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Segmentation and Inpainting of Color Images*

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Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

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We introduce and study a variational model for segmentation and inpainting of 2-dimensional color images. The model consists in the minimization of a functional dependent on second derivatives, free discontinuity and free gradient discontinuity. The competitors are piecewise C^2 vector valued functions, whose components represent the intensity of RGB channels.

Keywords: Calculus of variations, free discontinuity problems, regularity, inpainting, image segmentation, Gamma convergence.

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1. A variational approach to RGB image analysis and main results

Image segmentation (see [1], [8], [12], [16], [36], [37]), and inpainting (see [25], [26], [29], [34]) are relevant problems in both digital image processing and understanding of biological vision.

Segmenting an image means to find regions of interest in a picture, so that these regions can be enhanced and classified for further analysis: the process cuts the picture into the simplest shaped pieces possible while keeping smooth the color

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or luminance on each piece, with the aim of identifying different objects and finding the most significant edges in a given digital image.

In the book [8] a variational principle for image segmentation was introduced in the context of visual reconstruction: the Blake-Zisserman functional which depends on second derivatives, free discontinuities and free gradient discontinuities of the intensity levels. The Blake-Zisserman model faces the segmentation as an energy minimization problem. The segmentation it looks for provides a cartoon of the given image satisfying some requirements: the decomposition of the image is performed by choosing a pattern of lines of steepest discontinuity for light intensity, and this pattern will be called segmentation of the image ([24]).

In image restoration the term inpainting denotes the process of filling in the missing information over subdomains where a given image is damaged: these domains may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film or ageing of canvas and colors in a painting ([32]).

About minimization of the Blake-Zisserman functional for analysis of monochromatic images we refer to [12], [15], [18], [19], [23], [22]. In general uniqueness of minimizers for functionals of this kind fails due to lack of convexity. Nevertheless in the 1-D formulation, uniqueness of minimizer is a generic property with respect to admissible data: in [7] is proven that for a G_δ (countable intersection of dense open sets) set of admissible data the minimizer is unique. Hence the whole picture is coherent with the presence of unstable patterns, each of them corresponding to a bifurcation of optimal segmentation under variation of parameters related to contrast threshold, "luminance sensitivity", resistance to noise, crease detection, double edge detection.

In color image processing ([3]) the datum is given as a vector-valued input $\mathbf{g}\colon \mathcal{Q} \to \mathbf{R}^3$ (a noisy image, damaged somewhere and possibly with complete loss of information in small subregions) and the required output is a vector-valued piecewise C^2 image $\mathbf{u}\colon \mathcal{Q} \to \mathbf{R}^3$. Some preliminary results about the vector-valued case with Neumann boundary condition were stated in Section 4 of [15] for segmentation problem only.

Here we introduce and study a model for both segmentation and inpainting in the vector-valued case under Dirichlet boundary condition, by exploiting the higher order part of the Blake-Zisserman functional ([13], [14], [17], [18]): the strong formulation of Blake-Zisserman like functional for inpainting of vector-valued RGB images (functional (6) below). We show the existence of an optimal segmentation (Theorems 1.3 and 1.4) together with some numerical experiments (Section 4).

The Γ-convergence approximation via elliptic functional is achieved by a procedure analogous to the one introduced by Ambrosio, Faina and March in [2] for the scalar case of gray-valued images, and recently numerically tuned and implemented in image restoration by [38].

The Euler-Lagrange system (81) associated to the approximating functionals (79) in our approach seems closely related to the stationary/asymptotic version ([27], [28]) of the scheme for Bertozzi-Esedoglou-Gillette-Cahn-Hilliard model [6] as implemented by Cherfils, Fakhi, Miranville for proving the existence of finite dimensional attractors ([27], [28]). Another approach based on finite element method was introduced and analyzed in [9].

Our paper is a preliminary study of inpainting and segmentation of color images, by exploiting the Blake-Zisserman functional: the analysis of additional constraints on the RGB channels (the output belongs either to the unit cube [30] or to the unit sphere) is postponed to subsequent papers, where fractional derivatives ([4]) are exploited.

The main novelty is the detailed analysis in the inpainting region where the functional lacks the usual fidelity term $\int |v-g|^2 dx$. To this aim we introduce a small additional term enhancing the coercivity, analogously to [22], and we provide the complete proof of the blow-up at the boundary of inpainting region for local minimizers (see Section 3, and in particular Theorem 3.4); this last result together with a long but standard regularization procedure (inspired by [13], [18], [20] and shortly summarized here in Section 2) leads to the complete proofs of Theorems 1.3 and 1.4.

Segmentation in aerophotogrammetry via Blake and Zisserman functional was studied in [40]. Several accurate numerical experiments based on the Blake-Zisserman variational principle were performed by Zanetti, Ruggiero and Miranda (see [39]) on data sets including both real and synthetic images and also Digital Surfaces Models (DSMs), obtained from remote sensing LiDAR (Light Detection and Ranging) data (see [31]).

For different second order variational approaches to image analysis we mention also some recent contributions ([5], [10], [33])

We fix the notation for the image domain and the region where the information is lost, together with basic assumptions on the parameters: we denote by $\mathcal{Q}$ the open set representing the whole area of the image, by Ω_j, the portion of the image where the j-th channel has to be restored and by the pairs (α_j, β_j) suitable contrast parameters, possibly different for the various channels, for $j = 1, 2, 3$. We assume

$$\Omega_j \subset\subset \mathcal{Q} \subset\subset \mathbf{R}^2\,, \quad j = 1, 2, 3; \tag{1}$$

$$\Omega_j \text{ open set with piecewise } C^2 \text{ boundary, } j = 1, 2, 3, \ \ \mathcal{Q} \text{ open set;} \tag{2}$$

$$0 < \beta_j \le \alpha_j \le 2\beta_j\ . \tag{3}$$

We recall the definition of triplets in competition for segmentation and inpainting.

Definition 1.1. We say that (K_0, K_1, v) is an *essential triplet* if

$$\begin{cases} K_0\,,\ K_1 \ \text{ are Borel subsets of } \ \mathbf{R}^2\,,\ K_0 \cap K_1 = \emptyset\,, \\ K_0 \cup K_1 \text{ is the smallest closed set such that } \ v \in C^2\big(\mathcal{Q} \setminus (K_0 \cup K_1)\big), \\ \ \text{and } v \text{ is approximately continuous in } \mathcal{Q} \setminus K_0\,. \end{cases} \tag{4}$$

The notion of essential triplet selects those triplets which are cleansed of every spurious artifact that does not affect the functional value and are good representatives in equivalence classes of admissible triplets, which allows highly irregular gray level function v (see Remark 2.5 and Lemma 2.6 in [22]).

Definition 1.2. We say that $(\mathbf{K}_0, \mathbf{K}_1, \mathbf{v})$ is an *admissible triplet* if its components are essential triplets and fulfill the constraint

$$v_j = w_j \quad \text{a.e. on } \mathcal{Q} \setminus \overline{\Omega_j} \quad (j = 1, 2, 3). \tag{5}$$

where $\mathbf{w}$ is a given vector-valued function whose components w_j are defined only in the sets $\mathcal{Q} \setminus \overline{\Omega}_j,\ j = 1, 2, 3$.

The notion of admissible triplet restricts the selection of the RGB output image (with domain in $\mathcal{Q}$) to the class of the ones which coincide with the RGB input image in the noise-free subset.

The strong formulation of Blake-Zisserman like functionals for inpainting of vector-valued RGB images is

$$\begin{aligned} \mathbf{E}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{u}) := & \sum_{j=1}^{3} \left(\int_{\overline{\Omega_j} \setminus (K_0^j \cup K_1^j)} |D^2 u_j|^2 d\mathbf{x} + \int_{U_j} \delta_j |u_j|^2 d\mathbf{x} \right) + \\ & + \sum_{j=1}^{3} \Big(\alpha_j \mathcal{H}^1(K_0^j \cap \overline{\Omega}) + \beta_j \mathcal{H}^1((K_1^j \setminus K_0^j) \cap \overline{\Omega}) \Big)\,, \end{aligned} \tag{6}$$

where $\mathbf{K}_0 = \big(K_0^1, K_0^2, K_0^3\big)$, $\mathbf{K}_1 = \big(K_1^1, K_1^2, K_1^3\big)$, $\mathbf{u} = (u_1, u_2, u_3)$, $(\mathbf{K}_0, \mathbf{K}_1, \mathbf{u})$ is an admissible triplet, $\delta_j > 0 \ \ (j = 1, 2, 3)$ and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

About assumption (3) we emphasize that the inequalities $\beta_j \le \alpha_j \le 2\beta_j\,,\ j = 1, 2, 3$, are necessary for lower semi-continuity of $\boldsymbol{\mathcal{E}}$ (see (16)).

Moreover, we assume that the undamaged portion of the image (that plays the role of Dirichlet datum) enjoys the natural properties of a piecewise smooth image. More precisely, this means that we are given a triplet $(\mathbf{T}_0, \mathbf{T}_1, \mathbf{w})$ with components $\mathbf{T}_0 = \big(T_0^1, T_0^2, T_0^3\big)$, $\mathbf{T}_1 = \big(T_1^1, T_1^2, T_1^3\big)$, $\mathbf{w} = (w_1, w_2, w_3)$, such that

$$(\mathbf{T}_0, \mathbf{T}_1, \mathbf{w}) \quad \text{is an admissible triplet,} \tag{7}$$

$$\mathcal{H}^1\left(\big(T_0^j \cup T_1^j\big) \cap \mathcal{Q}\right) < +\infty, \tag{8}$$

$$T_0^j \cup T_1^j \text{ is a finite union of } C^1 \text{ curves, } \left(T_0^j \cup T_1^j\right) \cap \partial\Omega_j \text{ is a finite set,} \tag{9}$$

$$D^2 w_j \in L^\infty(\mathcal{Q} \setminus (T_0^j \cup T_1^j \cup \partial\Omega_j)) \,. \tag{10}$$

Theorem 1.3. (Strong inpainting for RGB color images)
Assume (1),(2),(3),(5),(8),(9),(10). *Then there is at least one triplet with finite energy minimizing functional* $\mathbf{E}$ *among admissible triplets* $(\mathbf{K}_0, \mathbf{K}_1, \mathbf{u})$, *according to* (6) *and Definitions* 1.1 *and* 1.2.

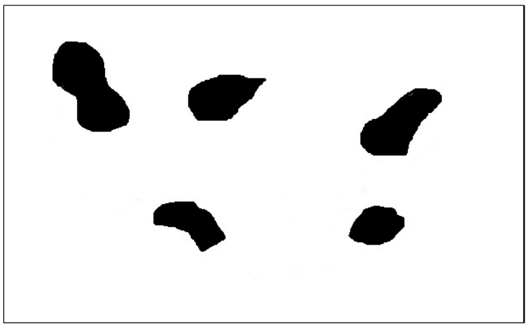

Figure 1.1: in Theorem 1.3 the image domain is the rectangle $\mathcal{Q}$, the blotches $\bigcup_j \Omega_j \subset\subset \mathcal{Q}$ with complete loss of information are the black region $\bigcup_j \overline{\Omega_j}$.

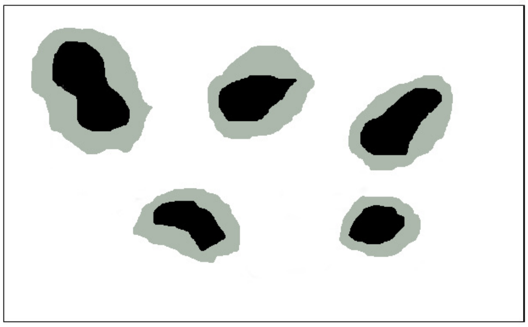

Figure 1.2: Theorem 1.4: the image domain is the rectangle $\mathcal{Q}$, the blotches $\Omega = \bigcup_j \Omega_j \subset\subset \mathcal{Q}$ with some loss of information, complete loss of information in the black region $\mathbf{U} = \bigcup_j U_j$, the partially damaged (noisy) image is given in the gray region $\Omega \setminus \mathbf{U}$.

We introduce the notation for the image domain and its noisy region, together with basic assumptions on fidelity term.

Henceforth we assume that

$$\begin{gathered} U_j \text{ are Lipschitz open sets with piecewise } C^2 \text{ boundary and} \\ U_j \subset \Omega_j \subset\subset \mathcal{Q} \subset\subset \mathbf{R}^2, \text{ and} \end{gathered} \tag{11}$$

$$\mu_j > 0\,, \;\; g_j \in L^\infty(\Omega_j \setminus U_j)\,, \;\; \mathbf{g} = (g_1, g_2, g_3)\,. \tag{12}$$

We define the strong formulation of Blake-Zisserman like functional for inpainting of noisy RGB images:

$$\mathbf{F}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{u}) := \mathbf{E}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{u}) + \sum_{j=1}^{3} \int_{\Omega_j \setminus U_j} \mu_j |u_j - g_j|^2 d\mathbf{x}\,, \tag{13}$$

where $(\mathbf{K}_0, \mathbf{K}_1, \mathbf{u})$ is an *admissible triplet* in the sense of Definition 1.2.

Theorem 1.4. (Strong inpainting for noisy RGB color images)
Assume (1), (2), (3), (5), (8), (9), (10), (11), (12). *Then there is at least one triplet with finite energy minimizing functional* $\mathbf{F}$ *among admissible triplets, according to* (13) *and Definitions* 1.1, 1.2. *The last term of any locally minimizing triplet fulfills the following system of partial differential equations:*

$$\Delta^2 u_j + \mu_j (u_j - g_j) = 0 \quad \textit{in } \Omega_j \setminus (U_j \cup K_0^j \cup K_1^j) \qquad j = 1, 2, 3\,,$$

and

$$\Delta^2 u_j = 0 \quad \textit{in } U_j \setminus (K_0^j \cup K_1^j) \qquad j = 1, 2, 3\,,$$

We refer to [17] for additional information concerning the copious set of necessary conditions fulfilled by minimizers.

2. Existence of strong solutions and Euler equations

Here the approximate limit of vector-valued functions is defined componentwise, and the singular set is the union of the singular sets of components.

We denote by $B_\varrho(\mathbf{x})$ the *open ball* $\{\, \mathbf{y} \in \mathbf{R}^2\,;\ |\mathbf{y} - \mathbf{x}| < \varrho \,\}$, and set $B_\varrho = B_\varrho(\mathbf{0})$, $B_\varrho^+ = B_\varrho \cap \{(x, y) :\ y > 0\}$, $B_\varrho^- = B_\varrho \cap \{(x, y) :\ y < 0\}$. We denote by χ_V the *characteristic function* of V for any $V \subset \mathbf{R}^2$, by $\mathcal{H}^1(V)$ its 1-*dimensional Hausdorff measure* and by $|V|$ its *Lebesgue outer measure.*

For any Borel function $v \colon \Omega \to \mathbf{R}$ and $\mathbf{x} \in \Omega$, $z \in \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$, we set $z = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ (the *approximate limit* of v at $\mathbf{x}$) if, for every $g \in C^0(\overline{\mathbf{R}})$,

$$g(z) \;=\; \lim_{\varrho \to 0} ⨍_{B_\varrho(\mathbf{0})} g(v(\mathbf{x} + \boldsymbol{\xi}))\, d\boldsymbol{\xi}\,;$$

the function $\widetilde{v}(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ is called *representative* of v; the *singular set* of v is $S_v = \{\mathbf{x} \in \Omega : \nexists z \ \text{ such that } \ \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y}) = z\}$.

A Borel function $v : \Omega \to \mathbf{R}$ is *approximately continuous at* $\mathbf{x} \in \Omega$ if and only if $\operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ exists and is a real number.

By referring to [1], [13], [17]: Dv denotes the *distributional gradient* of v, $\nabla v(\mathbf{x})$ denotes the *approximate gradient* of v. We say v is *approximately differentiable* at $\mathbf{x}$ if there exists a vector $\nabla v(\mathbf{x}) \in \mathbf{R}^2$ (the *approximate gradient* of v at $\mathbf{x}$) such that

$$\operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} \frac{|v(\mathbf{y}) - \widetilde{v}(\mathbf{x}) - \nabla v(\mathbf{x}) \cdot (\mathbf{y}-\mathbf{x})|}{|\mathbf{y}-\mathbf{x}|} = 0\,.$$

A function $u \in BV(\Omega)$ is *approximately differentiable a.e.* and for $\mathcal{H}^1$ almost all $\mathbf{x} \in S_u$ there exist $\nu(\mathbf{x}) \in \partial B_1$, $v_+(\mathbf{x}) \in \mathbf{R}$, $v_-(\mathbf{x}) \in \mathbf{R}$ with $v_+(\mathbf{x}) > v_-(\mathbf{x})$ such that

$$\lim_{\varrho\to 0} \varrho^{-2} \int_{\{\mathbf{y}\in B_\varrho;\mathbf{y}\cdot\nu(\mathbf{x})>0\}} |v(\mathbf{x}+\mathbf{y}) - v_+(\mathbf{x})|\, d\mathbf{y} = 0, \text{ and}$$

$$\lim_{\varrho\to 0} \varrho^{-2} \int_{\{\mathbf{y}\in B_\varrho;\mathbf{y}\cdot\nu(\mathbf{x})<0\}} |v(\mathbf{x}+\mathbf{y}) - v_-(\mathbf{x})|\, d\mathbf{y} = 0\,.$$

By $SBV(\Omega)$ we denote the *De Giorgi class of functions* $v \in BV(\Omega)$ such that

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, d\mathbf{x} + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1.$$

Moreover we set $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$,

$$SBV_{loc}(\Omega) \;:=\; \{v \in SBV(\Omega');\ \forall\,\Omega' \subset\subset \Omega\}\,,$$

$$GSBV(\Omega) \;:=\; \left\{ \begin{array}{c} v\colon \Omega \to \mathbf{R} \text{ Borel function and} \\ -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbb{N} \end{array} \right\}, \tag{14}$$

$$GSBV^2(\Omega) \;:=\; \big\{ v \in GSBV(\Omega),\ \nabla v \in \big(GSBV(\Omega)\big)^2 \big\}. \tag{15}$$

If $v \in GSBV(\Omega)$, then ∇v exists a.e., and for $v \in GSBV^2(\Omega)$ we set $\nabla^2 v = \nabla(\nabla v)$.

The weak formulation of Blake-Zisserman functional for vector-valued images (functional (16)) is, for all $\mathbf{v} \in \mathbf{X}(\mathcal{Q})$,

$$\boldsymbol{\mathcal{E}}(\mathbf{v}) := \sum_{j=1}^{3} \left(\int_{\Omega_j} |\nabla^2 v_j|^2\, d\mathbf{x} + \int_{U_j} \delta_j |v_j|^2 d\mathbf{x} + \alpha_j \mathcal{H}^1(S_{v_j}) + \beta_j \mathcal{H}^1(S_{\nabla v_j} \setminus S_{v_j}) \right), \tag{16}$$

and the weak formulation of Blake-Zisserman functional for inpainting of vector-valued noisy images:

$$\boldsymbol{\mathcal{F}}(\mathbf{v}) := \boldsymbol{\mathcal{E}}(\mathbf{v}) + \sum_{j=1}^{3} \int_{\Omega_j \setminus U_j} \mu_j |v_j - g_j|^2\, d\mathbf{x} \quad \forall\, \mathbf{v} \in \mathbf{X}(\mathcal{Q}), \tag{17}$$

where for any open set $\mathcal{Q} \subset \mathbb{R}^2$ the domain $\mathbf{X}(\mathcal{Q})$ is defined by

$$\mathbf{X}(\mathcal{Q}) = \big(X(\mathcal{Q})\big)^3, \text{ where } X(\mathcal{Q}) := \big\{ v\,;\, v \in GSBV^2(\mathcal{Q}),\, \nabla^2 v \in L^2(\mathcal{Q}) \big\}. \tag{18}$$

Theorem 2.1. (Weak inpainting for noisy RGB color images)
Assume (1),(2),(3),(5),(11),(12),(17) *and* $\mathbf{w} \in \mathbf{X}(\mathcal{Q})$ *with the notation* (18). *Then there is* $\mathbf{v} \in \mathbf{X}(\mathcal{Q})$ *such that* $\boldsymbol{\mathcal{F}}(\mathbf{v}) \leq \boldsymbol{\mathcal{F}}(\mathbf{z})$ *for any* $\mathbf{z} \in \mathbf{X}(\mathcal{Q})$.

Proof. Actually the assumptions (8)–(10) entail $w_j \in X(\mathcal{Q})$, $j = 1, 2, 3$. Since $\mathbf{w} \in X(\mathcal{Q} \setminus \overline{\Omega})$, we can extend its definition, without relabeling, by setting: $\mathbf{w} \equiv (0, 0, 1)$ in Ω. $\boldsymbol{\mathcal{F}}(\mathbf{v})$ is finite and $\boldsymbol{\mathcal{F}}(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in X(\mathcal{Q})$. Since $\mathbf{w} \in X(\mathcal{Q})$ we have $\mathbf{w} \in L^\infty$, $\boldsymbol{\mathcal{F}}(\mathbf{w}) < +\infty$ and $0 \leq \inf\{\boldsymbol{\mathcal{F}}(\mathbf{v}); \mathbf{v} \in X(\mathcal{Q})\} < +\infty$.

Let $\mathbf{v}_h \in X$ be a minimizing sequence for $\boldsymbol{\mathcal{F}}$. By Theorem 8 in [12] there is $\mathbf{v}_\infty$ in $X(\mathcal{Q})$ and a subsequence such that, without relabeling, $\mathbf{v}_h \to \mathbf{v}_\infty$ a.e. in $\mathcal{Q}$. Hence $\mathbf{v}_\infty \in X(\mathcal{Q})$.

The constraint $\mathbf{v}_h = \mathbf{w}$ in $\mathcal{Q} \setminus \Omega$ entails $\mathbf{v}_\infty = \mathbf{w}$ in $\mathcal{Q} \setminus \Omega$. By Theorem 10 in [12]:

$$\boldsymbol{\mathcal{F}}(\mathbf{v}_\infty) \;\leq\; \liminf_h \boldsymbol{\mathcal{F}}(\mathbf{v}_h)\,,$$

hence $\boldsymbol{\mathcal{F}}(\mathbf{v}_\infty) = \inf_{\mathbf{v} \in X(\mathcal{Q})} \boldsymbol{\mathcal{F}}(\mathbf{v})$. □

In the same way one can prove the statement below.

Theorem 2.2. (Weak inpainting for RGB color images)
Assume (1),(2),(3),(5) *and* $\mathbf{w} \in \mathbf{X}(\mathcal{Q})$, *with the notation* (16),(18).
Then there is $\mathbf{v} \in \mathbf{X}(\mathcal{Q})$ *such that* $\boldsymbol{\mathcal{E}}(\mathbf{v}) \leq \boldsymbol{\mathcal{E}}(\mathbf{z})$ *for any* $\mathbf{z} \in \mathbf{X}(\mathcal{Q})$.

Proof of Theorems 1.3, 1.4. (Strong inpainting for noisy RGB color images)
Actually the assumptions (8)–(10) entail $w_j \in X(\mathcal{Q})$, $j = 1, 2, 3$.
The proof of Theorem 1.3 follows from the one of Theorem 1.4, since no L^q estimate of minimizers is exploited in the whole proof.
In order to prove Theorem 1.4 we can repeat the analysis developed in [13], [18] and [20]: first we list a concise recap of all the steps of the procedure, then we provide the details only for the steps which are different or contain novelties: item 5.1 of the subsequent summary, say blow-up at points of ∂U (where $U = U_1 \cup U_2 \cup U_3$) which will be proved in Theorem 3.4.
The proof is achieved via direct methods by performing several steps which entail the partial regularity for a minimizer u of $\boldsymbol{\mathcal{F}}$ (weak Blake-Zisserman functional for inpainting defined by (17)). This is done following a scheme similar to the one used in [18], but here we have to deal also with the transmission condition at ∂U since the fidelity term $\mu \int |u-g|^2$ acts in $\Omega \setminus U$ and not in U. The regularity at points where a weak minimizer has vanishing 1-dimensional density of $\boldsymbol{\mathcal{F}}$ is proved by performing:

1. the procedure of [13] at points in $\Omega \setminus \overline{U}$;
2. the procedure of [13] at points in U;
3. the proof of partial regularity at points of $\partial\Omega$ via
    3.1 joining along lunulae filling half-disk in order to take into account Dirichlet condition at $\partial\Omega$;

- 3.2 blow-up at points of $\partial\Omega$ where a technical joining lemma between lunulae was exploited (Lemma 4.1 in [18] and Theorem 7.1 in [18]);
- 3.3 a decay estimate of the weak functional evaluated at local minimizers (Theorem 4.4 of [21]);

4. the proof of partial regularity at points *close to* $\partial\Omega$ via
   - 4.1 joining along lunulae filling half-disk in order to take into account Dirichlet condition at $\partial\Omega$;
   - 4.2 blow-up at points close to $\partial\Omega$ (Theorem 5.1 in [13])
   - 4.3 L^2 hessian decay for bi-harmonic functions in a portion of a disk (Theorem 3.4 and Figure 1 in [20]) taking into account the two parameters describing the lunulae;
   - 4.4 a decay estimate of the weak functional evaluated at local minimizers (Theorem 3.8 in [20]);
5. the proof of partial regularity at points of ∂U via
   - 5.1 standard joining along disks;
   - 5.2 blow-up at points of ∂U (stated in [21] and proved in Section 3 of the present paper: Theorem 3.4);
   - 5.3 the decay estimate of the weak functional evaluated at local minimizers which follows by the previous blow-up.

The blow-up argument in all the cases listed above, can be recapped as follows: if a sequence of local minimizers has vanishing length of jumps and creases, then a subsequence (which is provided by Theorem 3.4 of [21]), converges to a bi-harmonic function in the whole disk in cases $(1), (2), (4)$ and (5), and in a half-disk in case (3). If the decay property fails we can construct a sequence of local minimizers which contradicts the previous statement, thanks to the classical estimates of the hessian in cases $(1), (2), (4)$ and (5), while achieving the contradiction in case (3) is more difficult.

The usual approach to regularity at Dirichlet boundary points consists in smoothly extending the blown-up solution while keeping intact suitable energy estimates: this method is satisfactory for first order problems since in that case one can exploit the odd extension of a harmonic function. In case (3) performing regularity analysis at Dirichlet boundary points requires a smooth extension with suitable estimates of the blown-up solution. Yet the extension of bi-harmonic functions is quite different from extension of an harmonic function vanishing on the diameter, the last one is based on classical Schwarz reflection principle and doubles L^2 norm of the gradient in the whole disk: this doubling property was exploited in [11] to prove decay property for local minimizers of Mumford & Shah functional with Dirichlet boundary condition (see also [35]); unfortunately bi-harmonic extension lacks this doubling property. We overcame this difficulty by a new tool, precisely an L^2 decay estimate of hessian for a bi-harmonic function in a half-disk vanishing together with its normal derivative on the diameter (Theorem 6.1 in [18]): proving this decay requires a careful application (as in [18]) of Duffin extension formula and Almansi decomposition, since the bi-harmonic extension in the whole disk may increase a lot the L^2 norm of the hessian in the complementary half-disk.

In cases (1) and (2) we can conclude the proof as like as in the last section of [13]; in case (3) as in [18]; in case (4) as in [20]; in case (5) as in [18] but exploiting a different blow-up which takes into account transmission conditions at ∂U: the detailed procedure of blow-up suitable for this case is contained in Section 3 below.
In all cases we deduce that $\mathcal{H}^1\Big((S_u \cup S_{\nabla u}) \cap B_\varrho\Big)$ decays faster than ϱ.

By iterating the decay estimate of the functional in smaller and smaller balls, we get

$$\mathcal{H}^1\Big(\,\big(\overline{S_u \cup S_{\nabla u}} \setminus (S_u \cup S_{\nabla u})\big) \cap \mathcal{Q}\Big) = 0.$$

So we can define a minimizing triplet as follows:

$$K_0 = \overline{S_u}, \qquad K_1 = \overline{S_{\nabla u}} \setminus K_0, \qquad u = \widetilde{u}.$$

Hence we obtain

$$\Delta^2 v + \mu(v-g) = 0 \quad \text{in} \;\; \Omega \setminus (\overline{U} \cup K_0 \cup K_1)\,, \text{ and} \tag{19}$$

$$\Delta^2 v = 0 \quad \text{in} \;\; U \setminus (K_0 \cup K_1)\,. \tag{20}$$

together with many kind of integral and geometric relationships as like as minimizing triplet of Blake-Zisserman functional for image segmentation (see [17], [19]). □

Proof of Theorem 2.2.(Weak inpainting for RGB color images)
The proof can be achieved by repeating the same steps in the proof of Theorem 2.1: extension of $\mathbf{w}$ and selection of a minimizing sequence $\mathbf{v}_h$ for $\boldsymbol{\mathcal{E}}$. □

Proof of Theorem 1.3.(Strong inpainting for RGB color images)
This theorem is a straightforward consequence of Theorem 1.4. □

Moreover we get the additional properties of weak minimizers that are stated in the next remark.

Remark 2.3. Whenever $\mathbf{v}$ is a minimizer for $\boldsymbol{\mathcal{E}}$, the following construction provides a minimizing triplet for $\mathbf{E}$:

$$K_0^j = \overline{S}_{v_j}, \qquad K_1^j = \overline{S}_{\nabla v_j} \setminus K_0^j, \qquad u_j = \widetilde{v}_j \text{ in } \Omega \setminus \big(K_0^j \cup K_1^j\big).$$

Whenever $\mathbf{v}$ is a minimizer for $\boldsymbol{\mathcal{F}}$, the following construction provides a minimizing triplet for $\mathbf{F}$:

$$K_0^j = \overline{S}_{v_j} \qquad K_1^j = \overline{S}_{\nabla v_j} \setminus K_0^j, \qquad u_j = \widetilde{v}_j \text{ in } \Omega \setminus K_0^j.$$

3. Blow-up at boundary points of U

The analysis must be performed in each $U_j \subset \Omega_j$, since we analyze separately each U_j; for the sake of simplicity, we drop the suffix j; so we write δ, U and Ω instead of δ_j, U_j and Ω_j along this Section, where we analyze local minimizers of $\mathcal{F}$ at points in $\partial U \setminus (T_0 \cup T_1 \cup M)$, where $\mathcal{F}$ is the scalar version of $\boldsymbol{\mathcal{F}}$, and $\mathcal{E}$ is the scalar version of $\boldsymbol{\mathcal{E}}$, say, for all $v \in X(\mathcal{Q})$:

$$\mathcal{E}(v) := \int_\Omega |\nabla^2 v|^2 \, d\mathbf{x} + \delta \int_U |v|^2 \, d\mathbf{x} + \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v), \tag{21}$$

$$\mathcal{F}(v) := \mathcal{E}(v) + \int_{\Omega \setminus U} \mu |v-g|^2 \, d\mathbf{x} \, . \tag{22}$$

The complete proof of blow-up is available in our previous papers ([13], [18], [22]) at every kind of points in $\mathcal{Q}$ except the ones in $\overline{U}$; the blow-up in $U \cap \Omega$ can be done by repeating exactly the same steps as in [13]; in this section we provide all the details of the blow-up at points in ∂U.

In this section we show the details of item 5.1 (the blow-up at points on ∂U) by proving Theorem 3.4. We need a suitable statement about joining along annular sets filling a disk to take into account boundary effects: this is provided by Lemma 3.3, whose proof follows by Lemma 3.7 in [13].

We introduce a localization $\mathfrak{F}_\gamma$ of the functional $\mathcal{F}$ depending on suitably rescaled, strictly positive parameters $\alpha_h, \beta_h, \delta_h, \mu_h > 0$ and a function $\gamma \in L^\infty$:

$$\begin{aligned} \mathfrak{F}_\gamma(v, \mu_h, \delta_h, \alpha_h, \beta_h, A) \; := \; & \int_A |\nabla^2 v|^2 \, d\mathbf{x} + \delta_h \int_U |v|^2 \, d\mathbf{x} \; + \alpha_h \mathcal{H}^1(A \cap S_v) \; + \\ & + \beta_h \mathcal{H}^1\big(A \cap (S_{\nabla v} \setminus S_v)\big) + \mu_h \int_{A \setminus U} |v-\gamma|^2 \, d\mathbf{x} \quad \forall\, v \in X(\mathcal{Q}); \end{aligned} \tag{23}$$

we will shortly write $\mathfrak{F}(v, A) := \mathfrak{F}_\gamma(\, v \,, \mu, \delta, \alpha, \beta, A)$.

Definition 3.1. For any $\mathbf{x} \in \mathcal{Q}$ and r such that $0 < r < dist(\mathbf{x}, \partial \mathcal{Q})$, we say that v is an Ω *local minimizer of* $\mathfrak{F}_\gamma(\cdot, \mu, \alpha, \beta, A)$ if

$$\begin{cases} v \in GSBV^2(B_r(\mathbf{x}))) \colon \; v = w \; a.e. \; B_r(\mathbf{x}) \backslash \Omega, \; \mathfrak{F}_\gamma(v, \mu, \delta, \alpha, \beta, A) < +\infty, \\ \mathfrak{F}_\gamma(v, \mu, \delta, \alpha, \beta, A) \leq \mathfrak{F}_\gamma(v + \eta, \mu, \delta, \alpha, \beta, A) \\ \forall A \subset\subset B_r(\mathbf{x}), \eta \in GSBV^2(B_r(\mathbf{x})) \colon \operatorname{spt} \eta \subset A, \eta = 0 \text{ a.e. in } B_r(\mathbf{x}) \backslash \Omega. \end{cases} \tag{24}$$

The localized functional $\mathfrak{F}(v, A)$ is exploited in the blow-up procedure with the following choice of the scaling, which takes into account the different homogeneity of the various terms in the functional.

Lemma 3.2. (Scaling, see Lemma 3.4 of [13]) *Let* $v \in GSBV^2(B_r(\mathbf{x}_0))$*, where* $\mathbf{x}_0 = (x_0, y_0)$*. For* $\lambda > 0$ *and for every* $\mathbf{x} \in B_1$ *set*

$$v_r(\mathbf{x}) \;=\; \frac{v(\mathbf{x}_0 + r\mathbf{x})}{\lambda^{1/2}\, r^{3/2}}\,, \qquad g_r(\mathbf{x}) \;=\; \frac{g(\mathbf{x}_0 + r\mathbf{x})}{\lambda^{1/2}\, r^{3/2}}\,. \tag{25}$$

Then $v_r \in GSBV^2(B_1)$*,*

$$\mathcal{F}_g(v, \mu, \delta, \alpha, \beta, B_r(\mathbf{x}_0)) = \lambda r \mathcal{F}_{g_r}(v_r, \mu r^4, \delta r^4, \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, B_1) \tag{26}$$

and, by setting $K_{0r} = (K_0 - \mathbf{x}_0)/r,\ K_{1r} = (K_1 - \mathbf{x}_0)/r$*,*

$$F_g(K_0, K_1, v, \mu, \delta, \alpha, \beta, B_r(\mathbf{x}_0)) = \lambda r F_{g_r}(K_{0r}, K_{1r}, v_r, \mu r^4, \delta r^4, \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, B_1). \tag{27}$$

Lemma 3.3. (Joining along annular sets, see Lemma 3.7 of [13]) *Assume* (1), (3), (12)*,* z, u *in* $GSBV^2(\mathcal{Q})$*,* $\mathbf{x}_0 = (x_0, y_0) \in \partial U$*,* $0 < s < t < 1$ *such that* $\overline{B_t(\mathbf{x}_0)} \subset \mathcal{Q}$*,* $\partial U \cap B_t(\mathbf{x}_0) \in C^2$*. Then for every* $\theta \in (0,1)$ *there are* $c = c(\theta) > 0$ *and a cut-off function* Ψ *in* $C^2(B_t(\mathbf{x}_0))$ *such that* $\Psi \equiv 1$ *in a neighborhood of* $\overline{B_s(\mathbf{x}_0)}$ *and, by setting*

$$U = \Psi u + (1 - \Psi) z\,,$$

we have

$$\mathcal{F}(U, \overline{B_t}) \;\le\; (1+\theta)\,\big(\mathcal{F}(u, \overline{B_t}) \,+\, \mathcal{F}(z, \overline{B_t} \setminus B_s)\big) \,+$$
$$+\,\frac{c}{(t-s)^2}\left(\int_{B_t \setminus B_s} |\nabla(u-z)|^2 d\mathbf{x} \;+\; \frac{c}{\theta\, s^2\,(t-s)^2}\int_{B_t \setminus B_s} |u-z|^2 d\mathbf{x}\right).$$

Moreover, for any scalar function $v \in GSBV$ we define an affine polynomial correction such that both median and gradient median vanish. Let $B_r(\mathbf{x}) \subset \Omega$ and $v \in GSBV(B_r(\mathbf{x})$; for every $\mathbf{y} \in \mathbb{R}^n$ we set

$$(M_{\mathbf{x},r} v)(y) = \mathrm{med}(\nabla v, B_r(\mathbf{x}))\dot{(}\mathbf{y} - \mathbf{x}), \text{ and}$$

$$(\mathcal{P}_{\mathbf{x},r} v)(\mathbf{y}) = (M_{\mathbf{x},r} v)(y) - \mathrm{med}\,\big(v - M_{\mathbf{x},r} v, B_r(\mathbf{x})\big)\,,$$

where

$$\mathrm{med}(v, B) = \inf\{\ t \in \mathbf{R};\ |\{v < t\} \cap B| \ge |B|/2\ \}.$$

We emphasize that the nonlinear operator med is defined for every measurable, possibly non-integrable, function v.

We now define a suitable truncation operator T, which acts on $GSBV$ scalar functions as follows: for every $v \in GSBV(B)$ and $a \in \mathbf{R}$ with $(2\gamma\mathcal{H}^1(S_v))^2 \le a \le \frac{1}{2}|B|$, we set

$$\tau'(v, a, B) = \inf\{t \in \mathbf{R};\ |\{v < t\}| \;\ge\; a\}\,,$$
$$\tau''(v, a, B) = \inf\{t \in \mathbf{R};\ |\{v \ge t\}| \;\le\; a\}\,,$$

where c_2 is the isoperimetric constant relative to the balls of $\mathbf{R}^2$, i.e.

$$\min\{|E\cap B|^{\frac{1}{2}}, |B\setminus E|^{\frac{1}{2}}\} \leq c_2 P(E,B) \qquad \forall \text{ Borel set } E\,,$$

and $P(E,B)$ denotes the perimeter of E in B: $P(E,B)=\int_B |D\chi_E|$; by exploiting the previous tools we formulate the definition of the truncation operator as follows, for all $v\in GSBV$, $a\in\mathbf{R}$, and $\eta\geq 0$:

$$T(v,a,\eta) = (\tau'(v,a,B)-\eta)\vee v\wedge(\tau''(v,a,B)+\eta)\,. \tag{28}$$

We get easily $T(T(v,a,\eta),a,\eta)=T(v,a,\eta)$, $\mathrm{med}(T(v,a,\eta),B)=\mathrm{med}(v,B)$ and $T(\lambda v,a,\lambda\eta)=\lambda T(v,a,\eta)$ for every $\lambda>0$. Moreover $|\nabla T(v,a,\eta)|\leq|\nabla v|$ a.e. on B and

$$|\{v\neq T(v,a,\eta)\}|\leq 2a. \tag{29}$$

In case v is vector-valued the operators med and T are defined componentwise. Below we give a precise formulation and proof of a statement which was implicitly announced and exploited in [22]: we focus a generic point of ∂U such that ∂U is C^2 in a neighborhood.

Theorem 3.4. (Blow-up of the functional $\mathcal{F}$ at ∂U) *Assume* (11), (12) *and there exist a sequence of functions* $\gamma_h\in L^\infty(B_r(\mathbf{x}))$, *four sequences* α_h, β_h, μ_h, *and* $\delta_h=\frac{\delta}{\mu}\mu_h$ *of positive numbers with* $\beta_h\leq\alpha_h$ *and* $v_\infty\in H^2(B_r(\mathbf{x}))$ *with*

(i) v_h *are* Ω *local minimizers of* $\mathfrak{F}_{\gamma_h}(\,\cdot\,,\mu_h,\delta_h,\alpha_h,\beta_h,B_r(\mathbf{x}))$,

(ii) $\lim_h \mathcal{H}^1\left((S_{v_h}\cup S_{\nabla v_h})\cap B_r(\mathbf{x})\right) = 0$,

(iii) $\exists\ \lim_h \mathfrak{F}_{\gamma_h}(\,v_h,\mu_h,\delta_h,\alpha_h,\beta_h,\overline{B_\varrho}(\mathbf{x})\,) \stackrel{\text{def}}{=} \omega(\varrho)\ \leq\ 1$ *for a.e.* $\varrho\in(0,r)$.

(iv) $\lim_h v_h-\mathcal{P}_{\mathbf{x},r}v_h\ =\ v_\infty$ *a.e. in* $B_r(\mathbf{x})$,

(v) $\lim_h \mu_h=0$, $\lim_h \mu_h\|\gamma_h\|^2_{L^2(B_r(\mathbf{x})\setminus U)}=0$, $\lim_h \mu_h^{1/2}\mathcal{P}_{\mathbf{x},r}v_h=\boldsymbol{\xi}$ *uniformly in* $B_r(\mathbf{x})$, *where* $\boldsymbol{\xi}$ *is a polynomial of degree one.*

Then, for every $\varrho\in(0,r)$, v_∞ *minimizes the functional*

$$G_\varrho(w) := \int_{B_\varrho(\mathbf{x})} \left|\,D^2 w\,\right|^2 d\mathbf{y} + \frac{\delta}{\mu}\int_{B^-_\varrho(\mathbf{x})}|\boldsymbol{\xi}|^2\,d\mathbf{y} + \int_{B^+_\varrho(\mathbf{x})}|\boldsymbol{\xi}|^2\,d\mathbf{y} \tag{30}$$

over $\{w\in H^2(B_r(\mathbf{x})) : w=v_\infty \text{ in } B_r(\mathbf{x})\setminus B_\varrho(\mathbf{x})\}$.

Moreover, we have for a.e. ϱ *with* $0<\varrho<r$:

$$\omega(\varrho) = \int_{B_\varrho(\mathbf{x})} \left|\,D^2 v_\infty\,\right|^2 d\mathbf{y} + \frac{\delta}{\mu}\int_{B^-_\varrho(\mathbf{x})}|\boldsymbol{\xi}|^2 d\mathbf{y} + \int_{B^+_\varrho(\mathbf{x})}|\boldsymbol{\xi}|^2 d\mathbf{y}. \tag{31}$$

In particular $\Delta^2 v_\infty=0$ *in* $B_r(\mathbf{x})$.

Proof. Without loss of generality and up to a change of coordinates, we can assume $\mathbf{x} = \mathbf{0}$, and perform the analysis in $B_r := B_r(\mathbf{0})$ and:

$$\begin{cases} \mathbf{0} \in \partial U,\ B_r(\mathbf{0}) \subset \Omega,\ \psi_h \in C^2(-r,r),\ \psi_h(\mathbf{0}) = 0, \\ \psi_h'(\mathbf{0}) = 0,\ \mathrm{Lip}\,(\psi_h') \le 1,\ \psi_h \to 0 \ \text{ in } \ W^{2,\infty}(-r,+r), \\ B_\varrho^{\psi_h +} \overset{\text{def}}{:=} B_\varrho \cap \{y > \psi_h(x)\},\ B_\varrho^{\psi_h -} \overset{\text{def}}{:=} B_\varrho \cap \{y < \psi_h(x)\}, \\ B_\varrho^{+} = \{\mathbf{x} = (x,y) :\ |\mathbf{x}| < \varrho,\ y > 0)\} \ \text{for } 0 < \varrho < r\,, \\ B_\varrho^{-} = \{\mathbf{x} = (x,y) :\ |\mathbf{x}| < \varrho,\ y < 0)\} \ \text{for } 0 < \varrho < r\,. \end{cases} \tag{32}$$

We can assume r small enough such that $B_r \cap \partial\Omega = \emptyset$ and we set

$$u_h = v_h - \mathcal{P}_{\mathbf{0},r} v_h \qquad \text{in } B_r\,.$$

Hence no Dirichlet datum is imposed in B_r.

First we notice that, due to (32), ψ_h converges uniformly to a C^2 parametrization of the diameter of B, $B^{\psi_h +}$ converges to B^+ in Kuratowski sense and the indicator function of the symmetric difference of $B^{\psi_h +}$ and B^+ tends to 0 in L^1. By (iv), (v), $v_h - \mathcal{P}_{\mathbf{0},r} v_h \to v_\infty$ a.e. and $\mu_h \to 0$ we get, up to subsequences,

$$\lim_h \ {\mu_h}^{1/2}\, |v_h - \mathcal{P}_{\mathbf{0},r} v_h| \ = \ 0 \qquad \text{a.e. in } B_r\,;$$

so that, by Fatou's Lemma:

$$\int_{B_\varrho^\pm} |\boldsymbol{\xi}|^2 \ = \ \lim_h \mu_h \int_{B_\varrho^\pm} |\mathcal{P}_{\mathbf{0},r} v_h|^2 \ \le \ \liminf_h \ \mu_h \int_{B_\varrho^\pm} |v_h - \gamma_h|^2\,. \tag{33}$$

By (ii) (iii) sequence $u_h = v_h - \mathcal{P}_{\mathbf{0},r} v_h$ fulfills all the assumptions of Theorem 4.3 in [13], namely:

$$\sup_\varrho \sup_h \int_{B_\varrho} |\nabla^2 v_h|^2\, d\mathbf{x} \ \le \ 1\ , \qquad \lim_h L_h \ = \ 0\ . \tag{34}$$

Therefore we can proceed to build a new sequence z_h, by truncating and joining u_h. By setting $L_h = \mathcal{H}^1\left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(\mathbf{0})\right)$ and $a_h = 4{c_2}^2 {L_h}^2$, it turns out $a_h \le |B_r|/2$ for large h. Hence there is a constant c dependent on $\sup_h \|\nabla^2 v_h\|^2_{L^2}$ and there are $\eta_h^k \in (0,1)\,, h \in \mathbb{N},\ k = 1,2$, such that

$$\left|\,\{\, T(\nabla_k\, u_h, a_h, \eta_h^k) \ \ne \ \nabla_k\, u_h \,\}\,\right| \ \le \ c\, {L_h}^2 \tag{35}$$

$$P\left(\,\{\, T(\nabla_k\, u_h, a_h, \eta_h^k) \ \ne \ \nabla_k\, u_h \,\},\, B_r\,\right) \ \le \ c\left(L_h + \mathcal{H}^1(S_{\nabla_k\, u_h})\right) \tag{36}$$

Referring to definition (28) of truncating operator T, we set

$$E_h \ = \ \bigcup_{k=1,2} \{\, \mathbf{y} \in B_r \ :\ T(\nabla_k\, u_h, a_h, \eta_h^k) \ \ne \ \nabla_k\, u_h \,\}, \tag{37}$$

$$\xi_h \;=\; u_h\, \chi_{B_r \setminus E_h}, \tag{38}$$

$$b_h \;=\; 4\, {K_\vartheta}^2\, \left(\mathcal{H}^1(S_{\xi_h} \cup S_{\nabla \xi_h}) \right)^2 \;\le\; \frac{1}{2}\, |B_r|, \text{ and} \tag{39}$$

$$z_h \;=\; T(\xi_h, b_h, \eta_h). \tag{40}$$

Due to a compactness result (Theorem 5.4 of [18]), we can choose subsequences (without relabeling) z_h, u_h, $v_h = u_h + \mathcal{P}_{\mathbf{0},r} v_h$ and $u_\infty \in H^2$ and a sequence $z_h \in GSBV^2(B_r(\mathbf{x}))$ (whose explicit construction is given by (35)–(40)) such that, up to a finite number of indices,

$$|\,\{z_h \neq u_h\}\,| \le\; c\, {L_h}^2, \tag{41}$$

$$P(\,\{z_h \neq u_h\},\, B_r(\mathbf{0}))\;\le\; c\, L_h, \tag{42}$$

and there is a subsequence z_{h_k} such that

$$\lim_k\, z_{h_k} \;=\; u_\infty \;\text{ strongly in } L^p(B_r(\mathbf{0}))\,,\; \forall p \ge 1\,, \tag{43}$$

$$\lim_k\, \nabla\, z_{h_k} \;=\; Du_\infty \;\text{ strongly in } L^p(B_r(\mathbf{0}))\,,\; \forall p \ge 1\,, \tag{44}$$

$$\int_{B_r(\mathbf{0})} |D^2 u_\infty|^2\, d\mathbf{y} \le \liminf_k \int_{B_r(\mathbf{0})} |\nabla^2 z_{h_k}|^2\, d\mathbf{y} \le \liminf_k \int_{B_r(\mathbf{0})} |\nabla^2 u_{h_k}|^2\, d\mathbf{y}\,. \tag{45}$$

Since $\lim_h \mu_h^{1/2} \mathcal{P}_{\mathbf{0},r} v_h = \boldsymbol{\xi}$ uniformly in $B_r(\mathbf{0})$ and $v_h \to v_\infty$ a.e., we get

$$u_h \to u_\infty = v_\infty \;\text{ a.e. in } B_r(\mathbf{0})\,.$$

By (iii) and (v) we obtain, for a.e. $0 < \varrho < r$,

$$\begin{aligned}
&\int_{B_\varrho} \left|\, D^2 v_\infty \,\right|^2\, d\mathbf{y} + \frac{\delta}{\mu} \int_{B_\varrho^-} |\boldsymbol{\xi}|^2\, d\mathbf{y} + \int_{B_\varrho^+} |\boldsymbol{\xi}|^2\, d\mathbf{y} \le \\
&\le \liminf_h \int_{B_\varrho} \left|\, \nabla^2 u_h \,\right|^2\, d\mathbf{y} + \frac{\delta}{\mu} \int_{B_\varrho^{\psi_h -}} \mu_h\, |v_h|^2\, d\mathbf{y} + \int_{B_\varrho \setminus U} \mu_h |u_h - \gamma_h + \mathcal{P}_{\mathbf{0},r} v_h|^2\, d\mathbf{y} \\
&\le \liminf_h \int_{B_\varrho} \left(\left|\, \nabla^2 v_h \,\right|^2 + \delta_h\, |v_h|^2 \right)\, d\mathbf{y} \;+ \int_{B_\varrho \setminus U} \mu_h |v_h - \gamma_h|^2\, d\mathbf{y} \\
&\le \lim_h\, \mathfrak{F}_{\gamma_h}(v_h, \overline{B_\varrho}) \;=\; \omega(\varrho)\,.
\end{aligned}$$

Until now we have shown $G_\varrho(v_\infty) \le \omega(\varrho)$. In the sequel we prove the opposite inequality: $G_\varrho(v_\infty) \ge \omega(\varrho)$.

To achieve the proof of this last claim we have to show that for a.e. ϱ, $0 < \varrho < r$, for every $w \in H^2(B_r)$ with $w = v_\infty$ in $B_r(\mathbf{0}) \setminus B_\varrho$:

$$\int_{B_\varrho} \left|\, D^2 w \,\right|^2\, d\mathbf{y} + \frac{\delta}{\mu} \int_{B_\varrho^-} |\boldsymbol{\xi}|^2\, d\mathbf{y} + \int_{B_\varrho^+} |\boldsymbol{\xi}|^2\, d\mathbf{y} \;\ge\; \omega(\varrho)\,. \tag{46}$$

In fact (46) implies $\Delta^2 v_\infty = 0$, in B_ϱ.

Now we prove the inequality (46). Referring to (37), (39), we set

$$L_h = \mathcal{H}^1 \left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(\mathbf{0}) \right) , \tag{47}$$

$$\Xi_h \; = \; \{ \mathbf{y} \in B_r \; : \; z_h \neq \xi_h \} , \text{ and} \tag{48}$$

$$A_h \; = \; \{ z_h \neq u_h \} . \tag{49}$$

In particular $A_h \; = \; E_h \cup \Xi_h$.

In order to get a contradiction we will paste together u_h and z_h along the boundary of a suitably chosen annular set $\big(B_\varrho \setminus B_s$ as it will be chosen below, fulfilling (54), (53): the energy addition will tend to 0 as $h \to \infty$ due to (iii), (42)$\big)$ and a quantitative matching (Lemma 3.6 in [13]), then we will join such new function in a smaller annular set with u (which has less squared hessian energy). Before some preliminary estimates are needed. We emphasize that

$$\mathcal{H}^1(S_{v_h} \cap \partial B_\varrho) \; = \mathcal{H}^1(S_{\nabla v_h} \cap \partial B_\varrho) \; = \; 0 \qquad \text{for } a.e \; \varrho \in (0, r) \tag{50}$$

and by (42)

$$P(A_h, B_r) \; \leq \; c \, L_h \, . \tag{51}$$

By integrating in polar coordinates, taking into account the two dimensional isoperimetric inequality with constant c_2 and (51) we get

$$\alpha_h \int_0^r \mathcal{H}^1(A_h \cap \partial B_\varrho) \, d\varrho \; = \; \alpha_h |A_h \cap B_r| \; \leq \; \alpha_h (c_2 P(A_h, B_r))^2 \; \leq \; c^2 c_2^2 \alpha_h {L_h}^2 \tag{52}$$

Hence, by (52), we have, up to subsequences and without relabeling,

$$\exists \; \lim_h \; \alpha_h \, \mathcal{H}^1 \left(A_h \cap \partial B_t \right) \; = \; 0 \;\; \text{for a.e. } t \in (0, r) \, . \tag{53}$$

Since the map ω is monotone non decreasing in ϱ, by contradicting the claim that (46) holds true a.e., we can assume that there exist $s \in (0,r)$, $\varepsilon > 0$ and $u \in H^2(B_r)$ such that $u = v_\infty$ in $B_r \backslash B_s$, limit relation (53) holds true at $t = s$, ω is continuous at $\varrho = s$, and

$$G_s(u) \; = \; \int_{B_s} \left| \, D^2 u \, \right|^2 d\mathbf{y} \; + \; \frac{\delta}{\mu} \int_{B_s^-} |\boldsymbol{\xi}|^2 \, d\mathbf{y} + \int_{B_s^+} |\boldsymbol{\xi}|^2 \, d\mathbf{y} \; \leq \; \omega(s) - \varepsilon \, . \tag{54}$$

By (50), (53), (ii) and $\beta_h \leq \alpha_h$ we get, for a.e. $t \in (s, r)$,

$$\lim_h \big(\alpha_h \, \mathcal{H}^1 \left(S_{z_h} \cap \partial B_t \right) + \beta_h \, \mathcal{H}^1 \left((S_{\nabla z_h} \backslash S_{z_h}) \cap \partial B_t) \right) \big) = 0 \, . \tag{55}$$

By continuity of ω at $\varrho = s$ and (iii) we can choose $t \in (s, r)$ as close to s as needed (fixed in the following) and $\widetilde{h} \in \mathbb{N}$ such that, setting $A_{s,t} = \overline{B_t} \setminus B_s$, the following list of inequalities hold true:

$$\omega(t) - \omega(s) \; < \; \varepsilon/3 \, , \tag{56}$$

$$\lim_h \alpha_h \,\mathcal{H}^1\left(A_h \cap \partial B_t\right) \;=\; 0\,, \tag{57}$$

$$\alpha_h \,\mathcal{H}^1\left(A_h \cap\, \partial\, A_{s,t}\right) \;\leq\; \varepsilon/6 \quad \text{for} \quad h > \widetilde{h}\,, \tag{58}$$

$$\alpha_h \,\mathcal{H}^1\left(S_{z_h} \cap\, \partial\, A_{s,t}\right) + \beta_h \,\mathcal{H}^1\left(\left(S_{\nabla z_h} \setminus S_{z_h}\right) \cap\, \partial\, A_{s,t}\right) \;\leq\; \varepsilon/6 \;\; \text{for } h > \widetilde{h}\,, \tag{59}$$

$$\int_{B_t \setminus B_s} \left|\, D^2 u \,\right|^2 \, d\mathbf{y} \;<\; \varepsilon/6\,, \tag{60}$$

$$\mathfrak{F}\left(v_h, \overline{B_t} \setminus B_s\right) \leq\; 2\,\varepsilon/6 \;\;\text{for} \;\; h > \widetilde{h}\,. \tag{61}$$

In fact (56) holds true due to the continuity of ω at s; feasibility of choices (58), (59) follows by (54), (53) and (55); inequality (60) follows by the absolute continuity of $\int_A |D^2u|^2 d\mathbf{x}$ with respect to the Lebesgue measure of A, eventually (61) follows by

$$\lim_h \mathfrak{F}\left(v_h, \overline{B_t} \setminus B_s\right) \;=\; \omega(t) - \omega(s) \;+ \lim_h \mathfrak{F}\left(v_h, \partial B_s\right)$$

which is estimated by $2\,\varepsilon/6$ thanks to (56), (59). We fix the matching:

$$\zeta_h \;\;=\;\; u_h\, \chi_{B_r \setminus \overline{B_t}} \;+\; z_h\, \chi_{\overline{B_t}}\;, \tag{62}$$

hence (53) and Lemma 3.6 in [13] entail, for a.e. $\varrho \in (0, r)$,

$$\lim_h \mathfrak{F}_{\gamma_h}(\zeta_h, \mu_h, \alpha_h, \beta_h, B_\varrho) = \lim_h \mathfrak{F}_{\gamma_h}(v_h, \mu_h, \alpha_h, \beta_h, B_\varrho) = \omega(\varrho) \leq 1\,. \tag{63}$$

Eventually we perform the joining of $u + \mathcal{P}_{\mathbf{0},r} v_h$ and $\zeta_h + \mathcal{P}_{\mathbf{0},r} v_h$ along annular set $B_t \setminus B_s$: exploiting Lemma 3.3 we choose a sequence of cut-off functions $\Psi_h \in C^2(B_t(\mathbf{0}))$ such that $\Psi_h \equiv 1$ in a neighborhood of B_s and set, for every h:

$$\tau_h \;\;=\;\; \left(u + \mathcal{P}_{\mathbf{0},r} v_h\right) \Psi_h \;+\; \left(\zeta_h + \mathcal{P}_{\mathbf{0},r} v_h\right)\left(1 - \Psi_h\right), \tag{64}$$

so that

$$\tau_h \;=\; u_h + \mathcal{P}_{\mathbf{0},r} v_h = v_h \quad \text{in } B_r \setminus \overline{B_t}\,, \qquad \tau_h \;=\; u + \mathcal{P}_{\mathbf{0},r} v_h \quad \text{in } \overline{B_s}\,, \tag{65}$$

hence

$$\mathfrak{F}(\tau_h, B_r \setminus \overline{B_t}) \;\;=\;\; \mathfrak{F}(v_h, B_r \setminus \overline{B_t})\,. \tag{66}$$

Then, by Lemma 3.3, we obtain, for any $\theta > 0$,

$$\begin{aligned} \mathfrak{F}(\tau_h, \overline{B_t}) \;\leq\; & (1+\theta)\,\left(\mathfrak{F}(u + \mathcal{P}_{\mathbf{0},r} v_h, \overline{B_t}) \;+\; \mathfrak{F}(\zeta_h + \mathcal{P}_{\mathbf{0},r} v_h, \overline{B_t} \setminus B_s)\right) + \\ & + \frac{c}{(t-s)^2}\left(\int_{B_t \setminus B_s} |\nabla(u - \zeta_h)|^2 d\mathbf{x} \;+\; \frac{c}{\theta\, s^2\,(t-s)^2}\int_{B_t \setminus B_s} |u - \zeta_h|^2 d\mathbf{x}\right). \end{aligned} \tag{67}$$

Again by compactness (Theorem 4.3 in [13]), with our choice for t and s

$$\lim_h \int_{B_t \setminus B_s} |\nabla(v_\infty - \zeta_h)|^2 d\mathbf{x} \;=\; \lim_h \int_{B_t \setminus B_s} |v_\infty - \zeta_h|^2 d\mathbf{x} \;=\; 0\,, \tag{68}$$

therefore, thanks to $u = v_\infty$ in $B_r \setminus B_s$, possibly by extracting subsequences without relabeling and letting $h \to +\infty$ in (67) we obtain

$$\lim_h \mathfrak{F}(\tau_h, \overline{B_t}) \le \tag{69}$$
$$\le (1+\theta)\left(\lim_h \mathfrak{F}(u + \mathcal{P}_{\mathbf{0},r}v_h, \overline{B_t}) + \lim_h \mathfrak{F}(\zeta_h + \mathcal{P}_{\mathbf{0},r}v_h, \overline{B_t}\setminus B_s)\right).$$

By (60) there is $h_0 \ge \widetilde{h}$ such that

$$\left|\int_{B_t\setminus B_s} |D^2(u+\mathcal{P}_{\mathbf{0},r}v_h)|^2\,d\mathbf{x}\right| = \left|\int_{B_t\setminus B_s} |D^2u|^2 d\mathbf{x}\right| < \frac{\varepsilon}{6} \quad \text{for } h > h_0\,. \tag{70}$$

By (v) we have $\mu_h\,\mathcal{P}_{\mathbf{0},r}v_h \to \boldsymbol{\xi}$ in $W^{2,\infty}(B_r)$. By (38), (61), (65) and the convergence of $\mu_h\mathcal{P}_{\mathbf{0},r}v_h \to \boldsymbol{\xi}$ in $W^{2,\infty}$, we get

$$\lim_h \mathfrak{F}\left(\zeta_h + \mathcal{P}_{\mathbf{0},r}v_h, \overline{B_t}\setminus B_s\right) = \tag{71}$$
$$= \lim_h \mathfrak{F}\left(z_h + \mathcal{P}_{\mathbf{0},r}v_h, \overline{B_t}\setminus B_s\right) \le \lim_h \mathfrak{F}\left(v_h, \overline{B_t}\setminus B_s\right) \le 2\,\frac{\varepsilon}{6}. \tag{72}$$

By letting $\vartheta \to 0$ in (69), taking into account (58)–(62),(65),(70),(71), we get

$$\mathfrak{F}\left(\tau_h, \overline{B_t}\setminus B_s\right) \le 4\,\varepsilon/6 \quad \text{for} \quad h > h_0\,. \tag{73}$$

By exploiting (58),(59),(61),(73),(70),(55),(54), the equality $\tau_h = u + \mathcal{P}_{\mathbf{0},r}v_h$ in B_s and eventually the monotonicity of ω with respect to inclusion of sets, we get the contradiction:

$$\begin{aligned}
\omega(t) &= \lim_h \mathfrak{F}\left(v_h\,,\,\overline{B_t}\right) \overset{\text{minimality of } v_h}{\le} \lim_h \mathfrak{F}\left(\tau_h\,,\,\overline{B_t}\right)\\
&= \lim_h \mathfrak{F}\left(\tau_h\,,\,\overline{B_s}\right) + \lim_h \mathfrak{F}\left(\tau_h\,,\,\overline{B_t}\setminus B_s\right) - \lim_h \mathfrak{F}\left(\tau_h\,,\,\partial B_s\right)\\
&\overset{(73)}{\le} \lim_h \mathfrak{F}\left(\tau_h\,,\,\overline{B_s}\right) + 4\,\frac{\varepsilon}{6} = \lim_h \mathfrak{F}\left(u + \mathcal{P}_{\mathbf{0},r}v_h\,,\,\overline{B_s}\right) + 4\,\frac{\varepsilon}{6}\\
&\overset{(70),(55)}{\le} \lim_h \int_{B_s} |D^2u|^2\,d\mathbf{x} + \frac{\delta}{\mu}\int_{B_s^-(\mathbf{x})} |\boldsymbol{\xi}|^2\,d\mathbf{y} + \int_{B_s^+(\mathbf{x})} |\boldsymbol{\xi}|^2\,d\mathbf{y} + 4\,\frac{\varepsilon}{6}\\
&= G_s(u) + 4\,\frac{\varepsilon}{6} \overset{(54)}{\le} \omega(s) - \varepsilon + 5\,\frac{\varepsilon}{6} \le \omega(t) - \frac{\varepsilon}{6}\,.
\end{aligned} \tag{74}$$

□

4. Numerical experiments

We present some numerical experiments, which were performed by applying component-wise the numerical procedure that we have introduced in [22]. Here we shortly recall the details for a single channel of an RGB image: in order to approximate one component $\mathcal{F} = \mathcal{F}(v)$ of the weak functional $\boldsymbol{\mathcal{F}}$ defined in (17), we introduce another functional $\mathcal{G} = \mathcal{G}(v, s, \sigma)$ which depends also on two control functions s, σ and coincides with $\mathcal{F}$ for trivial choices ($s \equiv 1$, $\sigma \equiv 1$) of the control functions. We define the spaces:

$$Y\big(\mathcal{Q}\big) := \big(GSBV^2(\mathcal{Q}) \cap L^2(\mathcal{Q})\big) \times L^\infty\big(\mathcal{Q}; [0,1]\big) \times L^\infty\big(\mathcal{Q}; [0,1]\big), \tag{75}$$

$$\mathfrak{Y}(\mathcal{Q}) := H^2(\mathcal{Q}) \times H^1\big(\mathcal{Q}; [0,1]\big) \times H^1\big(\mathcal{Q}; [0,1]\big) \subset Y(\mathcal{Q}), \tag{76}$$

together with another space $\mathfrak{Y}_0(\mathcal{Q})$, larger than $\mathfrak{Y}(\mathcal{Q})$:

$$\mathfrak{Y}_0(\mathcal{Q}) := \big\{ (v, s, \sigma)\,;\ v, s, \sigma \in H^1(\mathcal{Q}), \sigma\,\nabla v \in W^{1,p}(\mathcal{Q}; \mathbf{R}^2) \big\} \subset Y\big(\mathcal{Q}\big), \tag{77}$$

where $p = 2\gamma/(\gamma + 2)$ and $\gamma \geq 2$. In the following they are shortly denoted respectively by $\mathfrak{Y}$ and $\mathfrak{Y}_0$. We define a functional $\mathcal{G}\colon Y(\mathcal{Q}) \to [0, +\infty]$ as follows:

$$\mathcal{G}(v, s, \sigma) := \begin{cases} \mathcal{F}(v) & \text{if } v \in X\big(\mathcal{Q}\big),\ s \equiv 1,\ \sigma \equiv 1 \text{ in } \mathcal{Q}, \\ +\infty & \text{otherwise.} \end{cases} \tag{78}$$

Obviously, by Theorem 2.2, there exists the minimum of $\mathcal{G}$ in $Y(\mathcal{Q})$. In order to obtain the variational approximation of the functional (78), for $\gamma \geq 2$ and suitable positive infinitesimals κ_k, ξ_k, ζ_k, we introduce a sequence of elliptic functionals $\mathcal{G}_k\colon Y(\mathcal{Q}) \to [0, +\infty]$ as follows:

Case 1: if $\{\kappa_k > 0$ and $(v, s, \sigma) \in \mathfrak{Y}\}$ or $\{\kappa_k = 0$ and $(v, s, \sigma) \in \mathfrak{Y}_0\}$, then:

$$\begin{aligned} \mathcal{G}_k(v, s, \sigma) := & \int_{\mathcal{Q}} \Big(\sigma^2 + \kappa_k\Big) |D^2 v|^2 \, d\mathbf{x} + \xi_k \int_{\mathcal{Q}} \Big(s^2 + \zeta_k\Big) |Dv|^\gamma \, d\mathbf{x} + \\ & + (\alpha - \beta) \int_{\mathcal{Q}} \Big(\frac{1}{k} |Ds|^2 + k\, \frac{(s-1)^2}{4}\Big) \, d\mathbf{x} + \\ & + \beta \int_{\mathcal{Q}} \Big(\frac{1}{k} |D\sigma|^2 + k\, \frac{(\sigma - 1)^2}{4}\Big) \, d\mathbf{x} + \mu \int_{\Omega \setminus U} |v - g|^2 \, d\mathbf{x} + \\ & + \delta \int_U |v - 1/2|^2 \, d\mathbf{x} + k \int_{\mathcal{Q} \setminus \Omega} |v - w|^2 \, d\mathbf{x}\,. \end{aligned} \tag{79}$$

Case 2: elsewhere in Y: $\quad \mathcal{G}_k(v, s, \sigma) := +\infty$.

The functionals $\mathcal{G}_k$ are to be minimized on triplets of functions (v, s, σ). We emphasize that the minimization acts not only on the restored image v but also on two auxiliary functions: s which is a control function for Dv and σ which is a control function of the Hessian of v.

To explain heuristically the effectiveness of this approximation, we observe that if (v_k, s_k, σ_k) is a sequence of minimizers of $\mathcal{G}_k$, then the function s_k assumes value 1 where v is continuous and it is close to 0 in a tubular neighborhood of discontinuity set S_v of thickness $1/k$. As $k \to +\infty$, this neighborhood shrinks and then, for k large enough, s_k yields an approximate representation of the jump set of v.

The function σ_k, instead, assumes values near 0 only in a tubular neighborhood of $S_v \cup S_{\nabla v}$ of thickness $1/k$. As $k \to +\infty$, this neighborhood shrinks and σ_k yields an approximate representation of the jump and crease sets of v. The last term in (79) forces v to assume the value w in $\mathcal{Q} \setminus \overline{\Omega_j}$, where label j denote the selected color component. The following term of Modica-Mortola type is inserted in the functional, to approximate the Hausdorff measure of the singular sets:

$$\mathcal{M}_k(z) := \int_{\widetilde{\Omega}} \left(\frac{1}{k} |Dz|^2 + k \frac{(z-1)^2}{4} \right) d\mathbf{x} \,. \tag{80}$$

These Γ-convergence approximations $\mathcal{G}_k$ of the extended Blake-Zisserman functional $\mathcal{G}$ provide an efficient numerical approximation of the minimizers for the functionals $\mathcal{G}_k$ and consequently of the minimizers for functional $\mathcal{F}$, too.

The approximation of second order functionals with free gradient discontinuity by elliptic functionals Γ-convergent has been stated and numerically implemented in the context of image segmentation and depth extraction from stereo images in [2], by adopting a discretization approach different from the one proposed here. For the sake of simplicity, we assume $\kappa_k = 0$, $\gamma = 2$ and that the open set $\mathcal{Q}$ is a rectangle.

The system of Euler-Lagrange equations, associated to the k-th approximating functional $\mathcal{G}_k$ in (79), when $\kappa_k = 0$, is given by:

$$\begin{cases} \sigma^2 \Delta^2 v + 2\sigma \Big(2D\sigma \cdot D(\Delta v) + D^2\sigma : D^2 v \Big) \\ \qquad + 2(D^2 v D\sigma) \cdot D\sigma - \xi_k \Big(s^2 \Delta v + 2s Dv \cdot Ds \Big) + \\ \qquad + \mu\,(v-g)\,\chi_{\Omega \setminus U} + \delta\left(v - \tfrac{1}{2}\right)\chi_U + k\,(v-w)\,\chi_{\mathcal{Q}\setminus\Omega} = 0, \\ 4(\alpha-\beta)\Delta s = 4\xi_k k\, s\,|Dv|^2 + (\alpha-\beta)k^2\,(s-1), \\ 4\beta\Delta\sigma = 4k\,\sigma\,|D^2 v|^2 + \beta k^2\,(\sigma - 1), \end{cases} \tag{81}$$

in the open set $\mathcal{Q}$. The Euler-Lagrange system can be discretized by a central differencing scheme. Hence we obtain a numerical approximation for the system (81), which we used in the numerical experiments. The solution of the nonlinear algebraic system is approximated through a Gauss-Seidel relaxation method (see [22] and the references therein).
In order to simulate the Γ-convergence in the discrete space, we generate a sequence of differential problems like (81), by changing the value of the parameter k according to a suitable rule $k(I)$, where I is the current iterate.

Figure 4.1: Image inpainting

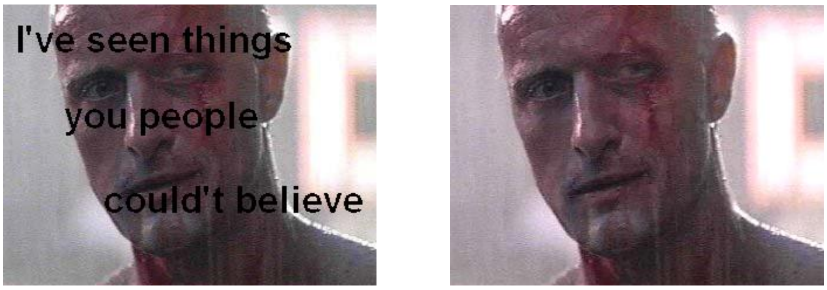

Figure 4.2: Image inpainting

We conclude by showing some pictures obtained in numerical experiments which exploit the variational approximation (79) with $\kappa_k = 0$ of the functional (78).

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