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Two Positive Solutions for Superlinear Neumann Problems with a Complete Sturm-Liouville Operator

Gabriele Bonanno
Department of Engineering, University of Messina, C.da Di Dio, Sant'Agata, 98166 Messina, Italy
bonanno@unime.it

Antonio Iannizzotto
Department of Mathematics and Computer Science, University of Cagliari, Viale L. Merello 92, 09123 Cagliari, Italy
antonio.iannizzotto@unica.it

Monica Marras
Department of Mathematics and Computer Science, University of Cagliari, Viale L. Merello 92, 09123 Cagliari, Italy
mmarras@unica.it

Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

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We establish existence of two positive solutions for a nonlinear Sturm-Liouville equation in a complete form, that is, involving the first derivative, with Neumann boundary conditions. The conclusion is obtained by assuming a suitable behaviour of the nonlinearity in a well determined interval and at infinity, requiring no condition at zero. Our approach is based on variational methods.

Keywords: Neumann problem, multiple solutions, variational methods.

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1. Introduction

Nonlinear second-order ordinary differential problems have a wide range of applications in various fields of applied sciences, such as physics, engineering, economics, and biology. This is due to the fact that many phenomena can be described by means of various differential equations involving a 'velocity' as well as an 'acceleration' term, represented by the first and second derivatives,

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respectively, of the unknown. Usually, such equations are coupled with some boundary conditions, and among these Neumann conditions play a prominent role, since they describe all those phenomena with a zero velocity at the edges (i.e., zero flux).

Existence and multiplicity of solutions for a nonlinear Neumann problem can be studied via several methods, e.g., fixed point methods, which in most cases require some monotonicity or sign assumptions on the nonlinear term (see [15], [17]). We mainly focus on variational methods, in which solutions are seen as the critical points of an energy functional related to the problem (see the monographs [11], [12], [14]). In most results based on variational methods, the first order term does not appear, and the authors assume convenient growth conditions on the nonlinearity both at infinity and near zero, or at $\pm\infty$ (see [9], [16]).

In the present paper we investigate a Neumann problem for an ordinary differential equation driven by the complete Sturm-Liouville operator (involving the first derivative), under the Ambrosetti-Rabinowitz condition (which implies superlinearity at infinity, see [2]) together with a local condition, but no growth assumption near zero. For other examples of results dealing with the complete Sturm-Liouville operator see [4], for the use of such local conditions in ordinary Neumann problems see [10] (see also [5], [6] for the elliptic case).

Precisely, we consider the following boundary value problem:

$$\begin{cases} -(p(x)u')' + r(x)u' + q(x)u = \lambda f(x,u) & \text{in } [a,b] \\ u'(a) = u'(b) = 0, \end{cases} \tag{1}$$

where $a < b$ are real numbers, $p \in C^1([a,b])$, $q, r \in C^0([a,b])$ are functions s.t. p, q are positive (while r has no prescribed sign), $f \in C^0([a,b] \times \mathbb{R}^+)$ is a non-negative function ($\mathbb{R}^+ = [0, +\infty[$), and $\lambda > 0$ is a parameter.

Before stating our main result, we introduce some notation: let $R \in C^1([a,b])$ be a primitive of $\frac{r}{p}$, and set

$$R_0 = \min_{x \in [a,b]} e^{-R(x)}, \quad R_1 = \max_{x \in [a,b]} e^{-R(x)},$$

and define three positive real numbers

$$p_0 = \min_{x \in [a,b]} p(x), \quad q_0 = \min_{x \in [a,b]} q(x), \quad m = R_0 \min\{p_0, q_0\}.$$

Besides, set

$$M = \begin{cases} \sqrt{\dfrac{2(b-a)}{m}} \max\left\{1, \dfrac{1}{b-a}\right\} & \text{if } \sqrt{2}-1 \leqslant b-a \leqslant \dfrac{1}{\sqrt{2}-1} \\ \sqrt{\dfrac{b-a}{m}} \left(1 + \dfrac{1}{b-a}\right) & \text{otherwise,} \end{cases} \tag{2}$$

and define $K \in (0,1)$ by setting

$$K = \frac{R_0}{M^2 \|q\|_1 R_1^2} \tag{3}$$

(by $\|\cdot\|_\nu$ we denote the $L^\nu([a,b])$-norm for $\nu \in [1,+\infty]$). Moreover, we set

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau \text{ for all } (x,t) \in [a,b] \times \mathbb{R}^+.$$

Our main results is the following:

Theorem 1.1. *Let $f \in C^0([a,b] \times \mathbb{R}^+)$ be a non-negative function satisfying the following conditions for all $(x,t) \in [a,b] \times [L,+\infty[$:*

(AR) *there exist* $\mu > 2$, $L > 0$ *such that* $0 < \mu F(x,t) \leqslant t f(x,t)$*;*

(LC) *there exist* $0 < d < c$, *such that* $\dfrac{1}{c^2}\displaystyle\int_a^b F(x,c)\,dx < \frac{K}{d^2}\int_a^b F(x,d)\,dx.$

Then, for all λ lying in the interval

$$\Lambda = \left] \frac{d^2}{2M^2 R_1 K} \Big(\int_a^b F(x,d)\,dx \Big)^{-1}, \frac{c^2}{2M^2 R_1} \Big(\int_a^b F(x,c)\,dx \Big)^{-1} \right[,$$

problem (1) *admits at least two positive classical solutions.*

Here (AR) is the classical Ambrosetti-Rabinowitz condition, and inequality (LC) is a local condition which is somewhat related to a sublinear growth of $f(x,\cdot)$ in some interval, while it does not prescribe the behavior of f near zero. The paper has the following structure: in Section 2 we collect some preliminary results; in Section 3 we prove Theorem 1.1; and in Section 4 we discuss some special cases and applications.

2. Preliminary results

Our main tool is a two critical point result, established in [7] (which comes from an appropriate combination of the mountain pass theorem of [2] and of a local minimum theorem of [3]). Before stating it, we introduce some notation: let $(X, \|\cdot\|)$ denote a real Banach space, $(X^*, \|\cdot\|_*)$ its topological dual, $I \in C^1(X)$ a continuously differentiable functional defined in X. We recall the classical Palais-Smale condition for I:

(PS) every sequence (u_n) in X, such that $(I(u_n))$ is bounded in $\mathbb{R}$ and $I'(u_n) \to 0$ in X^*, admits a (strongly) convergent subsequence.

Theorem 2.1. [7, Theorem 2.1] *Let $(X, \|\cdot\|)$ be a real Banach space, $\Phi, \Psi \in C^1(X)$, and $I_\lambda = \Phi - \lambda\Psi$ for all $\lambda > 0$. Let $r \in \mathbb{R}$, $\tilde{u} \in X$ satisfy the following conditions:*

(i) $\inf_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$;

(ii) $0 < \Phi(\tilde{u}) < r$;

(iii) $\dfrac{1}{r} \sup_{\Phi(u) \leqslant r} \Psi(u) < \dfrac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$;

(iv) *for all λ lying in the interval*

$$\tilde{\Lambda} = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, r\Big(\sup_{\Phi(u) \leqslant r} \Psi(u)\Big)^{-1}\right[,$$

the functional I_λ is unbounded from below and sasitsfies (PS).

Then, for all $\lambda \in \tilde{\Lambda}$ there exist $u_{1,\lambda}, u_{2,\lambda} \in X$ such that

$$I'_\lambda(u_{1,\lambda}) = I'_\lambda(u_{2,\lambda}) = 0, \ I_\lambda(u_{1,\lambda}) < 0 < I_\lambda(u_{2,\lambda}).$$

We have used the convention $0^{-1} = \infty$ and we will do the same in what follows.

Next, we provide problem (1) with a variational framework (for an alternative approach, see [8, p. 223]). Set $X = H^1([a, b])$ endowed with the norm

$$\|u\| = \Big(\int_a^b e^{-R(x)}\big(p(x)(u')^2 + q(x)u^2\big)\,dx\Big)^{\frac{1}{2}}.$$

So, X is a Hilbert space and it is compactly embedded in $(C^0([a, b]), \|\cdot\|_\infty)$ with

$$\|u\|_\infty \leqslant M\|u\| \text{ for all } u \in X, \tag{4}$$

where $M > 0$ is given by (2). Indeed, for all $u \in X$ we can find $\bar{x} \in [a, b]$ with

$$|u(\bar{x})| = \frac{1}{b-a}\int_a^b |u(\xi)|\,d\xi.$$

So, we have for all $x \in [a, b]$

$$\begin{aligned}
|u(x)| &= \Big|\int_{\bar{x}}^x u'(\xi)\,d\xi + u(\bar{x})\Big| \leqslant \int_a^b |u'(\xi)|\,d\xi + \frac{1}{b-a}\int_a^b |u(\xi)|\,d\xi \\
&\leqslant (b-a)^{\frac{1}{2}}\Big(\int_a^b u'(\xi)^2\,d\xi\Big)^{\frac{1}{2}} + \frac{1}{(b-a)^{\frac{1}{2}}}\Big(\int_a^b u(\xi)^2\,d\xi\Big)^{\frac{1}{2}} \\
&\leqslant \Big(\frac{b-a}{R_0 p_0}\Big)^{\frac{1}{2}}\Big(\int_a^b e^{-R(\xi)}p(\xi)u'(\xi)^2\,d\xi\Big)^{\frac{1}{2}} + \\
&\qquad + \Big(\frac{1}{(b-a)R_0 q_0}\Big)^{\frac{1}{2}}\Big(\int_a^b e^{-R(\xi)}q(\xi)u(\xi)^2\,d\xi\Big)^{\frac{1}{2}}.
\end{aligned}$$

Now, if $\sqrt{2}-1 \leqslant b-a \leqslant \frac{1}{\sqrt{2}-1}$, we have

$$\begin{aligned}\|u\|_\infty &\leqslant \Big(\frac{b-a}{m}\Big)^{\frac{1}{2}} \max\Big\{1, \frac{1}{b-a}\Big\}\Big[\Big(\int_a^b e^{-R(\xi)} p(\xi) u'(\xi)^2\, d\xi\Big)^{\frac{1}{2}} + \\ &\qquad\qquad + \Big(\int_a^b e^{-R(\xi)} q(\xi) u(\xi)^2\, d\xi\Big)^{\frac{1}{2}}\Big] \\ &\leqslant \Big(\frac{2(b-a)}{m}\Big)^{\frac{1}{2}} \max\Big\{1, \frac{1}{b-a}\Big\}\|u\|.\end{aligned}$$

Otherwise, we get

$$\|u\|_\infty \leqslant \Big(\frac{b-a}{m}\Big)^{\frac{1}{2}}\Big(1+\frac{1}{b-a}\Big)\|u\|,$$

thus proving (4).

Now we extend f, F to continuous functions on $[a,b]\times\mathbb{R}$ by setting $f(x,t) = f(x,0)$ for all $x \in [a,b]$, $t<0$ and keeping the same definition of F. For all $u \in X$ we set

$$\Phi(u) = \frac{\|u\|^2}{2},\ \ \Psi(u) = \int_a^b e^{-R(x)} F(x,u)\, dx,$$

and for all $\lambda > 0$ we define I_λ as in Theorem 2.1. Clearly, for all $\lambda > 0$ we have $I_\lambda \in C^1(X)$, besides for all $u, v \in X$ we have

$$I'_\lambda(u)(v) = \int_a^b e^{-R(x)}\big(p(x)u'v' + q(x)uv\big)\, dx - \lambda \int_a^b e^{-R(x)} f(x,u) v\, dx.$$

In particular, for all $w \in X$ we can plug $v(x) = e^{R(x)} w(x)$ into the above equality and get

$$I'_\lambda(u)(v) = \int_a^b \big(p(x)u'w' + r(x)u'w + q(x)uw\big)\, dx - \lambda \int_a^b f(x,u) w\, dx,$$

thus critical points of I_λ are exactly the weak solutions of (1). Furthermore, classical regularity theory for ordinary differential equations (see [8]) implies that, whenever $u \in X$ is a weak solution of (1), $u \in C^2([a,b])$ and it satisfies (1) pointwisely, i.e., it is indeed a classical solution (we will refer to such functions simply as *solutions*).

We prove now a strong maximum principle for problem (1):

Lemma 2.2. *Let $u \in X$ be a solution of* (1)*. Then*

(*i*) $u(x) \geqslant 0$ *for all* $x \in [a,b]$*;*

(*ii*) *if u is not identically zero, then* $u(x) > 0$ *for all* $x \in [a,b]$.

Proof. First we prove (i). Since $I'_\lambda(u) = 0$, taking $v = -u^- \in X$ as a test function we easily get

$$\|u^-\|^2 = -\lambda \int_a^b e^{-R(x)} f(x,u)u^- \, dx \leqslant 0,$$

hence $u^- = 0$ and (i) is achieved.

Now we prove (ii). Since $u \geqslant 0$, f is non-negative, and $p_0 > 0$, the following differential inequality is satisfied in $[a,b]$:

$$-u'' + \frac{r(x) - p'(x)}{p(x)} u' + \frac{q(x)}{p(x)} u \geqslant 0,$$

with $\frac{q}{p} > 0$. By [13, Theorem 3] we have $u(x) > 0$ for all $x \in]a,b[$. Moreover, since u is a classical solution, we also have $u'(a) = u'(b) = 0$. Hence, by [13, Theorem 4] we get $u(a), u(b) > 0$, which concludes the proof. $\square$

One crucial step in our study is the following result, which draws both unboundedness of the energy functional and (PS) from (AR). Such implication is not new (see [14]), but we recall it here for the reader's convenience.

Lemma 2.3. *Let f satisfy* (AR). *Then, for all $\lambda > 0$*

(i) $\inf_{u \in X} I_\lambda(u) = -\infty$;

(ii) I_λ *satisfies* (PS).

Proof. First we prove (i). From (AR) we have, for all $x \in [a,b]$ and $t \geqslant L$,

$$\int_L^t \frac{\mu}{\tau} \, d\tau \leqslant \int_L^t \frac{f(x,\tau)}{F(x,\tau)} \, d\tau,$$

which implies

$$F(x,t) \geqslant \frac{F(x,L)}{L^\mu} t^\mu \geqslant A t^\mu \ (A > 0).$$

Besides, by continuity, there exists $B \geqslant 0$ such that for all $x \in [a,b]$ and $t \in [0,L]$ we have $F(x,t) - At^\mu \geqslant -B$. Thus, we have

$$F(x,t) \geqslant At^\mu - B \text{ for all } (x,t) \in [a,b] \times \mathbb{R}^+. \tag{5}$$

Fix $u_0 \in X$ such that $u_0(x) > 0$ for all $x \in [a,b]$. For all $h > 0$ we have

$$\begin{aligned} I_\lambda(hu_0) &= \frac{\|hu_0\|^2}{2} - \lambda \int_a^b e^{-R(x)} F(x, hu_0) \, dx \\ &\leqslant h^2 \frac{\|u_0\|^2}{2} - \lambda \int_a^b e^{-R(x)} (A(hu_0)^\mu - B) \, dx \\ &\leqslant h^2 \frac{\|u_0\|^2}{2} - \lambda R_0 A h^\mu \|u_0\|_\mu^\mu + \lambda R_1 B(b-a), \end{aligned}$$

and the latter tends to $-\infty$ as $h \to +\infty$ (recall that $\mu > 2$), thus proving (i).

Now we prove (ii). Let (u_n) be a sequence in X, such that $|I_\lambda(u_n)| \leqslant C$ for all $n \in \mathbb{N}$ and $\|I'_\lambda(u_n)\|_* \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$ we have

$$\frac{\mu\|u_n\|^2}{2} = \mu I_\lambda(u_n) + \lambda \int_a^b \mu e^{-R(x)} F(x, u_n)\, dx,$$

$$\|u_n\|^2 = I'_\lambda(u_n)(u_n) + \lambda \int_a^b e^{-R(x)} f(x, u_n) u_n\, dx.$$

Summing up the above equalities, we have

$$\begin{aligned}
\frac{\mu-2}{2}\|u_n\|^2 &= \mu I_\lambda(u_n) - I'_\lambda(u_n)(u_n) + \lambda \int_a^b e^{-R(x)} \big(\mu F(x,u_n) - f(x,u_n)u_n\big)\, dx \\
&\leqslant \mu C + \|I'_\lambda(u_n)\|_* \|u_n\| + \lambda \int_{\{u_n \leqslant L\}} e^{-R(x)} \big(\mu F(x,u_n) - f(x,u_n)u_n\big)\, dx \\
&\leqslant \mu C + \|I'_\lambda(u_n)\|_* \|u_n\| + \lambda \int_{\{0 \leqslant u_n \leqslant L\}} e^{-R(x)} \big|\mu F(x,u_n) - f(x,u_n)u_n\big|\, dx \\
&\qquad\qquad + \lambda \int_{\{u_n \leqslant 0\}} (\mu - 1) f(x,0) u_n\, dx \\
&\leqslant \mu C + \|I'_\lambda(u_n)\|_* \|u_n\| + \lambda R_1 (b-a) \max_{(x,t)\in[a,b]\times[0,L]} \big|\mu F(x,t) - f(x,t)t\big| \\
&\qquad\qquad + \lambda(\mu-1) R_1 (b-a) \max_{x\in[a,b]} f(x,0) \|u_n\|_\infty \\
&\leqslant C_1 \|u_n\| + C_2,
\end{aligned}$$

where we have used (AR) and (4) and we have set

$$C_1 = \max_{n\in\mathbb{N}} \|I'_\lambda(u_n)\|_* + \lambda(\mu-1) R_1 (b-a) \max_{x\in[a,b]} f(x,0) M,$$

$$C_2 = \mu C + \lambda R_1 (b-a) \max_{(x,t)\in[a,b]\times[0,L]} \big|\mu F(x,t) - f(x,t)t\big|.$$

Since $\mu > 2$, from the above relations it follows that (u_n) is bounded in X. By reflexivity of X and the compact embedding $X \hookrightarrow C^0([a,b])$, passing to a subsequence we have $u_n \rightharpoonup u$ in X and $u_n \to u$ in $C^0([a,b])$, for some $u \in X$. In particular, there exists $C_3 > 0$ such that $\|u_n\|_\infty \leqslant C_3$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|u_n - u\|^2 &= \langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle \\
&= I'_\lambda(u_n)(u_n - u) + \lambda \int_a^b e^{-R(x)} f(x,u_n)(u_n - u)\, dx - \langle u, u_n - u\rangle \\
&\leqslant \|I'_\lambda(u_n)\|_* \|u_n - u\| + \\
&\qquad + \lambda R_1 (b-a) \max_{(x,t)\in[a,b]\times[-C_3,C_3]} |f(x,t)| \|u_n - u\|_\infty - \langle u, u_n - u\rangle,
\end{aligned}$$

and the latter tends to 0 as $n \to \infty$. So $u_n \to u$ in X, thus proving (PS) and concluding the argument. □

3. Proof of the main result

In this section we give the proof of Theorem 1.1. As said in Section 2, we are going to apply Theorem 2.1 to the space X and the functionals Φ, Ψ, and I_λ defined above. We set

$$r = \frac{c^2}{2M^2}, \ \tilde{u}(x) = d \text{ for all } x \in [a, b],$$

where $c, d > 0$ are as in hypothesis (LC) and $M > 0$ is defined by (2). Clearly $r > 0$ and $\tilde{u} \in X$. Now we will show that all assumptions of Theorem 2.1 are satisfied.

We check hypothesis (i). Clearly $\Phi(0) = \Psi(0) = 0$, while $\Phi(u) > 0$ for all $u \in X$, $u \neq 0$.

We check hypothesis (ii). We know that $d < c$, but by (LC) this inequality improves to

$$d < \sqrt{K}c, \tag{6}$$

where $K \in (0, 1)$ is defined by (3). Indeed, arguing by contradiction, assume $d < c \leqslant \frac{d}{\sqrt{K}}$. Then, recalling also that $F(x, \cdot)$ is non-decreasing for all $x \in [a, b]$, we would have

$$\frac{K}{d^2}\int_a^b F(x, d)\, dx \leqslant \frac{1}{c^2}\int_a^b F(x, c)\, dx,$$

against (LC). By (6) and (3) we have

$$\Phi(\tilde{u}) = \frac{1}{2}\int_a^b e^{-R(x)}q(x)d^2\, dx < \frac{R_1\|q\|_1 Kc^2}{2} = \frac{R_0c^2}{2R_1M^2} \leqslant r.$$

We check hypothesis (iii). For all $u \in X$, $\Phi(u) \leqslant r$, by (4) we have $\|u\|_\infty \leqslant c$, hence

$$\Psi(u) \leqslant \int_a^b e^{-R(x)} \max_{|t|\leqslant c} F(x, c)\, dx \leqslant R_1\int_a^b F(x, c)\, dx$$

By (LC) we have

$$\frac{1}{r}\sup_{\Phi(u)\leqslant r}\Psi(u) \leqslant \frac{2M^2R_1}{c^2}\int_a^b F(x, c)\, dx < \frac{2M^2R_1K}{d^2}\int_a^b F(x, d)\, dx.$$

Besides, by (6) and the above estimate of $\Phi(\tilde{u})$, we have

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} > R_0\int_a^b F(x, d)\, dx \left(\frac{R_0c^2}{2R_1M^2}\right)^{-1} = \frac{2M^2R_1K}{d^2}\int_a^b F(x, d)\, dx.$$

Comparing the above inequalities, we get

$$\frac{1}{r}\sup_{\Phi(u)\leqslant r}\Psi(u) < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}\ .$$

Finally we check (*iv*). For all $\lambda > 0$, we know from Lemma 2.3 that I_λ is unbounded from below and satisfies (PS) (note that $\tilde{\Lambda} \subseteq]0, +\infty[$).

Theorem 2.1 ensures, for all $\lambda \in \tilde{\Lambda}$, the existence of two critical points $u_{1,\lambda}, u_{2,\lambda} \in X$ of I_λ with $I_\lambda(u_{1,\lambda}) < 0 < I_\lambda(u_{2,\lambda})$, in particular $u_{1,\lambda} \neq u_{2,\lambda}$ and both are not zero. Let us check that $\Lambda \subseteq \tilde{\Lambda}$. Indeed, by the above inequalities we have

$$\begin{aligned} \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})} &\leqslant \frac{d^2}{2M^2R_1K}\Big(\int_a^b F(x,d)\,dx\Big)^{-1} \\ &< \frac{c^2}{2M^2R_1}\Big(\int_a^b F(x,c)\,dx\Big)^{-1} \leqslant r\Big(\sup_{\Phi(u)\leqslant r}\Psi(u)\Big)^{-1}. \end{aligned}$$

So, for all $\lambda \in \Lambda$ the functional I_λ admits at least two distinct critical points $u_{1,\lambda}, u_{2,\lambda} \in X \setminus \{0\}$. By classical regularity theory and Lemma 2.2, both are positive solutions of (1). $\square$

4. Special cases

This final section is devoted to highlighting some special cases and consequences of Theorem 1.1.

We begin by considering the typically variational case $r = 0$. Then, problem (1) rephrases as

$$\begin{cases} -(p(x)u')' + q(x)u = \lambda f(x,u) & \text{in } [a,b] \\ u'(a) = u'(b) = 0, \end{cases} \tag{7}$$

while obviously $R = 0$ produces $R_0 = R_1 = 1$ and M, K defined by (2), (3) are replaced by

$$M_0 = \begin{cases} \sqrt{\dfrac{2(b-a)}{m_0}}\max\Big\{1, \dfrac{1}{b-a}\Big\} & \text{if } \sqrt{2}-1 \leqslant b-a \leqslant \dfrac{1}{\sqrt{2}-1} \\ \sqrt{\dfrac{(b-a)}{m_0}}\Big(1+\dfrac{1}{b-a}\Big) & \text{otherwise,} \end{cases}$$

with $m_0 = \min\{p_0, q_0\}$, and $K_0 = \dfrac{1}{M_0\|q\|_1}$.

The following result is an immediate consequence of Theorem 1.1:

Corollary 4.1. *Let $f \in C^0([a,b] \times \mathbb{R}^+)$ be a non-negative function satisfying* (AR) *and*

(LC_0) *there exist* $0 < d < c$ *such that* $\dfrac{1}{c^2}\displaystyle\int_a^b F(x,c)\,dx < \dfrac{K_0}{d^2}\int_a^b F(x,d)\,dx.$

Then, for all λ *lying in the interval*

$$\Lambda_0 = \left]\frac{d^2}{2M_0^2 K_0}\left(\int_a^b F(x,d)\,dx\right)^{-1}, \frac{c^2}{2M_0^2}\left(\int_a^b F(x,c)\,dx\right)^{-1}\right[,$$

problem (7) *admits at least two positive classical solutions.*

Now we turn to the case when the nonlinearity $f(x,u) = f(u)$ is of autonomous type:

$$\begin{cases} -(p(x)u')' + r(x)u' + q(x)u = \lambda f(u) & \text{in } [a,b] \\ u'(a) = u'(b) = 0 \end{cases} \tag{8}$$

Here, M and K are defined as in (2), (3), respectively. From Theorem 1.1 we achieve the following result:

Corollary 4.2. *Let* $f \in C^0(\mathbb{R}^+)$ *be a non-negative function satisfying the following conditions:*

(AR_1) *there exist* $\mu > 2$, $L > 0$ *such that* $0 < \mu F(t) \leqslant t f(t)$ *for all* $t \geqslant L$;

(LC_1) *there exist* $0 < d < c$ *such that* $\dfrac{F(c)}{c^2} < \dfrac{KF(d)}{d^2}$.

Then, for all λ *lying in the interval*

$$\Lambda_1 = \left]\frac{d^2}{2M^2 R_1 K(b-a)F(d)}, \frac{c^2}{2M^2 R_1 (b-a)F(c)}\right[,$$

problem (8) *admits at least two positive classical solutions.*

The following example satisfies the assumptions of Corollary 4.2:

Example 4.3. Let us consider the problem

$$\begin{cases} -u'' - u' + (x^2+1)u = \lambda f(u) & \text{in } [0,1] \\ u'(0) = u'(1) = 0, \end{cases}$$

where $f \in C^0(\mathbb{R}^+)$ is defined by

$$f(t) = \begin{cases} \sin(t) & \text{if } t \in \left[0, \frac{\pi}{2}\right] \\ 1 & \text{if } t \in \left]\frac{\pi}{2}, 50\right] \\ (t-50)^2 + 1 & \text{if } t \in]50, +\infty[. \end{cases}$$

A straightforward computation shows that $M = \sqrt{2}$, $K = \frac{3}{8e^2}$, and (AR_1), (LC_1) hold with any $\mu \in]2,3[$, $L > 0$ big enough, and $c = 50$, $d > 0$ small enough. Thus, by Corollary 4.2, the above problem admits at least two positive solutions for any λ lying in the interval

$$\left]\frac{4e}{3}, \frac{625}{e(51-\frac{\pi}{2})}\right[.$$

In particular, condition (LC_1) holds with conveniently small $d > 0$ for nonlinearities with a sublinear behavior near zero:

Corollary 4.4. *Let $f \in C^0(\mathbb{R}^+)$ be a non-negative function satisfying* (AR_1) *and*

(LC_2) $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$.

Then, for all λ lying in the interval

$$\Lambda_2 = \left]0, \frac{1}{2M^2R_1(b-a)} \max_{t>0} \frac{t^2}{F(t)}\right[,$$

problem (8) *admits at least two positive classical solutions.*

Proof. By (AR_1) and (LC_2) we have that $t \mapsto \frac{t^2}{F(t)}$ tends to 0 both for $t \to 0^+$ and $t \to +\infty$, while it is positive in $]0, +\infty[$, so there exists $c > 0$ such that

$$\frac{c^2}{F(c)} = \max_{t>0} \frac{t^2}{F(t)}.$$

Pick $\lambda \in \Lambda_2$. Then, by (LC_2) there exists $d \in]0, c[$ such that

$$\frac{d^2}{2M^2R_1(b-a)KF(d)} < \lambda.$$

Thus, taking into account Corollary 4.2, the conclusion follows. □

By virtue of Corollary 4.4 we can handle the following case:

Example 4.5. Let us consider the problem

$$\begin{cases} -u'' + u' + (x+1)u = \lambda \max\{\sqrt{u}, u^2\} & \text{in } [0,1] \\ u'(0) = u'(1) = 0. \end{cases}$$

In this case we have $M = \sqrt{2e}$ and $K = \frac{1}{3e^2}$, besides condition (AR_1) holds with any $\mu \in]2, 3[$ and $L > 0$ big enough, and (LC_2) holds as well. By Corollary 4.4, the above problem admits at least two positive solutions for any λ lying in the interval

$$\left]0, \frac{1}{4^{\frac{2}{3}}e}\right[.$$

The situation depicted in Corollary 4.4 also fits with the well-known case of *concave-convex* nonlinearities, first studied in [1]:

$$\begin{cases} -(p(x)u')' + r(x)u' + q(x)u = \nu u^\alpha + u^\beta & \text{in } [a,b] \\ u'(a) = u'(b) = 0 \end{cases} \tag{9}$$

with $\nu > 0$ and $0 \leqslant \alpha < 1 < \beta < +\infty$ (note that the parameter ν only appears in the sublinear term). For simplicity of notation we set

$$A = \frac{(1-\alpha)(\beta+1)}{(\alpha+1)(\beta-1)},$$

and we denote

$$\nu^* = \left(2M^2 R_1(b-a) A^{\frac{\alpha-1}{\beta-\alpha}} \left(\frac{1}{\alpha+1} + \frac{A}{\beta+1}\right)\right)^{\frac{\beta-\alpha}{1-\beta}}. \tag{10}$$

We have the following result:

Corollary 4.6. *Let $0 \leqslant \alpha < 1 < \beta < +\infty$ be real numbers and $\nu^* > 0$ be defined by* (10)*. Then, for all $\nu \in]0, \nu^*[$ problem* (9) *admits at least two positive classical solutions.*

Proof. We shall apply Corollary 4.4. Fix $\nu \in]0, \nu^*[$ and set

$$f(t) = \nu t^\alpha + t^\beta \text{ for all } t \in \mathbb{R}^+.$$

Then we have

$$F(t) = \frac{\nu t^{\alpha+1}}{\alpha+1} + \frac{t^{\beta+1}}{\beta+1}.$$

Fix $\mu \in]2, \beta+1[$, then

$$\lim_{t \to +\infty} \frac{t f(t)}{\mu F(t)} = \frac{\beta+1}{\mu},$$

hence there exists $L > 0$ such that for all $t \geqslant L$

$$0 < \mu F(t) \leqslant t f(t),$$

i.e., f satisfies (AR_1). Besides, clearly f satisfies (LC_2). By Corollary 4.4 problem (8) has at least two positive solutions for any $\lambda \in \Lambda_2$.

We claim that $1 \in \Lambda_2$, i.e., that

$$\max_{t>0} \frac{t^2}{F(t)} > 2M^2 R_1(b-a). \tag{11}$$

Indeed, the mapping $t \mapsto \frac{t^2}{F(t)}$ attains its maximum at $c = (\nu A)^{\frac{1}{\beta-\alpha}}$, where we have

$$\frac{c^2}{F(c)} = \nu^{\frac{1-\beta}{\beta-\alpha}} A^{\frac{1-\alpha}{\beta-\alpha}} \Big(\frac{1}{\alpha+1} + \frac{A}{\beta+1}\Big)^{-1},$$

so (11) follows from $\nu < \nu^*$ and (10).

Thus, problem (9) is just (8) with $\lambda = 1$ and it admits at least two positive solutions. □

An explicit example:

Example 4.7. Let us consider the problem

$$\begin{cases} -u'' + e^x u = \nu\sqrt{u} + u^2 & \text{in } [0,1] \\ u'(0) = u'(1) = 0. \end{cases}$$

We have $M = \sqrt{2}$, $K = \frac{1}{2(e-1)}$, $A = 1$. So, by Corollary 4.6, the above problem admits at least two positive solutions for any ν lying in the interval

$$\Big]0, \frac{1}{8}\Big[\,.$$

Remark 4.8. We note that all problems examined in Examples 4.3, 4.5, and 4.7 *do not* admit positive constant solutions (such solutions are usually considered trivial in Neumann problems).

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