

Some Remarks on the (Non-) Attainment of the Boundary Data for Variational Problems in the Space BV

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We discuss the standard relaxed version of a minimization problem for variational integrals of linear growth together with prescribed Dirichlet boundary data u_0 and give estimates for the size of the set $\{x \in \partial\Omega : u(x) \neq u_0(x)\}$ for BV-minimizers u which imply $\mathcal{H}^{n-1}(\{x \in \partial\Omega : u(x) < u_0(x)\}) = \mathcal{H}^{n-1}(\{x \in \partial\Omega : u(x) > u_0(x)\})$ in the case of minimal surfaces u not attaining the boundary values u_0 on a subset of $\partial\Omega$ with positive measure.

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1. Introduction

To explain our results we first look at the classical variational problem for minimal surfaces

$$J[u] := \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx \rightarrow \min \quad \text{in } u_0 + \mathring{W}_1^1(\Omega), \quad (1)$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and u_0 denotes a given function from the Sobolev space $W_1^1(\Omega)$ (see, e.g., [1] for a definition). As it is outlined for example in [7], the variational problem (1) admits a natural extension to the class $BV(\Omega)$ consisting of all functions $u \in L^1(\Omega)$ having finite total variation (compare [8] or [2] for details) i.e. one studies the problem

$$K[u] := \int_{\Omega} \sqrt{1 + |\nabla u|^2} + \int_{\partial\Omega} |u - u_0| \, d\mathcal{H}^{n-1} \rightarrow \min \quad \text{in } BV(\Omega). \quad (2)$$

Note that the expression $\int_{\Omega} \sqrt{1 + |\nabla u|^2}$ has to be understood as a convex function of a measure (see [5]) which equals the quantity $J[u]$ from formula (1) in case that $u \in W_1^1(\Omega)$. Moreover, we recall that in $\int_{\partial\Omega} |u - u_0| d\mathcal{H}^{n-1}$ the trace of $u \in \text{BV}(\Omega)$ on $\partial\Omega$ is considered.

We also note that due to famous examples (see, e.g., Santis example [10], compare also [8]), in general the uniqueness of solutions and the attainment of the boundary data cannot be expected. We wish to mention that a discussion of the (non-) attainment of the boundary data can also be found in [3], Theorem 1.4 and 1.5.

2. Results

Here we can state and prove by elementary reasoning

Theorem 2.1. *Let $u \in \text{BV}(\Omega)$ denote a solution of the variational problem (2). We define the sets (up to subsets of $\partial\Omega$ with vanishing $\mathcal{H}^{n-1}$ -measure)*

$$\begin{aligned}\partial_+\Omega &:= \{x \in \partial\Omega : u(x) > u_0(x)\} , \\ \partial_-\Omega &:= \{x \in \partial\Omega : u(x) < u_0(x)\} , \\ \partial_0\Omega &:= \{x \in \partial\Omega : u(x) = u_0(x)\} .\end{aligned}$$

Then

$$|\mathcal{H}^{n-1}(\partial_+\Omega) - \mathcal{H}^{n-1}(\partial_-\Omega)| \leq \mathcal{H}^{n-1}(\partial_0\Omega). \quad (3)$$

Formula (3) admits the following nice interpretation:

Corollary 2.2. *If the generalized minimal surface u ignores the boundary data u_0 completely in the sense that $\mathcal{H}^{n-1}(\partial\Omega_0) = 0$, then – at least in measure – the sets $\partial_{\pm}\Omega$ are equally distributed.*

The proof of Theorem 2.1 will be obtained as a by-product of the following slightly more general investigations whose framework is collected in Appendix A.1 of [4], where the interested reader will also find a list of further references.

Let $N \geq 1$ and consider a strictly convex integrand $F: \mathbb{R}^{nN} \rightarrow [0, \infty)$ with (w.l.o.g.) $F(0) = 0$ being of linear growth in the sense that

$$a|Z| - b \leq F(Z) \leq A|Z| + B \quad \text{for all } Z \in \mathbb{R}^{nN} \quad (4)$$

holds with constants $a, A > 0, b, B \geq 0$. Letting

$$F_{\infty}(Z) := \lim_{t \rightarrow \infty} \frac{1}{t} F(tZ)$$

we consider for $u \in \text{BV}(\Omega, \mathbb{R}^N)$ the functional

$$\begin{aligned} L[u] := & \int_{\Omega} F(\nabla^a u) \, dx + \int_{\Omega} F_{\infty} \left(\frac{\nabla^s u}{|\nabla^s u|} \right) \, d|\nabla^s u| + \\ & + \int_{\partial\Omega} F_{\infty} ((u_0 - u) \otimes \nu) \, d\mathcal{H}^{n-1}, \end{aligned} \quad (5)$$

where now u_0 is a given function from the space $W_1^1(\Omega, \mathbb{R}^N)$, $\otimes$ is the tensor product of vectors and ν stands for the exterior normal of $\partial\Omega$. Moreover, we use the symbols $\nabla^a u$ ($\nabla^s u$) to denote the regular (singular) part of the tensor-valued measure ∇u w.r.t. Lebesgue's measure.

We have

Proposition 2.3. *Let F satisfy (4) and define L according to (5).*

1. *The minimization problem*

$$L[w] \rightarrow \min \quad \text{in } \text{BV}(\Omega, \mathbb{R}^N) \quad (6)$$

admits at least one solution.

2. *It holds*

$$\inf_{u_0 + \overset{\circ}{W}_1^1(\Omega, \mathbb{R}^N)} \int_{\Omega} F(\nabla w) \, dx = \inf_{\text{BV}(\Omega, \mathbb{R}^N)} L[w]$$

3. *The solutions of problem (6) are in one-to-one correspondence with the L^1 -limits of minimizing sequences for problem (6) restricted to the class of functions from $u_0 + \overset{\circ}{W}_1^1(\Omega, \mathbb{R}^N)$.*

With a given function $f: [0, \infty) \rightarrow [0, \infty)$, $f(0) = 0$, being strictly increasing and strictly convex we additionally impose the structure condition

$$F(Z) = f(|Z|), \quad Z \in \mathbb{R}^{nN}, \quad (7)$$

so that

$$F_{\infty}(Z) = f_{\infty}|Z|, \quad f_{\infty} := \lim_{t \rightarrow \infty} \frac{1}{t} f(t).$$

Then it holds

Theorem 2.4. *Let F satisfy (4) and (7) and consider a solution $u \in \text{BV}(\Omega, \mathbb{R}^N)$ of problem (6) with L defined in formula (5).*

Then we have the estimate (as above $\partial_0\Omega := \{x \in \partial\Omega : u(x) = u_0(x)\}$)

$$\left| \int_{\partial\Omega \cap [u \neq u_0]} \frac{u - u_0}{|u - u_0|} \, d\mathcal{H}^{n-1} \right| \leq \mathcal{H}^{n-1}(\partial_0\Omega). \quad (8)$$

Clearly (3) is a consequence of (8): if we are in the scalar case, then it holds

$$\int_{\partial\Omega \cap [u \neq u_0]} \frac{u - u_0}{|u - u_0|} d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial_+\Omega) - \mathcal{H}^{n-1}(\partial_-\Omega)$$

and we obtain the following

Corollary 2.5. *Let the assumptions of Theorem 2.4 hold together with $N = 1$. Then inequality (3) is true.*

Proof of Theorem 2.4. From $L[u] \leq L[u + \xi]$ for any $\xi \in \mathbb{R}^N$ we find that

$$\xi \mapsto \int_{\partial\Omega} F_\infty((u_0 - (u + \xi)) \otimes \nu) d\mathcal{H}^{n-1}$$

attains its minimum at $\xi = 0$, which by the structure condition (7) means that we just have to discuss the convex function

$$g : \mathbb{R}^N \rightarrow [0, \infty), \quad g(\xi) := \int_{\partial\Omega} |u - u_0 + \xi| d\mathcal{H}^{n-1}.$$

Let $\partial g(0)$ denote the subgradient of g at 0, i.e. the closed and convex subset ($\neq \emptyset$) of $\mathbb{R}^N$ consisting of those vectors $\eta \in \mathbb{R}^N$ such that

$$g(y) \geq g(0) + \eta \cdot y \quad \text{for all } y \in \mathbb{R}^N.$$

Since $g(y) \geq g(0)$ for any $y \in \mathbb{R}^N$, it holds

$$0 \in \partial g(0). \tag{9}$$

Let us write

$$\begin{aligned} g(\xi) &:= g_0(\xi) + g_1(\xi), \\ g_0(\xi) &:= |\xi| \mathcal{H}^{n-1}(\partial_0\Omega), \\ g_1(\xi) &:= \int_{\partial\Omega \cap [u \neq u_0]} |u + \xi - u_0| d\mathcal{H}^{n-1}. \end{aligned}$$

Now we observe referring to [6], [9] or [11]: let $h : \mathbb{R}^N \rightarrow [0, \infty)$ denote a convex function for which $\eta := \nabla h(0)$ exists. For a number $\alpha \geq 0$ consider the function $G := \alpha|\cdot| + h$.

Then it holds

$$\partial G(0) = \overline{B_\alpha(\eta)}.$$

This well known observation implies

$$\partial g(0) = \overline{B_\alpha(\eta)}, \tag{10}$$

where we have set

$$\alpha := \mathcal{H}^{n-1}(\partial_0\Omega), \quad \eta := \nabla g_1(0) = \int_{\partial\Omega \cap [u \neq u_0]} \frac{u - u_0}{|u - u_0|} d\mathcal{H}^{n-1}.$$

Note that the existence of the gradient of g_1 at the origin together with the formula follows by elementary calculations.

Combining (9) and (10) we find that $|\eta| \leq \alpha$ holds, hence (8) is established. $\square$

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