

Optimisation of the Lowest Robin Eigenvalue in the Exterior of a Compact Set

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We consider the problem of geometric optimisation of the lowest eigenvalue of the Laplacian in the exterior of a compact planar set, subject to attractive Robin boundary conditions. Under either a constraint of fixed perimeter or area, we show that the maximiser within the class of exteriors of convex sets is always the exterior of a disk. We also argue why the results fail without the convexity constraint and in higher dimensions.

Keywords: Robin Laplacian, negative boundary parameter, exterior of a convex set, lowest eigenvalue, spectral isoperimetric inequality, spectral isochoric inequality, parallel coordinates.

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1. Introduction

The *isoperimetric inequality* states that among all planar sets of a given perimeter, the disk has the largest area. This is equivalent to the *isochoric inequality* stating that among all planar sets of a given area, the disk has the smallest perimeter. These two classical geometric optimisation problems were known to ancient Greeks, but a first rigorous proof appeared only in the 19th century (see [5] for an overview).

Going from geometric to spectral quantities, the *spectral isochoric inequality* states that among all planar membranes of a given area and with fixed edges, the circular membrane produces the lowest fundamental tone. It was conjectured by Lord Rayleigh in 1877 in his famous book *The theory of sound* [22], but proved only by Faber [14] and Krahn [21] almost half a century later. Using scaling, it is easily seen that this result implies the *spectral isoperimetric*

inequality as well: among all planar membranes of a given perimeter and with fixed edges, the circular membrane produces the lowest fundamental tone.

The same spectral inequalities under the area or perimeter constraints extend to elastically supported membranes. To be more precise, let $\Omega \subset \mathbb{R}^2$ be any smooth bounded open set. Given a real number α , consider the spectral problem for the Laplacian, subject to Robin boundary conditions,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \alpha u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where n is the *outer* unit normal to Ω . It is well known that (1) induces an infinite number of eigenvalues $\lambda_1^\alpha(\Omega) \leq \lambda_2^\alpha(\Omega) \leq \lambda_3^\alpha(\Omega) \leq \dots$ diverging to infinity. The lowest eigenvalue admits the variational characterisation

$$\lambda_1^\alpha(\Omega) = \inf_{\substack{u \in W^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2 + \alpha \int_{\partial\Omega} |u|^2}{\int_{\Omega} |u|^2}, \quad (2)$$

from which it is clear that $\lambda_1^\alpha(\Omega)$ is non-negative if, and only if, α is non-negative. In this elastic regime, the spectral isoperimetric and isochoric inequalities for the Robin Laplacian can be respectively stated as follows:

$$\alpha \geq 0 : \quad \min_{|\partial\Omega|=c_1} \lambda_1^\alpha(\Omega) = \lambda_1^\alpha(B_{R_1}) \quad \text{and} \quad \min_{|\Omega|=c_2} \lambda_1^\alpha(\Omega) = \lambda_1^\alpha(B_{R_2}). \quad (3)$$

Here B_{R_1} and B_{R_2} are disks of perimeter $|\partial B_{R_1}| = c_1$ and area $|B_{R_2}| = c_2$, respectively, with c_1 and c_2 being two arbitrary positive constants. With an abuse of notation, we denote by $|\Omega|$ and $|\partial\Omega|$ the 2-dimensional Lebesgue measure of Ω and the 1-dimensional Hausdorff measure of its boundary $\partial\Omega$, respectively. In the Dirichlet case (formally corresponding to setting $\alpha = \infty$), the second optimisation problem in (3) is just the Rayleigh-Faber-Krahn inequality mentioned above. Both the statements in (3) are trivial for Neumann boundary conditions ($\alpha = 0$). If $\alpha > 0$, the second result in (3) is due to Bossel [6] from 1986 (and Daners [12] from 2006, also for higher dimensions), while the first identity again follows by scaling.

Summing up, going from the ancient isoperimetric inequality to the most recent spectral results on the Robin problem with positive boundary coupling parameter, the disk turns out to be always the extremal set for the optimisation problems under the area or perimeter constraints. Moreover, the isoperimetric and isochoric optimisation problems are closely related. In the last two years, however, it has been noticed that the optimisation of the Robin eigenvalue in the case of *negative* α is quite different. Here the story goes back to 1977 when Bareket conjectured that a *reverse spectral isochoric inequality* should hold:

Conjecture 1.1. (Bareket's reverse spectral isochoric inequality [3])

For all negative α , we have

$$\alpha < 0 : \quad \max_{|\Omega|=c_2} \lambda_1^\alpha(\Omega) = \lambda_1^\alpha(B_{R_2}), \quad (4)$$

where the maximum is taken over all bounded open connected sets Ω of a given area $c_2 > 0$ and B_{R_2} is the disk of the same area as Ω , i.e. $|B_{R_2}| = c_2$.

Notice that, contrary to (3), it is natural to maximise the eigenvalue if α is negative. The conjecture was supported by its validity for sets close to the disk [3, 15] and revived both in [8] and [13]. However, Freitas and one of the present authors have recently disproved the conjecture [16]: While (4) holds for small negative values of α , it cannot hold for all values of the boundary parameter. In fact, the annulus provides a larger value of the lowest Robin eigenvalue as $\alpha \rightarrow -\infty$. This provides the first known example where the extremal domain for the lowest eigenvalue of the Robin Laplacian is not a disk. On the other hand, in the most recent publication [2], it was shown that the *reverse spectral isoperimetric inequality* does hold:

Theorem 1.2. (Reverse spectral isoperimetric inequality [2])

For all negative α , we have

$$\alpha < 0 : \quad \max_{|\partial\Omega|=c_1} \lambda_1^\alpha(\Omega) = \lambda_1^\alpha(B_{R_1}), \quad (5)$$

where the maximum is taken over all smooth bounded open connected sets Ω of a given perimeter $c_1 > 0$ and B_{R_1} is the disk of the same perimeter as Ω , i.e. $|\partial B_{R_1}| = c_1$.

Summing up, when α is negative, the disk is still the optimiser of the reverse spectral isoperimetric problem, while it stops to be the optimiser of the isochoric problem for large negative values of α . Whether the optimiser becomes the annulus for the larger negative values of α and whether Conjecture 1.1 holds under some geometric restrictions on Ω represent just a few hot open problems in the recent study (see [2, Section 5.3] for conjectures supported by numerical experiments).

In this paper, we show that both the reverse spectral isochoric and isoperimetric inequalities hold in the dual setting of the Robin problem in the *exterior of a convex set*. To this purpose, for any open set $\Omega \subset \mathbb{R}^2$, we define $\Omega^{\text{ext}} := \mathbb{R}^2 \setminus \overline{\Omega}$.

Theorem 1.3. For all negative α , we have

$$\alpha < 0 : \quad \max_{\substack{|\partial\Omega|=c_1 \\ \Omega \text{ convex}}} \lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_{R_1}^{\text{ext}}) \quad \text{and} \quad \max_{\substack{|\Omega|=c_2 \\ \Omega \text{ convex}}} \lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_{R_2}^{\text{ext}}), \quad (6)$$

where the maxima are taken over all convex smooth bounded open sets Ω of a given perimeter $c_1 > 0$ or area $c_2 > 0$, respectively, and B_{R_1} and B_{R_2} are disks of perimeter $|\partial B_{R_1}| = c_1$ and area $|B_{R_2}| = c_2$.

It is important to mention that $\lambda_1^\alpha(\Omega^{\text{ext}})$ when defined by (2) indeed represents a (negative) *discrete* eigenvalue of a self-adjoint realisation of the Robin Laplacian in $L^2(\Omega^{\text{ext}})$. It is not obvious because there is also the essential spectrum $[0, \infty)$, but it can be shown by using the criticality of the Laplacian in $\mathbb{R}^2$ and the fact that α is negative (cf. Section 2). In fact, $\lambda_1^\alpha(\Omega^{\text{ext}})$ equals zero (the lowest point in the essential spectrum) for any domain Ω if α is non-negative, so the optimisation problems (6) are trivial in this case. At the same time, because of the existence of the Hardy inequality in higher dimensions, $\lambda_1^\alpha(\Omega^{\text{ext}})$ is also zero for all small (negative) values of α if the dimension is greater than or equal to three, so the identities (6) become trivial in this regime, too. This is the main reason why we mostly (but not exclusively) restrict to planar domains in this paper. As a matter of fact, in Section 5.3 we argue that an analogue of Theorem 1.3 can not hold in space dimensions greater than or equal to three.

We point out that the disks are the only optimisers in Theorem 1.3 (see Section 5.2 for the respective argument). It is worth mentioning that the identities (6) are no longer valid if the condition of connectedness of Ω is dropped (see Section 5.1 for a counterexample). However, it is still unclear at the moment whether the convexity of Ω in (6) can be replaced by a weaker assumption.

The organisation of this paper is as follows. In Section 2 we provide an operator-theoretic framework for the eigenvalue problem of Robin type (1) and establish basic spectral properties in the exterior of a compact set. Section 3 is devoted to more specific results on the lowest eigenvalue in the case of the compact set being a disk. Theorem 1.3 is proved in Section 4: The isoperimetric part of the theorem follows quite straightforwardly by using test functions with level lines parallel to the boundary $\partial\Omega$, while the isochoric result is established with help of scaling and a monotonicity result of Section 3. The method of parallel coordinates was employed also in [16] to establish Conjecture 1.1 for small values of α , however, the reader will notice that the present implementation of the technique is quite different and in principal does not restrict to planar sets (cf. Section 5.4). The paper is concluded by Section 5 with more comments on our results and methods.

2. The spectral problem in the exterior of a compact set

Throughout this section, Ω is an arbitrary bounded open set in $\mathbb{R}^2$, not necessarily connected or convex. A standing assumption is that the exterior Ω^{ext} is connected. Occasionally, we adopt the shorthand notation $\Sigma := \partial\Omega$. For simplicity, we assume that Ω is smooth (i.e. C^∞ -smooth), but less regularity is evidently needed for the majority of the results to hold. At the same time, α is an arbitrary real number, not necessarily negative (unless otherwise stated).

We are interested in the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega^{\text{ext}}, \\ -\frac{\partial u}{\partial n} + \alpha u = 0 & \text{on } \partial\Omega^{\text{ext}}. \end{cases} \quad (7)$$

We recall that n is the *outer* unit normal to Ω , that is why we have the flip of sign with respect to (1). As usual, we understand (7) as the spectral problem for the self-adjoint operator $-\Delta_\alpha^{\Omega^{\text{ext}}}$ in $L^2(\Omega^{\text{ext}})$ associated with the closed quadratic form

$$Q_\alpha^{\Omega^{\text{ext}}}[u] := \|\nabla u\|_{L^2(\Omega^{\text{ext}})}^2 + \alpha \|u\|_{L^2(\Sigma)}^2, \quad \mathcal{D}(Q_\alpha^{\Omega^{\text{ext}}}) := W^{1,2}(\Omega^{\text{ext}}). \quad (8)$$

The boundary term is understood in the sense of traces $W^{1,2}(\Omega^{\text{ext}}) \hookrightarrow L^2(\Sigma)$ and represents a relatively bounded perturbation of the Neumann form $Q_0^{\Omega^{\text{ext}}}$ with the relative bound equal to zero. Since Ω is smooth, the operator domain of $-\Delta_\alpha^{\Omega^{\text{ext}}}$ consists of functions $u \in W^{2,2}(\Omega^{\text{ext}})$ which satisfy the Robin boundary conditions of (7) in the sense of traces and the operator acts as the distributional Laplacian (cf. [4, Thm. 3.5] for the $W^{2,2}$ -regularity). We call $-\Delta_\alpha^{\Omega^{\text{ext}}}$ the *Robin Laplacian* in Ω^{ext} .

Since Ω is bounded, the embedding $W^{1,2}(\Omega^{\text{ext}}) \hookrightarrow L^2(\Omega^{\text{ext}})$ is *not* compact. In fact, the Robin Laplacian possesses a non-empty essential spectrum which equals $[0, \infty)$. This property is expected because the (essential) spectrum of the Laplacian in the whole space $\mathbb{R}^2$ (i.e. $\Omega = \emptyset$) equals $[0, \infty)$ and removing Ω can be understood as a compact perturbation. In order to keep the paper self-contained, we provide a proof of this claim which relies on an explicit construction of singular sequences and a Neumann bracketing argument.

Proposition 2.1. *We have $\sigma_{\text{ess}}(-\Delta_\alpha^{\Omega^{\text{ext}}}) = [0, \infty)$.*

Proof. First, we show the inclusion $\sigma_{\text{ess}}(-\Delta_\alpha^{\Omega^{\text{ext}}}) \supset [0, \infty)$ by constructing a suitable singular sequence. For any positive integer n , let us set $u_n(x) := \varphi_n(x) e^{ik \cdot x}$ with an arbitrary vector $k \in \mathbb{R}^2$ and $\varphi_n(x) := n^{-1} \varphi((x - nx_0)/n)$, where φ is a function from $C_0^\infty(\mathbb{R}^2)$ normalised to 1 in $L^2(\mathbb{R}^2)$ and $x_0 := (1, 0)$. The prefactor in the definition of φ_n is chosen in such a way that φ_n is normalised to 1 in $L^2(\mathbb{R}^2)$ for each n . In fact, we have $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$,

$$\|\nabla \varphi_n\|_{L^2(\mathbb{R}^2)} = n^{-1} \|\nabla \varphi\|_{L^2(\mathbb{R}^2)}, \quad \text{and} \quad \|\Delta \varphi_n\|_{L^2(\mathbb{R}^2)} = n^{-2} \|\Delta \varphi\|_{L^2(\mathbb{R}^2)}. \quad (9)$$

At the same time, the support of φ_n leaves any bounded set for all sufficiently large n . Consequently, for all sufficiently large n , we have $u_n \in C_0^\infty(\Omega^{\text{ext}}) \subset \mathcal{D}(-\Delta_\alpha^{\Omega^{\text{ext}}})$ and $\|u_n\|_{L^2(\Omega^{\text{ext}})} = 1$. A direct computation yields

$$\left| -\Delta_\alpha^{\Omega^{\text{ext}}} u_n - |k|^2 u_n \right| = \left| (-\Delta \varphi_n + 2ik \cdot \nabla \varphi_n) e^{ik \cdot x} \right| \leq |\Delta \varphi_n| + 2|k| |\nabla \varphi_n|.$$

Using (9), we therefore have

$$\left\| -\Delta_\alpha^{\Omega^{\text{ext}}} u_n - |k|^2 u_n \right\|_{L^2(\Omega^{\text{ext}})}^2 \leq 2 \|\Delta \varphi_n\|_{L^2(\mathbb{R}^2)}^2 + 8 |k|^2 \|\nabla \varphi_n\|_{L^2(\mathbb{R}^2)}^2 \xrightarrow{n \rightarrow \infty} 0.$$

Since k is arbitrary, we conclude that $[0, \infty) \subset \sigma(-\Delta_\alpha^{\Omega^{\text{ext}}})$ by [24, Thm. 7.22]. It is clear that $[0, \infty)$ actually belongs to the *essential* spectrum, because the interval has no isolated points.

Second, to show the opposite inclusion $\sigma_{\text{ess}}(-\Delta_\alpha^{\Omega^{\text{ext}}}) \subset [0, \infty)$, we use a Neumann bracketing argument. Let H_n be the operator that acts as $-\Delta_\alpha^{\Omega^{\text{ext}}}$ but satisfies an extra Neumann condition on the circle $\mathcal{C}_n := \{x \in \mathbb{R}^2 : |x| = n\}$ of radius $n > 0$. More specifically, H_n is the operator associated with the form

$$h_n[u] := \|\nabla u\|_{L^2(\Omega^{\text{ext}})}^2 + \alpha \|u\|_{L^2(\Sigma)}^2, \quad \mathcal{D}(h_n) := W^{1,2}(\Omega^{\text{ext}} \setminus \mathcal{C}_n).$$

Because of the domain inclusion $\mathcal{D}(h_n) \supset \mathcal{D}(Q_\alpha^{\Omega^{\text{ext}}})$, we have $-\Delta_\alpha^{\Omega^{\text{ext}}} \geq H_n$ and, by the min-max principle, $\inf \sigma_{\text{ess}}(-\Delta_\alpha^{\Omega^{\text{ext}}}) \geq \inf \sigma_{\text{ess}}(H_n)$ for all n . Assuming that n is sufficiently large so that Ω is contained in the disk $B_n := \{x \in \mathbb{R}^2 : |x| < n\}$, H_n decouples into an orthogonal sum of two operators, $H_n = H_n^{(1)} \oplus H_n^{(2)}$ with respect to the decomposition $L^2(\Omega^{\text{ext}}) = L^2(\Omega^{\text{ext}} \cap B_n) \oplus L^2(\Omega^{\text{ext}} \setminus \overline{B_n})$. Here $H_n^{(1)}$ and $H_n^{(2)}$ are respectively the operators in $L^2(\Omega^{\text{ext}} \cap B_n)$ and $L^2(\Omega^{\text{ext}} \setminus \overline{B_n})$ associated with the forms

$$\begin{aligned} h_n^{(1)}[u] &:= \|\nabla u\|_{L^2(\Omega^{\text{ext}} \cap B_n)}^2 + \alpha \|u\|_{L^2(\Sigma)}^2, & \mathcal{D}(h_n^{(1)}) &:= W^{1,2}(\Omega^{\text{ext}} \cap B_n), \\ h_n^{(2)}[u] &:= \|\nabla u\|_{L^2(\Omega^{\text{ext}} \setminus \overline{B_n})}^2, & \mathcal{D}(h_n^{(2)}) &:= W^{1,2}(\Omega^{\text{ext}} \setminus \overline{B_n}). \end{aligned}$$

Since $\Omega^{\text{ext}} \cap B_n$ is a smooth bounded open set, the spectrum of $H_n^{(1)}$ is purely discrete. Consequently, $\inf \sigma_{\text{ess}}(-\Delta_\alpha^{\Omega^{\text{ext}}}) \geq \inf \sigma_{\text{ess}}(H_n^{(2)}) \geq \inf \sigma(H_n^{(2)}) \geq 0$, where the last inequality follows by the fact that $H_n^{(2)}$ is non-negative. $\square$

Despite the presence of an essential spectrum, it still makes sense to define the lowest point in the spectrum of $-\Delta_\alpha^{\Omega^{\text{ext}}}$ by the variational formula (cf. (2))

$$\lambda_1^\alpha(\Omega^{\text{ext}}) := \inf_{\substack{u \in W^{1,2}(\Omega^{\text{ext}}) \\ u \neq 0}} \frac{Q_\alpha^{\Omega^{\text{ext}}}[u]}{\|u\|_{L^2(\Omega^{\text{ext}})}^2}. \quad (10)$$

However, it is not evident that it represents a discrete eigenvalue of $-\Delta_\alpha^{\Omega^{\text{ext}}}$. Obviously, it is not the case if α is non-negative, in which case $-\Delta_\alpha^{\Omega^{\text{ext}}}$ is non-negative and therefore its spectrum is purely essential. The following result shows that the situation of negative α is different.

Proposition 2.2. *If $\alpha < 0$ and Ω is not empty, then $\sigma_{\text{disc}}(-\Delta_\alpha^{\Omega^{\text{ext}}}) \neq \emptyset$. More specifically, $\lambda_1^\alpha(\Omega^{\text{ext}})$ is a negative discrete eigenvalue.*

Proof. By Proposition 2.1 and (10), it is enough to find a test function $u \in W^{1,2}(\Omega^{\text{ext}})$ such that $Q_\alpha^{\Omega^{\text{ext}}}[u]$ is negative. For any positive number n , we

introduce a function $u_n: \mathbb{R}^2 \rightarrow [0, 1]$ by setting $u_n(x) := \varphi_n(|x|)$ with

$$\varphi_n(r) := \begin{cases} 1 & \text{if } r < n, \\ \frac{\log n^2 - \log r}{\log n^2 - \log n} & \text{if } n < r < n^2, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check that the restriction of u_n to Ω^{ext} (that we shall denote by the same symbol) belongs to $W^{1,2}(\Omega^{\text{ext}})$ for every n . By employing polar coordinates, we have

$$\begin{aligned} \|\nabla u_n\|_{L^2(\Omega^{\text{ext}})}^2 &\leq \|\nabla u_n\|_{L^2(\mathbb{R}^2)}^2 = 2\pi \int_0^\infty |\varphi'_n(r)|^2 r \, dr \\ &= 2\pi \int_n^{n^2} \frac{1}{(\log n)^2 r} \, dr = \frac{2\pi}{\log n} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

On the other hand, $\|u_n\|_{L^2(\Sigma)}^2 = |\Sigma| > 0$ for all sufficiently large n . Since α is assumed to be negative, it follows that $Q_\alpha^{\Omega^{\text{ext}}}[u_n]$ can be made negative for n large enough. $\square$

Summing up, if α is negative, the essential spectrum of $-\Delta_\alpha^{\Omega^{\text{ext}}}$ equals the interval $[0, \infty)$ and there is at least one discrete eigenvalue below 0. In particular, the lowest point $\lambda_1^\alpha(\Omega^{\text{ext}})$ in the spectrum is always a negative discrete eigenvalue. By standard methods (see, e.g., [17, Thm. 8.38]), it is possible to show that $\lambda_1^\alpha(\Omega^{\text{ext}})$ is simple and that the corresponding eigenfunction u_1^α can be chosen to be positive in Ω^{ext} (recall that we always assume that Ω^{ext} is connected).

It is straightforward to verify that $\{Q_\alpha^{\Omega^{\text{ext}}}\}_{\alpha \in \mathbb{R}}$ is a holomorphic family of forms of type (a) in the sense of Kato [18, Sec. VII.4]. In fact, recalling that the boundary term in (8) is relatively bounded with respect to the Neumann form $Q_0^{\Omega^{\text{ext}}}$ with the relative bound equal to zero, one can use [18, Thm. 4.8] to get the claim. Consequently, $-\Delta_\alpha^{\Omega^{\text{ext}}}$ forms a self-adjoint holomorphic family of operators of type (B). Because of the simplicity, it follows that $\alpha \mapsto \lambda_1^\alpha(\Omega^{\text{ext}})$ and $\alpha \mapsto u_1^\alpha$ with $\|u_1^\alpha\| = 1$ are real-analytic functions on $(-\infty, 0)$.

Proposition 2.3. *Let $\alpha < 0$ and $\Omega \neq \emptyset$. Then $\alpha \mapsto \lambda_1^\alpha(\Omega^{\text{ext}})$ is a strictly concave increasing function.*

Proof. For simplicity, let us set $\lambda_1^\alpha := \lambda_1^\alpha(\Omega^{\text{ext}})$. The eigenvalue equation $-\Delta_\alpha^{\Omega^{\text{ext}}} u_1^\alpha = \lambda_1^\alpha u_1^\alpha$ means that

$$\forall \varphi \in W^{1,2}(\Omega^{\text{ext}}), \quad Q_\alpha^{\Omega^{\text{ext}}}(\varphi, u_1^\alpha) = \lambda_1^\alpha(\varphi, u_1^\alpha)_{L^2(\Omega^{\text{ext}})}. \quad (11)$$

Differentiating the identity (11) with respect to α , we easily arrive at the formula

$$\begin{aligned} (\nabla\varphi, \nabla\dot{u}_1^\alpha)_{L^2(\Omega^{\text{ext}})} + (\varphi, u_1^\alpha)_{L^2(\Sigma)} + \alpha(\varphi, \dot{u}_1^\alpha)_{L^2(\Sigma)} &= \\ &= \dot{\lambda}_1^\alpha(\varphi, u_1^\alpha)_{L^2(\Omega^{\text{ext}})} + \lambda_1^\alpha(\varphi, \dot{u}_1^\alpha)_{L^2(\Omega^{\text{ext}})}, \end{aligned} \quad (12)$$

where the dot denotes the derivative with respect to α . Notice that the differentiation below the integral signs is permitted because $\dot{u}_1^\alpha \in W^{1,2}(\Omega^{\text{ext}})$ by standard elliptic regularity theory. Moreover, differentiating the normalisation condition $\|u_1^\alpha\| = 1$, we get the orthogonality property

$$(u_1^\alpha, \dot{u}_1^\alpha)_{L^2(\Omega^{\text{ext}})} = 0. \quad (13)$$

Now, substituting $\varphi = u_1^\alpha$ into (12) and $\varphi = \dot{u}_1^\alpha$ into (11) and taking the difference of the resulting equations, we get a formula for the eigenvalue derivative

$$\dot{\lambda}_1^\alpha = \|u_1^\alpha\|_{L^2(\Sigma)}^2 > 0. \quad (14)$$

The above inequality is strict because otherwise u_1^α would be an eigenfunction of the Dirichlet Laplacian on Ω^{ext} corresponding to a negative eigenvalue λ_1^α , which is a contradiction to the non-negativity of the latter operator. This proves that $\alpha \mapsto \lambda_1^\alpha$ is strictly increasing.

Next, we differentiate equation (14) with respect to α ,

$$\begin{aligned} \ddot{\lambda}_1^\alpha &= \frac{d}{d\alpha} \left(\|u_1^\alpha\|_{L^2(\Sigma)}^2 \right) = 2(u_1^\alpha, \dot{u}_1^\alpha)_{L^2(\Sigma)} \\ &= 2\lambda_1^\alpha \|\dot{u}_1^\alpha\|_{L^2(\Omega^{\text{ext}})}^2 - 2Q_\alpha^{\Omega^{\text{ext}}}[\dot{u}_1^\alpha] < 0. \end{aligned} \quad (15)$$

Here the last equality employs (12) with the choice $\varphi = \dot{u}_1^\alpha$ and (13). The inequality follows from the fact that λ_1^α is the lowest eigenvalue of $-\Delta_\alpha^{\Omega^{\text{ext}}}$. The above inequality is indeed strict since otherwise $\dot{u}_1^\alpha$ would be either another eigenfunction of $-\Delta_\alpha^{\Omega^{\text{ext}}}$ corresponding to λ_1^α , which is impossible because of the simplicity, or a constant multiple of u_1^α , which would imply $\dot{u}_1^\alpha = 0$ due to (13). In the latter case (12) gives

$$\forall \varphi \in C_0^\infty(\Omega^{\text{ext}}), \quad (\varphi, u_1^\alpha)_{L^2(\Omega^{\text{ext}})} = 0,$$

and therefore $u_1^\alpha = 0$, which is also a contradiction. From (15) we therefore conclude that $\alpha \mapsto \lambda_1^\alpha$ is strictly concave. $\square$

As a consequence of Proposition 2.3, we get

$$\lim_{\alpha \rightarrow -\infty} \lambda_1^\alpha(\Omega^{\text{ext}}) = -\infty. \quad (16)$$

3. The lowest eigenvalue in the exterior of a disk

In this section, we establish some properties of $\lambda_1^\alpha(B_R^{\text{ext}})$, where B_R is an open disk of radius $R > 0$. Without loss of generality, we can assume that B_R is centred at the origin of $\mathbb{R}^2$. We always assume that α is negative.

Using the rotational symmetry, it is easily seen that $\lambda_1^\alpha(B_R^{\text{ext}}) = -k^2 < 0$ is the smallest solution of the ordinary differential spectral problem

$$\begin{cases} -r^{-1}[r\psi'(r)]' = \lambda\psi(r), & r \in (R, \infty), \\ -\psi'(R) + \alpha\psi(R) = 0, \\ \lim_{r \rightarrow \infty} \psi(r) = 0. \end{cases} \quad (17)$$

The general solution of the differential equation in (17) is given by

$$\psi(r) = C_1 K_0(kr) + C_2 I_0(kr), \quad C_1, C_2 \in \mathbb{C}, \quad (18)$$

where K_0, I_0 are modified Bessel functions [1, Sec. 9.6]. The solution $I_0(kr)$ is excluded because it diverges as $r \rightarrow \infty$, whence $C_2 = 0$. Requiring ψ to satisfy the Robin boundary condition at R leads us to the implicit equation

$$kK_1(kR) + \alpha K_0(kR) = 0 \quad (19)$$

that k must satisfy as a function of α and R .

First of all, we state the following upper and lower bounds.

Proposition 3.1. *We have*

$$-\alpha^2 < \lambda_1^\alpha(B_R^{\text{ext}}) < -\alpha^2 - \frac{\alpha}{R}$$

for all negative α .

Proof. For simplicity, let us set $\lambda_R := \lambda_1^\alpha(B_R^{\text{ext}})$ and recall the notation $\lambda_R = -k^2$. Using (19) and [23, Eq. 74] (with $\nu = 0$), we have

$$\lambda_R = -k^2 = -\alpha^2 \left(\frac{K_0(kR)}{K_1(kR)} \right)^2 > -\alpha^2.$$

This establishes the lower bound of the proposition. In the case $\alpha \in (-R^{-1}, 0)$, we obtain the upper bound from the elementary estimate

$$\lambda_R + \alpha^2 + \frac{\alpha}{R} < \alpha^2 + \frac{\alpha}{R} = \alpha \left(\alpha + \frac{1}{R} \right) < 0,$$

where we have used the fact that λ_R is negative. In the other case $\alpha \in (-\infty, -R^{-1}]$, we get by [23, Thm. 1] (with $\nu = 1/2$) that

$$k^2 = \alpha^2 \left(\frac{K_0(kR)}{K_1(kR)} \right)^2 > \frac{\alpha^2(kR)^2}{1/2 + (kR)^2 + \sqrt{1/4 + (kR)^2}}.$$

The latter inequality implies

$$\sqrt{\frac{1}{4R^4} + \frac{k^2}{R^2}} > \alpha^2 - \frac{1}{2R^2} - k^2. \quad (20)$$

If the right-hand side of (20) is negative, then

$$-k^2 < -\alpha^2 + \frac{1}{2R^2} = -\alpha^2 + \frac{1}{R} \frac{1}{2R} \leq -\alpha^2 - \frac{\alpha}{2R} < -\alpha^2 - \frac{\alpha}{R},$$

which is the desired inequality. If the right-hand side of (20) is non-negative, we take the squares of both the right- and left-hand sides of (20) and obtain

$$\frac{1}{4R^4} + \frac{k^2}{R^2} > \left(\alpha^2 - \frac{1}{2R^2} - k^2 \right)^2 = \left(\alpha^2 - \frac{1}{2R^2} \right)^2 - 2k^2 \left(\alpha^2 - \frac{1}{2R^2} \right) + k^4.$$

This inequality is equivalent to $0 > (\alpha^2 - k^2)^2 - \frac{\alpha^2}{R^2}$.

Consequently, $\alpha^2 - k^2 < -\frac{\alpha}{R}$, which again yields the desired upper bound. $\square$

We notice that analogous upper and lower bounds for $\lambda_1^\alpha(B_R)$ have been recently established in [2, Thm. 3]. Moreover, it has been shown in [2, Thm. 5] that $R \mapsto \lambda_1^\alpha(B_R)$ is strictly increasing. Now we have a reversed monotonicity result.

Proposition 3.2. *If α is negative, then $R \mapsto \lambda_1^\alpha(B_R^{\text{ext}})$ is strictly decreasing.*

Proof. We follow the strategy of the proof of [2, Thm. 5]. Computing the derivative of $\lambda_R := \lambda_1^\alpha(B_R^{\text{ext}})$ using the differential equation that λ_R satisfies, one finds (cf. [2, Lem. 2])

$$\frac{\partial \lambda_R}{\partial R} = -\frac{2}{R} \lambda_R + \alpha \frac{\psi_R(R)^2}{\int_R^\infty \psi_R(r)^2 r \, dr}, \quad (21)$$

where $\psi_R(r) := K_0(kr)$ is the eigenfunction corresponding to $\lambda_R = -k^2$. Employing the formula

$$\begin{aligned} \int_R^\infty K_0(kr)^2 r \, dr &= \frac{r^2}{2} [K_0(kr)^2 - K_1(kr)^2] \Big|_{r=R}^{r=\infty} \\ &= \frac{R^2}{2} [K_1(kR)^2 - K_0(kR)^2] \end{aligned} \quad (22)$$

and (19), we eventually arrive at the equivalent identity for the eigenvalue derivative

$$\frac{\partial \lambda_R}{\partial R} = -\frac{2}{R} \lambda_R \frac{\lambda_R + \alpha^2 + \frac{\alpha}{R}}{\lambda_R + \alpha^2}. \quad (23)$$

The proof is concluded by recalling Proposition 3.1, which implies that the right-hand side is negative. $\square$

4. Proof of Theorem 1.3

Now we are in a position to establish Theorem 1.3. Throughout this section, $\Omega \subset \mathbb{R}^2$ is a *convex* bounded open set with smooth boundary $\Sigma := \partial\Omega$; then Ω^{ext} is necessarily connected. We also assume that α is negative.

The main idea of the proof is to parameterise Ω^{ext} by means of the *parallel coordinates*

$$\mathcal{L} : \Sigma \times (0, \infty) \rightarrow \Omega^{\text{ext}} : \{(s, t) \mapsto s + n(s)t\}, \quad (24)$$

where n is the outer unit normal to Ω as above. Notice that $\mathcal{L}$ is indeed a diffeomorphism because of the convexity and smoothness assumptions. To be more specific, the metric induced by (24) acquires the diagonal form

$$d\mathcal{L}^2 = (1 - \kappa(s)t)^2 ds^2 + dt^2, \quad (25)$$

where $\kappa := -dn$ is the *curvature* of $\partial\Omega$. By our choice of n , the function κ is non-positive because Ω is convex (cf. [20, Thm. 2.3.2]). Consequently, the Jacobian of (24) given by $1 - \kappa(s)t$ is greater than or equal to 1. In particular, it is positive and therefore $\mathcal{L}$ is a local diffeomorphism by the inverse function theorem (see, e.g., [20, Thm. 0.5.1]). To see that it is a global diffeomorphism, notice that $\mathcal{L}$ is injective, because of the convexity assumption, and that $\mathcal{L}$ is surjective thanks to the smoothness of Ω .

Summing up, Ω^{ext} can be identified with the product manifold $\Sigma \times (0, \infty)$ equipped with the metric (25). Consequently, the Hilbert space $L^2(\Omega^{\text{ext}})$ can be identified with

$$\mathcal{H} := L^2(\Sigma \times (0, \infty), (1 - \kappa(s)t) ds dt).$$

The identification is provided by the unitary transform

$$U : L^2(\Omega^{\text{ext}}) \rightarrow \mathcal{H} : \{u \mapsto u \circ \mathcal{L}\}. \quad (26)$$

It is thus natural to introduce $H_\alpha := U(-\Delta_\alpha^{\Omega^{\text{ext}}})U^{-1}$, which is the unitarily equivalent operator associated to the transformed form $h_\alpha[\psi] := Q_\alpha^{\Omega^{\text{ext}}}[U^{-1}\psi]$, $D(h_\alpha) := UD(Q_\alpha^{\Omega^{\text{ext}}})$. Of course, we have the equivalent characterisation of the lowest eigenvalue

$$\lambda_1^\alpha(\Omega^{\text{ext}}) = \inf_{\substack{\psi \in D(h_\alpha) \\ \psi \neq 0}} \frac{h_\alpha[\psi]}{\|\psi\|_{\mathcal{H}}^2}. \quad (27)$$

The set of restrictions of functions from $C_0^\infty(\mathbb{R}^2)$ to Ω^{ext} is a core of $Q_\alpha^{\Omega^{\text{ext}}}$. Taking u from this core, it is easily seen that $\psi := Uu$ is a restriction of a function $C_0^\infty(\partial\Omega \times \mathbb{R})$ to $\partial\Omega \times (0, \infty)$ and that

$$\begin{aligned} h_\alpha[\psi] &= \int_{\Sigma \times (0, \infty)} \left(\frac{|\partial_s \psi(s, t)|^2}{1 - \kappa(s)t} + |\partial_t \psi(s, t)|^2 (1 - \kappa(s)t) \right) ds dt + \alpha \int_{\Sigma} |\psi(s, 0)|^2 ds, \\ \|\psi\|_{\mathcal{H}}^2 &= \int_{\Sigma \times (0, \infty)} |\psi(s, t)|^2 (1 - \kappa(s)t) ds dt. \end{aligned}$$

Restricting (27) to test functions with level lines parallel to Σ , i.e. taking ψ independent of s , we obtain

$$\lambda_1^\alpha(\Omega^{\text{ext}}) \leq \inf_{\substack{\psi \in C_0^\infty([0, \infty)) \\ \psi \neq 0}} \frac{\int_0^\infty |\psi'(t)|^2 (|\Sigma| + 2\pi t) dt + \alpha |\Sigma| |\psi(0)|^2}{\int_0^\infty |\psi(t)|^2 (|\Sigma| + 2\pi t) dt}. \quad (28)$$

Here we have used the geometric identity $\int_\Sigma \kappa = -2\pi$ (see, e.g., [20, Cor. 2.2.2]).

Now, assume that the perimeter is fixed, i.e. $|\Sigma| = c_1$. Since the perimeter is the only geometric quantity on which the right-hand side of (28) depends and since the eigenfunction corresponding to $\lambda_1^\alpha(B_{R_1}^{\text{ext}})$ is radially symmetric (therefore independent of s in the parallel coordinates), we immediately obtain

$$\lambda_1^\alpha(\Omega^{\text{ext}}) \leq \lambda_1^\alpha(B_{R_1}^{\text{ext}}) \quad (29)$$

for any Ω of the fixed perimeter c_1 . This proves the isoperimetric optimisation result of Theorem 1.3.

To establish the isochoric optimisation result of Theorem 1.3, we notice that since $|\Sigma| = |\partial B_{R_1}|$, the classical (geometric) isoperimetric inequality implies $|\Omega| \leq |B_{R_1}|$, with equality if and only if Ω is the disk. Hence, there exists $B_{R_2} \subset B_{R_1}$ such that $|B_{R_2}| = |\Omega|$. By (29), it is then enough to show that $\lambda_1^\alpha(B_{R_1}^{\text{ext}}) \leq \lambda_1^\alpha(B_{R_2}^{\text{ext}})$. This inequality (and therefore the second result of Theorem 1.3) is a consequence of the more general monotonicity result of Proposition 3.2. $\square$

5. Conclusions

5.1. Necessity of convexity

The assumption on Ω to be convex is necessary in view of the following simple counterexample. Let $\Omega \subset \mathbb{R}^2$ be the union of two disks $B_{R_3}(x_1)$ and $B_{R_3}(x_2)$ of the same radius $R_3 > 0$ whose centres x_1 and x_2 are chosen in such a way that the closures of the disks in $\mathbb{R}^2$ are disjoint. According to [19, Thm. 1.1] (see also [16, Thm. 3]), we have

$$\begin{aligned} \lambda_1^\alpha(\Omega^{\text{ext}}) &= -\alpha^2 - \frac{\alpha}{R_3} + o(\alpha), & \alpha \rightarrow -\infty, \\ \lambda_1^\alpha(B_R^{\text{ext}}) &= -\alpha^2 - \frac{\alpha}{R} + o(\alpha), & \alpha \rightarrow -\infty. \end{aligned}$$

The constraints $|\partial\Omega| = |\partial B_{R_1}|$ and $|\Omega| = |B_{R_2}|$ yield that $R_1 = 2R_3$ and $R_2 = \sqrt{2}R_3$, respectively. Taking into account the above large coupling asymptotics and the relations between the radii, we observe that for $\alpha < 0$ with sufficiently large $|\alpha|$ the reverse inequalities $\lambda_1^\alpha(B_{R_1}^{\text{ext}}) < \lambda_1^\alpha(\Omega^{\text{ext}})$ and $\lambda_1^\alpha(B_{R_2}^{\text{ext}}) < \lambda_1^\alpha(\Omega^{\text{ext}})$ are satisfied.

We point out that, while the domain Ω of the above counterexample is disconnected, its exterior Ω^{ext} is still connected. We leave it as an open question whether there exists a counterexample in the class of connected non-convex domains Ω .

5.2. Uniqueness of the optimiser

In this subsection, we demonstrate that the exterior of the disk is the *unique* maximiser for both the isochoric and isoperimetric spectral optimisation problems of Theorem 1.3.

Theorem 5.1. *Let α be negative. For all convex smooth bounded open sets $\Omega \subset \mathbb{R}^2$ of a fixed perimeter (respectively, fixed area) different from a disk B_{R_1} of the same perimeter (respectively, from a disk B_{R_2} of the same area), we have a strict inequality*

$$\alpha < 0 : \quad \lambda_1^\alpha(\Omega^{\text{ext}}) < \lambda_1^\alpha(B_{R_1}^{\text{ext}}) \quad (\text{respectively, } \lambda_1^\alpha(\Omega^{\text{ext}}) < \lambda_1^\alpha(B_{R_2}^{\text{ext}})). \quad (30)$$

Proof. As usual, $\lambda_1^\alpha := \lambda_1^\alpha(\Omega^{\text{ext}})$ and u_1^α denote respectively the lowest eigenvalue and the corresponding eigenfunction of $-\Delta_\alpha^{\Omega^{\text{ext}}}$ with $\alpha < 0$. Without loss of generality, we assume that u_1^α is positive everywhere in Ω^{ext} . Furthermore, we introduce the auxiliary function $\psi := Uu_1^\alpha$ where the unitary transform U is as in (26). In view of Theorem 1.3 and inequality (28) in its proof, non-uniqueness of the optimiser for the spectral isoperimetric problem would necessarily imply the existence of a non-circular domain Ω for which the function ψ is independent of s ; i.e. its level lines are parallel to $\Sigma := \partial\Omega$. In the sequel, with a slight abuse of notation, we use the same symbol ψ to denote the function $t \mapsto \psi(t)$ of a single variable.

Restricting ourselves to test functions with support lying inside Ω^{ext} , the variational characterisation of ψ implies

$$\forall \varphi \in C_0^\infty(\Sigma \times (0, \infty)), \quad h_\alpha(\varphi, \psi) = \lambda_1^\alpha(\varphi, \psi)_\mathcal{H}.$$

It is enough to consider real-valued test functions φ only. Taking into account that ψ is independent of s , we end up with the identity

$$\int_{\Sigma \times (0, \infty)} \partial_t \varphi(s, t) \psi'(t) (1 - \kappa(s)t) \, ds \, dt = \lambda_1^\alpha \int_{\Sigma \times (0, \infty)} \varphi(s, t) \psi(t) (1 - \kappa(s)t) \, ds \, dt$$

valid for all real-valued $\varphi \in C_0^\infty(\Sigma \times (0, \infty))$. Now we restrict our attention to test functions of the type $\varphi(s, t) = \varphi_1(s)\varphi_2(t) \in C_0^\infty(\Omega^{\text{ext}})$ with $\varphi_1 \in C^\infty(\Sigma)$ and $\varphi_2 \in C_0^\infty((0, \infty))$. Then the above displayed equation reduces to

$$\int_{\Sigma} \varphi_1(s) \int_0^\infty \varphi_2'(t) \psi'(t) (1 - \kappa(s)t) \, dt \, ds = \lambda_1^\alpha \int_{\Sigma} \varphi_1(s) \int_0^\infty \varphi_2(t) \psi(t) (1 - \kappa(s)t) \, dt \, ds.$$

The density of $C^\infty(\Sigma)$ in $L^1(\Sigma)$ gives us, for all $s \in \Sigma$ and $\varphi_2 \in C_0^\infty((0, \infty))$,

$$\int_0^\infty \varphi_2'(t) \psi'(t) (1 - \kappa(s)t) dt = \lambda_1^\alpha \int_0^\infty \varphi_2(t) \psi(t) (1 - \kappa(s)t) dt. \quad (31)$$

Since Ω is not a disk, there exist $s_1, s_2 \in \Sigma$ such that $\kappa(s_1) \neq \kappa(s_2)$ (see, e.g., [20, Prop. 1.4.3]). Taking the difference of (31) with $s = s_1$ and with $s = s_2$, we get

$$\forall \varphi_2 \in C_0^\infty((0, \infty)), \quad \int_0^\infty \varphi_2'(t) \psi'(t) t dt = \lambda_1^\alpha \int_0^\infty \varphi_2(t) \psi(t) t dt. \quad (32)$$

We fix a function $\eta \in C_0^\infty((0, \infty))$ which satisfies the following properties:

- (i) $0 \leq \eta \leq 1$,
- (ii) $\eta(t) = 1$ for all $t \in [1, 2]$,
- (iii) $\text{supp } \eta \subset [0, 3]$.

Then, for every positive integer n , we define a function $\eta_n \in C_0^\infty((0, \infty))$ by

$$\eta_n(t) := \begin{cases} \eta(nt), & t \in (0, \frac{2}{n}), \\ 1, & t \in (\frac{2}{n}, n+1), \\ \eta(t-n), & t \in (n+1, \infty). \end{cases} \quad (33)$$

Now we plug $\varphi_2 = \eta_n \psi \in C_0^\infty((0, \infty))$ into (32). By the dominated convergence theorem (using that $t \mapsto |\psi(t)|^2 t$ is integrable), we obtain

$$\int_0^\infty |\psi(t)|^2 \eta_n(t) t dt \xrightarrow{n \rightarrow \infty} \int_0^\infty |\psi(t)|^2 t dt. \quad (34)$$

The left-hand side in (32) with $\varphi_2 = \eta_n \psi$ can be rewritten as

$$\begin{aligned} I_n &:= \int_0^\infty (\eta_n \psi)'(t) \psi'(t) t dt \\ &= \int_0^\infty |\psi'(t)|^2 \eta_n(t) t dt + \int_0^\infty \psi'(t) \psi(t) \eta_n'(t) t dt. \end{aligned} \quad (35)$$

For the first term on the right-hand side in (35) we get

$$I_n^{(1)} := \int_0^\infty |\psi'(t)|^2 \eta_n(t) t dt \xrightarrow{n \rightarrow \infty} \int_0^\infty |\psi'(t)|^2 t dt, \quad (36)$$

by the dominated convergence (using that $t \mapsto |\psi'(t)|^2 t$ is integrable). The

second term on the right-hand side in (35) can be further transformed as

$$\begin{aligned} & \int_0^\infty \psi'(t) \psi(t) \eta'_n(t) t \, dt = \\ &= n \int_0^{2/n} \psi'(t) \psi(t) \eta'(nt) t \, dt + \int_{n+1}^\infty \psi'(t) \psi(t) \eta'(t-n) t \, dt \\ &= \int_0^2 \psi'\left(\frac{r}{n}\right) \psi\left(\frac{r}{n}\right) \frac{\eta'(r)r}{n} \, dr + \int_1^3 \psi'(r+n) \psi(r+n) \eta'(r) (r+n) \, dr. \end{aligned} \quad (37)$$

Again making use of the dominated convergence theorem, we obtain

$$I_n^{(2)} := \int_0^2 \psi'\left(\frac{r}{n}\right) \psi\left(\frac{r}{n}\right) \frac{\eta'(r)r}{n} \, dr \xrightarrow{n \rightarrow \infty} 0; \quad (38)$$

here we implicitly employed that the integrand is uniformly bounded in $n \in \mathbb{N}$. Observe that

$$\left| \sum_{n=1}^\infty \int_1^3 \psi'(r+n) \psi(r+n) \eta'(r) (r+n) \, dr \right| \leq 2 \|\eta'\|_\infty \int_0^\infty |\psi(t) \psi'(t)| t \, dt < \infty,$$

where finiteness of the latter integral follows from the fact that $\psi \in D(h_\alpha)$. Therefore, we conclude

$$I_n^{(3)} := \int_1^3 \psi'(r+n) \psi(r+n) \eta'(r) (r+n) \, dr \xrightarrow{n \rightarrow \infty} 0. \quad (39)$$

Combining the decompositions (35), (37) with the limits (36), (38), (39), we arrive at

$$I_n = I_n^{(1)} + I_n^{(2)} + I_n^{(3)} \xrightarrow{n \rightarrow \infty} \int_0^\infty |\psi'(t)|^2 t \, dt. \quad (40)$$

The limits (34), (40) and the condition (32) imply

$$\int_0^\infty |\psi'(t)|^2 t \, dt = \lambda_1^\alpha \int_0^\infty |\psi(t)|^2 t \, dt.$$

Finally, taking into account that λ_1^α is negative, we get a contradiction. This completes the proof of the first strict inequality in (30).

To show that disk is the unique optimiser for the spectral isochoric inequality is much simpler than in the isoperimetric case. Suppose that there exists a non-circular domain Ω for which $\lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_{R_2}^{\text{ext}})$ with $|\Omega| = |B_{R_2}|$. Note that for B_{R_1} with $|\partial B_{R_1}| = |\partial \Omega|$ we get $R_2 < R_1$ using the standard geometric isoperimetric inequality. Thus, Theorem 1.3 and Proposition 3.2 imply

$$\lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_{R_2}^{\text{ext}}) > \lambda_1^\alpha(B_{R_1}^{\text{ext}}) \geq \lambda_1^\alpha(\Omega^{\text{ext}}),$$

which is obviously a contradiction. $\square$

Remark 5.2. As a consequence of the first claim in Theorem 1.3, we obtain the following quantitative improvement upon the second inequality of (30)

$$\lambda_1^\alpha(\Omega^{\text{ext}}) \leq \lambda_1^\alpha(B_{R_2}^{\text{ext}}) - [\lambda_1^\alpha(B_{R_2}^{\text{ext}}) - \lambda_1^\alpha(B_{R_1}^{\text{ext}})], \quad (41)$$

where the radii R_1 and R_2 can be easily expressed through $|\partial\Omega|$ and $|\Omega|$, respectively, by virtue of the relations $|B_{R_1}| = |\partial\Omega|$ and $|B_{R_2}| = |\Omega|$. In view of the inequality $R_2 < R_1$, the difference $\lambda_1^\alpha(B_{R_2}^{\text{ext}}) - \lambda_1^\alpha(B_{R_1}^{\text{ext}})$ is positive by Proposition 3.2, so (41) indeed represents a quantified version of the reverse spectral isochoric inequality in the spirit of [7, 9]. More careful analysis of the derivative in (23) can be used to get a positive lower bound on this difference in terms of R_1 and R_2 .

5.3. Higher dimensions

We have already noticed that $\lambda_1^\alpha(\Omega^{\text{ext}})$ is not necessarily a discrete eigenvalue in higher dimensions. For any dimension $d \geq 3$, however, there exists a critical value $\alpha_0 < 0$ depending on Ω such that $\lambda_1^\alpha(\Omega^{\text{ext}})$ is a discrete eigenvalue if, and only if, $\alpha < \alpha_0$, so the optimisation problem in the exterior of a compact set becomes non-trivial in this regime. In this subsection, we argue that no analogue of Theorem 1.3 can be expected if $d \geq 3$.

To this aim we construct a simple counterexample which relies on the large coupling asymptotics for the lowest eigenvalue. First, we fix a ball $B_R \subset \mathbb{R}^d$ of arbitrary radius $R > 0$. Further, let $\Omega_0 \subset \mathbb{R}^d$ be the union of two disjoint balls $B_r(x_1)$ and $B_r(x_2)$ of the same sufficiently small radius $r > 0$ whose centers x_1 and x_2 are located at a distance $L > 0$. Finally, we define the domain Ω as the convex hull of Ω_0 . By choosing $L > 0$ large enough, we can satisfy either of the constraints $|\partial\Omega| = |\partial B_R|$ or $|\Omega| = |B_R|$. It can be easily checked that the domain Ω has a $C^{1,1}$ boundary and that the mean curvature of $\partial\Omega$ is piecewise constant, being equal to $-1/r$ on the hemispheric cups and to $-\frac{d-2}{(d-1)r}$ on the cylindrical face (in agreement with the rest of this paper, we compute the mean curvature with respect to the outer normal to the bounded set Ω). Applying [19, Thm. 1.1], we arrive at

$$\begin{aligned} \lambda_1^\alpha(\Omega^{\text{ext}}) &= -\alpha^2 - \frac{\alpha(d-2)}{r} + o(\alpha), & \alpha \rightarrow -\infty, \\ \lambda_1^\alpha(B_R^{\text{ext}}) &= -\alpha^2 - \frac{\alpha(d-1)}{R} + o(\alpha), & \alpha \rightarrow -\infty. \end{aligned}$$

In view of the above asymptotics, we infer that for $r < \frac{d-2}{d-1}R$ and for $\alpha < 0$ with sufficiently large $|\alpha|$ the reverse inequality $\lambda_1^\alpha(B_R^{\text{ext}}) < \lambda_1^\alpha(\Omega^{\text{ext}})$ holds.

We expect that a counterexample based on a (C^∞) -smooth domain can also be constructed with additional technical efforts.

5.4. More on dimension three

The previous subsection demonstrates that, contrary to the two-dimensional situation, the exterior of the ball can be a global maximiser neither for the isoperimetric nor isochoric problems. Let us look at where the technical approach of the present paper fails in dimension three.

Let Ω be a convex smooth bounded open set in $\mathbb{R}^3$. In this case, the usage of parallel coordinates based on $\Sigma := \partial\Omega$ and restricting to test functions depending only on the distance to the boundary yield

$$\begin{aligned} & \lambda_1^\alpha(\Omega^{\text{ext}}) \\ & \leq \inf_{\substack{\psi \in C_0^\infty([0, \infty)) \\ \psi \neq 0}} \frac{\int_{\Sigma \times (0, \infty)} |\psi'(t)|^2 (1 - 2M(s)t + K(s)t^2) \, d\Sigma \, dt + \alpha \int_{\Sigma} |\psi(0)|^2 \, d\Sigma}{\int_{\Sigma \times (0, \infty)} |\psi(t)|^2 (1 - 2M(s)t + K(s)t^2) \, d\Sigma \, dt}. \end{aligned} \quad (42)$$

Here $d\Sigma := |g|^{1/2}(s) \, ds$ is the surface measure of Σ , with g being the Riemannian metric of Σ induced by the embedding of Σ in $\mathbb{R}^3$, and K and M denote respectively the *Gauss curvature* and the *mean curvature* of Σ (see [11] for more geometric details). K is an intrinsic quantity, while M is non-positive when computed with respect to our choice (outer to Ω) of the normal vector field n .

By definition, $\int_{\Sigma} 1 \, d\Sigma$ equals the total area $|\Sigma|$ of Σ , while $\int_{\Sigma} K(s) \, d\Sigma = 4\pi$ by the Gauss-Bonnet theorem for closed surfaces diffeomorphic to the sphere (see [20, Thm. 6.3.5]). The quantity $\mathcal{M}_{\Sigma} := \int_{\Sigma} |M(s)| \, d\Sigma$ is known as the half of the *total mean curvature* of Σ (see [10, § 28.1.3]). Moreover, we have [10, § 19] $\mathcal{M}_{\Sigma} = 2\pi b(\Omega)$, where $b(\Omega)$ is the *mean width* of Ω . Consequently,

$$\begin{aligned} & \lambda_1^\alpha(\Omega^{\text{ext}}) \\ & \leq \inf_{\substack{\psi \in C_0^\infty([0, \infty)) \\ \psi \neq 0}} \frac{\int_0^\infty |\psi'(t)|^2 (|\Sigma| + 2\mathcal{M}_{\Sigma}t + 4\pi t^2) \, dt + \alpha |\Sigma| |\psi(0)|^2}{\int_0^\infty |\psi(t)|^2 (|\Sigma| + 2\mathcal{M}_{\Sigma}t + 4\pi t^2) \, dt}. \end{aligned} \quad (43)$$

To get a reverse spectral isoperimetric inequality as in the planar case above, we would need in addition to the constraint $|\partial\Omega| = c_1$ also require that the mean width $b(\Omega)$ is fixed. However, the Minkowski quadratic inequality for cross-sectional measures (cf. [10, § 20.2]), $\mathcal{M}_{\Sigma}^2 \geq 4\pi |\Sigma|$, with equality only if Ω is a ball, implies that the two simultaneous constraints are possible only if either the class of admissible domains excludes the ball or the class of admissible domains consists of the ball only. In the first case our method is not applicable, while in the second case the method can be applied but it yields a trivial statement.

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