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Convex Optimization of Second Order Discrete and Differential Inclusions with Inequality Constraints

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The paper deals with a Bolza problem of optimal control theory given by second order convex differential inclusions (*DFIs*) with second order state variable inequality constraints (*SVICs*). The main problem is to derive sufficient conditions of optimality for second order *DFIs* with *SVICs*. According to the proposed discretization method, problems with discrete-approximation inclusions and inequalities are investigated. Necessary and sufficient conditions of optimality including distinctive "transversality" condition are proved in the form of Euler-Lagrange inclusions. Construction of Euler-Lagrange type adjoint inclusions is based on the presence of equivalence relations of locally adjoint mappings (*LAMs*). Moreover, in the application of these results, we consider the second order "linear" differential inclusions.

Keywords: Euler-Lagrange inclusions, adjoint mappings, set-valued, approximation, second order, transversality.

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1. Introduction

Optimal control problems with ordinary and partial *DFIs* represent one of the most intensively developing areas in mathematical theory of optimal processes (see [2, 3],[5]–[9],[11]–[26],[22] and their references). Convex optimization problems with *DFIs* or inequalities appear naturally in several areas of applied mathematics, such as in mathematical economics in problems of resource allocation and in the study of planning procedures, in mechanics in the study of elastoplastic systems, and in differential games (constraint optimization has been used in many application areas, too numerous to cite here).

It should be noted that the qualitative properties of the second-order *DFIs* and *SVICs* as a whole have been investigated extensively in the literature (see [1, 4, 10, 14, 21, 27] and references therein). In the paper [1], it was shown that the second-order set contingent to a closed convex set coincides with the

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contingent cone to this set. Furthermore, sufficient conditions were given to prove the existence of a viable solution for second-order *DFIs*.

The paper [4] gives necessary and sufficient conditions ensuring the existence of solutions to the second-order *DFIs* with state constraints. Moreover, second-order interior tangent sets are introduced and studied to obtain such conditions. In the work [10], three-point boundary value problems (BVP) for second-order DFI were considered in a separable Banach space. The existence of solutions for the set-valued function which is unbounded and satisfies a pseudo-Lipschitz property was investigated. Then, a Lipschitz case was derived and the associated relaxed problem was studied.

In the paper [14] the concept of a weak solution for second-order *DFIs* of the form $u''(t) \in Au(t) + f(t)$ was introduced, where A is a maximal monotone operator in a Hilbert space H. The existence and uniqueness of weak solutions to two-point boundary value problems associated with such kind of equations were proved. Furthermore, existence of (strong and weak) solutions to the above equation which are bounded on the positive half axis was proved under the optimal condition $f(t) \in L^1(0, \infty; H)$. The treatment regarding weak solutions is similar to the corresponding theory related to the first-order *DFIs* of the form $f(t) \in Au(t) + u'(t)$ which has already been well developed.

In spite of the presence of the above mentioned qualitative studies, optimal control of problems with the second-order *DFIs* has not been studied in the literature; up to my best knowledge, there are only two papers devoted to such studies [18, 21].

The paper [18] studies approximation of the Bolza problem of optimal control theory with a fixed time interval given by convex and non-convex second order *DFIs*. The main goal is to derive necessary and sufficient conditions of optimality for a Cauchy problem of second-order discrete inclusions (*DSIs*).

The paper [21] is devoted to second-order polyhedral optimization described by ordinary discrete *DFIs*. The stated second-order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and necessary and sufficient conditions of optimality are derived in terms of the polyhedral Euler-Lagrange inclusions. Derivation of the sufficient conditions for the second-order polyhedral *DFIs* is based on the discrete-approximation method.

In [27] the necessary conditions of optimality are derived for optimal control problems with path-wise state constraints, in which the dynamic constraint is modelled as a DFI. Existence of homoclinic solutions to a nonlinear second-order *ODE* is presented. To this end, a method based on differential inequalities is used. Note that the existing papers devoted to optimal control problems with constraints deal with the “usual” state constraints of “zero-order” (see [27] and references therein) i.e. when the inequality constraints do not involve the derivative of a feasible trajectory.

In our present paper, we discuss a special kind of optimization problem with second-order discrete *DFIs* in which the constraints are defined by second-order *DSIs* and *DFIs* with *SVICs*. In fact, the difficulty in the problems with higher-order *DFIs* (or differential inequalities) is to construct the Euler-Lagrange type higher-order adjoint inclusions and suitable transversality conditions.

The stated problems and obtained optimality conditions in our paper are new. The novelty of our approach is to use the discretization method to establish necessary and sufficient conditions of optimality for the second-order *DSIs* and *DFIs* with *SVICs*. The paper can be formally divided into three parts.

In the first part of the paper, we formulate an optimal control problem in which the system dynamics are described by so-called second-order discrete *SVICs* and second-order discrete inclusions. Second part of the paper is devoted to the construction of discrete-approximation problem for second-order *DFIs* with differential $SVIC$. On the basis of the results for second-order *DSIs* with discrete *SVICs* we formulate sufficient conditions of optimality for *DFIs* with differential *SVICs*. In the third part of the paper optimization of second-order *DFIs* with *SVICs* involving first- and second-order derivatives of searched functions is formulated.

In Section 2 the necessary notions and results such as LAM properties, cones generated by a nonempty subset of finite-dimensional Euclidean spaces, Hamiltonian function, argmaximum sets, equivalence theorems for functions and set-valued mappings are summarized for the reader's convenience, adopted from the monograph of Mahmudov [20] and references [16, 21]. Then the Cauchy problems for second-order *DSIs* and *DFIs* with second-order $SVICs$ are formulated.

In Section 3 optimality problem for the posed second-order *DSIs* with $SVICs$ is reduced to the problem with finite number of geometric constraints. By constructions of convex analysis and under the non-degeneracy condition, necessary and sufficient conditions of optimality for these problems are proved.

In Section 4 we associate the second-order discrete-approximation problem with the problem for second-order DFI and $SVIC$ by using first- and second-order difference operators. Here we use the equivalence Theorems 3.1 [16] and 4.1 [18] of $LAMs$ connecting the main results of discrete *(PD)* and discrete-approximate (PDA) problems. It is obvious that this method, which is certainly of independent interest from the qualitative point of view, can play an important role also in numerical procedures.

In Section 5 we derive sufficient conditions of optimality for second-order *DFIs* with *SVICs*. Formulation of these conditions is based on formal limiting procedure in the optimality conditions for second-order *DSIs* with *SVICs*. Thus, employing LAM and the discrete-approximations method, we derive sufficient optimality conditions for second-order adjoint *DFIs* in Euler-Lagrange form. In fact, by using the functional analysis approach in the convex problems, necessity of these conditions for optimality can be justified. However, proof of necessary conditions is a separate subject of discussion and it is omitted from the present paper.

2. Needed facts and problem statement

Auxiliary notions can be found in detail in [20]. As usually, we denote by $\mathbb{R}^n$ n-dimensional Euclidean space. Then $\langle x, y\rangle$ is an inner product of elements $x, y \in \mathbb{R}^n$, (x, y) is a pair of x, y. Besides if $P(\mathbb{R}^n)$ is a family of subsets of $\mathbb{R}^n$ then $F\colon \mathbb{R}^{2n} \to P(\mathbb{R}^n)$ is a set-valued (multivalued) mapping from $\mathbb{R}^{2n}$ into $P(\mathbb{R}^n)$. A set-valued mapping $F\colon \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is *convex* if its graph $gphF = \{(x, y, z) : z \in F(x, y)\}$ is a convex subset of $\mathbb{R}^{3n}$. A set-valued mapping F is *convex-closed* if its $gphF$ is a convex-closed set in $\mathbb{R}^{3n}$. It is *convex-valued* if $F(x, y)$ is a convex set for each $x, y \in domF = \{(x, y) : F(x, y) \neq \emptyset\}$.

Let us introduce the *Hamiltonian function* and *argmaximum set* for a set-valued mapping F as

$$H_F(x, y, z^*) = \sup_z \Big\{\langle z, z^*\rangle : z \in F(x, y)\Big\}\,, z^* \in \mathbb{R}^n,$$
$$F(x, y; z^*) = \Big\{z \in F(x, y) : \langle z, z^*\rangle = H_F(x, y, z^*)\Big\},$$

respectively. For convex F we set $H_F(x, y, z^*) = -\infty$ if $F(x, y) = \emptyset$. Clearly, for the convex set-valued mapping F the Hamiltonian function $H_F(\cdot, \cdot, z^*)$ is concave.

Let $intA$ be the *interior* of the set $A \subset \mathbb{R}^{3n}$ and riA be the *relative interior* of the set A, i.e. the set of interior points of A with respect to its affine hull $Aff A$.

Definition 2.1. The convex cone $K_A(\omega_0)$, $\omega = (x, y, z)$ is called the *cone of tangent directions* at a point $\omega_0 \in A$ to the set A if from $\overline{\omega} = (\overline{x}, \overline{y}, \overline{z}) \in K_A(\omega_0)$ it follows that $\overline{\omega}$ is a tangent vector to the set A at point $\omega_0 \in A$, i.e., there exists such function $\gamma(\varepsilon) \in \mathbb{R}^{3n}$ that $\omega_0 + \varepsilon\overline{\omega} + \gamma(\varepsilon) \in A$ for sufficiently small $\varepsilon > 0$ and $\varepsilon^{-1}\gamma(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

For a convex mapping F at a point $(x, y, z) \in gphF$ setting $\gamma(\varepsilon) \equiv 0$ we have

$$\begin{aligned} K_{gphF}(x, y, z) &= cone\Big[gphF - (x, y, z)\Big] \\ &= \Big\{(\overline{x}, \overline{y}, \overline{z}) : \overline{x} = \varepsilon(x_1 - x),\ \overline{y} = \varepsilon(y_1 - y),\ \overline{z} = \varepsilon(z_1 - z)\Big\}, \end{aligned}$$

for all $(x_1, y_1, z_1) \in gph\, F$.

Definition 2.2. For a convex mapping F a multifunction defined by

$$F^*(z^*; (x, y, z)) := \Big\{(x^*, y^*) : (x^*, y^*, -z^*) \in K^*_{gphF}(x, y, z)\Big\}$$

is called a *locally adjoint multifunction* (LAM) to F at a point $(x,y,z)\in gphF$, where $K^*_{gphF}(x,y,z)$ is the dual to a cone of tangent vectors $K_{gphF}(x,y,z)$.

Definition 2.3. For an arbitrary nonempty subset $P\subset\mathbb{R}^{3n}$ the set

$$coneP=\Big\{(\overline{x},\overline{y},\overline{z}):\ \overline{x}=\alpha x,\ \overline{y}=\alpha y,\overline{z}=\alpha z,\ (x,y,z)\in P,\alpha>0\Big\}$$

is called the *cone generated by* P.

The next lemma follows from Theorem 2.1 [20].

Lemma 2.4. *Let $F\colon\mathbb{R}^{2n}\to P(\mathbb{R}^n)$ be convex multivalued mapping. Then*

$$F^*\big(z^*;(x,y,z)\big)=\begin{cases}\partial_{(x,y)}H_F(x,y,z^*), & z\in F(x,y;z^*),\\ \emptyset\ , & z\notin F(x,y;z^*)\end{cases}$$

where $\partial_{(x,y)}H_F(x,y,z^*):=-\partial_{(x,y)}[-H_F(x,y,z^*)]$ is the classical sub-differential of $H_F(\cdot,\cdot,z^*)$ at a point (x,y).

Let us now recall the Theorem 3.1 [16] crucial in what follows.

Theorem 2.5. *Let Φ be proper convex function defined by relation $\Phi(x,y,z)\equiv\varphi[x,(y-x)/h,(z-2y+x)/h^2]$. Then the following inclusions are equivalent:*

(1) $\Big(\overline{x}^*\ ,\ \overline{y}^*\ ,\ \overline{z}^*\Big)\in\partial_\omega\Phi(x,y,z)\ ,\ (x,y,z)\in dom\Phi,$

(2) $\Big(\overline{x}^*+\overline{y}^*+\overline{z}^*\ ,\ h\overline{y}^*+2h\overline{z}^*\ ,\ h^2\overline{z}^*\Big)\in\partial_\omega\varphi\Big[x,\dfrac{y-x}{h},\dfrac{z-2y+x}{h^2}\Big].$

Besides, let us formulate the important Theorem 4.1 from [18] for the future development.

Theorem 2.6. *If Q is a convex set-valued mapping defined by $Q(x,y)=2y-x+h^2F[x,(y-x)/h]$ then the following inclusions are equivalent:*

(1) $(x^*,y^*)\in Q^*(z^*;(x,y,z))\ ,\ v\in Q(x,y;z^*),$

(2) $\Big(\dfrac{x^*+y^*-z^*}{h^2},\ \dfrac{y^*-2z^*}{h}\Big)\in F^*\Big(z^*;\big(x,\dfrac{y-x}{h},\dfrac{x-2y+z}{h^2}\big)\Big),$
$\dfrac{x-2y+z}{h^2}\in F\Big(x,\dfrac{y-x}{h};z^*\Big),\ z^*\in\mathbb{R}^n.$

Optimization of second-order *DSIs* with *SVICs* is the object of investigation in Section 3, which deals with the following second-order discrete model labelled as *(PD)*:

$$\text{minimize} \sum_{t=2}^{T-1} g(x_t, x_{t+1}, t), \tag{1}$$

$$(PD) \qquad x_{t+2} \in F(x_t, x_{t+1}),\ \varphi\big(x_t, x_{t+1}, x_{t+2}\big) \le 0 \tag{2}$$

$$\text{for } \ t = 0, \dots, T-2,\ x_0 = \overline{\theta}_0\ ,\ x_1 = \overline{\theta}_1, \tag{3}$$

where $x_t \in \mathbb{R}^n, g(\cdot, t)$ are real-valued functions, $g(\cdot, t)\colon \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, F is a set-valued mapping: $F\colon \mathbb{R}^{2n} \to P(\mathbb{R}^n)$, $\varphi(x, y, z)$, $(x, y, z) \in \mathbb{R}^{3n}$ is convex continuous function, T is a fixed natural number, and $\overline{\theta}_0, \overline{\theta}_1$ are fixed vectors.

Condition (3) is a discrete analogous of Cauchy initial conditions for second-order *DFIs* with *SVICs.* A sequence $\{x_t\}_{t=0}^T = \{x_t : t = 0, 1, \dots, T\}$ is called a feasible trajectory for the stated problem (1)–(3).

The problem (1)–(3) is said to be *convex* if F is a convex set-valued mapping and $g(\cdot, t), \varphi$ are proper convex functions, that is, do not assume the value $-\infty$ and are not identically equal to $+\infty$. We note that a convex function is continuous on the relative interior of its domain; it may have discontinuities only on its relative boundary (see, for example, Theorem 1.18 [20])

Definition 2.7. We say that the convex problem *(PD)* satisfies the *non-degeneracy condition* if for some feasible solution $\{x_t\}_{t=0}^T$ we have either (i) or (ii):

(i) $\big(x_t, x_{t+1}, x_{t+2}\big) \in int[gph\, F \cap D]$ for $t = 0, \dots, T-2$ (with the possible exception of one fixed t) and $g(\cdot, t)$ are continuous at x_t, $D = \{(x, y, z) : \varphi(x, y, z) \le 0\}$.

(ii) $\big(x_t, x_{t+1}, x_{t+2}\big) \in ri[gph\, F \cap D]$, $x_t \in ri[domg(\cdot, t)]$ (note that if $rigphF \cap riD \neq \emptyset$ then $ri[gphF \cap D] = ri\ gphF \cap ri\ D$).

As a consequence of the non-degeneracy condition we know that if $\{\widetilde{x}_t\}_{t_0}^T$ is the optimal trajectory in the problem *(PD)*, then the cones of tangent directions $K_{gphF \cap D}\big(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}\big)$ are not separable and, consequently, the requirement of Theorem 3.2 [20] on necessary conditions for an extremum in convex programming problem is satisfied.

In Section 4 we are mainly interested in the convex problem for second-order *DFIs* with *SVICs* labelled as *PC*:

$$\text{minimize } J[x(\cdot)] = \int_0^1 g(x(t), t)dt + q(x(1)), \tag{4}$$

$$(PC) \qquad x''(t) \in F(x(t), x'(t)),\ \varphi(x(t), x'(t), x''(t)) \le 0 \tag{5}$$

$$\text{for a.e. } t \in [0, 1],\ x(0) = \theta_0\ ,\ x'(0) = \theta_1. \tag{6}$$

Here $F\colon \mathbb{R}^{2n} \to P(\mathbb{R}^n)$ is convex set-valued mapping, $g(\cdot, t)$, q and φ are continuous and convex functions $g(\cdot, t)\colon \mathbb{R}^n \to \mathbb{R}$ and θ_0, θ_1 are fixed vectors. The problem is to find an arc trajectory) $\widetilde{x}(t)$ of the Cauchy problem *PC* satisfying the second-order *DFIs* and *SVICs* almost everywhere (a.e.) on $[0, 1]$ and the initial conditions (6) that minimize the Bolza functional $J[x(\cdot)]$. To this end, our further strategy is as follows: first, we derive necessary and sufficient conditions of optimality for discrete-approximation problem (PDA) and then, we establish, by passing to the limit (formal) as $h \to 0$ (h is the discrete step), sufficient conditions of optimality to the original optimal control problem PC described by *DFIs* and *SVICs*. Here, a feasible trajectory $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[0,1]$ together with the first-order derivative whose second derivative belongs to the space L_1^n, that is $x''(\cdot) \in L_1^n$. Notice that such class of functions is a Banach space, endowed with the different equivalent norms.

3. Necessary and sufficient conditions for DSIs

In this section we begin with the convex problem *(PD)* for *DSIs* with *SVICs*. By reducing problem *(PD)* to an equivalent form of convex programming and using subdifferential calculus we derive an optimality condition for it. With this aim, let us introduce a vector $w = (x_0, x_1, \dots, x_T) \in \mathbb{R}^{n(T+1)}$ and define in the space $\mathbb{R}^{n(T+1)}$ the following convex sets

$$\begin{aligned} N_t &= \Big\{ w = (x_0, \dots, x_T) : \big(x_t, x_{t+1}, x_{t+2}\big) \in gphF \Big\},\ t = 0, 1, \dots, T-2, \\ \Omega_t &= \Big\{ w = (x_0, \dots, x_T) : \varphi\big(x_t, x_{t+1}, x_{t+2}\big) \le 0 \Big\},\ t = 0, 1, \dots, T-2, \qquad (7) \\ Q_0 &= \Big\{ w = (x_0, \dots, x_T) : x_0 = \overline{\theta}_0 \Big\},\ Q_1 = \Big\{ w = (x_0, \dots, x_T) : x_1 = \overline{\theta}_1 \Big\}. \end{aligned}$$

It is easy to see that the problem *(PD)* can be reduced to the following problem with geometric constraints

$$\begin{aligned} &\text{minimize} \ \ g_+(w) \ \ \text{subject to} \\ &w \in Q = Q_0 \cap Q_1 \cap \Big(\bigcap_{t=0}^{T-2} \Omega_t \Big) \cap \Big(\bigcap_{t=0}^{T-2} N_t \Big), \quad g_+(w) = \sum_{t=2}^{T-1} g(x_t, t) \end{aligned} \qquad (8)$$

First let us calculate the dual cone $K^*_{\Omega_t}(w)$ to cone of tangent directions $K_{\Omega_t}(w)$.

Lemma 3.1. *Let cone* $\partial_\omega \varphi\big(x_t, x_{t+1}, x_{t+2}\big)$, $\omega = (x, y, z)$ *be the cone generated by the subdifferential set* $\partial_\omega \varphi\big(x_t, x_{t+1}, x_{t+2}\big)$. *Then under the non-degeneracy condition we have*

$$K^*_{\Omega_t}(w) = \left\{ w^* = (x_0^*, \dots, x_T^*) \ \middle| \ \begin{array}{l} (x_t^*, x_{t+1}^*, x_{t+2}^*) \in \big[- cone\, \partial_\omega \varphi\big(x_t, x_{t+1}, x_{t+2}\big)\big], \\ x_k^* = 0,\ k \ne t,\ t+1,\ t+2 \end{array} \right\}.$$

Proof. One finds easily that

$$K_{\Omega_t}(w) = \big\{ \overline{w} : (\overline{x}_t, \overline{x}_{t+1}, \overline{x}_{t+2}) \in K_D(x_t, x_{t+1}, x_{t+2}) \big\},$$

where

$$K_D(x_t, x_{t+1}, x_{t+2}) \equiv cone\big[D - (x_t, x_{t+1}, x_{t+2})\big],\ D = \big\{(x, y, z) : \varphi(x, y, z) \leq 0\big\}.$$

Now we recall that by Theorem 1.34 [20]

$$K_D^*\big(x_t, x_{t+1}, x_{t+2}\big) = \begin{cases} \{0\} \in \mathbb{R}^{3n}, & \text{if } \varphi\big(x_t, x_{t+1}, x_{t+2}\big) < 0, \\ -cone\, \partial_\omega \varphi\big(x_t, x_{t+1}, x_{t+2}\big), & \text{if } \varphi\big(x_t, x_{t+1}, x_{t+2}\big) = 0 \end{cases}$$

Then the proof of the lemma follows immediately from the arbitrariness of the components $x_k,\ k \neq t, t+1, t+2$ of the vectors $\overline{w}$. □

In the sense of the terminology of first-order *DSIs* [20] we are ready to give the necessary and sufficient conditions for problem *(PD)*.

Theorem 3.2. *Let F be convex set-valued mapping and $g(\cdot, \cdot, t)$ and φ be proper convex functions such that the non-degeneracy condition is satisfied. Then for optimality of the trajectory $\{\widetilde{x}_t\}_{t=0}^T$ in the problem (PD), it is necessary and sufficient that there exist numbers $\alpha_t \geq 0,\ t = 0, 1, \ldots, T-2$ and vectors $x_t^*, x_T^*, \psi_t^*, \psi_T^*, \beta_t^*,\ t = 0, \ldots, T-1$ all simultaneously not equal to zero satisfying the Euler-Lagrange second-order adjoint DSIs (i) and the transversality condition (ii):*

(i) $$\big(x_t^* - \beta_t^* + \psi_t^*\ ,\ \beta_{t+1}^*\ ,\ -\psi_{t+2}^*\big) \in F^*\big(x_{t+2}^*; \big(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}\big)\big) \times \{0\} -$$
$$-\partial_x g(\widetilde{x}_t, t) \times \{0\} \times \{0\} - \alpha_t \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}),\ t = 0, 1, \ldots, T-2,$$
$$\alpha_t \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) = 0,\ \alpha_t \geq 0,\ \beta_0^* = 0,\ \partial_x g(\widetilde{x}_0, 0) = \partial_x g(\widetilde{x}_1, 1) = 0,$$

(ii) $$-x_{T-1}^* + \beta_{T-1}^* - \psi_{T-1}^* \in \partial_x g(\widetilde{x}_{T-1}, T-1)\ ,\ x_T^* + \psi_T^* = 0.$$

Proof. If $\{\widetilde{x}_t\}_{t=0}^T$ is an optimal trajectory we claim that $w = (\widetilde{x}_0, \ldots, \widetilde{x}_N)$ is a solution of the convex mathematical programming problem (8). On the basis of results concerning convex mathematical programming [12, 20, 23] we can prove necessary optimality conditions for problem (8) with geometric constraints. Thus, by these results there exist non-zero vectors $w^*(t) \in K_{N_t}^*(\widetilde{w})$, $\overline{w}^*(t) \in K_{\Omega_t}^*(\widetilde{w})$, $t = 0, 1, \ldots, T-2$, $w_0^* \in K_{Q_0}^*(\widetilde{w})$, $w_1^* \in K_{Q_1}^*(\widetilde{w})$, $\hat{w}^* \in \partial_w g_+(\widetilde{w})$, and the number $\lambda \in \{0, 1\}$ such that

$$\lambda \hat{w}^* = \sum_{t=0}^{T-2} w^*(t) + \sum_{t=0}^{T-2} \overline{w}^*(t) + w_0^* + w_1^*,\ \hat{w}^* \in \partial_w g_+(\widetilde{w}). \tag{9}$$

We claim that the vector $\hat{w}^* \in \partial_w g_+(\widetilde{w})$ has a form $\hat{w}^* = (\hat{x}_0^*, \hat{x}_1^*, \ldots, \hat{x}_T^*)$, where $\hat{x}_t^* \in \partial_x g(\widetilde{x}_t, t)$. According to relation $w^*(t) \in K_{N_t}^*(\widetilde{w})$ and Lemma 3.1 we have

$$w^*(t) = \big(0, \ldots, 0, x_t^*(t), x_{t+1}^*(t), x_{t+2}^*(t), 0, \ldots, 0\big),$$
$$\big(x_t^*(t), x_{t+1}^*(t), x_{t+2}^*(t)\big) \in K_{gphF}^*\big(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}\big)\ ,\ t = 2, \ldots, T-2, \tag{10}$$

$$w_0^* = \big(x_{0i}^*, 0, \dots, 0\big),\ w_1^* = \big(0, x_{1i}^*, 0, \dots, 0\big),$$
$$\overline{w}^*(t) = \big(0, \dots, 0, \overline{x}_t^*(t), \overline{x}_{t+1}^*(t), \overline{x}_{t+2}^*(t), 0, \dots, 0\big),$$
$$\big(\overline{x}_t^*(t), \overline{x}_{t+1}^*(t), \overline{x}_{t+2}^*(t)\big) \in \big[-\mathit{cone}\,\partial_\omega \varphi(x_t, x_{t+1}, x_{t+2})\big]\ , t = 2, \dots, T-2, \quad (11)$$

where x_{0i}^*, x_{1i}^* are arbitrary vectors.

Now we recall that by Theorem 3.3 [20] under the non-degeneracy condition, the equality (9) holds with parameter $\lambda = 1$ and so from the component-wise representation of (9) we derive for $t = 2, \dots, T-2$

$$\hat{x}_t^* = x_t^*(t) + x_t^*(t-1) + x_t^*(t-2) + \overline{x}_t^*(t) + \overline{x}_t^*(t-1) + \overline{x}_t^*(t-2)\ ,$$
$$x_{1i}^* + x_1^*(1) + x_1^*(0) + \overline{x}_1^*(1) + \overline{x}_1^*(0) = 0, \quad (12)$$
$$x_{0i}^* + x_0^*(0) + \overline{x}_0^*(0) = 0.$$

Furthermore, from formula (10) and Definition 2.2 of LAM we deduce that

$$\big(x_t^*(t), x_{t+1}^*(t)\big) \in F^*(-x_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\ ,\ t = 2, \dots, T-2. \quad (13)$$

On the other hand, by Definition 2.7 of generated cone we have

$$\mathit{cone}\,\partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) = \Big\{\big(\alpha_t \overline{x}_t^*, \alpha_t \overline{x}_{t+1}^*, \alpha_t \overline{x}_{t+2}^*\big) :$$
$$(\overline{x}_t^*, \overline{x}_{t+1}^*, \overline{x}_{t+2}^*) \in \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}),\ \alpha_t \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) = 0\ ,\ \alpha_t \geq 0\Big\}$$

whence (see (11))

$$\Big(-\overline{x}_t^*(t), -\overline{x}_{t+1}^*(t), -\overline{x}_{t+2}^*(t)\Big) \in \alpha_t \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}),$$
$$\alpha_t \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) = 0,\ \alpha_t \geq 0,\ t = 2, \dots, T-2. \quad (14)$$

Combining relation (13) with the first relation of (14) we deduce that

$$\big(x_t^*(t) + \overline{x}_t^*(t), x_{t+1}^*(t) + \overline{x}_{t+1}^*(t), \overline{x}_{t+2}^*(t)\big) \in$$
$$\in F^*(-x_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \times \{0\} - \alpha_t \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}), \quad (15)$$

for $t = 2, \dots, T-2$. Now we provide the preliminary notations that are relevant for further development; denoting $x_{t+1}^*(t) + \overline{x}_{t+1}^*(t) \equiv \beta_{t+1}^*$, $-x_{t+2}^*(t) \equiv x_{t+2}^*$, $-\overline{x}_{t+2}^*(t) \equiv \psi_{t+2}^*$, $t = 1, \dots, T-2$ we derive from (12) and (15) that

$$\Big(\hat{x}_t^* + x_t^* - \beta_t^* + \psi_t^*\ ,\ \beta_{t+1}^*\ ,\ -\psi_{t+2}^*\big) \in$$
$$\in F^*\Big(x_{t+2}^*; (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\Big) \times \{0\} - \alpha_t \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}),$$

for $t = 2, \dots, T-2$, or in more detail

$$\begin{aligned} &\big(x_t^* - \beta_t^* + \psi_t^* \ , \ \beta_{t+1}^* \ , \ -\psi_{t+2}^*\big) \in F^*\Big(x_{t+2}^*; (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\Big) \times \{0\} - \\ &- \partial_x g(\widetilde{x}_t, t) \times \{0\} \times \{0\} - \alpha_t \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}), \ t = 2, \dots, T-2. \end{aligned} \tag{16}$$

On the other hand, it should be noted that setting $g(x_0, 0) = g(x_1, 1) = 0$, $\beta_0^* \equiv 0$, $x_{0i}^* = -x_0^* - \psi_0^*$, and $x_{1i}^* = -x_1^* - \psi_1^*$ in the second and third relations of (12), respectively we can generalize the formula (16) to the case $t = 0, 1$. In addition, for $t = T-1$ we have

$$\hat{x}_{T-1}^* = x_{T-1}^*(T-2) + x_{T-1}^*(T-3) + \overline{x}_{T-1}^*(T-2) + \overline{x}_{T-1}^*(T-3)$$

or by using our notations we can write

$$-x_{T-1}^* + \beta_{T-1}^* - \psi_{T-1}^* = \hat{x}_{T-1}^*. \tag{17}$$

Finally, since $\partial_x g(\widetilde{x}_T, T) \equiv 0$ it follows that

$$x_T^* + \psi_T^* = 0. \tag{18}$$

Thus, taking into account the formulas (16), (17), (18) we deduce that the indicated conditions of the theorem are necessary and sufficient for optimality of the trajectory $\{\widetilde{x}_t\}_{t=0}^T$. The proof of the theorem is completed. $\square$

Note that since x_{0i}^*, x_{1i}^* are arbitrary the first and second equations of formulas (12) are always valid. Consequently, without loss of generality the conditions $\beta_0^* = 0, \partial g(\widetilde{x}_0, 0) = \partial g(\widetilde{x}_1, 1) = 0$ can be neglected and only the adjoint second-order *DSIs* should be preserved $t = 2, \dots, T-2$.

Remark 3.3. The result of Theorem 3.2 can be extended to the problem *(PD)* in the nonconvex case; assume that in the problem *(PD)* a mapping F is such that the cones of tangent directions $K_{gphF\cap D}(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})$ are local tents [20], where $\widetilde{x}_t$ are the points of the optimal trajectory $\{\widetilde{x}_t\}_{t=0}^T$. Moreover, assume that the functions $g(\cdot, t)$ admit a continuous convex upper approximations [20] at the points $\widetilde{x}_t$. In this case these conditions ensure the conditions of Theorem 3.24 [20] for the problem (8) in non-convex case. Then for optimality of the trajectory $\{\widetilde{x}_t\}_{t=0}^T$ it is necessary that there exist numbers $\lambda \in \{0, 1\}$, $\alpha_t \geq 0$, $t = 0, 1, \dots, T-2$ and triple of vectors $\{x_t^*\}, \{\psi_t^*\}, \{\beta_t^*\}$, simultaneously not all equal to zero, satisfying the conditions of Theorem 3.2 in the “non-convex” case with terms $\lambda \partial_x g(\widetilde{x}_t, t)$.

Remark 3.4. We give an alternative way to derive the second-order adjoint *DSI* of Theorem 3.2. We have seen that in the proof of Theorem 3.2 the relation (15) plays exclusive role; however, this basic inclusion can be obtained by using the results of calculus of *LAM* [20]. Indeed, using an auxiliary set-valued mapping the original optimization problem *(PD)* in term of set-valued mappings can be transformed to another equivalent form. Let us introduce a new set-valued convex mapping G which is an intersection of the set-valued convex mappings F and $\wedge$ defined as follows

$$G(x,y) = F(x,y) \cap \wedge(x,y), \; gphG \equiv gphF \cap gph\wedge, \; \wedge(x,y) \equiv \{z : \varphi(x,y,z) \leq 0\}.$$

Then it is obvious that in terms of convex set-valued mapping G the problem (7), (8) has a new formulation:

$$\text{minimize } g_+(w), \quad \text{subject to} \tag{19}$$

$$w \in Q_0 \cap Q_1 \cap \Big(\bigcap_{t=0}^{T-2} P_t \Big) = Q_0 \cap Q_1 \cap \Big(\bigcap_{t=0}^{T-2} M_t \Big) \cap \Big(\bigcap_{t=0}^{T-2} N_t \Big),$$

where

$$P_t = \Big\{ w = (x_0, \ldots, x_T) : (x_t, x_{t+1}, x_{t+2}) \in gphG \Big\},$$

$$M_t = \Big\{ w = (x_0, \ldots, x_T) : (x_t, x_{t+1}, x_{t+2}) \in gph \wedge \Big\}, \; t = 0, 1, \ldots, T-2.$$

Here it is taken into account that $P_t = M_t \cap N_t$, where N_t is defined as in (7).

In a similar way in terms of LAM for the set-valued mapping G for a problem of the form (19) we can write

$$\Big(\xi_t^*(t) \; , \; \xi_{t+1}^*(t) \Big) \in G^* \Big(-\xi_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big), \quad t = 2, \ldots, T-2.$$

Suppose now that either $gphF \cap intgph\wedge \neq \emptyset$ or $ri \; gphF \cap ri \; gph\wedge \neq \emptyset$ is satisfied. Then from calculus of LAM of Mahmudov (Theorem 2.8 [20]) it is known that

$$G^* \Big(-\xi_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big) =$$

$$= F^* \Big(-x_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big) + \wedge^* \Big(-\overline{x}_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big),$$

where $\xi_{t+2}^*(t) = x_{t+2}^*(t) + \overline{x}_{t+2}^*(t)$. Thus we have two inclusions:

$$\Big(x_t^*(t), x_{t+1}^*(t) \Big) \in F^* \Big(-x_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big),$$

$$\Big(\overline{x}_t^*(t), \overline{x}_{t+1}^*(t) \Big) \in \wedge^* \Big(-\overline{x}_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big).$$

Summing these inclusions we conclude that $\xi_t^*(t) = x_t^*(t) + \overline{x}_t^*(t)$ and $\xi_{t+1}^*(t) = x_{t+1}^*(t) + \overline{x}_{t+1}^*(t) = \beta_{t+1}^*$. Consequently, for $t = 2, \ldots, T-2$,

$$\Big(x_t^*(t) + \overline{x}_t^*(t), \; \beta_{t+1}^* \Big) \in$$

$$\in F^* \Big(-x_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big) + \wedge^* \Big(-\overline{x}_{t+2}^*(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \Big).$$

Furthermore, similar to the proof of Theorem 2.5 using the LAM $\wedge^*$ for the set-valued mapping $\wedge$

$$\wedge^*\Big(-\overline{x}^*_{t+2}(t);(\widetilde{x}_t,\widetilde{x}_{t+1},\widetilde{x}_{t+2})\Big)=\Big\{\big(\overline{x}^*_t(t),\overline{x}^*_{t+1}(t)\big):\ \big(\overline{x}^*_t(t),\overline{x}^*_{t+1}(t),\overline{x}^*_{t+2}(t)\big)\in$$

$$\in-\alpha_t\partial_\omega\varphi(\widetilde{x}_t,\widetilde{x}_{t+1},\widetilde{x}_{t+2}),\ \alpha_t\varphi(\widetilde{x}_t,\widetilde{x}_{t+1},\widetilde{x}_{t+2})=0,\ \alpha_t\geq 0,\ t=2,\dots,T-2\Big\}$$

we can easily derive (16). □

Let us consider the problem PC in the case where $\varphi\big(x(t),x'(t),x''(t)\big)\equiv\varphi_0\big(x(t),x'(t)\big)$, that is φ does not depend on the third argument x'' and label this problem by (POC):

$$\begin{aligned}&\text{minimize}\ \ J[x(\cdot)]=\int_0^1 g(x(t),t)dt+q(x(1))\\ (POC)\qquad &x''(t)\in F(x(t),x'(t)),\ \varphi_0(x(t),x'(t))\leq 0,\\ &\text{for a.e. } t\in[0,1],\ x(0)=\theta_0,\ x'(0)=\theta_1.\end{aligned}$$

According to problem POC we can formulate the following problem (POD) with second-order $DSIs$ and $SVICs$

$$\begin{aligned}&\text{minimize}\ \sum_{t=2}^{T-1}g(x_t,t),\\ (POD)\qquad &x_{t+2}\in F(x_t,x_{t+1}),\ \varphi(x_t,x_{t+1})\leq 0\\ &\text{for } t=0,\dots,T-2,\ x_0=\overline{\theta}_0,\ x_1=\overline{\theta}_1.\end{aligned}$$

Theorem 3.5. *Let F be a convex set-valued mapping and $g(\cdot,t)$ and φ_0 be proper convex functions such that the non-degeneracy condition is satisfied. Then for the $\{\widetilde{x}_t\}_{t=0}^T$ to be an optimal trajectory of the problem (POD), it is necessary and sufficient that there exist numbers $\alpha_t\geq 0$, $t=0,\dots,T-2$ and vectors $x^*_t,x^*_T,\beta^*_t,t=0,\dots,T-1$ simultaneously not all equal to zero satisfying the second-order adjoint Euler-Lagrange DSIs and the transversality condition:*

(i) $\Big(x^*_t-\beta^*_t\ ,\ \beta^*_{t+1}\Big)\in F^*\big(x^*_{t+2};(\widetilde{x}_t,\widetilde{x}_{t+1},\widetilde{x}_{t+2})\big)-\partial_x g(\widetilde{x}_t,t)\times\{0\}-$

$-\alpha_t\partial_{(x,y)}\varphi_0(\widetilde{x}_t,\widetilde{x}_{t+1}),\ t=0,1,\dots,T-2,\ \alpha_t\varphi_0(\widetilde{x}_t,\widetilde{x}_{t+1})=0,$

$\alpha_t\geq 0,\ \beta^*_0=0,\ \partial_x g(\widetilde{x}_0,0)=\partial_x g(\widetilde{x}_1,1)=0,$

(ii) $-x^*_{T-1}+\beta^*_{T-1}\in\partial_x g(\widetilde{x}_{T-1},T-1)\ ,\ x^*_T=0.$

Proof. In this case, it can be shown easily that the relation (15) consists of the following relations

$$\Big(x^*_t(t),x^*_{t+1}(t)\Big)\in F^*\Big(-x^*_{t+2}(t);(\widetilde{x}_t,\widetilde{x}_{t+1},\widetilde{x}_{t+2})\Big),$$

$$\Big(-\overline{x}^*_t(t), -\overline{x}^*_{t+1}(t)\Big) \in \alpha_t \partial_{(x,y)}\varphi_0(\widetilde{x}_t, \widetilde{x}_{t+1}), \tag{20}$$
$$\alpha_t\varphi_0(\widetilde{x}_t, \widetilde{x}_{t+1}) = 0,\ \alpha_t \geq 0,\ t = 2, \dots, T-2.$$

Summing the first and second inclusion of (20) we obtain by analogy that

$$\begin{aligned} &\big(x^*_t(t) + \overline{x}^*_t(t), x^*_{t+1}(t) + \overline{x}^*_{t+1}(t)\big) \in \\ &\in F^*\big(-x^*_{t+2}(t); (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\big) - \alpha_t\partial_{(x,y)}\varphi_0(\widetilde{x}_t, \widetilde{x}_{t+1}), \end{aligned} \tag{21}$$

for $t = 2, \dots, T-2$. Taking into account in (21) the relations

$$\hat{x}^*_t = x^*_t(t) + x^*_t(t-1) + x^*_t(t-2) + \overline{x}^*_t(t) + \overline{x}^*_t(t-1),\ t = 2, \dots, T-2$$

and notations $x^*_{t+1}(t) + \overline{x}^*_{t+1}(t) \equiv \beta^*_{t+1},\ -x^*_{t+2}(t) \equiv x^*_{t+2},\ t = 1, \dots, T-2$ we have

$$\Big(\hat{x}^*_t + x^*_t - \beta^*_t, \beta^*_{t+1}\Big) \in F^*\Big(x^*_{t+2}; (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\Big) \times \{0\} - \alpha_t\partial_{(x,y)}\varphi_0(\widetilde{x}_t, \widetilde{x}_{t+1}),$$

whence for $t = 2, \dots, T-2$

$$\Big(x^*_t - \beta^*_t, \beta^*_{t+1}\Big) \in F^*\Big(x^*_{t+2}; (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\Big) - \partial_x g(\widetilde{x}_t, t) \times \{0\} - \alpha_t\partial_{(x,y)}\varphi_0(\widetilde{x}_t, \widetilde{x}_{t+1}),$$

and the proof of the theorem is complete. □

Remark 3.6. Consider the following problem without *SVICs*

$$(PF) \qquad \text{minimize} \sum_{t=2}^{T-1} g(x_t, t) \ \text{ subject to}$$
$$x_{t+2} \in F(x_t, x_{t+1}) \text{ for } t = 0, \dots, T-2,\ x_0 = \overline{\theta}_0\ ,\ x_1 = \overline{\theta}_1.$$

Since in this case without loss of generality we can take $\varphi(x_t, x_{t+1}, x_{t+2}) \equiv 0$, it follows that $\partial_{(x,y,z)}\varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) \equiv \{0, 0, 0\}$ and so (see (13)), $\overline{x}^*_t(t) = \overline{x}^*_{t+1}(t) = \overline{x}^*_{t+2}(t) = 0$ that is $x^*_{t+1}(t) + \overline{x}^*_{t+1}(t) = x^*_{t+1}(t) = \beta^*_{t+1}$ and $\psi^*_{t+2} = 0$. As a result, by the third relation in (12) $\hat{x}^*_t = x^*_t(t) + x^*_t(t-1) + x^*_t(t-2)$ and the necessary conditions of Theorem 3.2 are simplified as follows (see [18]) for $t = 0, 1, \dots, T-2$:

$$\big(x^*_t - \beta^*_t, \beta^*_{t+1}\big) \in F^*\big(x^*_{t+2}; (\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2})\big) - \partial_x g(\widetilde{x}_t, t) \times \{0\},\ \beta^*_0 = 0,$$
$$\partial_x g(\widetilde{x}_0, 0) = \partial_x g(\widetilde{x}_1, 1) = 0,\ \beta^*_{T-1} - x^*_{T-1} \in \partial_x g(\widetilde{x}_{T-1}, T-1),\ x^*_T = 0.$$

Remark 3.7. Now consider the following problem with *SVICs*

$$(P_\varphi) \qquad \text{minimize} \sum_{t=2}^{T-1} g(x_t, t) \ \text{ subject to}$$
$$\varphi(x_t, x_{t+1}, x_{t+2}) \leq 0 \text{ for } t = 0, \dots, T-2,\ x_0 = \overline{\theta}_0,\ x_1 = \overline{\theta}_1.$$

In the present case $domF(\cdot,\cdot) \equiv \mathbb{R}^{2n}$ and $F(\cdot,\cdot) \equiv \mathbb{R}^n$. Therefore, it follows that $K_{gphF}(x,y,z) = \mathbb{R}^{3n}$ and $K^*_{gphF}(x,y,z) \equiv \{0,0,0\}$. Hence

$$F^*(z^*;(x,y,z)) = \begin{cases} (0,0) \in \mathbb{R}^{2n}, & \text{if } z^* = 0, \\ \emptyset, & \text{if } z^* \neq 0. \end{cases}$$

Then, obviously, $F^* \equiv \{(0,0)\}$ so that $x^* = y^* = z^* = 0 \in \mathbb{R}^n$. Then $x^*_{t+1}(t) \equiv 0$ and so (see(12)) $\overline{x}^*_{t+1}(t) = \beta^*_{t+1}(t)$, $-x^*_{t+2}(t) \equiv x^*_{t+2} = 0$. Finally, the necessary and sufficient conditions for problem (P_φ) consist of the following relations

$$\begin{gathered} \left(\psi^*_t - \beta^*_t\ ,\ \beta^*_{t+1}, -\psi^*_{t+2}\right) \in -\partial_x g(\widetilde{x}_t,t) \times \{0\} \times \{0\} - \alpha_t \partial_\omega \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}), \\ \alpha_t \varphi(\widetilde{x}_t, \widetilde{x}_{t+1}, \widetilde{x}_{t+2}) = 0,\ \alpha_t \geq 0,\ t = 0,1,\ldots,T-2, \\ \psi^*_{T-1} - \beta^*_{T-1} \in \partial_x g(\widetilde{x}_{T-1}, T-1)\ ,\ \psi^*_T = 0. \end{gathered}$$

4. Approximation of the continuous problem; necessary and sufficient optimality conditions for discrete-approximation problems

In what follows in this section we describe an idea of construction of the discrete-approximation problem for PC that will be important later. At first, we introduce the first- and second-order difference operators

$$\Delta x(t) = \frac{1}{h}\big[x(t+h) - x(t)\big], \Delta^2 x(t) = \frac{1}{h^2}\big[\Delta x(t+h) - \Delta x(t)\big],$$

for $t = 0, h, \ldots, 1-h$, where h is a step on the t-axis and $x(t) \equiv x_h(t)$ is a grid function on a uniform grid on $[0,1]$.

We define the following discrete-approximation problem associated with the problem PC

$$\text{minimize}\ \ J_h[x(\cdot)] = \sum_{t=2h,\ldots,1-2h} hg(x(t),t) + q(x(1-h)), \tag{22}$$

$$\begin{aligned} &\Delta^2 x(t) \in F(x(t), \Delta x(t)),\ \ \varphi(x(t), \Delta x(t), \Delta^2 x(t)) \leq 0 \qquad (23) \\ &\text{for}\ \ t = 0, h, 2h, \ldots, 1-h,\ x(0) = \theta_0,\ \Delta x(0) = \theta_1. \end{aligned}$$

We apply the results of Theorem 3.2 to the problem (22), (23). To this end, we reduce the problem (22), (23) to a problem of the form (PDA) consisting of the following relations:

$$(PDA) \qquad \begin{aligned} &\text{minimize}\ \ J_h[x(\cdot)] = \sum_{t=2h,\ldots,1-2h} hg(x(t),t) + hq(x(1-h)), \\ &x(t+2h) \in Q(x(t), x(t+h)),\ \Phi\big(x(t), x(t+h), x(t+2h)\big) \leq 0, \\ &x(0) = \theta_0,\ x(h) = \theta_0 + h\theta_1, \end{aligned}$$

where

$$\Phi\big(x(t), \Delta x(t+h), x(t+2h)\big) \equiv \varphi\big(x(t), \Delta x(t), \Delta^2 x(t)\big),$$

$$Q(x,y) = 2y - x + h^2 F\big(x, \frac{y-x}{h}\big).$$

By Theorem 3.2 we have already seen that for optimality of the trajectory $\{\widetilde{x}(t)\} : \{\widetilde{x}(t) : t = 0, h, \dots, 1\}$ in problem *(PDA)* it is necessary and sufficient that there exist numbers $\alpha(t) \geq 0$, $t = 0, \dots, 1-2h$ and a triple of vectors $\{x^*(t), \psi^*(t), \beta^*(t)\}$ not all equal to zero, such that for $t = 0, h, \dots, 1-2h$,

$$\begin{aligned}
&\big[x^*(t) - \beta^*(t) + \psi^*(t)\ ,\ \beta^*(t+h),\ -\psi^*(t+2h)\big] \in \\
&\quad \in Q^*\big(x^*(t+2h); (\widetilde{x}(t), \widetilde{x}(t+h), \widetilde{x}(t+2h))\big) \times \{0\} - \\
&\quad - h\partial_x g(\widetilde{x}(t), t) \times \{0\} \times \{0\} - \alpha(t)\partial_\omega \Phi\big(\widetilde{x}(t), \widetilde{x}(t+h), \widetilde{x}(t+2h)\big), \\
&\quad \alpha(t)\Phi(\widetilde{x}(t), \widetilde{x}(t+h), \widetilde{x}(t+2h)) = 0,\ \alpha(t) \geq 0, \\
&\quad \beta^*(0) = 0,\ \partial_x g(\widetilde{x}(0), 0) = \partial_x g(\widetilde{x}(h), h) = 0,\ x^*(1) + \psi^*(1) = 0, \\
&\quad - x^*(1-h) + \beta^*(1-h) - \psi^*(1-h) \in \partial q(\widetilde{x}(1-h)).
\end{aligned} \tag{24, 25}$$

Now, in the *DSIs* (24) we should express the *LAM* Q^* and $\partial\Phi$ in terms of *LAM* F^* and $\partial\varphi$, correspondingly.

Lemma 4.1. *If $Q(x,y) = 2y - x + h^2 F(x, (y-x)/h)$ is a convex set-valued mapping and $\Phi(x,y,z) \equiv \varphi\Big[x, (y-x)/h, (z-2y+x)/h^2\Big]$ a proper convex function then the following inclusions are equivalent:*

(1) $\big(x^* + \overline{x}^*, y^* + \overline{y}^*, \overline{z}^*\big) \in Q^*(z^*; (x,y,z)) \times \{0\} - \alpha_1 \partial_\omega \Phi(x,y,z)$,

(2) $$\Big(\frac{x^* + \overline{x}^* + y^* + \overline{y}^* - z^* + \overline{z}^*}{h^2}\ ,\ \frac{y^* - 2z^* + \overline{y}^* + 2\overline{z}^*}{h}, \overline{z}^*\Big) \in$$
$$\in F^*\big(z^*; \big(x, \frac{y-x}{h}, \frac{z-2y+x}{h^2}\big)\big) \times \{0\} - \alpha\partial_\omega\varphi\big[x, \frac{y-x}{h}, \frac{z-2y+x}{h^2}\big],$$

where $(x^*, y^*) \in Q^*(z^*; (x,y,z)), (\overline{x}^*, \overline{y}^*, \overline{z}^*) \in \partial_\omega \Phi(x,y,z)$, $(x-2y+z)/h^2 \in F(x, (y-x)/h; z^*)$, $v \in Q(x,y;z^*)$, $(x,y,z) \in dom\Phi$, $\alpha_1 = \alpha h^2$, $\alpha_1 > 0$.

Proof. Let us recall Theorem 3.1 [16] (Theorem 2.1 of Section 2) and rewrite it in more relevant form; obviously, by this theorem the inclusions

$$(\overline{x}^*, \overline{y}^*, \overline{z}^*) \in -\alpha_1 \partial_\omega \Phi(x,y,z),\ (x,y,z) \in dom\Phi,\ \alpha_1 > 0$$

and

$$\Big(\frac{-\overline{x}^* - \overline{y}^* - \overline{z}^*}{h^2}\ ,\ \frac{-\overline{y}^* - 2\overline{z}^*}{h}, -\overline{z}^*\Big) \in \alpha\partial\varphi\Big[x, \frac{y-x}{h}, \frac{z-2y+x}{h^2}\Big],\ \alpha = \frac{\alpha_1}{h^2}$$

are equivalent. Then taking into account Theorem 4.1 [18] (Theorem 2.2 of Section 2) from Section 2 we immediately have the desired result. □

Below in Theorem 4.2 the second-order adjoint Euler-Lagrange *DSIs* and transversality condition for the original problem *PC*, described naturally in term of *LAM* F^* and function q, are established using Lemma 4.1.

Theorem 4.2. *Let F be convex set-valued mapping and $\varphi, q, g(\cdot, t)$ be proper convex functions. such that the non-degeneracy condition is satisfied. Then for the $\{\widetilde{x}(t)\}$ to be an optimal trajectory in the discrete approximation problem (PDA), it is necessary and sufficient that there exist numbers $\alpha(t) \geq 0$, $t = 0, \ldots, 1-2h$ and a triple of vectors $\{x^*(t), \psi^*(t), \beta^*(t)\}$ $t = 0, \ldots, 1-h$ all simultaneously not all equal to zero satisfying the approximate adjoint Euler-Lagrange DSIs and transversality conditions:*

$$\begin{aligned}&\Bigg(\frac{x^*(t)+\psi^*(t)-\beta^*(t)+\beta^*(t+h)-x^*(t+2h)-\psi^*(t+2h)}{h^2},\\&\qquad\frac{-2x^*(t+2h)+\beta^*(t+h)-2\psi^*(t+2h)}{h}, -\psi^*(t+2h)\Bigg) \in \end{aligned} \tag{26}$$

$$\begin{gathered}\in F^*\big(x^*(t+2h); \big(x(t), \Delta x(t), \Delta^2 x(t)\big)\big) \times \{0\} - \partial g(\widetilde{x}(t), t) \times \{0\} \times \{0\}\\ -\alpha(t)\partial_\omega \varphi\big(\widetilde{x}(t), \Delta\widetilde{x}(t), \Delta^2\widetilde{x}(t)\big),\ \beta^*(0)=0,\ \partial_x g(\widetilde{x}(0), 0) = \partial_x g(\widetilde{x}(h), h) = 0,\\ \alpha(t)\varphi\big(x(t), \Delta x(t), \Delta^2 x(t)\big) = 0,\ \alpha(t) \geq 0,\ x^*(1)+\psi^*(1) = 0,\\ \frac{1}{h}\Big[-x^*(1-h)+\beta^*(1-h)-\psi^*(1-h)\Big] \in \partial q(\widetilde{x}(1-h)). \end{gathered}\tag{27}$$

Proof. Indeed, by Lemma 4.1 the conditions (24), (25) for convex problem take the form (26), (27) respectively. It only needs to be taken into account that *LAM* is positive homogeneous on the first argument and $hx^*(t)$, $h\beta^*(t)$, $h\psi^*(t)$, $\alpha(t)/h$ are denoted again by $x^*(t)$, $\beta^*(t)$, $\psi^*(t)$, $\alpha(t)$, respectively. □

Let us rewrite the conditions (26), (27) in a more convenient form.

Lemma 4.3. *The approximate second-order adjoint Euler-Lagrange DSIs* (26) *and transversality condition* (27) *of Theorem* 4.2 *in term of first- and second-order difference operators can be rewritten as follows:*

$$\begin{gathered}\Big[\Delta^2\Big(x^*(t)+\psi^*(t)\Big)+\Delta u^*(t), u^*(t+h), -\psi^*(t+2h)\Big] \in\\ \in F^*\Big(x^*(t+2h); \big(x(t), \Delta x(t), \Delta^2 x(t)\big)\Big) \times \{0\} - \partial g(\widetilde{x}(t), t) \times \{0\} \times \{0\} -\\ -\alpha(t)\partial_{(x,y,z)}\varphi\Big(\widetilde{x}(t), \Delta\widetilde{x}(t), \Delta^2\widetilde{x}(t)\Big),\ t = 2h, 3h, \ldots, 1-2h,\\ u^*(1-h)+\Delta\Big[x^*(1-h)+\psi^*(1-h)\Big] \in \partial q(\widetilde{x}(1-h));\ x^*(1)+\psi^*(1) = 0,\end{gathered}$$

respectively, where $u^*(t) = \Big[\beta^*(t) - 2\psi^*(t+h) - 2x^*(t+h)\Big]/h$.

Proof. From the notation $u^*(t+h) = \Big[\beta^*(t+h) - 2x^*(t+2h) - 2\psi^*(t+2h)\Big]/h$ it follows that $\beta^*(t+h) = hu^*(t+h) + 2x^*(t+2h) + 2\psi^*(t+2h)$ and so in term of Δ, Δ^2 we have the important representation

$$\begin{aligned}
&\frac{x^*(t) + \psi^*(t) - \beta^*(t) + \beta^*(t+h) - x^*(t+2h) - \psi^*(t+2h)}{h^2} = \\
&= \frac{x^*(t+2h) - 2x^*(t+h) + x^*(t) - hu^*(t) + hu^*(t+h)}{h^2} + \\
&+ \frac{\psi^*(t+2h) - 2\psi^*(t+h) + \psi^*(t)}{h^2} = \Delta^2 x^*(t) + \Delta^2 \psi^*(t) + \Delta u^*(t).
\end{aligned} \tag{28}$$

In turn, using $\beta^*(1-h) = hu^*(1-h) + 2x^*(1) + 2\psi^*(1)$ from the transversality condition (27) we have

$$\frac{1}{h}\Big[- x^*(1-h) + hu^*(1-h) + 2x^*(1) + 2\psi^*(1) - \psi^*(1-h)\Big] \in \partial q(\widetilde{x}(1-h)),$$

or more concisely

$$u^*(1-h) + \Delta x^*(1-h) + \Delta\psi^*(1-h) \in \partial q(\widetilde{x}(1-h)). \tag{29}$$

Substituting (28), (29) into conditions (26), (27) respectively, we have the desired result. $\square$

Recall that we can associate the following second-order discrete-approximation problem with the problem *POC*:

$$\begin{aligned}
&\text{minimize} \quad J_h[x(\cdot)] = \sum_{t=2h,\dots,1-2h} hg(x(t),t) + q(x(1-h)),\\
&\Delta^2 x(t) \in F\big(x(t), \Delta x(t)\big), \quad \varphi_0\big(x(t), \Delta x(t)\big) \le 0\\
&\text{for } t = 0, h, 2h, \dots, 1-h, \ x(0) = \theta_0, \ \Delta x(0) = \theta_1.
\end{aligned} \tag{30}$$

Corollary 4.4. *The necessary and sufficient conditions* (26), (27) *of Theorem* 4.2 *for a problem POC have the form*

$$\begin{aligned}
&\big[\Delta^2 x^*(t) + \Delta u^*(t), u^*(t)\big] \in F^*\big(x^*(t+2h); \big(x(t), \Delta x(t), \Delta^2 x(t)\big)\big)\\
&-\partial g(\widetilde{x}(t), t) \times \{0\} - \alpha(t)\partial_{(x,y)}\varphi_0(\widetilde{x}(t), \Delta\widetilde{x}(t)), \ t = 2h, 3h, \dots, 1-2h,\\
&u^*(1-h) + \Delta x^*(1-h) \in \partial q(\widetilde{x}(1-h)), \ x^*(1) = 0,
\end{aligned}$$

respectively.

Proof. Indeed, taking into account the results of Theorems 3.5 and 4.2 we deduce that the approximate adjoint Euler-Lagrange *DSIs* and transversality conditions for problem (30) consist of the following relations:

$$\left(\frac{x^*(t) - \beta^*(t) + \beta^*(t+h) - x^*(t+2h)}{h^2}, \frac{\beta^*(t+h) - 2x^*(t+2h)}{h}\right) \in$$

$$\in F^*\Big(x^*(t+2h); \big(x(t), \Delta x(t), \Delta^2 x(t)\big)\Big) -$$
$$-\partial g(\widetilde{x}(t), t) \times \{0\} - \alpha(t)\partial_{(x,y)}\varphi_0(\widetilde{x}(t), \Delta\widetilde{x}(t)).$$

Along with these relations we have

$$\alpha(t)\varphi_0(x(t), \Delta x(t)) = 0,\ \alpha(t) \geq 0,$$

$$\frac{1}{h}\Big[-x^*(1-h) + \beta^*(1-h)\Big] \in \partial q(\widetilde{x}(1-h)),\ x^*(1) = 0.$$

Using $u^*(t) = \big(\beta^*(t) - 2x^*(t+h)\big)/h$ (recall that in this case $\psi^*(t) \equiv 0$) further proof of the corollary is similar to that of Lemma 4.3. □

5. Sufficient optimality conditions for second-order DFIs

On the basis of Section 4, we formulate a sufficient optimality condition for the continuous problem PC; by passing to the formal limit in the conditions of Lemma 4.3 as $h \to 0$, we establish the so-called second-order adjoint Euler-Lagrange *DFI* for problem PC:

(a) $$\Big[\frac{d^2(x^*(t)+\psi^*(t))}{dt^2} + \tfrac{du^*(t)}{dt}\ ,\ u^*(t)\ ,\ -\psi^*(t)\Big] \in$$
$$\in F^*\Big(x^*(t); \big(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)\big)\Big) \times \{0\} - \partial g(\widetilde{x}(t), t) \times \{0\} \times \{0\} -$$
$$-\alpha(t)\partial_\omega \varphi\big(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)\big),\ \text{a.e. } t \in [0,1]$$

(b) $\alpha(t)\varphi(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)) = 0,\ \alpha(t) \geq 0,\ \text{a.e. } t \in [0,1]$

and the transversality condition at $t = 1$:

(c) $$u^*(1) + \frac{d\big(x^*(1) + \psi^*(1)\big)}{dt} \in \partial q(\widetilde{x}(1));\quad x^*(1) + \psi^*(1) = 0.$$

In what follows our notations and terminologies are generally consistent with those in Mordukhovich [23] and Mahmudov [20] for the first-order *DFIs*. Suppose that $x^*(t), \psi^*(t), t \in [0,1]$ are absolutely continuous functions together with the first-order derivatives whose second derivatives belong to the space L_1^n. Moreover, we assume that $u^*(t), t \in [0,1]$ is absolutely continuous and $u^{*\prime}(\cdot) \in L_1^n$.

We get one more condition providing that the *LAM* F^* is nonempty at a given point:

(d) $$\frac{d^2\widetilde{x}(t)}{dt^2} \in F(\widetilde{x}(t), \widetilde{x}'(t); x^*(t)),\ \text{a.e. } t \in [0,1].$$

It appears that the following assertion is true.

Theorem 5.1. *Let F be convex set-valued mapping and φ, $g(\cdot,t), q$ be convex continuous functions. Then for optimality of the trajectory $\widetilde{x}(t)$ in the problem PC with DFIs and SVICs it is sufficient that there exist a triple of absolutely continuous functions $\{x^*(t), \psi^*(t), u^*(t)\}$, $t \in [0,1]$ and function $\alpha(t) \geq 0$, $t \in [0,1]$ satisfying a.e. the second-order adjoint Euler-Lagrange DFI (a) and conditions (b)–(d).*

Proof. By Theorem 2.1 [20] (Lemma 2.1 of Section 2) $F^*(z^*, (x,y,z)) = \partial_{(x,y)} H_F(x,y,z^*)$, $z \in F(x,y;z^*)$. Then, by using the Moreau-Rockafellar theorem [12, 20] and convenience $-\partial g(\cdot,t) = \partial(-g(\cdot,t))$ from condition (a) we obtain the second-order adjoint *DFI*

$$\left[\frac{d^2(x^*(t)+\psi^*(t))}{dt^2} + \frac{du^*(t)}{dt},\ u^*(t),\ -\psi^*(t)\right] \in \partial_{(x,y)} H_F(\widetilde{x}(t), \widetilde{x}'(t), x^*(t)) \times \{0\} -$$

$$-\partial_x g(\widetilde{x}(t),t) \times \{0\} \times \{0\} - \alpha(t)\partial_{(x,y,z)}\varphi(\widetilde{x}(t), \widetilde{x}(t), \widetilde{x}''(t))$$

or more compactly

$$\left[\frac{d^2(x^*(t)+\psi^*(t))}{dt^2} + \frac{du^*(t)}{dt}\ ,\ u^*(t)\ ,\ -\psi^*(t)\right] \in$$

$$\in \partial_\omega \big[H_F(\widetilde{x}(t), \widetilde{x}'(t), x^*(t)) - g(\widetilde{x}(t),t) - \alpha(t)\varphi(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t))\big].$$

We rewrite the last relation in the form:

$$H_F\big(x(t), x'(t), x^*(t)\big) - H_F\big(\widetilde{x}(t), \widetilde{x}'(t), x^*(t)\big) - g(x(t),t) + g(\widetilde{x}(t),t) -$$

$$-\alpha(t)\varphi\big(x(t), x'(t), x''(t)\big) + \alpha(t)\varphi\big(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)\big) \leq \big\langle u^*(t), x'(t) - \widetilde{x}'(t)\big\rangle +$$

$$+\left\langle \frac{d^2\big(x^*(t)+\psi^*(t)\big)}{dt^2} + \frac{du^*(t)}{dt}, x(t) - \widetilde{x}(t)\right\rangle - \big\langle \psi^*(t)\ ,\ x''(t) - \widetilde{x}''(t)\big\rangle. \tag{31}$$

In turn by using the definition of the Hamiltonian function, (31) can be reduced to the inequality relation

$$\left\langle \frac{d^2x(t)}{dt^2}\ ,\ x^*(t)\right\rangle - \left\langle \frac{d^2\widetilde{x}(t)}{dt^2}\ ,\ x^*(t)\right\rangle - g(x(t),t) + g(\widetilde{x}(t),t) -$$

$$-\alpha(t)\varphi\big(x(t), x'(t), x''(t)\big) + \alpha(t)\varphi\big(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)\big) \leq \frac{d}{dt}\Big\langle u^*(t), x(t) - \widetilde{x}(t)\Big\rangle +$$

$$+\left\langle \frac{d^2\big(x^*(t)+\psi^*(t)\big)}{dt^2}, x(t) - \widetilde{x}(t)\right\rangle - \left\langle \psi^*(t), \frac{d^2(x(t) - \widetilde{x}(t))}{dt^2}\right\rangle.$$

For further convenience, rewrite the latter inequality as follows

$$g(x(t),t) - g(\widetilde{x}(t),t) \geq \left\langle \frac{d^2\big(x(t) - \widetilde{x}(t)\big)}{dt^2}\ ,\ x^*(t)\right\rangle - \frac{d}{dt}\Big\langle u^*(t), x(t) - \widetilde{x}(t)\Big\rangle -$$

$$-\left\langle \frac{d^2\big(x^*(t)+\psi^*(t)\big)}{dt^2}, x(t) - \widetilde{x}(t)\right\rangle + \left\langle \psi^*(t), \frac{d^2(x(t) - \widetilde{x}(t))}{dt^2}\right\rangle -$$

$$-\alpha(t)\varphi(x(t), x'(t), x''(t)) + \alpha(t)\varphi(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)).$$

Now since by condition (b) $\alpha(t)\varphi\big(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)\big) = 0$ and $\alpha(t)\varphi\big(x(t), x'(t), x''(t)\big) \le 0$ $(\alpha(t) \ge 0)$ for all feasible solutions $x(\cdot)$ we conclude from the last inequality

$$g(x(t),t) - g(\widetilde{x}(t),t) \ge \Big\langle \frac{d^2\big(x(t)-\widetilde{x}(t)\big)}{dt^2}\ ,\ x^*(t)\Big\rangle - \frac{d}{dt}\Big\langle u^*(t), x(t)-\widetilde{x}(t)\Big\rangle -$$
$$- \Big\langle \frac{d^2\big(x^*(t)+\psi^*(t)\big)}{dt^2}, x(t)-\widetilde{x}(t)\Big\rangle + \Big\langle \psi^*(t), \frac{d^2(x(t)-\widetilde{x}(t))}{dt^2}\Big\rangle. \tag{32}$$

Integrating (32) over the interval $[0,1]$ and taking into account that $x(\cdot)$, $\widetilde{x}(\cdot)$ are feasible, $x(0) = \widetilde{x}(0) = \theta_0$, $x'(0) = \widetilde{x}'(0) = \theta_1$ one has

$$\int_0^1 \Big[g(x(t),t) - g(\widetilde{x}(t),t)\Big]dt \ge \int_0^1 \Big[\Big\langle \frac{d^2\big(x(t)-\widetilde{x}(t)\big)}{dt^2}\ ,\ x^*(t)\Big\rangle -$$
$$- \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\widetilde{x}(t)\Big\rangle\Big]dt - \Big\langle u^*(1), x(1)-\widetilde{x}(1)\Big\rangle +$$
$$+ \int_0^1 \Big[\Big\langle \psi^*(t), \frac{d^2\big(x(t)-\widetilde{x}(t)\big)}{dt^2}\Big\rangle - \Big\langle \frac{d^2\psi^*(t)}{dt^2}, x(t)-\widetilde{x}(t)\Big\rangle\Big]dt. \tag{33}$$

Obviously, we can transform the first integral on the right hand side of (33) as follows

$$\int_0^1 \Big[\Big\langle \frac{d^2\big(x(t)-\widetilde{x}(t)\big)}{dt^2}\ ,\ x^*(t)\Big\rangle - \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\widetilde{x}(t)\Big\rangle\Big]dt =$$
$$= \Big\langle \frac{d\big(x(1)-\widetilde{x}(1)\big)}{dt}, x^*(1)\Big\rangle - \Big\langle \frac{d\big(x(0)-\widetilde{x}(0)\big)}{dt}, x^*(0)\Big\rangle -$$
$$- \Big\langle \frac{dx^*(1)}{dt}, x(1)-\widetilde{x}(1)\Big\rangle + \Big\langle \frac{dx^*(0)}{dt}, x(0)-\widetilde{x}(0)\Big\rangle. \tag{34}$$

Since $x(\cdot)$ and $\widetilde{x}(\cdot)$ are feasible solutions $x(0) = \widetilde{x}(0) = \theta_0$, $x'(0) = \widetilde{x}'(0) = \theta_1)$ it follows from (34) that

$$\int_0^1 \Big[\Big\langle \frac{d^2\big(x(t)-\widetilde{x}(t)\big)}{dt^2}\ ,\ x^*(t)\Big\rangle - \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\widetilde{x}(t)\Big\rangle\Big]dt =$$
$$= \Big\langle \frac{d\big(x(1)-\widetilde{x}(1)\big)}{dt}, x^*(1)\Big\rangle - \Big\langle \frac{dx^*(1)}{dt}, x(1)-\widetilde{x}(1)\Big\rangle. \tag{35}$$

In a similar way the second integral on the right hand side of (33) is transformed as follows

$$\int_0^1 \Big[\Big\langle \psi^*(t), \frac{d^2\big(x(t)-\widetilde{x}(t)\big)}{dt^2}\Big\rangle - \Big\langle \frac{d^2\psi^*(t)}{dt^2}, x(t)-\widetilde{x}(t)\Big\rangle\Big]dt =$$
$$= \Big\langle \frac{d\big(x(1)-\widetilde{x}(1)\big)}{dt}, \psi^*(1)\Big\rangle - \Big\langle \frac{d\psi^*(1)}{dt}, x(1)-\widetilde{x}(1)\Big\rangle. \tag{36}$$

Then substituting (35) and (36) into the inequality (33) we get

$$\int_0^1 \big[g(x(t),t) - g(\widetilde{x}(t),t)\big]dt \geq \Big\langle \frac{d\big(x(1) - \widetilde{x}(1)\big)}{dt}, x^*(1) + \psi^*(1) \Big\rangle - \\ - \Big\langle \frac{dx^*(1) + d\psi^*(1)}{dt}, x(1) - \widetilde{x}(1) \Big\rangle - \Big\langle u^*(1), x(1) - \widetilde{x}(1) \Big\rangle. \quad (37)$$

Therefore, by the second condition in (c) of Theorem 5.1 we have $x^*(1) + \psi^*(1) = 0$, and from (37) we obtain

$$\int_0^1 \big[g(x(t),t) - g(\widetilde{x}(t),t)\big]dt \geq \\ \geq -\Big\langle \frac{d\big(x^*(1) + \psi^*(1)\big)}{dt} + u^*(1), x(1) - \widetilde{x}(1) \Big\rangle. \quad (38)$$

On the other hand, by the first condition in (c) of Theorem 5.1 we get for an arbitrary feasible solution $x(t)$ the inequality

$$q(x(1)) - q(\widetilde{x}(1)) \geq \Big\langle \frac{d\big(x^*(1) + \psi^*(1)\big)}{dt} + u^*(1), x(1) - \widetilde{x}(1) \Big\rangle. \quad (39)$$

At last, adding the inequalities (38) and (39) we derive that

$$\int_0^1 \big[g(x(t),t) - g(\widetilde{x}(t),t)\big]dt + q(x(1)) - q(\widetilde{x}(1)) \geq 0$$

that is, for all feasible solution $x(\cdot)$ the inequality $J[x(\cdot)] \geq J[\widetilde{x}(\cdot)]$ holds and so $\widetilde{x}(t)$ is optimal. □

Remark 5.2. Note that for simplicity of representation we have studied the Bolza problem PC with objective functional $J[x(\cdot)] = \int_0^1 g(x(t),t)dt + q(x(1))$. In a similar way it is easy to show that for a problem PC with more general objective functional $J_0[x(\cdot)] = \int_0^1 g(x(t),x'(t),t)dt + q(x(1),x'(1))$ the second-order adjoint Euler-Lagrange DFI and transversality condition at point $t = 1$ of Theorem 5.1 consist of the following

$$(a_0) \quad \left[\frac{d^2(x^*(t) + \psi^*(t))}{dt^2} + \tfrac{du^*(t)}{dt}\ ,\ u^*(t)\ ,\ -\psi^*(t)\right] \in \\ \in F^*\big(x^*(t); \big(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)\big)\big) \times \{0\} - \partial_{(x,y)} g(\widetilde{x}(t), \widetilde{x}'(t), t) \times \{0\} - \\ -\alpha(t)\partial_\omega \varphi(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)), \text{ a.e. } t \in [0,1]$$

$$(c_0) \quad \left[u^*(1) + \tfrac{d\left(x^*(1) + \psi^*(1)\right)}{dt}, -x^*(1) - \psi^*(1)\right] \in \partial_{(x,y)} q(\widetilde{x}(1), \widetilde{x}'(1)).$$

It remains to emphasize that for the construction of the transversality condition we use a particular case of the Equivalence Theorem 2.5 from Section 2:

(1) $\left(\overline{x}^*, \overline{y}^*\right) \in \partial_{(x,y)} g\ ,\ (x,y) \in dom\, g,$

(2) $\left(\overline{x}^* + \overline{y}^*, h\overline{y}^*\right) \in \partial_{(x,y)} q\left(x, \dfrac{y-x}{h}\right),\ g(x,y) \equiv q\left(x, \dfrac{y-x}{h}\right).$

Then the transversality condition of the corresponding discrete approximate problem

$$\left[-x^*(1-h)+\beta^*(1-h)-\psi^*(1-h)\ ,\ -x^*(1)-\psi^*(1)\right] \in \partial_{(x,y)} g(\widetilde{x}(1-h), \widetilde{x}(1))$$

is equivalent to

$$\frac{1}{h}\Big[-x^*(1-h)+\beta^*(1-h)-\psi^*(1-h)-x^*(1)-\psi^*(1)\ ,\ -hx^*(1)-h\psi^*(1)\Big] \in$$
$$\in \partial_{(x,y)} q(\widetilde{x}(1-h), \Delta\widetilde{x}(1-h)).$$

Now, taking into account $\beta^*(1-h) = hu^*(1-h) + 2x^*(1) + 2\psi^*(1)$ (see the proof of Lemma 4.3) and by passing to the formal limit as $h \to 0$, we obtain the transversality condition (c_0).

Corollary 5.3. *Let us consider a problem PC without SVICs, that is F be convex mapping and $g(\cdot,t), q$ be convex continuous functions. Then for $\widetilde{x}(t)$ to be an optimal trajectory in the problem PC with DFIs it is sufficient that there exists a pair of absolutely continuous functions $\{x^*(t), u^*(t)\}$, $t \in [0,1]$ satisfying a.e. the second-order Euler-Lagrange adjoint DFI:*

$$\left[\frac{d^2x^*(t)}{dt^2} + \frac{du^*(t)}{dt}, u^*(t)\right] \in F^*(x^*(t); (x(t), \widetilde{x}'(t), \widetilde{x}''(t))) - \partial g(\widetilde{x}(t), t) \times \{0\}$$

a.e. $t \in [0,1]$, and the transversality condition at $t = 1$:

$$u^*(1) + \frac{dx^*(1)}{dt} \in \partial q(\widetilde{x}(1));\ x^*(1) = 0.$$

Proof. Setting $\varphi(x(t), x'(t), x''(t)) \equiv 0$ in the adjoint Euler-Lagrange DFI (a) of Theorem 5.1 we immediately obtain $\psi^*(t) \equiv 0$, $t \in [0,1]$. In turn it follows that in the transversality condition (c) of Theorem 5.1 $\psi^*(1) = 0$ and $d\psi^*(1)/dt = 0$. □

Let us consider the problem PC in the case where $\varphi(x(t), x'(t), x''(t)) \equiv \varphi_0(x(t), x'(t))$, that is φ does not depend on the third argument x'' and label this problem by POC:

$$(POC) \qquad \begin{aligned} &\text{minimize}\ \ J[x(\cdot)] = \int_0^1 g(x(t),t)dt + q(x(1)), \\ &x''(t) \in F(x(t), x'(t)),\ \varphi_0(x(t), x'(t)) \le 0 \\ &\text{a.e. } t \in [0,1],\ x(0) = \theta_0,\ x'(0) = \theta_1. \end{aligned}$$

Taking into account Corollary 4.4 and Theorem 5.1 we can establish easily the sufficient condition of optimality for problem POC.

Theorem 5.4. *Let F be convex mapping and $g(\cdot,t), q$ and φ be convex continuous functions such that $\varphi(x(t),x'(t),x''(t)) \equiv \varphi_0(x(t),x'(t))$ that is φ does not depend on the third argument x''. Then for the $\widetilde{x}(t)$ to be an optimal trajectory in the problem POC with DFIs and SVICs it is sufficient that there exists a pair of absolutely continuous functions $\{x^*(t),u^*(t)\}$, $t \in [0,1]$ and function $\alpha(t) \geq 0$ satisfying a.e. the second-order adjoint Euler-Lagrange DFI (d), the transversality condition (f) and the condition (e):*

$$(d) \quad \left[\frac{d^2x^*(t)}{dt^2} + \frac{du^*(t)}{dt}, u^*(t)\right] \in F^*(x^*(t);(x(t),\widetilde{x}'(t),\widetilde{x}''(t))) -$$
$$-\partial g(\widetilde{x}(t),t) \times \{0\} - \alpha(t)\partial_{(x,y)}\varphi_0(\widetilde{x}(t),\widetilde{x}'(t)), \text{ a.e. } t \in [0,1]$$

$$(e) \quad \alpha(t)\varphi_0(\widetilde{x}(t),\widetilde{x}'(t)) = 0,\ \alpha_t \geq 0, \text{ a.e. } t \in [0,1]$$

$$(f) \quad u^*(1) + \frac{dx^*(1)}{dt} \in \partial q(\widetilde{x}(1));\ x^*(1) = 0.$$

Proof. By Corollary 4.4 as a result of limiting procedure we have the condition (d), (e) and (f) of theorem. Further proof of the theorem is similar to the one given in Theorem 3.2. □

Now consider the following problem with *SVICs*

$$(PC_\varphi) \qquad \begin{aligned} &\text{minimize} \quad J[x(\cdot)] = \int_0^1 g(x(t),t)dt + q(x(1)), \\ &\varphi(x(t),x'(t),x''(t)) \leq 0, \text{ a.e. } t \in [0,1],\ x(0) = \overline{\theta}_0,\ x'(0) = \overline{\theta}_1. \end{aligned}$$

Corollary 5.5. *The sufficient optimality conditions of Theorem* 5.1 *for a problem* (PC_φ) *have the form*

$$(a) \quad \left[\frac{d^2\psi^*(t)}{dt^2} + \tfrac{du^*(t)}{dt},\ u^*(t),\ -\psi^*(t)\right] \in -\alpha(t)\partial_\omega\varphi(\widetilde{x}(t),\widetilde{x}'(t),\widetilde{x}''(t)) -$$
$$-\partial_x g(\widetilde{x}(t),t) \times \{0\} \times \{0\} \text{ a.e. } t \in [0,1],$$

$$(b) \quad \alpha(t)\varphi(\widetilde{x}(t),\widetilde{x}'(t),\widetilde{x}''(t)) = 0,\ \alpha(t) \geq 0, \text{ a.e. } t \in [0,1],$$

$$(c) \quad u^*(1) + \tfrac{d\psi^*(1)}{dt} \in \partial q(\widetilde{x}(1));\ \psi^*(1) = 0.$$

Proof. In the present case $dom\, F(\cdot,\cdot) \equiv R^{2n}$ and $F(\cdot,\cdot) \equiv \mathbb{R}^n$. Therefore as shown in Remark 3.7 $F^* \equiv \{(0,0)\}$ which means that in the adjoint Euler-Lagrange DFI $x^*(t) \equiv 0$. As a consequence, in the transversality condition (c) of Theorem 5.1 $dx^*(1)/dt = 0$ and $x^*(1) = 0$. The proof of the corollary is completed. □

Example 5.6. We consider now the so-called "linear" continuous problem for second-order differential inclusions:

$$
\begin{aligned}
&\text{minimize} \;\; J[x(\cdot)] = \int_0^1 g(x(t),t)dt + q(x(1)), \\
&x''(t) \in F(x(t), x'(t)), \; \text{a.e. } t \in [0,1], \; x(0) = \theta_0 \;, \; x'(0) = \theta_1, \\
&F(x,y) = A_0 x + A_1 y + BU,
\end{aligned} \tag{40}
$$

where g and q are continuously differentiable function in x, A_i, $i = 0,1$ and B are $n \times n$ and $n \times r$ matrices, respectively, U is a convex closed subset of $\mathbb{R}^r$. The problem is to find a controlling parameter $\widetilde{u}(t) \in U$ such that the arc $\widetilde{x}(t)$ corresponding to it minimizes $J[x(\cdot)]$.

By elementary computations we find that

$$
F^*(z^*; (x,y,z)) = \begin{cases} (A_0^* z^*, A_1^* z^*), & -B^* z^* \in K_U^*(u), \\ \emptyset, & -B^* z^* \notin K_U^*(u) \end{cases} \tag{41}
$$

where $z = A_0 x + A_1 y + Bu$, $u \in U$, $A_i^* (i = 0,1)$ and B^* are transposed matrices. We proceed on the basis of Theorem 5.1. Thus the second-order adjoint Euler-Lagrange *DFI (a)* of Theorem 5.1 for problem (40) has the form

$$
\begin{aligned}
&\left[\frac{d^2\big(x^*(t) + \psi^*(t)\big)}{dt^2} + \frac{du^*(t)}{dt} \;,\; u^*(t) \;,\; -\psi^*(t) \right] \in \\
&\in F^*\Big(x^*(t); (x(t), \widetilde{x}'(t), \widetilde{x}''(t))\Big) \times \{0\} - g'(\widetilde{x}(t),t) \times \{0\} \times \{0\}
\end{aligned}
$$

for a.e. $t \in [0,1]$, where it is taken into account that $\varphi(x,y,z) \equiv 0$ and so $\partial_{(x,y,z)} \varphi(\widetilde{x}(t), \widetilde{x}'(t), \widetilde{x}''(t)) \equiv \{0,0,0\}$. Then, we conclude that $-\psi^*(t) \equiv 0$ for $t \in [0,1]$ identically and by formula (41) of *LAM* we have

$$
\frac{d^2 x^*(t)}{dt^2} + \frac{du^*(t)}{dt} \;\; = A_0^* x^*(t) - g'(\widetilde{x}(t),t), \quad u^*(t) \;\; = A_1^* x^*(t) \tag{42}
$$

Differentiating the second equation in (42) and substituting into the first equation we immediately get the second-order adjoint differential equation for the problem (40)

$$
\frac{d^2 x^*(t)}{dt^2} = -A_1^* \frac{dx^*(t)}{dt} + A_0^* x^*(t) - g'(\widetilde{x}(t),t), \;\text{a.e. } t \in [0,1]. \tag{43}
$$

From condition (c) of Theorem 5.1 the transversality condition at the point $t = 1$ follows immediately:

$$
\frac{dx^*(1)}{dt} = q'(\widetilde{x}(1)); \; x^*(1) = 0, \tag{44}
$$

where it is taken into account that $u^*(1) = A_1^* x^*(1) = 0$. Finally, by the definition of $F(x, y; z^*)$, formula (41) and condition (d) of Theorem 5.1 we obtain

$$\left\langle B\widetilde{u}(t), x^*(t) \right\rangle = \sup_{u \in U} \left\langle Bu, x^*(t) \right\rangle. \tag{45}$$

Thus, the arc $\widetilde{x}(t)$ corresponding to the controlling parameter $\widetilde{u}(t)$ minimizes $J(x(\cdot))$ in problem (40) if there exists an absolutely continuous function $x^*(t)$ $(x^{*\prime\prime}(\cdot) \in L_1^n)$ satisfying the second-order adjoint differential equation (43), the transversality and Weierstrass-Pontryagin conditions (44) and (45). $\square$

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