb{K}$ is locally compact if and only if for every Banach space E over $\mathbb{K}$ the space E_w has countable tightness if and only if for every Banach space E over $\mathbb{K}$ the space $E^*_{w^*}$ has the Lindelöf property.*

Theorem 1.3. *A non-archimedean valued separable field $\mathbb{K}$ is spherically complete if and only if every Banach space E over $\mathbb{K}$ for which E_w has the Lindelöf property must be separable if and only if every Banach space E over $\mathbb{K}$ for which $E^*_{w^*}$ has countable tightness must be separable.*

2. Preliminaries

Let V be an ultrametric space, i.e. a metric space (V, d) where d satisfies the *strong triangle inequality* $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in V$. Let $x \in V$ and $r > 0$; recall that the set $B_r(x) = \{y \in V : d(y, x) \leq r\}$ is called a *closed ball* in V and $B^-_r(x) = \{y \in V : d(y, x) < r\}$ is called an *open* *ball* in V, respectively. Note that both balls are clopen (closed and open in the topological sense) and two balls in V are either disjoint, or one is contained in the other.

By a *non-archimedean valued field* we mean a non-trivially valued field $\mathbb{K}$ that is complete under the metric induced by its valuation $|\,.\,| : \mathbb{K} \to [0, \infty)$, which satisfies the *strong triangle inequality* $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

Recall that $|\mathbb{K}^*| = \{|\lambda| : \lambda \in \mathbb{K}\setminus\{0\}\}$ is the *value group* of $\mathbb{K}$ and $\mathbb{k} = B_{\mathbb{K}}/B_{\mathbb{K}}^-$ is the *residue class field* of $\mathbb{K}$, where $B_{\mathbb{K}}$ and $B_{\mathbb{K}}^-$ are the closed and open unit ball in $\mathbb{K}$ centered at zero, respectively. $\mathbb{K}$ is said to be *discretely valued* if 0 is the only accumulation point of $|\mathbb{K}^*|$ (then, there exists an *uniformizing element* $\rho \in \mathbb{K}$ with $0 < |\rho| < 1$ such that $|\mathbb{K}^*| = \{|\rho|^n : n \in \mathbb{Z}\}$); otherwise, we say that $\mathbb{K}$ is *densely valued* (then, $|\mathbb{K}^*|$ is a dense subset of $[0, \infty)$).

We say that $\mathbb{K}$ is *spherically complete* if every shrinking sequence of balls in $\mathbb{K}$ has a non-empty intersection; otherwise, $\mathbb{K}$ is *non-spherically complete.* Every locally compact field is discretely valued and separable; every discretely valued field is spherically complete.

For any prime number p the field $\mathbb{Q}_p$ of p-adic numbers is non-archimedean and locally compact. On the other hand, the valued field $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, is separable and non-spherically complete.

By a *non-archimedean Banach space* over $\mathbb{K}$ we mean a complete normed space E over $\mathbb{K}$ whose norm $\|\,.\,\|$ satisfies the *strong triangle inequality* $||x + y|| \leq \max\{||x||, ||y||\}$ for all $x, y \in E$. For $A \subset E$, $[A]$ denotes the linear hull of A.

The *topological dual of* E is denoted by E^*. Also, $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak and weak* topology on E and E^*, respectively; and $E_w := (E, \sigma(E, E^*))$, $E^*_{w^*} := (E^*, \sigma(E^*, E))$. For a set $A \subset E$ (resp. $A \subset E^*$), $\overline{A}^w$ is the closure of A in E_w (resp. $\overline{A}^{w^*}$ is the closure of A in $E^*_{w^*}$).

If $x^* \in E^*$, then $\ker x^* := \{x \in E : x^*(x) = 0\}$ is the kernel of x^*. If D is a subspace of E, by $x^*|D$ we mean the restriction of x^* to D. Analogously, $\sigma(E, E^*)\,|D$ denotes the restriction to D of the weak topology on E; same procedure to denote the restriction of $\sigma(E^*, E)$ to a subspace of E^*.

By B_E and B_{E^*} we mean the *closed unit ball* in E and E^* centered at zero, respectively.

We say that E is *normpolar* (or the norm of E is *polar*) if, for each $x \in E$, $\|x\| = \sup\{|x^*(x)| : x^* \in B_{E^*}\}$. E is called *polar* if its norm topology is defined by a polar norm. If $\mathbb{K}$ is spherically complete every Banach space E over $\mathbb{K}$ is polar. For non-spherically complete ground fields, the most popular examples of non-archimedean Banach spaces are polar, see [12, Section 2.5].

A continuous linear map $T\colon E \to F$ between two non-archimedean Banach spaces E, F over $\mathbb{K}$ is called an *isomorphism* if T is bijective and its inverse T^{-1} is also continuous; in this case we say that E and F are *isomorphic.* Then, the adjoint of T, $T^*\colon F^* \to E^*$, $y^* \mapsto y^* \circ T$ $(y^* \in F^*)$, is also an isomorphism with $(T^*)^{-1} = (T^{-1})^*$; if, in addition, E, F are normpolar, then $\|T\| = \|T^*\|$.

Let I be an infinite set. $\ell^\infty(I)$ denotes the (normpolar) non-archimedean Banach space over $\mathbb{K}$ consisting of all bounded maps $I \to \mathbb{K}$, equipped with the usual supremum norm given by $\left\|(\lambda_i)_{i\in I}\right\| = \sup_{i\in I} |\lambda_i|$. $c_0(I)$ is the closed subspace of $\ell^\infty(I)$ formed by the $(\lambda_i)_{i\in I} \in \ell^\infty(I)$ such that for every $\varepsilon > 0$ there exists a finite $J \subset I$ for which $|\lambda_i| < \varepsilon$ for all $i \in I\backslash J$. By $c_{00}(I)$ we denote the linear hull of $\{e_i : i \in I\}$, where $(e_i)_{i\in I}$ are the unit vectors of $c_0(I)$. In particular, $\ell^\infty := \ell^\infty(\mathbb{N})$, $c_0 := c_0(\mathbb{N})$ and $c_{00} := c_{00}(\mathbb{N})$. We have $c_0(I)^* = \ell^\infty(I)$. For each $y \in \ell^\infty(I)$ we denote by y^* the element of $c_0(I)^*$ defined by y. When $\mathbb{K}$ is not spherically complete and I is small, $c_0(I)$ and $\ell^\infty(I)$ are reflexive, so $\ell^\infty(I)^* = c_0(I)$. Recall that a set I is called *small* if it has non-measurable cardinality (the sets we meet in daily mathematical life are small; see [14, p. 31-33] for further discussions and references on small sets).

A family $(x_i)_{i\in I}$ in E is a *base* of E if each $x \in E$ has a unique expansion $x = \sum_{i\in I} \lambda_i x_i$, where $\lambda_i \in \mathbb{K}$ for all $i \in I$. The unit vectors of $c_0(I)$ form a base of this space. Even more, if E has a base $\{x_i\}_{i\in I}$, then E is isomorphic to $c_0(I)$, hence E is polar. For any infinite set I, $\ell^\infty(I)$ has a base if and only if $\mathbb{K}$ is discretely valued.

Let $t \in (0,1]$. A countable set $\{x_1, x_2, ...\} \subset E\backslash\{0\}$ is called *t-orthogonal* if for each finite subset J of $\mathbb{N}$ and all $\{\lambda_i\}_{i\in J} \subset \mathbb{K}$ we have $\left\|\sum_{i\in J} \lambda_i x_i\right\| \geq t \cdot \max_{i\in J} \left\|\lambda_i x_i\right\|$.

E is of *countable type* if it contains a countable set whose linear hull is dense in E. If $\mathbb{K}$ is separable, then a Banach space is of countable type if and only if it is separable. If E is of countable type it has, for each $t \in (0,1)$, a *t-orthogonal base*, i.e. a t-orthogonal set $\{x_1, x_2, ...\} \subset E$ that is a base of E; hence, if E is infinite-dimensional, it is isomorphic to c_0. For any infinite set I, $\ell^\infty(I)$ is not of countable type.

Throughout this paper $\mathbb{K}$ will be a non-archimedean valued field. All the Banach spaces over $\mathbb{K}$, denoted by $E, F, ...$, considered in the sequel are assumed to be non-archimedean and infinite-dimensional.

For more background on normed spaces over non-archimedean valued fields we refer the reader to [12] and [14].

The following two basic Lemmas will be used along the paper.

Lemma 2.1. *Let E be normpolar. Then, for each $t \in (0,1)$ there exist t-orthogonal sequences $x_1, x_2, \ldots$ in E and $x_1^*, x_2^*, \ldots$ in E^* such that*

$$t \leq \|x_n\| \leq 1 \leq \|x_n^*\| \leq \frac{1}{t} \quad \text{and} \quad x_n^*(x_m) = \delta_{nm} \quad \text{for all } n, m \in \mathbb{N}.$$

Proof. Let $t_1, t_2, \ldots \in (t,1)$ with $t_1^2 \cdot t_2^2 \cdots > t$. We are done as soon as we construct $x_1, x_2, \ldots$ in E and $x_1^*, x_2^*, \ldots$ in E^* such that

(a) $t_n \leq \|x_n\| \leq 1 \leq \|x_n^*\| \leq \frac{1}{t_1^2 \cdots t_n^2}$ and $x_n^*(x_m) = \delta_{nm}$ for all $n, m \in \mathbb{N}$.

(b) For each $n \geq 2$, $x_1, \ldots, x_n$ and $x_1^*, \ldots, x_n^*$ are $(t_1^2 \cdot \ldots \cdot t_{n-1}^2)$−orthogonal in E and E^*, respectively.

We proceed inductively for this construction. For $n = 1$, choose $x_1 \in E$ with $t_1 \leq ||x_1|| \leq 1$. Let y_1^* be a linear functional defined on $[x_1]$, given by $y_1^*(x_1) = 1$. Then, $1 \leq ||y_1^*|| = \frac{1}{\|x_1\|} \leq \frac{1}{t_1}$. By normpolarity and [12, Theorem 4.4.5], we can extend y_1^* to $x_1^* \in E^*$ with $1 \leq ||x_1^*|| \leq \frac{1}{t_1^2}$.

For the step $n \to n+1$, suppose that we have constructed $x_1, x_2, \ldots, x_n$ in E and $x_1^*, x_2^*, \ldots, x_n^*$ in E^* satisfying (a) and (b). Choose $x_{n+1} \in \bigcap_{i=1}^n \ker x_i^*$ with $t_{n+1} \leq \|x_{n+1}\| \leq 1$. Let us see that $\{x_1, \ldots, x_{n+1}\}$ is a $(t_1^2 \cdot \ldots \cdot t_n^2)$−orthogonal set in E. For that, let $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{K}$. For each $i \in \{1, \ldots, n\}$, we have

$$|\lambda_i| = |x_i^*(\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1})| \leq \|x_i^*\| \, \|\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1}\|,$$

from which

$$\|\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1}\| \geq (t_1^2 \cdots t_i^2)\|\lambda_i x_i\| \geq (t_1^2 \cdots t_n^2)\|\lambda_i x_i\|,$$

and by [14, Lemma 3.2], we are done.

Now, let y_{n+1}^* be a linear functional defined on $[x_1, \ldots, x_{n+1}]$ by $y_{n+1}^*(\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1}) = \lambda_{n+1}$ $(\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{K})$. It is easily seen that $1 \leq \|y_{n+1}^*\| \leq \frac{1}{t_1^2 \cdots t_n^2 \cdot t_{n+1}}$. Applying normpolarity and [12, Theorem 4.4.5] again, we can extend y_{n+1}^* to $x_{n+1}^* \in E^*$ with $1 \leq \left\|x_{n+1}^*\right\| \leq \frac{1}{t_1^2 \cdots t_{n+1}^2}$.

Finally, proceeding similarly as above for $x_1, \ldots, x_{n+1}$, it can be proved that $x_1^*, \ldots, x_{n+1}^*$ are $(t_1^2 \cdot \ldots \cdot t_n^2)$−orthogonal in E^*. □

Lemma 2.2. *Suppose either E has a base or $\mathbb{K}$ is spherically complete and separable. Then E is isomorphic to $c_0(I)$ for some I.*

Proof. When E has a base the conclusion follows from [14, Corollary 3.8]. Now, let $\mathbb{K}$ be spherically complete and separable. By [15, Theorem 20.5], $\mathbb{K}$ is discretely valued and by [12, Theorems 2.1.9 and 2.5.4] E is isomorphic to $c_0(I)$ for some I. □

3. Countable tightness

The main results about countable tightness of E_w for the case when $\mathbb{K}$ may not be locally compact (see Problem 1.1) are provided by Theorems 3.7 and 3.11. To prove them we need a few preparing lemmas.

Lemma 3.1. *Let (V, d) be an ultrametric space. Then, for every $r > 0$ there exists a partition of V consisting of closed (open) balls with radius equal to r.*

Proof. We prove the result for closed balls. Similarly can be done for open balls. Let $r > 0$. The formula $x \sim y$ if $|x - y| \le r$, defines an equivalence relation on V. Its equivalence classes form a partition of V consisting of closed balls with radius equal to r. $\square$

Recall that if $\mathbb{K}$ is separable, then $\Bbbk$ and $|\mathbb{K}^*|$ are both countable, but the converse is not true (see [15, Exercise 19.B]). We also get the following.

Lemma 3.2. *Let $\mathbb{K}$ be non-separable. If the residue class field $\Bbbk$ and the value group $|\mathbb{K}^*|$ of $\mathbb{K}$ are both countable, then we have the following.*

(1) *$\mathbb{K}$ is densely valued.*

(2) *For every $r \in (0,1) \cap |\mathbb{K}^*|$ there exists a partition of $B_{\mathbb{K}} \backslash B_{\mathbb{K}}^-$, consisting of uncountable many closed balls with radius equal to r.*

Proof. (1): Assume that $\mathbb{K}$ is discretely valued; we will arrive at a contradiction. If $\rho \in \mathbb{K}$ is an uniformizing element, then $B_{\mathbb{K}}^- = \{\lambda \in \mathbb{K} : |\lambda| \le |\rho|\}$. Since, by assumption, $\Bbbk$ is countable, $B_{\mathbb{K}}$ has a countable partition formed by closed balls with radius equal to $|\rho|$. Hence, setting $n \in \mathbb{N}$, we imply that every closed ball contained in $B_{\mathbb{K}}$ with radius equal to $|\rho^n|$ has a countable partition consisting of closed balls with radius equal to $|\rho^{n+1}|$. Thus, we conclude that, for every $n \in \mathbb{N}$, $B_{\mathbb{K}}$ has a countable partition composed of closed balls with radius equal to $|\rho^n|$. This implies that $B_{\mathbb{K}}$, hence $\mathbb{K}$, is separable, a contradiction.

(2): Denote $V = B_{\mathbb{K}} \backslash B_{\mathbb{K}}^-$ $(= \{x \in \mathbb{K} : |x| = 1\})$. Since $|\mathbb{K}^*|$ is countable and $\mathbb{K}$ is non-separable, V is also non-separable. This, together with Lemma 3.1, implies that the set

$$\mathcal{R} := \left\{ r \in (0,1) \cap |\mathbb{K}^*| \;\middle|\; \begin{array}{l} V \text{ has an uncountable partition consisting} \\ \text{of closed balls with radius equal to } r \end{array} \right\}$$

is non-empty.

Let $p = \sup \mathcal{R}$. Assume $p < 1$; we will arrive at a contradiction. As we proved in (1), $\mathbb{K}$ is densely valued, thus, we can find $r_1 \in (p,1) \cap |\mathbb{K}^*|$. Also, there exists $r_2 \in (r_1 p, p) \cap \mathcal{R}$. Since $r_1 > p$, by Lemma 3.1 there exists a countable partition of V, $\{B_{r_1}(x_n) : n \in \mathbb{N}\}$. Furthermore, since $r_2 \in \mathcal{R}$ there is a partition $\{B_{r_2}(y_i) : i \in I\}$ of V for some uncountable I. Hence, there are $m \in \mathbb{N}$ and uncountable $J \subset I$ such that $B_{r_1}(x_m) = \bigcup_{i \in J} B_{r_2}(y_i)$.

Since $r_1 \in |\mathbb{K}^*|$ there is $\mu_1 \in \mathbb{K}$ with $|\mu_1| = r_1$. Define $T\colon B_{r_1}(x_m) \to B_{\mathbb{K}}$ by setting $T(z) := \frac{1}{\mu_1}(z - x_m)$. Then $B_{\mathbb{K}}$ has an uncountable partition $\{B_{r_2/r_1}(T(y_i))\}_{i \in J}$. Let $J_v = \{i \in J : B_{r_2/r_1}(T(y_i)) \subset V\}$. We show that J_v is uncountable. Assume for a contradiction that J_v is countable. Then, for every $\lambda \in B_{\mathbb{K}}$

$$\{B_{\frac{r_2}{r_1}|\lambda|}(\lambda T(y_i))\}_{i \in J_v}$$

is a countable partition of the set $V_\lambda = \{x \in B_{\mathbb{K}} : |x| = |\lambda|\}$. Fix $\lambda \in B_{\mathbb{K}}$ such that $|\lambda| > \frac{r_2}{r_1}$ and consider the family $\{B_{r_2/r_1}(\lambda T(y_i))\}_{i\in J_v}$. Then,

$$\bigcup_{i\in J_v} B_{r_2/r_1}(\lambda T(y_i)) = V_\lambda.$$

Indeed, if $x \in V_\lambda$ then there is $i \in J_v$ such that $x \in B_{\frac{r_2}{r_1}|\lambda|}(\lambda T(y_i))$; thus, $x \in B_{r_2/r_1}(\lambda T(y_i))$. On the other hand, assume that $x \in B_{r_2/r_1}(\lambda T(y_i))$ for some $i \in J_v$. Clearly $\lambda T(y_i) \in V_\lambda$ and $|x| = |x - \lambda T(y_i) + \lambda T(y_i)| = |\lambda T(y_i)|$ since $|x - \lambda T(y_i)| < \frac{r_2}{r_1} < |\lambda| = |\lambda T(y_i)|$. Thus, $x \in V_\lambda$.

Note that if $|\lambda T(y_i) - \lambda T(y_j)| \leq \frac{r_2}{r_1}$ then $B_{r_2/r_1}(\lambda T(y_i)) = B_{r_2/r_1}(\lambda T(y_j))$. Hence, from J_v we can select a subset J_v' such that $\{B_{r_2/r_1}(\lambda T(y_i))\}_{i\in J_v'}$ is a partition (obviously countable) of V_λ.

By assumption, $|\mathbb{K}^*|$ is countable; thus, we can find a countable subset $\{\lambda_1, \lambda_2, ...\} \subset B_{\mathbb{K}}$, $|\lambda_n| \neq |\lambda_m|$ if $n \neq m$, such that $|\mathbb{K}^*| \cap (\frac{r_2}{r_1}, 1] = \{|\lambda_1|, |\lambda_2|, ...\}$. Then,

$$B_{\mathbb{K}} = B_{r_2/r_1}(0) \cup \bigcup_{n=1}^{\infty} V_{\lambda_n}.$$

As we proved above, V_{λ_n} has a countable partition consisting of closed balls with radius equal to $\frac{r_2}{r_1}$ for every $n \in \mathbb{N}$. Since $V_{\lambda_n} \cap V_{\lambda_m} = \varnothing$ if $n \neq m$, we imply that $B_{\mathbb{K}}$ has a countable partition consisting of closed balls with radius equal to $\frac{r_2}{r_1}$, either. Thus, $\frac{r_2}{r_1} \in \mathcal{R}$. However, $\frac{r_2}{r_1} > \frac{r_1 p}{r_1} = p$, a contradiction. Therefore, $\sup \mathcal{R} = 1$, from which (2) follows easily. □

The next two lemmas, which will be used in the sequel, show that if $\mathbb{K}$ is not separable, c_0 contains subsets which do not have countable tightness with respect to the restricted weak topology and weak* topology, respectively.

Lemma 3.3. *Assume that $\mathbb{K}$ is not separable. Let $E = c_0$ and $S_0 = \{x \in c_{00} : ||x|| = 1\}$. Then $0 \in \overline{S_0}^w$, but there is no countable set $T \subset S_0$ such that $0 \in \overline{T}^w$.*

Proof. First, we prove that $0 \in \overline{S_0}^w$ (also true in the real case, see [5, Exercise 3.8]). Take a weak zero-neighborhood

$$W = \{x \in E : |x_i^*(x)| < \varepsilon, i = 1, ..., n\},$$

where $\varepsilon > 0$, $x_1^*, ..., x_n^* \in E^*$, $n \in \mathbb{N}$. Then, the map $f \colon E \to \mathbb{K}^n$, $x \mapsto (x_1^*(x), ..., x_n^*(x))$, is linear and, by infinite-dimensionality of c_{00}, there is a non-zero $x \in c_{00}$ in $\ker f = \bigcap_{i=1}^n \ker x_i^*$. Let $\lambda \in \mathbb{K}$ with $|\lambda| = ||x||$. Then, $\frac{x}{\lambda} \in S_0 \cap \ker f$ and so $\frac{x}{\lambda} \in S_0 \cap W$.

Now, assume that there is a countable set $T \subset S_0$ such that $0 \in \overline{T}^w$; we will construct a contradiction.

Write $T = \{u_1, u_2, ...\}$, where $u_k = (u_k^1, u_k^2, ...) \in S_0$, $k \in \mathbb{N}$. Clearly, for each $k \in \mathbb{N}$, the set $M_k := \{n \in \mathbb{N} : |u_k^n| = 1\}$ is non-empty. Let $M = M_1 \cup M_2 \cup ....$

First, suppose that M is finite, say $M = \{m_1, m_2, ..., m_p\}$. Let $W = \{x \in E : |e_{m_i}^*(x)| < 1,\ i = 1, ..., p\}$. Then, for every $k \in \mathbb{N}$ there is $n \in \{m_1, m_2, ..., m_p\}$ such that $|u_k^n| = 1$, i.e. $|e_n^*(u_k)| = 1$. Thus, $T \cap W = \varnothing$, a contradiction.

Next, assume that M is infinite. We will construct inductively a bounded sequence $v_1, v_2, \ldots$ in $\mathbb{K}$ such that

$$\left|\sum_{i=1}^{\infty} v_i u_k^i\right| > \frac{1}{2} \quad \text{for every } k \in \mathbb{N}. \tag{1}$$

Once this sequence is constructed the proof is finished. Indeed, the formula

$$v^*(x) := \sum_{i=1}^{\infty} v_i x_i \quad (x = (x_1, x_2, ...) \in E)$$

defines an element of E^*. Then, setting the weak zero-neighborhood $W := \{x \in E : |v^*(x)| \leq \frac{1}{2}\}$ and applying (1), we obtain that $T \cap W = \varnothing$; again a contradiction.

For the construction of $v_1, v_2, \ldots$ we distinguish three cases.

1. $\Bbbk$ is uncountable. Define, for each $n \in \mathbb{N}$,

$$L_n := \left\{k \in \mathbb{N} : |u_k^n| = 1 \text{ and } \left|u_k^i\right| < 1 \text{ if } i > n\right\}$$

Then, $\{L_1, L_2, \ldots\}$ is a partition of $\mathbb{N}$ and, since M is infinite, the set $\mathcal{L} := \{n \in \mathbb{N} : L_n \neq \varnothing\}$ is also infinite. To simplify notations we assume $\mathcal{L} = \mathbb{N}$ (otherwise, take $v_n = 0$ if $L_n = \varnothing$).

In this case we construct $(v_n)_n$ in $B_{\mathbb{K}} \backslash B_{\mathbb{K}}^-$ such that

$$\left|\sum_{i=1}^{n} v_i u_k^i\right| = 1 \quad \text{for each } n \in \mathbb{N} \text{ and each } k \in L_n. \tag{2}$$

Set $v_1 := 1$. For the step $(n-1) \to n$, assume $v_1, \ldots, v_{n-1}$ are already constructed. For each $k \in L_n$, we set

$$z_k := -\frac{1}{u_k^n} \sum_{i=1}^{n-1} v_i u_k^i.$$

Each z_k belongs to $B_{\mathbb{K}}$ and by assumption $\Bbbk$ is uncountable, so we can select $v_n \in B_{\mathbb{K}} \backslash B_{\mathbb{K}}^-$ such that $|v_n - z_k| = 1$ for all $k \in L_n$. Thus,

$$\left|\sum_{i=1}^{n} v_i u_k^i\right| = |u_k^n| \cdot \left|\frac{1}{u_k^n} \sum_{i=1}^{n-1} v_i u_k^i + v_n\right| = |v_n - z_k| = 1,$$

and so (2) holds.

Next we will get (1). Fix $k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ with $k \in L_n$. Then $|u_k^i| < 1$ if $i > n$, so that $\left|\sum_{i=n+1}^{\infty} v_i u_k^i\right| < 1$. Hence, by (2) we obtain

$$\left|\sum_{i=1}^{\infty} v_i u_k^i\right| = \left|\sum_{i=1}^{n} v_i u_k^i\right| = 1 > \frac{1}{2}.$$

2. $|\mathbb{K}^*|$ is uncountable. Choose $\lambda \in \mathbb{K}$ with $|\lambda| > 1$. Let $\Gamma_0 = [1, |\lambda|) \cap |\mathbb{K}^*|$. Observe that Γ_0 is uncountable; otherwise $|\mathbb{K}^*| = \bigcup_{m\in\mathbb{Z}} |\lambda|^m \Gamma_0$ would be countable, which contradicts the assumption.

In this case we construct $(v_n)_n$ in $\mathbb{K}$ with $|v_n| \in \Gamma_0$ and such that

$$\left|\sum_{i=1}^{n} v_i u_k^i\right| = \max_{i=1,\ldots,n} \left|v_i u_k^i\right| \quad \text{for each } n, k \in \mathbb{N}. \tag{3}$$

Set $v_1 := 1$. For the step $n-1 \to n$, assume $v_1, \ldots, v_{n-1}$ are already constructed. For the k with $u_k^n = 0$ it is obvious that (3) holds for each $v_n \in \mathbb{K}$. So, we also can assume that $u_k^n \neq 0$ for each $k \in \mathbb{N}$.

Let $z_k = \sum_{i=1}^{n-1} v_i u_k^i$. Since Γ_0 is uncountable we can find $v_n \in \mathbb{K}$ with $|v_n| \in \Gamma_0$ such that $|v_n u_k^n| \neq |z_k|$ for every $k \in \mathbb{N}$. Thus,

$$\left|\sum_{i=1}^{n} v_i u_k^i\right| = |z_k + v_n u_k^n| = \max\{|z_k|, |v_n u_k^n|\} = \max_{i=1,\ldots,n} \left|v_i u_k^i\right|,$$

and so (3) holds.

Next we will get (1). Fix $k \in \mathbb{N}$. Since $u_k \in S_0 \subset c_{00}$, there exists $n \in \mathbb{N}$ such that $u_k^i = 0$ if $i > n$. Hence, by (3) we obtain

$$\left|\sum_{i=1}^{\infty} v_i u_k^i\right| = \left|\sum_{i=1}^{n} v_i u_k^i\right| = \max_{i=1,\ldots,n} \left|v_i u_k^i\right| \geq 1 > \frac{1}{2}.$$

3. $\Bbbk$ and $|\mathbb{K}^*|$ are both countable. By Lemma 3.2, $\mathbb{K}$ is densely valued. Choose a sequence $(\lambda_n)_n$ in $\mathbb{K}$ such that $1 > |\lambda_1| > |\lambda_2| > \ldots > \frac{1}{2}$. For every $n \in \mathbb{N}$ define $r_n := |\lambda_n|$ and

$$J_n = \{k \in \mathbb{N} : |u_k^n| \geq r_n\}.$$

Then $J_1 \cup J_2 \cup \ldots = \mathbb{N}$. As in the first case we may assume that $\{n \in \mathbb{N} : J_n \neq \varnothing\} = \mathbb{N}$. We now construct $(v_n)_n$ in $B_\mathbb{K} \backslash B_\mathbb{K}^-$ such that

$$\left|\sum_{i=1}^{n} v_i u_k^i\right| > r_{n+1} \quad \text{for each } n \in \mathbb{N} \text{ and each } k \in J_n. \tag{4}$$

Set $v_1 := 1$. For the step $n-1 \to n$, assume $v_1, \ldots, v_{n-1}$ are already constructed. For each $k \in J_n$, we set

$$z_k := -\frac{1}{u_k^n} \sum_{i=1}^{n-1} v_i u_k^i.$$

By Lemma 3.2, there exists a partition of $B_{\mathbb{K}} \backslash B_{\mathbb{K}}^-$ consisting of uncountable many closed balls with radius equal to $\frac{r_{n+1}}{r_n}$. So, we can select $v_n \in \mathbb{K}$ with $|v_n| = 1$, such that $|v_n - z_k| > \frac{r_{n+1}}{r_n}$ for all $k \in J_n$. Thus,

$$\left| \sum_{i=1}^{n} v_i u_k^i \right| = |u_k^n| \cdot \left| \frac{1}{u_k^n} \sum_{i=1}^{n-1} v_i u_k^i + v_n \right| = |u_k^n| \cdot |v_n - z_k| > r_n \cdot \frac{r_{n+1}}{r_n} = r_{n+1}, \quad (5)$$

and so (4) holds.

Next we will get (1). Fix $k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ with $k \in J_n$. Let $n_0 = \max\{ n \in \mathbb{N} : k \in J_n\}$. Then, $|v_i u_k^i| = |u_k^i| < r_{n_0+1}$ if $i > n_0$. Hence, by (4) we obtain

$$\left| \sum_{i=1}^{\infty} v_i u_k^i \right| = \left| \sum_{i=1}^{n_0} v_i u_k^i \right| > r_{n_0+1} > \frac{1}{2}. \qquad \square$$

Since $\ell^\infty = c_0^*$, $\sigma(\ell^\infty, c_0)$ is the weak* topology on ℓ^∞. Considering c_0 as a subspace of ℓ^∞, by w_0 we will denote the restricted weak* topology $\sigma(\ell^\infty, c_0)\,|c_0$ on c_0.

The next lemma shows that c_0 contains unbounded sets which do not have countable tightness with respect to the topology w_0. Also, it is worth mentioning that, by [16, Proposition 6.1], all bounded subsets of c_0 are metrizable in the topology w_0; thus, they have countable tightness.

Lemma 3.4. *Let $\mathbb{K}$ be non-separable and let $E = c_0$. Then, there exists a set $G \subset c_{00}$ for which $0 \in \overline{G}^w$ and there is no countable set $T_0 \subset G$ such that $0 \in \overline{T_0}^{w_0}$.*

Proof. Let $S_0 = \{x \in c_{00} : ||x|| = 1\}$. Fix $\lambda \in \mathbb{K}$ with $|\lambda| > 1$. Define

$$G := \left\{ (y_1, y_2, ...) \in c_{00} : \left(\frac{y_1}{\lambda}, \frac{y_2}{\lambda^2}, \ldots \right) \in S_0 \right\}.$$

Then, $0 \in \overline{G}^w$. Indeed, let $V = \{x \in E : |x_i^*(x)| < \varepsilon,\, i = 1, \ldots, n\}$ be a weak zero-neighborhood in E, where $\varepsilon > 0$ and $x_1^*, ..., x_n^* \in \ell^\infty$ $(= E^*)$, $n \in \mathbb{N}$. Applying the argumentation contained at the beginning of the proof of Lemma 3.3, we imply that there exists $x = (x_1, x_2, ...) \in c_{00} \backslash \{0\}$ such that $x \in \bigcap_{i=1}^n \ker x_i^*$.

Choose $\alpha \in \mathbb{K}$ with $|\alpha| = \max_n |\lambda^{-n} x_n|$. Clearly $\alpha^{-1} x \in \bigcap_{i=1}^n \ker x_i^*$. Also, it is easily seen that $\alpha^{-1} x \in G$. Thus, $V \cap G \neq \varnothing$, and we are done.

Now, suppose that there is a countable subset $T_0 \subset G$ such that $0 \in \overline{T_0}^{w_0}$; we will arrive at a contradiction. The map $c_0 \to c_0$, $(x_1, x_2, ...) \mapsto (\lambda^{-1}x_1, \lambda^{-2}x_2, ...)$ is a continuous linear injection $(c_0, w_0) \to (c_0, \sigma(c_0, \ell^\infty))$ and $f(G) = S_0$. So, $f(T_0)$ is a countable subset of S_0. By Lemma 3.3, we can select W_0, a weak zero-neighborhood in c_0, such that $W_0 \cap f(T_0) = \varnothing$. Thus, $f^{-1}(W_0)$ is a w_0-neighborhood of zero in c_0 with $f^{-1}(W_0) \cap T_0 = \varnothing$, a contradiction. □

The next result shows that in most cases the countable tightness of E_w implies separability of $\mathbb{K}$.

Proposition 3.5. *Let E be polar. If E_w has countable tightness then $\mathbb{K}$ is separable.*

Proof. It suffices to prove the result when E is normpolar. Assume that $\mathbb{K}$ is not separable and let us see that E_w does not have countable tightness.

Let $t \in (0,1)$ and let $x_1, x_2, \ldots \in E$ and $x_1^*, x_2^*, \ldots \in E^*$ be the t-orthogonal sequences in E and E^*, respectively, considered in Lemma 2.1. Clearly, $x_1, x_2, \ldots$ is a t-orthogonal base of $D := \overline{[x_1, x_2, \ldots]}$. Then, $T\colon D \to c_0$, $x_n \mapsto e_n$ $(n \in \mathbb{N})$ is an isomorphism for which $\|T\| \leq \frac{1}{t^2}$ and $\|T^{-1}\| \leq 1$. The adjoint $T^*\colon c_0^* \to D^*$ is also an isomorphism with $T^*(e_n^*) = x_n^*|D$ for all $n \in \mathbb{N}$. By normpolarity of c_0 and D, $\|T^*\| = \|T\|$ and $\|(T^*)^{-1}\| = \|T^{-1}\|$. Thus, $x_1^*|D, x_2^*|D, \ldots$ is a t-orthogonal sequence in D^* (hence, a t-orthogonal base of its closed linear hull in D^*), with $1 \leq \|x_n^*|D\| \leq \|x_n^*\| \leq \frac{1}{t}$ for all $n \in \mathbb{N}$.

Let w_0 be the topology on c_0 considered in Lemma 3.4 and let τ be the topology on D inherited by w_0 through T^{-1}. From the above facts we get that

$$\tau \leq \sigma(E, E^*)|D \leq \sigma(D, D^*).$$

Since $\mathbb{K}$ is not separable, by Lemma 3.4 there exists $G \subset D$ with $0 \in \overline{G}^{\sigma(D,D^*)}$, so $0 \in \overline{G}^{\sigma(E,E^*)|D}$, and such that for each countable set $T_0 \subset G$, $0 \notin \overline{T}^{\tau}$, so $0 \notin \overline{T}^{\sigma(E,E^*)|D}$. Therefore, we conclude that $(D, \sigma(E, E^*)|D)$, hence E_w, does not have countable tightness. □

As a last step before giving Theorem 3.7, let us recall the following result.

Proposition 3.6. ([8, Proposition 2]) *If $\mathbb{K}$ is locally compact then, for every Banach space E over $\mathbb{K}$, E_w has countable tightness.*

Now, we are ready to prove the first main theorem of this section.

Theorem 3.7. *Suppose either E has a base or $\mathbb{K}$ is spherically complete. Then, E_w has countable tightness if and only if one of the following conditions is satisfied.*

(1) *$\mathbb{K}$ is locally compact.*

(2) *E is separable, i.e. E is of countable type and $\mathbb{K}$ is separable.*

Proof. If (1) holds then E_w has countable tightness by Proposition 3.6.

If (2) holds then every set $A \subset E$ is separable. Thus, there exists a countable set $T \subset A$ with $\overline{T} = \overline{A}$, from which we have that $\overline{T}^w = \overline{A}^w$. Hence, E_w has countable tightness.

Next, let $\mathbb{K}$ be non-locally compact and assume that E_w has countable tightness; let us get (2). Firstly, since E is polar, it follows from Proposition 3.5 that $\mathbb{K}$ is separable.

Secondly, assume that E is not of countable type; we will arrive at a contradiction. By Lemma 2.2 we can take $E = c_0(I)$ with I uncountable. Let $\mathcal{F}$ be the collection of all two-point subsets of I. For every $J \in \mathcal{F}$, $J = \{i_1, i_2\}$, define $x_J := e_{i_1} - e_{i_2}$.

Let $M = \{x_J : J \in \mathcal{F}\}$. Then, $0 \in \overline{M}^w$. Indeed, let

$$W = \{x \in E : |z_k^*(x)| < \varepsilon, k = 1, ..., m\}$$

be a weak zero-neighborhood in E, where $0 < \varepsilon < 1$, $z_1^*, \ldots, z_m^* \in B_{E^*}$, $m \in \mathbb{N}$. Since $E^* = \ell^\infty(I)$, for each $k \in \{1, ..., m\}$, we can write $z_k^* = (z_k^i)_{i \in I}$, $z_k^i \in B_{\mathbb{K}}$ $(i \in I)$.

By non-local compactness of $\mathbb{K}$ and [12, Lemma 3.7.52] there is a partition $U_1, U_2, \ldots$ of $B_{\mathbb{K}}$ consisting of non-empty clopen sets. Also, as $\mathbb{K}$ is separable then so is each U_n and, by [15, Theorem 19.3] and Lemma 3.1, we obtain that $B_{\mathbb{K}}$ has a partition $V_1, V_2, \ldots$ consisting of open balls with radius equal to ε.

Since I is uncountable, we can choose an uncountable subset I_0 of I such that for each $k \in \{1, ..., m\}$ there exists $j_k \in \mathbb{N}$ with $z_k^i \in V_{j_k}$ if $i \in I_0$. Take any $J \in \mathcal{F}$, say $J = \{i_1, i_2\}$ such that $J \subset I_0$. Then, for each $k \in \{1, ..., m\}$, we obtain $|z_k^*(x_J)| = \left|z_k^{i_1} - z_k^{i_2}\right| < \varepsilon$. Hence, $x_J \in W \cap M$.

By countable tightness of E_w, there exists a countable set $M_0 \subset M$, say $M_0 = \{x_1, x_2, ...\}$, $x_k = (x_k^i)_{i \in I}$, $k \in \mathbb{N}$, such that $0 \in \overline{M_0}^w$.

Let $J_k = \{i \in I : x_k^i \neq 0\}$, $k \in \mathbb{N}$, and $J_0 = \bigcup_k J_k$. Clearly J_0 is countable, say $J_0 = \{i_1, i_2, ...\}$. Select a sequence $(\lambda_j)_j$ in $B_{\mathbb{K}}$ such that $\lambda_j \in V_j$ $(j \in \mathbb{N})$ and define $z^* := (z^i)_{i \in I} \in \ell^\infty(I)$, setting $z^{i_k} := \lambda_k$, $k \in \mathbb{N}$, and $z^i := 0$ if $i \in I \backslash J_0$. Then, $W_0 = \{x \in E : |z^*(x)| < \varepsilon\}$ is a weak zero-neighborhood in E. Also, if $x = (x^i)_{i \in I} \in M_0$, then there are $i_{j_1}, i_{j_2} \in J_0$ such that $x = e_{i_{j_1}} - e_{i_{j_2}}$. As $V_{j_1} \cap V_{j_2} = \varnothing$,

$$|z^*(x)| = \left|z^{i_{j_1}} - z^{i_{j_2}}\right| = |\lambda_{j_1} - \lambda_{j_2}| \geq \varepsilon,$$

so $x \notin W_0$, and we derive that $W_0 \cap M_0 = \varnothing$, a contradiction. □

The infinite-dimensional Banach spaces $\ell^\infty(I)$ are some of the most popular examples of Banach spaces without a base when $\mathbb{K}$ is not discretely valued. Theorem 3.11, preceded by a few preliminary results, provides the answer to Problem 1.1 for these spaces.

Lemma 3.8. *Let $B_{\ell^\infty(I)}$ be equipped with $\sigma(\ell^\infty(I), c_0(I))|_{B_{\ell^\infty(I)}}$, the restricted topology. Then the map $B_{\ell^\infty(I)} \to B^I_{\mathbb{K}}$, $f \longmapsto (f(e_i))_{i\in I}$, is a bijective homeomorphism.*

Proof. Proceed as in $(\alpha) \Longrightarrow (\beta)$ of [16, Theorem 8.1]. □

The following gives the weak* version of Proposition 3.5.

Proposition 3.9. *Let E be polar. If $E^*_{w^*}$ has countable tightness then $\mathbb{K}$ is separable.*

Proof. It suffices to prove the result when E is normpolar. Assume that $\mathbb{K}$ is not separable and let us see that $E^*_{w^*}$ does not have countable tightness. Let $t \in (0,1)$, $x_1, x_2, \ldots \in E$, $x^*_1, x^*_2, \ldots \in E^*$ and D be as in the proof of Proposition 3.5. Looking at the second paragraph of that proof we see that $T\colon D \to c_0$, $x_n \mapsto e_n$, as well as its adjoint $T^*\colon \ell^\infty(= c^*_0) \to D^*$, are norm-isomorphisms with $T^*(e^*_n) = x^*_n|D$ for all $n \in \mathbb{N}$; and also that $x^*_1|D, x^*_2|D, \ldots$ and $x^*_1, x^*_2, \ldots$ are t-orthogonal bases of

$$\mathcal{D}_D := \overline{[x^*_1|D, x^*_2|D, \ldots]} \subset D^* \ \text{ and } \ \mathcal{D} := \overline{[x^*_1, x^*_2, \ldots]} \subset E^*,$$

respectively, with $1 \le \|x^*_n|D\| \le \|x^*_n\| \le \frac{1}{t}$ for all $n \in \mathbb{N}$. The last implies that the map $S\colon \mathcal{D}_D \to \mathcal{D}$, $x^*_n|D \mapsto x^*_n$, is again a norm-isomorphism.

Let $w^{\mathcal{D}}_0$ be the topology on $\mathcal{D}_D$ that is image by T^* of the topology w_0 on $\overline{[e^*_1, e^*_2, \ldots]}$ $(\cong c_0)$ considered in Lemma 3.4 and let $\tau^{\mathcal{D}}_0$ be the topology on $\mathcal{D}$ that is image by S of the topology $w^{\mathcal{D}}_0$ on $\mathcal{D}_D$. Then,

$$\tau^{\mathcal{D}}_0 \le \sigma(E^*, E)|\mathcal{D} \le \sigma(\mathcal{D}, \mathcal{D}^*).$$

Since $\mathbb{K}$ is not separable, by Lemma 3.4 there exists $G \subset \mathcal{D}$ with $0 \in \overline{G}^{\sigma(\mathcal{D},\mathcal{D}^*)}$, so $0 \in \overline{G}^{\sigma(E^*,E)|\mathcal{D}}$, and such that for each countable set $T_0 \subset G$, $0 \notin \overline{T}^{\tau^{\mathcal{D}}_0}$, so $0 \notin \overline{T}^{\sigma(E^*,E)|\mathcal{D}}$. Therefore, we conclude that $(\mathcal{D}, \sigma(E^*, E)|\mathcal{D})$, hence $E^*_{w^*}$, does not have countable tightness. □

Proposition 3.10. *Let $F = \ell^\infty(I)$ and let F_{w^*} denote the space $\ell^\infty(I)$ equipped with its weak* topology $\sigma(\ell^\infty(I), c_0(I))$. Then, F_{w^*} has countable tightness if and only if I is countable and $\mathbb{K}$ is separable.*

Proof. Assume that I is countable and $\mathbb{K}$ is separable. By Lemma 3.8, B_F, equipped with the restricted topology $\sigma(\ell^\infty(I), c_0(I))|_{B_F}$, is metrizable and separable. Now, let A be a non-empty subset of F and let $\lambda_1, \lambda_2, \ldots$ be a sequence in $\mathbb{K}$ with $\lim_n |\lambda_n| = \infty$. Since $A = \bigcup_n (\lambda_n B_F \cap A)$, we derive that A is separable in F_{w^*}. Hence, F_{w^*} has countable tightness.

Conversely, let F_{w^*} have countable tightness. Since F is polar, from Proposition 3.9 we deduce that $\mathbb{K}$ is separable.

Now, assume that I is uncountable; we will construct at a contradiction.

Let $y^* = (y^i)_{i\in I} \in \ell^\infty(I)$, where $y^i = 1$ for all $i \in I$. Let $S_0(I) = \{x^* \in c_{00}(I) : \|x^*\| = 1\} \subset \ell^\infty(I)$. First we prove that $y^* \in \overline{S_0(I)}^{w^*}$. For that, let V be a zero-neighborhood in F_{w*} of the form

$$V = \{\, z^* \in F : |z^*(x_j)| < \varepsilon,\, j = 1, \ldots, n\},$$

where $\varepsilon > 0$, $x_1, \ldots, x_n \in c_0(I)$ and $n \in \mathbb{N}$.

For each $j \in \{1, \ldots, n\}$ we have $y^*(x_j) = \sum_{i\in I} x_j^i$, where $x_j = (x_j^i)_{i\in I}$. So, there is a finite set $J_j \subset I$ such that

$$|y^*(x_j) - \sum_{i\in K} e_i^*(x_j)| = |y^*(x_j) - \sum_{i\in K} x_j^i| < \varepsilon \tag{6}$$

for every finite subset K of I that contains J_j. Thus, setting $K := J_1 \cup \ldots \cup J_n$, (6) holds for this finite set K and all $j \in \{1, \ldots, n\}$. Then, $y^* - \sum_{i\in K} e_i^* \in V$, so that $\sum_{i\in K} e_i^* \in S_0(I) \cap (y^* - V)$, and we are done.

By assumption, there exists a countable set $T \subset S_0(I)$ such that $y^* \in \overline{T}^{w^*}$; say $T = \{u_1^*, u_2^*, \ldots\}$ where $u_n^* = (u_n^i)_{i\in I}$, $n \in \mathbb{N}$. Then, we can find a countable set $J \subset I$ such that $u_n^i = 0$ for all $n \in \mathbb{N}$, $i \in I\backslash J$. Choosing $i \in I\backslash J$, we derive that

$$|(u_n^* - y^*)(e_i)| = \left|u_n^i - y^i\right| = 1$$

for every $n \in \mathbb{N}$. Therefore, setting $\delta < 1$, we obtain that

$$T \cap \{z^* \in F : |(z^* - y^*)(e_i)| < \delta\} = \varnothing,$$

a contradiction. □

Finally, we have the machinery at hand to prove the second main theorem of this section.

Theorem 3.11. *Let $E = \ell^\infty(I)$, where I is a small set. Then, E_w has countable tightness if and only if one of the following conditions is satisfied.*

(1) $\mathbb{K}$ *is locally compact.*

(2) *I is countable and $\mathbb{K}$ is separable and non-spherically complete.*

Proof. If (1) holds then E_w has countable tightness by Proposition 3.6. If (2) holds then E is reflexive, by [12, Theorem 7.4.3], and the conclusion follows from Proposition 3.10.

Next, assume that E_w has countable tightness and $\mathbb{K}$ is not locally compact. As E is not of countable type, $\mathbb{K}$ is non-spherically complete, by Theorem 3.7. Hence, E is reflexive and (2) follows from Proposition 3.10. □

4. Countable tightness and the Lindelöf property

The main result of this section, Theorem 4.3, extends [8, Theorem 7] and [10, Theorem 3] and completes the two main theorems of Section 3. This result characterizes when E_w has countable tightness or the Lindelöf property in terms of the weak*-dual of E and some separability properties.

For the basic facts on topological spaces having the Lindelöf property, some of which will be used in this section, we refer to [4, Section 3.8]. Also, recall the well-known fact that a metric space has the Lindelöf property if and only if it is separable, see [4, Corollary 4.1.16].

Proposition 4.1. *Let F and F_{w^*} be as in Proposition 3.10. Then, F_{w^*} has the Lindelöf property if and only if one of the following conditions is satisfied.*

(1) *$\mathbb{K}$ is locally compact.*

(2) *I is countable and $\mathbb{K}$ is separable.*

Proof. Through this proof we consider B_F equipped with the restricted weak* topology.

First assume that (1) holds, i.e. $B_{\mathbb{K}}$ is compact. From the Tychonoff Theorem (see [4, Theorem 3.2.4]) and Lemma 3.8 we obtain that B_F is also compact, so it has the Lindelöf property. Now, let $\lambda_1, \lambda_2, \dots$ be a sequence in $\mathbb{K}$ with $\lim_n |\lambda_n| = \infty$. Then $F = \bigcup_n \lambda_n B_F$, so F_{w^*} has the Lindelöf property.

Next, assume that (2) holds. Again by Lemma 3.8, B_F is metrizable and separable, so it has the Lindelöf property. Proceeding as above we conclude that F_{w^*} also has this property.

Finally, suppose that $\mathbb{K}$ is not locally compact and F_{w^*} has the Lindelöf property. Then, B_F, so $B_{\mathbb{K}}^I$ by Lemma 3.8, and thus $B_{\mathbb{K}}$, also have this property. So, $B_{\mathbb{K}}$, hence $\mathbb{K}$, is separable. To have $B_{\mathbb{K}}^I$ the Lindelöf property also implies that it is normal. From the Stone Theorem (see [4, Problem 5.5.6]) and non-compactness of $B_{\mathbb{K}}$, it follows that I is countable, and we get (2). □

Since every locally compact $\mathbb{K}$ is spherically complete and separable, as a direct consequence of Lemma 2.2 and Proposition 4.1, we derive the following.

Corollary 4.2. *If $\mathbb{K}$ is locally compact then, for every Banach space E over $\mathbb{K}$, $E^*_{w^*}$ has the Lindelöf property.*

Now we are ready to prove the main theorem of this section. Recall that a topological space X is called *hereditary separable* if every subset of X is separable.

Theorem 4.3. *Suppose either E has a base or $\mathbb{K}$ is spherically complete. Then the following statements are equivalent:*

(1) *E is separable, i.e. E is of countable type and $\mathbb{K}$ is separable.*

(2) *E_w is separable.*
(3) *E_w is hereditary separable.*
(4) *E_w has the Lindelöf property.*
(5) *$E^*_{w^*}$ is hereditary separable.*
(6) *$E^*_{w^*}$ has countable tightness.*
If, in addition, $\mathbb{K}$ is not locally compact then $(1)-(6)$ are equivalent to
(7) *E_w has countable tightness.*
(8) *$E^*_{w^*}$ has the Lindelöf property.*

Proof. (1) $\iff$ (2) $\iff$ (4) $\iff$ (5): Any of the properties involved in these equivalences implies that $\mathbb{K}$ is separable. Indeed, for (1), (4) and (5), just note that $\mathbb{K}$ is isomorphic to every one-dimensional subspace of E, E_w and $E^*_{w^*}$, respectively. For (2), separability of $\mathbb{K}$ follows from the fact that, as $E^* \neq \{0\}$, $\mathbb{K}$ is the image of E under a continuous map $E_w \to \mathbb{K}$. By Lemma 2.2, E is isomorphic to $c_0(I)$ for some I. Now, the equivalences follow [10, Theorem 3].

For (1) $\Longrightarrow$ (3) proceed as in the second paragraph of the proof of Theorem 3.7. Also, (3) $\Longrightarrow$ (2) and (5) $\Longrightarrow$ (6) are obvious.

(6) $\Longrightarrow$ (1): Since E is polar then, by Proposition 3.9, $\mathbb{K}$ is separable. Then (1) follows from Lemma 2.2 and Proposition 3.10.

Finally, if $\mathbb{K}$ is not locally compact, then (1) $\iff$ (7) follows from Theorem 3.7. Also, (8) implies that $\mathbb{K}$ is separable, so (1) $\iff$ (8) follows from Lemma 2.2 and Proposition 4.1. $\square$

Remark 4.4. (a) Item (5) in Theorem 4.3 cannot be replaced only by separability of $E^*_{w^*}$. Indeed, let $\mathbb{K}$ be separable and let $E = c_0(I)$, where I is an uncountable set with cardinality equals to $2^{\aleph_0}$. By Lemma 3.8, B_{E^*}, equipped with the restricted weak* topology, is homeomorphic to $B^I_{\mathbb{K}}$, hence B_{E^*} is separable, by [4, Theorem 2.3.15]. Thus, $E^*_{w^*}$ is also separable. However, $E^*_{w^*}$ does not have countable tightness by Theorem 4.3.

(b) Let $\mathbb{K}$ be locally compact. Then, the equivalence of $(1)-(6)$ and $(7), (8)$ in Theorem 4.3 fails. For an example, let $E = c_0(I)$, where I is uncountable. Then E is a non-separable space such that, by Proposition 3.6 and Corollary 4.2, E_w has countable tightness and $E^*_{w^*}$ has the Lindelöf property, respectively.

(c) If the assumptions in Theorem 4.3 are dropped then the conclusions of this result fail. Indeed, let $F = \ell^\infty$ over a non-spherically complete separable $\mathbb{K}$ (e.g. $\mathbb{K} = C_p$). ℓ^∞ is a non-separable space, it does not even have a base, so that (1) of Theorem 4.3 fails for ℓ^∞. However, as $F_w = E^*_{w^*}$ and $F^*_{w^*} = E_w$ with $E = c_0$, applying Theorem 4.3 for $E = c_0$ we deduce that $(2)-(8)$ of Theorem 4.3 hold for ℓ^∞.

(d) Also, for every non-spherically complete $\mathbb{K}$ there exists a non-archimedean Banach space E such that $E^* = \{0\}$ (e.g. $E = \ell^\infty/c_0$). Trivially, the conclusions of Theorem 4.3 and Proposition 3.9 fail for such spaces.

(e) Let X be a zero-dimensional and compact topological space. By [12, Theorem 2.5.22], the Banach space $C(X, \mathbb{K})$ (of all $\mathbb{K}$-valued continuous maps on X, equipped with the canonical maximum norm) has a base. Hence, by Theorem 4.3 and [12, Theorem 2.5.24], if $\mathbb{K}$ is not locally compact, $C(X, \mathbb{K})$ equipped with the weak topology has countable tightness if and only if X is ultrametrizable and $\mathbb{K}$ is separable. In particular, let $X = [0, \omega_1]$. Then, $C(X, \mathbb{K})$ with respect to the weak topology has countable tightness only if $\mathbb{K}$ is locally compact. However, $C_p(X, \mathbb{K})$, the locally convex space $C(X, \mathbb{K})$ endowed with the pointwise topology, has countable tightness (even Fréchet-Urysohn property) for any $\mathbb{K}$, see [9, Theorem 16].

Now, we are ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. If $\mathbb{K}$ is locally compact then, for every Banach space E over $\mathbb{K}$, E_w has countable tightness, by Proposition 3.6, and $E^*_{w^*}$ has the Lindelöf property, by Corollary 4.2.
Conversely, assume that $\mathbb{K}$ is not locally compact. For any uncountable set I, $E := c_0(I)$ is not of countable type. From Theorem 4.3 we obtain that E_w does not have countable tightness and $E^*_{w^*}$ does not have the Lindelöf property. □

Proof of Theorem 1.3. If $\mathbb{K}$ is spherically complete, the conclusion follows from Theorem 4.3. Assume now that $\mathbb{K}$ is non-spherically complete and separable. Let $E = \ell^\infty$. Then, E_w has the Lindelöf property and $E^*_{w^*}$ has countable tightness, but E is not separable, see Remark 4.4(c). □

We finish the paper with an open problem, which arises naturally after looking at Theorem 4.3 and the Remarks 4.4(c) and (d).

Problem 4.5. *Let $\mathbb{K}$ be non-spherically complete and separable. Let E be a polar Banach space over $\mathbb{K}$ without a base. Suppose that E_w (resp. $E^*_{w^*}$) has countable tightness or (and) the Lindelöf property. Does it imply that E_w (resp. $E^*_{w^*}$) is separable, even hereditary separable?*

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