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On Minimax Theorems for Lower Semicontinuous Functions in Hilbert Spaces

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Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

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We prove minimax theorems for lower semicontinuous functions defined on a Hilbert space. The main tools are the theory of Φ-convex functions and sufficient and necessary conditions for the minimax equality for general Φ-convex functions. The conditions we propose are expressed in terms of abstract Φ-subgradients.

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1. Introduction

Let X and Y be nonempty sets and let $a\colon X \times Y \to \hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, be an extended real-valued function defined on $X \times Y$. Minimax theorems provide conditions for the minimax equality

$$\sup_{y\in Y} \inf_{x\in X} a(x,y) = \inf_{x\in X} \sup_{y\in Y} a(x,y)$$

to hold for a. An exhaustive survey of minimax theorems is given e.g. in [14].

In this paper we prove minimax theorems for lower semicontinuous functions defined on a Hilbert space and bounded from below by a quadratic function.

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The novelty of the results presented is that we do not exploit any compactness and/or connectedness assumptions. Our approach is based on minimax theorems for abstract convex functions obtained in [1, 16, 17, 18].

We start with definitions related to abstract convexity. Let X be a set. Let Φ be a set of real-valued functions $\varphi : X \to \mathbb{R}$.

For any $f, g : X \to \hat{\mathbb{R}}$ we define $f \le g \;\Leftrightarrow\; f(x) \le g(x) \;\; \forall x \in X$.

Let $f : X \to \hat{\mathbb{R}}$. The set $\operatorname{supp}(f, \Phi) := \{\varphi \in \Phi \;:\; \varphi \le f\}$ is called the *support* of f with respect to Φ. We will use the notation $\operatorname{supp}(f)$ if the class Φ is clear from the context.

Definition 1.1. ([4, 9, 12]) A function $f : X \to \hat{\mathbb{R}}$ is called Φ-*convex* if

$$f(x) = \sup\{\varphi(x) \;:\; \varphi \in \operatorname{supp}(f)\} \;\; \forall\, x \in X.$$

By convention, $\operatorname{supp}(f) = \emptyset$ if and only if $f \equiv -\infty$. In this paper we always assume that $\operatorname{supp}(f) \neq \emptyset$, i.e. we limit our attention to functions $f : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$.

We say that a function $f : X \to \bar{\mathbb{R}}$ is *proper* if the effective domain of f is nonempty, i.e.

$$\operatorname{dom}(f) := \{x \in X \;:\; f(x) < +\infty\} \neq \emptyset.$$

If, for every $y \in Y$, the function $a(\cdot, y)$ is Φ-convex, then a sufficient and necessary condition for a to satisfy the minimax equality is the so called intersection property introduced in [1] and investigated in [16, 17, 18]. Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be any functions from the set Φ_{lsc} defined below and $\alpha \in \mathbb{R}$. We say that *the intersection property* holds for φ_1 and φ_2 on X at the level α if and only if

$$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset,$$

where $[\varphi < \alpha] := \{x \in X \;:\; \varphi(x) < \alpha\}$ is the strict lower level set of function $\varphi : X \to \mathbb{R}$. Analogously we define the intersection property for functions from the class Φ_{conv} defined below.

From now on we consider classes Φ which are closed under vertical shift, i.e. $\varphi + c \in \Phi$ for any $\varphi \in \Phi$ and $c \in \mathbb{R}$. General minimax theorems for Φ-convex functions are proved in Theorem 1.2 of [18] and Theorem 3.3.3 of [16]. In the case where $\Phi = \Phi_{lsc}$ or $\Phi = \Phi_{conv}$ Theorem 3.3.3 of [16] can be rewritten in the following form.

Theorem 1.2. *Let X be a nonempty set and Y be a real vector space and $\Phi = \Phi_{lsc}$ or $\Phi = \Phi_{conv}$. Let $a : X \times Y \to \bar{\mathbb{R}}$. Assume that for any $y \in Y$ the function $a(\cdot, y) : X \to \bar{\mathbb{R}}$ is proper Φ-convex on X and for any $x \in X$ the function $a(x, \cdot) : Y \to \bar{\mathbb{R}}$ is concave on Y. The following conditions are equivalent:*

(i) *for every* $\alpha \in \mathbb{R}$, $\alpha < \inf_{x\in X}\sup_{y\in Y} a(x,y)$, *there exist* $y_1, y_2 \in Y$ *and* $\varphi_1 \in \operatorname{supp} a(\cdot, y_1)$, $\varphi_2 \in \operatorname{supp} a(\cdot, y_2)$ *such that the intersection property holds for* φ_1 *and* φ_2 *on* X *at the level* α,

(ii) $\sup_{y\in Y}\inf_{x\in X} a(x,y) = \inf_{x\in X}\sup_{y\in Y} a(x,y)$.

Condition (i) of Theorem 1.2 requires the existence of two functions from support sets $\operatorname{supp} a(\cdot, y_1)$ and $\operatorname{supp} a(\cdot, y_2)$ for which the intersection property holds on X at the level $\alpha < \inf_{x\in X}\sup_{y\in Y} a(x,y)$. A natural question is whether it is possible to choose these two functions as Φ-subgradients of $a(\cdot, y_1)$ and $a(\cdot, y_2)$ respectively, at some points.

In the present paper we investigate the possibility of expressing condition (i) of Theorem 1.2 with the help of Φ-subgradients. We consider Φ-convex functions $a(\cdot, y)$ for two classes Φ. The first class is defined as

$$\Phi_{conv} := \{\varphi\colon X \to \mathbb{R},\ \varphi(x) = \langle \ell, x\rangle + c,\quad x\in X,\ \ell \in X^*,\ c\in\mathbb{R}\},$$

where X is a topological vector space, X^* is the topological dual to X. It is a well known fact (see for example Proposition 3.1 of [5]) that a proper convex lower semicontinuous function $f : X \to \bar{\mathbb{R}}$ is Φ_{conv}-convex.

The second class which is considered in this paper is defined as

$$\Phi_{lsc} := \{\varphi\colon X \to \mathbb{R},\ \varphi(x) = -a\|x\|^2 + \langle \ell, x\rangle + c,\ \ x\in X,\ \ell\in X^*,\ a\geq 0,\ c\in\mathbb{R}\},$$

where X is a normed space.

In the following theorem we recall a characterization of Φ_{lsc}-convex functions.

Theorem 1.3. ([12], Example 6.2) *Let* X *be a Hilbert space. Let* $f\colon X \to \bar{\mathbb{R}}$ *be lower semicontinuous on* X*. If there exists* $\bar\varphi \in \Phi_{lsc}$ *such that* $\bar\varphi < f$*, then* f *is* $\Phi_{lsc}-$*convex.*

Let us note that when $X = \mathbb{R}^n$ the class of Φ_{lsc}-convex functions contains the class of prox-bounded functions f (Definition 1.23 in [10]), which possess proximal subgradients (Definition 8.45 of [10]) at each $x \in \mathbb{R}^n$ (if $\operatorname{dom} f = \mathbb{R}^n$) (Proposition 8.46f [10]). See also [2].

The organization of the paper is as follows. In section 2 we collect basic facts on Φ-subdifferentials of Φ-convex functions. In section 3 we provide characterizations of the intersection property in terms of Φ-ε-subdifferentials. In section 4 we discuss the intersection property in the class Φ_{lsc}, where our main tool is the Brønsted-Rockafellar-type theorem for Φ_{lsc}-convex functions. In section 5 we discuss the intersection property in the class Φ_{conv}. Section 6 contains our main results.

2. Φ-subdifferentials

Let X be a set. The Φ-*subdifferential* of a function $f : X \to \bar{\mathbb{R}}$ at a point $\bar{x} \in \mathrm{dom}(f)$ is defined as

$$\partial_\Phi f(\bar{x}) := \{\varphi \in \Phi \ : \ \varphi(x) - \varphi(\bar{x}) \leq f(x) - f(\bar{x}), \ \forall \ x \in X\}.$$

The elements φ of $\partial_\Phi f(\bar{x})$ are called Φ-*subgradients* of f at $\bar{x}$ as defined in [11]. Some results related to Φ-subdifferentials can also be found e.g. in [9, 12, 15].

Let $\varepsilon > 0$. The Φ-ε-*subdifferential* of f at the point $\bar{x} \in \mathrm{dom}(f)$ is defined as follows

$$\partial^\varepsilon_\Phi f(\bar{x}) := \{\varphi \in \Phi \ : \ \varphi(x) - \varphi(\bar{x}) - \varepsilon \leq f(x) - f(\bar{x}), \ \forall \ x \in X\}.$$

Whenever the class Φ is clear from the context we use the notation ε-subdifferential. The Φ-ε-subdifferential was defined in [7], which is a direct adaptation of the definition from [3]. The elements φ of $\partial^\varepsilon_\Phi f(\bar{x})$ are called Φ-ε-*subgradients* (ε-*subgradients*) of f at $\bar{x}$.

In the proposition below we give a characterization of Φ-subdifferential and Φ-ε-subdifferential of f at a given point $\bar{x}$. Similar results can be found in [12] (Proposition 1.2, Corollary 1.2) and [12] (Proposition 5.1).

Proposition 2.1. *Let $f : X \to \bar{\mathbb{R}}$ be a proper Φ-convex functions, $\varepsilon \geq 0$ and $\bar{x} \in \mathrm{dom}(f)$. Then*

$$\partial^\varepsilon_\Phi f(\bar{x}) = \{\varphi \in \mathrm{supp}(f) \ : \ f(\bar{x}) = \varphi(\bar{x}) + \varepsilon\} \tag{1}$$

Proof. Let $\varphi \in \partial^\varepsilon_\Phi f(\bar{x})$. By definition, $f(x) \geq \varphi(x) - \varphi(\bar{x}) + f(\bar{x}) - \varepsilon$. Let $\bar{\varphi} := \varphi - \varphi(\bar{x}) + f(\bar{x}) - \varepsilon$, then $\bar{\varphi} \in \mathrm{supp}(f)$ and $\bar{\varphi}(\bar{x}) + \varepsilon = f(\bar{x})$.

Let $\bar{\varphi} \in \mathrm{supp}(f)$ be such that $f(\bar{x}) = \bar{\varphi}(\bar{x}) + \varepsilon$. By the fact that $\bar{\varphi} \in \mathrm{supp}(f)$ we have $f(x) \geq \bar{\varphi}(x)$, for all $x \in X$. Hence the following inequality holds

$$\bar{\varphi}(x) - \bar{\varphi}(\bar{x}) - \varepsilon \leq f(x) - f(\bar{x}), \ \forall \ x \in X,$$

which means $\bar{\varphi} \in \partial^\varepsilon_\Phi f(\bar{x})$. □

3. The intersection property via Φ-ε-subgradients

Since $[\bar{\varphi} < \alpha] \subset [\varphi < \alpha]$ whenever $\varphi \leq \bar{\varphi}$, we have the following proposition.

Proposition 3.1. ([16], Proposition 2.1.3) *Let $\alpha \in \mathbb{R}$ and $\varphi_1, \varphi_2, \bar{\varphi_1}, \bar{\varphi_2} \colon X \to \mathbb{R}$ be such that $\bar{\varphi_1} \geq \varphi_1$ and $\bar{\varphi_2} \geq \varphi_2$. If φ_1 and φ_2 have the intersection property on X at the level α, then $\bar{\varphi_1}$ and $\bar{\varphi_2}$ have the intersection property on X at the level α.*

For ε-subdifferentials the following theorem holds.

Theorem 3.2. *Let X be a set. Let Φ be a class of functions $\varphi : X \to \mathbb{R}$. Let $f, g : X \to \bar{\mathbb{R}}$ be proper Φ-convex functions and $\alpha \in \mathbb{R}$.*

The following conditions are equivalent:

(i) *there exist* $\varphi_1 \in \operatorname{supp}(f)$ *and* $\varphi_2 \in \operatorname{supp}(g)$ *such that* φ_1, φ_2 *have the intersection property on* X *at the level* α*,*

(ii) *for any* $\varepsilon > 0$ *there exist* $x_1 \in \operatorname{dom}(f)$*,* $x_2 \in \operatorname{dom}(g)$ *and* $\bar{\varphi_1} \in \partial_\Phi^\varepsilon f(x_1)$ *and* $\bar{\varphi_2} \in \partial_\Phi^\varepsilon g(x_2)$ *such that* $\bar{\varphi_1}, \bar{\varphi_2}$ *have the intersection property on* X *at the level* α*.*

Proof. (ii)⇒(i). Follows immediately from Proposition 2.1.

(i)⇒(ii). Since $\varphi_1 \in \operatorname{supp}(f)$ and $\varphi_2 \in \operatorname{supp}(g)$, we have

$$\inf_{x\in X}\{f(x) - \varphi_1(x)\} =: c_1 \geq 0 \quad \text{and} \quad \inf_{x\in X}\{g(x) - \varphi_2(x)\} =: c_2 \geq 0.$$

Let $\varepsilon > 0$. There exist $x_1 \in \operatorname{dom}(f)$ and $x_2 \in \operatorname{dom}(g)$ such that

$$f(x_1) - \varphi_1(x_1) < c_1 + \varepsilon \quad \text{and} \quad g(x_2) - \varphi_2(x_2) < c_2 + \varepsilon.$$

Define $\bar{\varphi_1}(x) := \varphi_1(x) + c_1$ and $\bar{\varphi_2}(x) := \varphi_2(x) + c_2$.

The functions $\bar{\varphi_1}, \bar{\varphi_2}$ belong to Φ and

$$-f(x_1) > -\bar{\varphi_1}(x_1) - \varepsilon \quad \text{and} \quad - g(x_2) > -\bar{\varphi_2}(x_2) - \varepsilon.$$

Notice that

$$\bar{\varphi_1}(x) \leq f(x) \text{ and } \bar{\varphi_2}(x) \leq g(x) \quad \text{for all} \quad x \in X.$$

So, $\bar{\varphi_1} \in \partial_\Phi^\varepsilon f(x_1)$ and $\bar{\varphi_2} \in \partial_\Phi^\varepsilon g(x_2)$. It is obvious that $\bar{\varphi_1} \geq \varphi_1$ and $\bar{\varphi_2} \geq \varphi_2$ and by Proposition 3.1, $\bar{\varphi_1}$ and $\bar{\varphi_2}$ have the intersection property on X at the level α. □

A characterization of the intersection property in terms of Φ-subgradients requires more advanced tools as well as additional knowledge about properties of functions φ from a given class Φ. We provide characterizations of the intersection property in terms of Φ-subgradients for the class Φ_{lsc} (section 4) and for the class Φ_{conv} (section 5).

4. The intersection property for Φ_{lsc}-subgradients

In this section we express the intersection property for Φ_{lsc}-convex functions in terms of Φ_{lsc}-subgradients. The main result of the section is Theorem 4.5.

To this aim we use the Borwein-Preiss variational principle in the following form.

Theorem 4.1. ([13], Theorem 8.3.3) *Assume that* X *is a Hilbert space and the function* $f : X \to \bar{\mathbb{R}}$ *is proper lower semicontinuous on* X*. Let* $\varepsilon > 0$ *and let* $y \in X$ *be such that* $f(y) \leq \inf_{x\in X} f(x) + \varepsilon$*. Then for any* $\lambda > 0$ *there exist* $z \in X$*,*

$$\textit{such that} \quad \|z - x\| \leq \lambda \quad \textit{and} \quad f(z) \leq f(x) + \frac{\varepsilon}{\lambda^2}\|x - z\|^2 \quad \textit{for all} \quad x \in X.$$

With the help of this variational principle we prove the following variant of the Brønsted-Rockafellar theorem for Φ_{lsc}-convex functions. The proof we present below is an adaptation of the proof of Proposition 5.3 of [7] to our definition of subgradients.

Theorem 4.2. *Let X be a Hilbert space. Let $\varepsilon > 0$, $f : X \to \bar{\mathbb{R}}$ be a proper Φ_{lsc}-convex function, $y \in \mathrm{dom}(f)$ and $\varphi(\cdot) = -a\|\cdot\|^2 + \langle \ell, \cdot\rangle + c \in \partial^{\varepsilon}_{\Phi_{lsc}} f(y)$. For every $\lambda > 0$ there exist $\bar{y} \in \mathrm{dom}(f)$ and $\bar{\varphi}(\cdot) = -\bar{a}\|\cdot\|^2 + \langle \bar{\ell}, \cdot \rangle + \bar{c} \in \partial_{\Phi_{lsc}} f(\bar{y})$ such that*

$$\|y - \bar{y}\| \le \lambda, \quad \|\ell - \bar{\ell}\| \le \frac{2\varepsilon}{\lambda^2}(\lambda + \|y\|), \quad \bar{a} - a = \frac{\varepsilon}{\lambda^2} \quad \textit{and} \quad c - \bar{c} \le \frac{\varepsilon}{\lambda^2}\|\bar{y}\|^2.$$

Proof. Since $\varphi \in \partial^{\varepsilon}_{\Phi_{lsc}} f(y)$, by Proposition 2.1, we have

$$f(x) - f(y) \ge -a\|x\|^2 + \langle \ell, x\rangle + a\|y\|^2 - \langle \ell, y\rangle - \varepsilon \quad \text{for all} \quad x \in X.$$

which can be rewritten as

$$W(y) \le \inf_{x \in X} W(x) + \varepsilon,$$

where $W(x) := f(x) + a\|x\|^2 - \langle \ell, x\rangle$. By Theorem 4.1, for every $\lambda > 0$ there exists $\bar{y} \in X$ such that $\|y - \bar{y}\| \le \lambda$ and

$$W(\bar{y}) \le W(x) + \frac{\varepsilon}{\lambda^2}\|x - \bar{y}\|^2 \quad \text{for all} \quad x \in X.$$

Let $\bar{\varphi}(x) = -(a + \frac{\varepsilon}{\lambda^2})\|x\|^2 + \left\langle \ell + \frac{2\varepsilon}{\lambda^2}\bar{y}, x \right\rangle + c - \frac{\varepsilon}{\lambda^2}\|\bar{y}\|^2 - \varphi(\bar{y}) + f(\bar{y})$. We have

$$\begin{aligned} f(x) - f(\bar{y}) &\ge -a\|x\|^2 + \langle \ell, x\rangle + a\|\bar{y}\|^2 - \langle \ell, \bar{y}\rangle - \frac{\varepsilon}{\lambda^2}\|x - \bar{y}\|^2 \\ &\ge -a\|x\|^2 + \langle \ell, x\rangle + a\|\bar{y}\|^2 - \langle \ell, \bar{y}\rangle - \frac{\varepsilon}{\lambda^2}\|x\|^2 + \frac{2\varepsilon}{\lambda^2}\langle x, \bar{y}\rangle - \frac{\varepsilon}{\lambda^2}\|\bar{y}\|^2 \\ &\ge \left\langle \ell + \frac{2\varepsilon}{\lambda^2}\bar{y}, x \right\rangle - (a + \frac{\varepsilon}{\lambda^2})\|x\|^2 - \left\langle \ell + \frac{2\varepsilon}{\lambda^2}\bar{y}, \bar{y} \right\rangle + (a + \frac{\varepsilon}{\lambda^2})\|\bar{y}\|^2 \\ &= \bar{\varphi}(x) - \bar{\varphi}(\bar{y}). \end{aligned}$$

Thus $\bar{\varphi} \in \partial_{\Phi_{lsc}} f(\bar{y})$. Moreover, $\|\ell - \bar{\ell}\| = \| - \frac{2\varepsilon}{\lambda^2}\bar{y}\| = \frac{2\varepsilon}{\lambda^2}\|\bar{y}\| \le \frac{2\varepsilon}{\lambda^2}(\lambda + \|y\|)$,

$c - \bar{c} = c - c + \frac{\varepsilon}{\lambda^2}\|\bar{y}\|^2 + \varphi(\bar{y}) - f(\bar{y}) \le \frac{\varepsilon}{\lambda^2}\|\bar{y}\|$ and $\bar{a} - a = \frac{\varepsilon}{\lambda^2}$. □

The domain of Φ_{lsc}-subdifferential is defined as follows

$$\mathrm{dom}(\partial_{\Phi_{lsc}} f) := \{x \in X \ : \ \partial_{\Phi_{lsc}} f(x) \ne \emptyset\}.$$

The following theorem is an immediate corollary from Theorem 4.2 and Proposition 2.1.

Theorem 4.3. *Let X be a Hilbert space and $f : X \to \bar{\mathbb{R}}$ be a proper Φ_{lsc}-convex function. Then the set* $\mathrm{dom}(\partial_{\Phi_{lsc}} f)$ *is dense in* $\mathrm{dom}(f)$.

The following technical fact is used below.

Proposition 4.4. *Let X be a Hilbert space, Z be a subset of X, $\alpha \in \mathbb{R}$ and $\varphi_1, \varphi_2 : X \to \mathbb{R}$. If φ_1, φ_2 have the intersection property on X at the level α, then φ_1, φ_2 have the intersection property on Z at the level α.*

Theorem 4.2 together with Theorem 3.2 allow to prove the main result of this section.

Theorem 4.5. *Let X be a Hilbert space and let Z be a bounded subset of X. Let $f, g : X \to \bar{\mathbb{R}}$ be proper Φ_{lsc}-convex functions and $\alpha \in \mathbb{R}$.*

Assume that there exist $\varphi_1 \in \mathrm{supp}(f)$ and $\varphi_2 \in \mathrm{supp}(g)$ such that φ_1, φ_2 have the intersection property on X at the level α.

Then, for any $\eta > 0$, there exist $x_1 \in \mathrm{dom}(f)$, $x_2 \in \mathrm{dom}(g)$ and $\bar{\varphi}_1 \in \partial_{\Phi_{lsc}} f(x_1)$ and $\bar{\varphi}_2 \in \partial_{\Phi_{lsc}} g(x_2)$ such that $\bar{\varphi}_1, \bar{\varphi}_2$ have the intersection property on Z at the level $\alpha - \eta$.

Proof. By the boundedness of Z, there exists $\gamma > 0$ such that $Z \subset \gamma B(0,1)$, where $B(0,1)$ is the closed unit ball. Let φ_1, φ_2 satisfy the assumptions of the theorem. Let $\eta > 0$ and $\varepsilon = \frac{\eta}{\gamma}$.

By Theorem 3.2, there exist $x_1 \in \mathrm{dom}(f)$, $x_2 \in \mathrm{dom}(g)$ and $\tilde{\varphi}_1(x) = -\tilde{a}_1\|x\|^2 + \left\langle \tilde{\ell}_1, x \right\rangle + \tilde{c}_1 \in \partial^{\varepsilon}_{\Phi_{lsc}} f(x_1)$ and $\tilde{\varphi}_2(x) = -\tilde{a}_2\|x\|^2 + \left\langle \tilde{\ell}_2, x \right\rangle + \tilde{c}_2 \in \partial^{\varepsilon}_{\Phi_{lsc}} g(x_2)$ such that $\tilde{\varphi}_1, \tilde{\varphi}_2$ have the intersection property on X at the level α.

Let $\lambda_1 = 1 + \sqrt{1 + 2\|x_1\| + \gamma + \frac{1}{\gamma}\|x_1\|^2}$ and $\lambda_2 = 1 + \sqrt{1 + 2\|x_2\| + \gamma + \frac{1}{\gamma}\|x_2\|^2}$.

By Theorem 4.2 there exist $\bar{x}_1 \in \mathrm{dom}(f)$, $\bar{x}_2 \in \mathrm{dom}(g)$, $\bar{\varphi}_1(x) = -\bar{a}_1\|x\|^2 + \left\langle \bar{\ell}_1, x \right\rangle + \bar{c}_1 \in \partial_{\Phi_{lsc}} f(\bar{x}_1)$ and $\bar{\varphi}_2(x) = -\bar{a}_2\|x\|^2 + \left\langle \bar{\ell}_2, x \right\rangle + \bar{c}_2 \in \partial_{\Phi_{lsc}} g(\bar{x}_2)$ such that

$$\|\tilde{\ell}_1 - \bar{\ell}_1\| \leq \frac{2\varepsilon}{\lambda_1^2}(\lambda_1 + \|x_1\|), \quad \bar{a}_1 - \tilde{a}_1 = \frac{\varepsilon}{\lambda_1^2} \text{ and } \tilde{c}_1 - \bar{c}_1 \leq \frac{\varepsilon}{\lambda_1^2}\|x_1\|^2 \tag{2}$$

$$\|\tilde{\ell}_2 - \bar{\ell}_2\| \leq \frac{2\varepsilon}{\lambda_2^2}(\lambda_2 + \|x_2\|), \quad \bar{a}_2 - \tilde{a}_2 = \frac{\varepsilon}{\lambda_2^2} \text{ and } \tilde{c}_2 - \bar{c}_2 \leq \frac{\varepsilon}{\lambda_2^2}\|x_2\|^2. \tag{3}$$

By (2) and (3) and by the boundedness of Z, for every $x \in Z$, we get

$$\begin{cases} \tilde{\ell}_1(x) \leq \bar{\ell}_1(x) + \dfrac{2\gamma\varepsilon}{\lambda_1^2}(\lambda_1 + \|x_1\|), \\ -\tilde{a}_1\|x\|^2 = -\bar{a}_1\|x\|^2 + \dfrac{\varepsilon}{\lambda_1^2}\|x\|^2 \leq -\bar{a}_1\|x\|^2 + \dfrac{\varepsilon}{\lambda_1^2}\gamma^2, \end{cases} \tag{4}$$

$$\begin{cases} \tilde{\ell}_2(x) \leq \bar{\ell}_2(x) + \dfrac{2\gamma\varepsilon}{\lambda_2^2}(\lambda_2 + \|x_2\|), \\ -\tilde{a}_2\|x\|^2 = -\bar{a}_2\|x\|^2 + \dfrac{\varepsilon}{\lambda_2^2}\|x\|^2 \leq -\bar{a}_2\|x\|^2 + \dfrac{\varepsilon}{\lambda_1^2}\gamma^2. \end{cases} \tag{5}$$

By (2), (3), (4), (5) and the definitions of ε, λ_1 and λ_2 we get

$$\tilde{\varphi_1}(x) \leq \bar{\varphi_1}(x) + \eta, \quad \tilde{\varphi_2}(x) \leq \bar{\varphi_2}(x) + \eta \quad \text{for every} \quad x \in Z. \tag{6}$$

Since the intersection property holds for $\tilde{\varphi_1}, \tilde{\varphi_2}$ on X at the level α, then by Proposition 4.4, the intersection property holds for $\tilde{\varphi_1}, \tilde{\varphi_2}$ on Z at the level α. By Proposition 3.1, the intersection property holds for $\bar{\varphi_1} + \eta$ and $\bar{\varphi_2} + \eta$ on Z at the level α, i.e. the intersection property holds for $\bar{\varphi_1}$ and $\bar{\varphi_2}$ on Z at the level $\alpha - \eta$. □

The example below shows that, in general, we cannot expect, the intersection property for Φ_{lsc}-subgradients on the whole space X.

Example 4.6. Let $X = \mathbb{R}$, $f(x) := 2^x$, $g(x) := -|x| + 2$ and $\alpha = 0$. The functions f and g are Φ_{lsc}-convex. Let $\varphi_1(x) \equiv 0$, then $\varphi_1 \in \Phi_{lsc}$ and $\varphi_1 \in \operatorname{supp}(f)$. The set $[\varphi_1 < 0]$ is empty, then for every $\varphi_2 \in \operatorname{supp}(g)$ functions φ_1 and φ_2 have the intersection property on X at the level 0.

On the other hand, for every $\bar{x} \in X$ and $\bar{\varphi} \in \partial_{\Phi_{lsc}} g(\bar{x})$ it must be

$$\bar{\varphi}(x) = -ax^2 + bx + c,$$

where $a > 0$, $b, c \in \mathbb{R}$. Since all Φ_{lsc}-subgradients of f are affine functions, there exist no Φ_{lsc}-subgradient φ_f of f and a Φ_{lsc}-subgradient φ_g of g such that φ_f and φ_g have the intersection property on X at the level 0.

In view of Theorem 4.5, for every bounded subset $Z \subset X$ there exist subgradients of f and subgradients of g which have the intersection property on Z at the level 0.

5. The intersection property for Φ_{conv}-subgradients

Let X be a topological vector space. The *subdifferential* ∂f of the function $f\colon X \to \bar{\mathbb{R}}$ at the point $\bar{x} \in \operatorname{dom}(f)$ is defined as follows

$$\partial f(\bar{x}) := \{\ell \in X^* \ : \ f(x) - f(\bar{x}) \geq \langle \ell, x - \bar{x} \rangle, \ \forall x \in X\}.$$

There is a one-to-one correspondence between the Φ_{conv}-subdifferential and the subdifferential of a convex lower semicontinuous function f, i.e. $\varphi = \ell + c \in \partial_{\Phi_{conv}} f(\bar{x})$ if and only if $\ell \in \partial f(\bar{x})$.

We recall that the normal cone to a convex set $C \subset X$ at the point $\bar{x} \in C$ is defined as

$$N_C(\bar{x}) := \{\ell \in X^* \ : \ \langle \ell, \bar{x} - x \rangle \geq 0, \ \forall x \in C\},$$

These definitions can be found e.g. in [5].

Proposition 5.1. ([8], p. 200, Proposition 2) *Let X be a topological vector space. Let $\alpha \in \mathbb{R}$. Let $f : X \to \bar{\mathbb{R}}$ be proper convex and continuous and $[f < \alpha] \neq \emptyset$. Then*

$$N_{[f\leq\alpha]}(\bar{x}) = \mathbb{R}_{+}\partial f(\bar{x}), \text{ where } f(\bar{x}) = \alpha.$$

Now we are ready to prove the main result of this section.

Theorem 5.2. *Let X be a topological vector space. Let $f, g \colon X \to \bar{\mathbb{R}}$ be proper continuous and convex functions and $\alpha \in \mathbb{R}$. Assume that f and g have the intersection property on X at the level α and*

$$[f \leq \alpha] \cap [g \leq \alpha] \neq \emptyset.$$

Then there exist $x_1 \in \mathrm{dom}(f)$, $x_2 \in \mathrm{dom}(g)$ and $\varphi_1 \in \partial_{\Phi_{conv}} f(x_1)$, $\varphi_2 \in \partial_{\Phi_{conv}} g(x_2)$ such that φ_1, φ_2 have the intersection property on X at the level α.

Proof. Assume first that $[f < \alpha] = \emptyset$, i.e. $f(x) \geq \alpha$ for all $x \in X$.

By the assumption that $[f \leq \alpha] \cap [g \leq \alpha] \neq \emptyset$, there exists $x_1 \in \mathrm{dom}(f)$ such that $f(x_1) = \alpha$, and thus $\bar{\varphi} \equiv \alpha$ belongs to $\partial_{\Phi_{conv}} f(x_1)$. Let $x_2 \in \mathrm{dom}(\partial_{\Phi_{conv}} g)$. Then

$$[\bar{\varphi} < \alpha] \cap [\varphi < \alpha] = \emptyset \quad \text{for all} \quad \varphi \in \partial_{\Phi_{conv}} g(x_2),$$

which gives the assertion of the theorem. Analogously, by assuming that $[g < \alpha] = \emptyset$ we get the assertion of the theorem.

Assume now that $[f < \alpha] \neq \emptyset$ and $[g < \alpha] \neq \emptyset$. By the intersection property of f and g on X at the level α, we have $[f < \alpha] \cap [g < \alpha] = \emptyset$.

We show that there exists $\ell \in X^* \setminus \{0\}$ such that

$$\sup_{y\in[f\leq\alpha]} \langle \ell, y \rangle = \langle \ell, \bar{x} \rangle = \inf_{z\in[g\leq\alpha]} \langle \ell, z \rangle\,, \tag{7}$$

where $\bar{x} \in [f \leq \alpha] \cap [g \leq \alpha]$.

By the convexity and continuity of the functions f, g, and by the fact that $[f < \alpha] \neq \emptyset$, $[g < \alpha] \neq \emptyset$ we get $[f < \alpha] \cap [g \leq \alpha] = \emptyset$, $\mathrm{int}[f \leq \alpha] = [f < \alpha]$ and $f(\bar{x}) = g(\bar{x}) = \alpha$. By the separation theorem ([13],Theorem 1.5.3), there exist $\ell \in X^* \setminus \{0\}$ and $r \in \mathbb{R}$ such that

$$\langle \ell, y \rangle \leq r \leq \langle \ell, z \rangle \quad \text{for all} \quad y \in [f \leq \alpha],\ z \in [g \leq \alpha].$$

Let $\bar{x} \in [f \leq \alpha] \cap [g \leq \alpha]$. On the one hand

$$\langle \ell, y \rangle \leq r \leq \langle \ell, \bar{x} \rangle \quad \forall\, y \in [f \leq \alpha],$$

hence $\sup_{y\in[f\leq\alpha]} \langle \ell, y \rangle \leq r \leq \langle \ell, \bar{x} \rangle$. On the other hand

$$\langle \ell, \bar{x} \rangle \leq r \leq \langle \ell, z \rangle \quad \forall z \in [g \leq \alpha],$$

thus $\langle \ell, \bar{x} \rangle \leq r \leq \inf_{z \in [g \leq \alpha]} \langle \ell, z \rangle$. Then

$$\langle \ell, \bar{x} \rangle \leq \sup_{y \in [f \leq \alpha]} \langle \ell, y \rangle \leq r \leq \langle \ell, \bar{x} \rangle \leq r \leq \inf_{z \in [g \leq \alpha]} \langle \ell, z \rangle \leq \langle \ell, \bar{x} \rangle ,$$

which proves (7), and consequently $\sup_{y \in [f \leq \alpha]} \langle \ell, y \rangle = \langle \ell, \bar{x} \rangle$. This implies for all $y \in [f \leq \alpha]$, $\langle \ell, \bar{x} \rangle \geq \langle \ell, y \rangle$, or equivalently, $\langle \ell, \bar{x} - y \rangle \geq 0$. By Proposition 5.1

$$\ell \in N_{[f \leq \alpha]}(\bar{x}) = \mathbb{R}_+ \partial f(\bar{x}).$$

Hence, there exists $k > 0$ such that $\ell \in \partial k f(\bar{x})$, i.e.

$$f(x) \geq \frac{1}{k} \langle \ell, x - \bar{x} \rangle + f(\bar{x}), \quad \text{for all} \quad x \in X.$$

Let $\varphi_1(x) := \frac{1}{k} \langle \ell, x - \bar{x} \rangle + f(\bar{x})$. Then $\varphi_1 \in \partial_{\Phi_{conv}} f(\bar{x})$ and $\varphi_1 \in \text{supp}(f)$ and

$$[\varphi_1 < \alpha] = \{x \in X \ : \ \langle \ell, x - \bar{x} \rangle < 0\}.$$

By (7), $\langle \ell, \bar{x} \rangle = \inf_{z \in [g \leq \alpha]} \langle \ell, z \rangle$, which is equivalent to

$$- \langle \ell, \bar{x} \rangle = \sup_{z \in [g \leq \alpha]} - \langle \ell, z \rangle,$$

i.e. $$- \langle \ell, \bar{x} \rangle \geq - \langle \ell, z \rangle \quad \text{for all} \quad z \in [g \leq \alpha],$$

or $$- \langle \ell, \bar{x} - z \rangle \geq 0 \quad \text{for all} \quad z \in [g \leq \alpha].$$

Again, by Proposition 5.1

$$-\ell \in N_{[g \leq \alpha]}(\bar{x}) = \mathbb{R}_+ \partial g(\bar{x}).$$

There exists $\lambda > 0$ such that $-\ell \in \partial \lambda g(\bar{x})$. We have

$$g(x) \geq -\frac{1}{\lambda} \langle \ell, x - \bar{x} \rangle + g(\bar{x}), \quad \text{for all} \quad x \in X.$$

Let $\varphi_2(x) := -\frac{1}{\lambda} \langle \ell, x - \bar{x} \rangle + g(\bar{x})$. Then $\varphi_2 \in \partial_{\Phi_{conv}} g(\bar{x})$ and $\varphi_2 \in \text{supp}(g)$ and

$$[\varphi_2 < \alpha] = \{x \in X \ : \ \langle \ell, x - \bar{x} \rangle > 0\}.$$

Thus $$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset.$$ □

The following simple fact was proved in [17].

Proposition 5.3. ([17], Proposition 2.1) *If φ_1, φ_2 have the intersection property on X at the level $\alpha \in \mathbb{R}$, then φ_1, φ_2 have the intersection property on X at any level $\beta \leq \alpha$, $\beta \in \mathbb{R}$.*

Taking into account that above proposition we can formulate the following corollary of Theorem 5.2.

Corollary 5.4. *Let $f, g : X \to \bar{\mathbb{R}}$ be proper continuous and convex and $\alpha \in \mathbb{R}$. If f and g have the intersection property on X at the level α and*

$$[f \le \alpha] \cap [g \le \alpha] \neq \emptyset$$

then there exists $x_1 \in \mathrm{dom}(f)$, $x_2 \in \mathrm{dom}(g)$ and $\varphi_1 \in \partial_{\Phi_{conv}} f(x_1)$, $\varphi_2 \in \partial_{\Phi_{conv}} g(x_2)$ such that φ_1, φ_2 have the intersection property on X at every level $\beta \le \alpha$.

6. Main results

In this section we prove minimax theorems for functions $a : X \times Y \to \bar{\mathbb{R}}$ which are Φ_{lsc}-(Φ_{conv}-)convex with respect to variable x for every $y \in Y$. In these theorems we do not use any compactness and/or connectedness assumptions.

For Φ_{lsc}-functions we prove the following sufficient conditions for the minimax equality to hold.

Theorem 6.1. *Let X be a Hilbert space and Y be a real vector space. Let $a\colon X \times Y \to \bar{\mathbb{R}}$. Assume that for any $y \in Y$ the function $a(\cdot, y)\colon X \to \bar{\mathbb{R}}$ is proper Φ_{lsc}-convex on X and for any $x \in X$ the function $a(x, \cdot)\colon Y \to \bar{\mathbb{R}}$ is concave on Y. If for every $\alpha \in \mathbb{R}$, $\alpha < \inf_{x\in X} \sup_{y\in Y} a(x, y)$, there exist $y_1, y_2 \in Y$, $x_1 \in \mathrm{dom}(a(\cdot, y_1))$, $x_2 \in \mathrm{dom}(a(\cdot, y_2))$, $\varphi_1 \in \partial_{\Phi_{lsc}} a(\cdot, y_1)(x_1)$, and, furthermore, $\varphi_2 \in \partial_{\Phi_{lsc}} a(\cdot, y_2)(x_2)$ such that the intersection property holds for φ_1 and φ_2 on X at the level α, then*

$$\sup_{y\in Y} \inf_{x\in X} a(x, y) = \inf_{x\in X} \sup_{y\in Y} a(x, y).$$

Proof. Follows immediately from Proposition 2.1 and Theorem 1.2. □

Remark 6.2. Let us observe that if for some $y_1, y_2 \in Y$ and $x_1 \in \mathrm{dom}(a(\cdot, y_1))$, $x_2 \in \mathrm{dom}(a(\cdot, y_2))$ and $\varphi_1(\cdot) = -a_1\|\cdot\|^2 + \langle \ell_1, \cdot\rangle + c_1 \in \partial_{\Phi_{lsc}} a(\cdot, y_1)(x_1)$, $\varphi_2(\cdot) = -a_2\|\cdot\|^2 + \langle \ell_2, \cdot\rangle + c_2 \in \partial_{\Phi_{lsc}} a(\cdot, y_2)(x_2)$ the intersection property holds for φ_1 and φ_2 on X at the level α, then exactly one of the conditions hold

(i) $\varphi_1 := \mathit{const} \ge \alpha$ or $\varphi_2 := \mathit{const} \ge \alpha$,

(ii) $\varphi_1 := \langle \ell_1, \cdot\rangle + c_1$ and $\varphi_2 := \langle \ell_2, \cdot\rangle + c_2$, i.e. $a_1 = a_2 = 0$.

To show the necessity of the intersection property of Theorem 6.1 we restrict our attention to bounded subsets of the Hilbert space X.

Theorem 6.3. *Let X be a Hilbert space, Y be a real vector space and let $a : X \times Y \to \bar{\mathbb{R}}$. Assume that for any $y \in Y$ the function $a(\cdot, y) : X \to \bar{\mathbb{R}}$ is proper Φ_{lsc}-convex on X. If $\sup_{y\in Y} \inf_{x\in X} a(x, y) = \inf_{x\in X} \sup_{y\in Y} a(x, y)$, then for every $\alpha \in \mathbb{R}$, $\alpha < \inf_{x\in X} \sup_{y\in Y} a(x, y)$, and every bounded set $Z \subset X$ there exist*

$y_1, y_2 \in Y$, $x_1 \in \mathrm{dom}(a(\cdot, y_1))$, $x_2 \in \mathrm{dom}(a(\cdot, y_2))$ and $\varphi_1 \in \partial_{\Phi_{lsc}} a(\cdot, y_1)(x_1)$, $\varphi_2 \in \partial_{\Phi_{lsc}} a(\cdot, y_2)(x_2)$ such that the intersection property holds for φ_1 and φ_2 on Z at the level α.

Proof. Take any $\alpha < \inf\limits_{x\in X} \sup\limits_{y\in Y} a(x,y)$. We have $\sup\limits_{y\in Y} \inf\limits_{x\in X} a(x,y) > \alpha$ from the minimax inequality, i.e. there exists $\bar{y} \in Y$ such that for every $x \in X$ we have $a(x, \bar{y}) > \alpha$. The function $\bar{\varphi} := \alpha$ belongs to supp $a(\cdot, \bar{y})$. We have $[\bar{\varphi} < \alpha] = \emptyset$, thus

$$[\varphi < \alpha] \cap [\bar{\varphi} < \alpha] = \emptyset \quad \text{for all} \quad \varphi \in \Phi_{lsc},$$

i.e. for every $\varphi \in \Phi_{lsc}$ the functions $\bar{\varphi}$ and φ have the intersection property on X at the level α.

Take any $\inf\limits_{x\in X} \sup\limits_{y\in Y} a(x,y) > \beta > \alpha$ and set $\eta := \beta - \alpha > 0$. By the same arguments as above there exist $y_1, y_2 \in Y$, $\varphi_1 \in \mathrm{supp}\ a(\cdot, y_1)$ and $\varphi_2 \in \mathrm{supp}\ a(\cdot, y_2)$ such that the intersection property holds for functions φ_1 and φ_2 on X at the level β. Let Z be a bounded subset of X. By Theorem 4.5, there exist $x_1 \in \mathrm{dom}(a(\cdot, y_1))$, $x_2 \in \mathrm{dom}(a(\cdot, y_2))$ and $\bar{\varphi}_1 \in \partial_{\Phi_{lsc}} a(\cdot, y_1)(x_1)$, $\bar{\varphi}_2 \in \partial_{\Phi_{lsc}} a(\cdot, y_2)(x_2)$ such that the intersection property holds for $\bar{\varphi}_1$ and $\bar{\varphi}_2$ on Z at the level $\beta - \eta$, then the intersection property holds for $\bar{\varphi}_1$ and $\bar{\varphi}_2$ on Z at the level $\beta - (\beta - \alpha) = \alpha$. □

In the class of Φ_{conv}-convex functions we get the following sufficient and necessary conditions for the minimax equality.

Theorem 6.4. *Let X be a reflexive Banach space and Y be a real vector space. Let $a\colon X \times Y \to \bar{\mathbb{R}}$ be a function such that for any $y \in Y$ the function $a(\cdot, y)\colon X \to \mathbb{R}$ is proper continuous and convex on X and for any $x \in X$ the function $a(x, \cdot)\colon Y \to \mathbb{R}$ is concave on Y. Assume that there exists a point $\bar{y} \in Y$ such that $\sup\limits_{y\in Y} \inf\limits_{x\in X} a(x,y) = \inf\limits_{x\in X} a(x, \bar{y})$ and there exists $\tilde{y} \in Y$ and $\gamma > \sup\limits_{y\in Y} \inf\limits_{x\in X} a(x,y)$ such that the set $[a(\cdot, \tilde{y}) \le \gamma]$ is bounded in X.*

The following conditions are equivalent:

(i) *for every $\alpha \in \mathbb{R}$, $\alpha < \inf\limits_{x\in X} \sup\limits_{y\in Y} a(x,y)$ there exist $y_1, y_2 \in Y$, $x_1 \in \mathrm{dom}(a(\cdot, y_1))$, $x_2 \in \mathrm{dom}(a(\cdot, y_2))$ and $\varphi_1 \in \partial_{\Phi_{conv}} a(\cdot, y_1)(x_1)$, $\varphi_2 \in \partial_{\Phi_{conv}} a(\cdot, y_2)(x_2)$ such that the intersection property holds for φ_1 and φ_2 on X at the level α,*

(ii) $\sup\limits_{y\in Y} \inf\limits_{x\in X} a(x,y) = \inf\limits_{x\in X} \sup\limits_{y\in Y} a(x,y)$.

Proof. (i)⇒(ii): follows immediately from Proposition 2.1 and Theorem 1.2.

(ii)⇒(i): Let $\beta := \inf\limits_{x\in X} \sup\limits_{y\in Y} a(x,y) = \sup\limits_{y\in Y} \inf\limits_{x\in X} a(x,y)$. By assumption, $a(x, \bar{y}) \ge \beta$ for every $x \in X$. Hence, for every $\hat{y} \in Y$ the functions $a(\cdot, \bar{y}), a(\cdot, \hat{y})\colon X \to \bar{\mathbb{R}}$

have the intersection property on X at the level β. Moreover, for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that

$$a(x_\varepsilon, y) < \beta + \varepsilon \quad \text{for every} \quad y \in Y. \tag{8}$$

Let $\tilde{y} \in Y$ and $\tilde{\varepsilon}$ be such that the set $[a(\cdot, \tilde{y}) \leq \beta + \tilde{\varepsilon}]$ is bounded. Consider the sets

$$A_n := [a(\cdot, \tilde{y}) \leq \beta + \frac{1}{n}] \cap [a(\cdot, \bar{y}) \leq \beta + \frac{1}{n}], \quad \text{for all} \;\; n \in \mathbb{N}.$$

The sets A_n are nonempty (by (8)), $A_{n+1} \subset A_n$, convex, closed and bounded for all $n \geq n_0$. By the Cantor's intersection theorem, there exists $\bar{x} \in X$ such that

$$\bar{x} \in \bigcap_{n \geq n_0}^{\infty} A_n,$$

thus, $\bar{x} \in [a(\cdot, \tilde{y}) \leq \beta] \cap [a(\cdot, \bar{y}) \leq \beta]$. Hence, by Theorem 5.2, there exist $x_1 \in \operatorname{dom}(a(\cdot, \bar{y}))$, $x_2 \in \operatorname{dom}(a(\cdot, \tilde{y}))$ and $\varphi_1 \in \partial_{\Phi_{conv}} a(\cdot, \bar{y})(x_1)$, $\varphi_2 \in \partial_{\Phi_{conv}} a(\cdot, \tilde{y})(x_2)$ such that the intersection property holds for φ_1 and φ_2 on X at the level β.

Let $\alpha < \inf\limits_{x \in X} \sup\limits_{y \in Y} a(x, y) = \beta$. By Proposition 5.3, the functions φ_1 and φ_2 have the intersection property on X at the level α. □

Taking into account Theorem 3.2 we can formulate the following sufficient and necessary conditions for the minimax equality in terms of Φ-ε-subgradients for Φ-convex functions.

Theorem 6.5. *Let X be a set and Y be a vector space. Let $a\colon X \times Y \to \bar{\mathbb{R}}$ be a function such that for any $y \in Y$ the function $a(\cdot, y)\colon X \to \mathbb{R}$ is proper Φ-convex on X and for any $x \in X$ the function $a(x, \cdot)\colon Y \to \mathbb{R}$ is concave on Y. The following conditions are equivalent*

(i) *for every $\alpha \in \mathbb{R}$, $\alpha < \inf\limits_{x \in X} \sup\limits_{y \in Y} a(x, y)$ and for every $\varepsilon > 0$ there exist $y_1, y_2 \in Y$, $x_1 \in \operatorname{dom}(a(\cdot, y_1))$, $x_2 \in \operatorname{dom}(a(\cdot, y_2))$ and $\varphi_1 \in \partial_\Phi^\varepsilon a(\cdot, y_1)(x_1)$, $\varphi_2 \in \partial_\Phi^\varepsilon a(\cdot, y_2)(x_2)$ such that the intersection property holds for φ_1 and φ_2 on X at the level α,*

(ii) $\sup\limits_{y \in Y} \inf\limits_{x \in X} a(x, y) = \inf\limits_{x \in X} \sup\limits_{y \in Y} a(x, y)$.

Proof. Follows immediately from Theorem 3.2 and Theorem 1.2. □

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