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A Convex Optimization Model for Business Management

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Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

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We first present a supply chain network model with four different tiers of decision makers (suppliers of raw materials, manufacturers, retailers, demand markets), we derive the optimality conditions and the associated variational inequality problem for the representatives of each level and for the total supernetwork and focusing our attention on the behavior of the manufacturers. Then, in a more detailed model, we introduce the distinction by brand of the products of manufacturers and we add the e-commerce to the traditional physical links for the shipments from manufacturers to demand markets. Moreover, to the forward chain we add a reverse chain model where manufacturers, using the unsold product given back from retailers, after reworking, produce a new commodity which will be sold to new retailers. Also in this case we study the behavior of manufacturers obtaining their optimality conditions and the governing variational formulation. Finally, we apply our model to a concrete company (Valle del Dittaino, Italy), obtaining, after introducing additional constraints, the optimal amount of raw material, the optimal shipment of new product as well as the optimal production periods.

Keywords: Supply chains, reverse logistics, Lagrange multipliers, variational inequalities.

2010 Mathematics Subject Classification: 65K10, 65K15, 90C30, 90C35, 90C46.

1. Introduction

The network framework is usually associated with transportation problems, electrical power transmission, telecommunications, etc., that is with physical networks, where nodes and links are tangible (see [4]). However, networks can be applied also to a very large class of problems where these concepts have no physical characteristics, such as problems arising from economics, finance,

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information technologies, Internet, etc. As stated in the book by Nagurney and Dong (see [9]), *supernetworks* are networks that are above and beyond existing networks, which consist of nodes, links, and flows, with nodes corresponding to the locations in space, links to connections in the form of roads, cables, etc., and flows to vehicles, data, etc.

Supply chain networks, consisting of different tiers of decision-makers provide an effective framework for the production of goods, their distribution, and their consumption in today's globalized economies and societies. The representatives of each level may have various objectives: they may aim at maximizing their own profit, or minimizing the risk, or the environmental emissions, or the total costs. In general, a supply chain is a set of activities that includes purchasing, manufacturing, logistics, distribution, marketing (see [13] and [5]). Since the interest in sustainable economics has been growing in the last two decades, the theory of closed-loop supply chains has been deeply developed. Closed-loop supply chains are designed and managed to explicitly consider the reverse and forward supply chain activities over the entire life cycle of the product.

A traditional supply chain aims to lower the cost and to maximize the economic benefits. A closed-loop supply chain also seeks to maximize economic benefits, to decrease the consumption of resources and energy and to reduce the emissions of pollutants, all in an effort to create a socially responsible environment, and to balance the economic, social and environmental effects. The traditional supply chain starts with suppliers and ends with users. In the closed-loop supply chain thinking, product flow is circular and reversible and all products must be managed throughout the entire life cycle, and beyond so that waste finds a second life or becomes raw material available for new production or other purposes (see [11]). Closed-loop supply chains have been deeply studied in the case of electronic waste (see [10] and the references therein, [12], [13]), but we shall adapt them to an agribusiness company.

In this paper, we present a general supply chain network model with four different tiers of decision makers, represented by suppliers of raw materials, manufacturers, retailers, demand markets. In particular, we study a more comprehensive and extensive model than the one in [9] and [10], since we complete the forward chain by adding a reverse chain model where manufacturers, using the unsold product given back from retailers, after reworking, produce a new commodity which will be sold to new retailers (some of them could be the same as the retailers in the forward chain). Moreover, in this new structure we also include electronic commerce. E-commerce has had an enormous effect on the manner in which businesses as well as consumers order goods and have them transported. The primary benefit of the Internet for business is its open access to potential suppliers and customers both within a particular country and beyond national boundaries. Consumers, on the other hand, may obtain goods which they physically could not locate otherwise. In order to derive the optimality conditions for the typical manufacturer and the subsequent variational inequality problem governing the equilibrium of all manufacturers similtaneously, we need to guarantee the convexity of the constraint set and the convexity of the cost functions. An existence result, based on the classical theory due to Stampacchia, is stated.

Finally, we apply our model to a concrete company (Valle del Dittaino, Italy), obtaining, after introducing additional suitable constraints, the optimal amount of raw material, the optimal shipment of new product as well as the optimal production periods.

The purpose of this paper is to analyze the multiple features of business management by improving existing networks models, where only a few aspects have been taken under consideration.

The paper is organized as follows: In Section 2 we present a model where we allow for a distinction (by brand, small changes, ...) between a product of a manufacturer and the same good produced by other manufacturers. We introduce the reverse logistics and e-commerce and study the optimality conditions of the manufacturer. In Section 3 we apply our model to a well-known agribusiness company, the Cooperativa Agricola Valle del Dittaino (www.pandittaino.it), which is located in the heart of Sicily, Italy. Taking into account the company reality, we add some suitable constraints related to the capacity of a single tray and their number, as well as the production time. After analyzing the optimization problem and determining the sales forecast and the production periods, while minimizing the production and the storing costs, we obtain the optimal amount of product that all retailers must sell to all demand markets. We also establish, for every day, the amount of raw materials needed for the products. Section 4 is dedicated to the conclusions.

2. The Model

We consider a supply chain network with four tiers of decision-makers:

- H suppliers of raw materials;
- G manufacturers, who produce a new product using raw materials;
- S retailers;
- K demand markets.

We assume there are also V carriers, who are not decision-makers in our network since they do not decide on the transactions, but only make the shipment to retailers. Their role will be represented by parallel links connecting the nodes of the second tier with the ones of the third. Moreover, we shall add a new arc representing the possibility for retailers to buy directly the products, using no carriers. Further, it is also allowed to consumers at demand markets to buy the products directly from manufacturers.

Let L be the number of raw materials, I the number of new products. We assume suppliers are spatially distributed in different parts of the world and can be of different types (private suppliers, companies, institutions, local governments, ...).

The framework of the network is depicted in Fig. 2.1: the first tier represents the combination between suppliers and the kind of raw material (see [7]), the second one is the combination between manufacturers and products, in the third tier we represent retailers and, finally, in the last level we depict the demand markets.

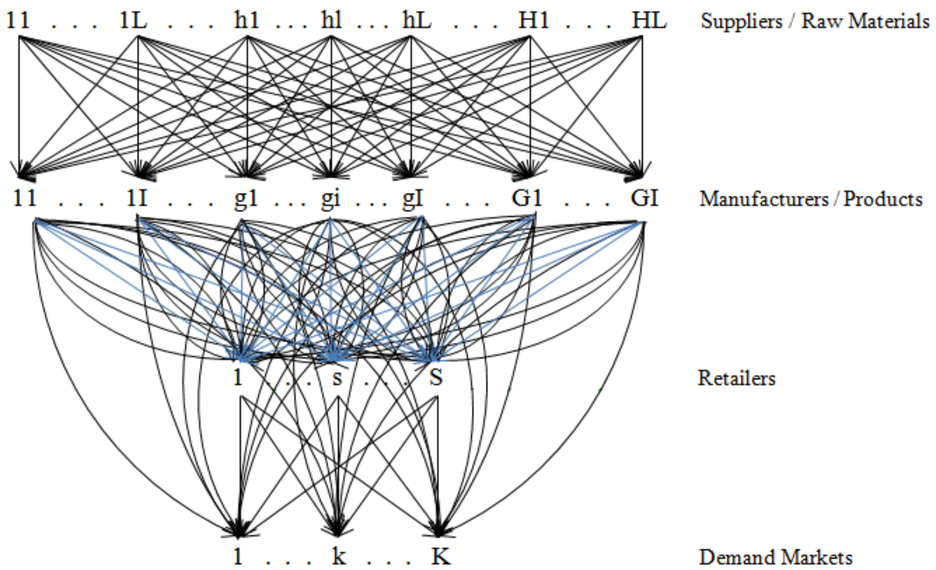

Figure 2.1: Network topology

The aim of suppliers, manufacturers and retailers is to maximize their own profit, while consumers at demand markets aim at minimizing their expenses.

In particular, in this paper we shall focus only on the behavior of manufacturers. A detailed presentation of the optimality conditions and the characterization by means of variational inequality problems can be found in [5], [7], [9], [10].

2.1. Supply chain network with e-commerce and excesses

We allow there can be a distinction (by brand, small changes, ...) between the product i of manufacturer g and the same good produced by other manufacturers. We shall find the optimality conditions of manufacturer g, who purchases raw materials from suppliers, and, after the production process, sells the new product to retailers (with or without carriers) both through traditional links or via Internet and to demand markets just via Internet (*forward chain*). Moreover, the same manufacturer receives the products unsold by all retailers and, after reworking, sells the new products (which generally differ from the previous ones) to new buyers (*reverse chain*).

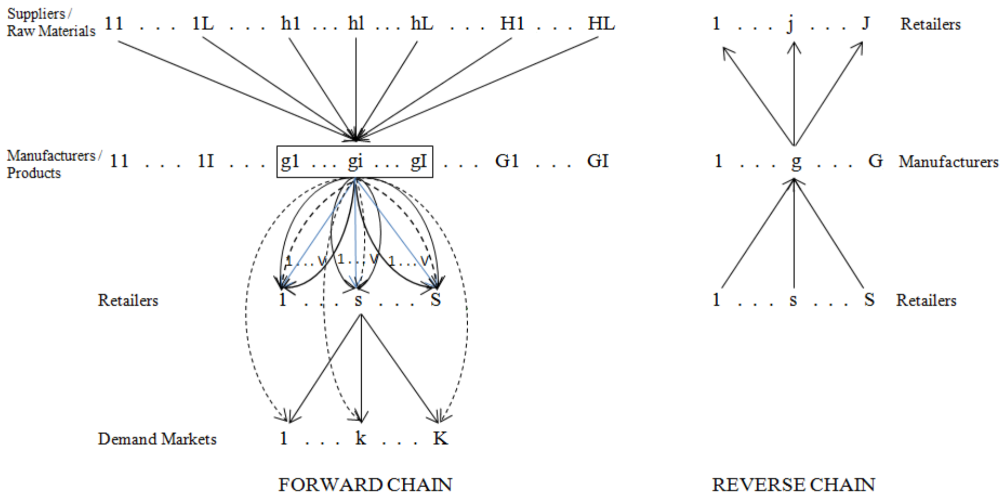

Figure 2.2: Network with e-commere and reverse logistics

The topology of the network is depicted in Fig. 2.2. Let:

- $q_{hlgi} \geq 0$ be the amount of raw material l shipped from supplier h to manufacturer g for the product i and let us group all these quantities into the vector $Q_0 = (q_{hlgi})_{\substack{h=1,\dots,H,\, l=1,\dots,L \\ g=1,\dots,G\, i=1,\dots,I}} \in \mathbb{R}_+^{HLGI}$;
- c_{hlgi} be the transaction cost of the materials from supplier h to manufacturer g for the product i (such costs could also be the same for every i), and let us assume c_{hlgi} is a function of $q_{hg} = (q_{hlgi})_{\substack{l=1,\dots,L \\ i=1,\dots,I}} \in \mathbb{R}_+^{LI}$:

$$c_{hlgi} = c_{hlgi}(q_{hg}) \quad \forall h = 1,\dots,H;\ \forall l = 1,\dots,L;$$
$$\forall g = 1,\dots,G;\ \forall i = 1,\dots,I;$$

- ρ^*_{0hlgi} be the price of raw material l in the transaction between supplier h and manufacturer g;
- $q^v_{gise} \geq 0$ be the amount of product i sold by manufacturer g to retailer s directly or throught the carrier v in e mode, where:
  - $e = 1$ means phisical link,
  - $e = 2$ means Internet link

  and let us group such quantities into the vector

$$Q_1 = (q^v_{gise})_{\substack{g=1,\dots,G\, i=1,\dots,I \\ s=1,\dots,S\, e=1,2\, v=1,\dots,V+1}} \in \mathbb{R}_+^{GIS2(V+1)};$$

- c^v_{gise} be the transaction costs of product i from manufacturer g to retailer s directly or throught the carrier v in e mode and let us assume c^v_{gise} is a function of q^v_{gise}:

$$c^v_{gise} = c^v_{gise}(q^v_{gise}),\ \forall g = 1,\dots,G,\ \forall i = 1,\dots,I,\ \forall s = 1,\dots,S,$$
$$\forall e = 1,2,\ \forall v = 1,\dots,V+1;$$

- ρ^{v*}_{1gise} be the price of product i charged by g in the transaction with s in e mode, directly or throught v;
- $q_{gik} \geq 0$ be the amount of product i sold by manufacturer g to demand market k and let us group such quantities into the vector

$$Q_2 = (q_{gik})_{\substack{g=1,\dots,G,\\ i=1,\dots,I,\, k=1,\dots,K}} \in \mathbb{R}^{GIK}_{+};$$

- c_{gik} be the transaction costs of product i from manufacturer g to demand market k and let us assume c_{gik} is a function of q_{gik}:

$$c_{gik} = c_{gik}(q_{gik}) \ \forall g = 1,\dots,G,\ \forall i = 1,\dots,I,\ \forall k = 1,\dots,K;$$

- ρ^{*}_{1gik} be the price of product i charged by g in the transaction with k;
- c^i_g be the handling cost, associated to product i, of manufacturer g and let us assume:

$$c^i_g = c^i_g(Q_0, Q_1, Q_2) \quad \forall g = 1,\dots,G,\ \forall i = 1,\dots,I;$$

- C^i be the fix costs, associated to product i, of manufacturer g, which do not depend on the production, $\forall i = 1,\dots,I$;
- p_{gi} be the production cost of i and let us assume:

$$p_{gi} = p_{gi}(Q_1, Q_2) \quad \forall g = 1,\dots,G,\ \forall i = 1,\dots,I;$$

- α^{gi}_{hl} be the transformation rate of raw material l bought by g from supplier h to produce one unit of i;
- β_{li} be the portion of raw material l which is transformed in one unit of raw material needed to produce i;
- τ be the unit tax charged to manufacturer g by the authorities in order to produce a certain amount of product; so, the total taxes are given by:

$$\tau\Big[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2} q^v_{gise} + \sum_{k=1}^{k} q_{gik}\Big];$$

- q^{gi}_{sk} be the amount of product i, produced by manufacturer g, shipped from retailer s to demand market k and let us group such quantities into the vector

$$Q_4 = (q^{gi}_{sk})_{\substack{s=1,\dots,S,\, k=1,\dots,K\\ g=1,\dots,G,\, i=1,\dots,I}} \in \mathbb{R}^{SKGI}_{+};$$

When we are in the presence of supply excesses, we remark that the amount of product i bought by retailer s is greater than or equal to the sold amount, that is:

$$\sum_{v=1}^{V+1}\sum_{e=1}^{2} q^v_{gise} \geq \sum_{k=1}^{K} q^{gi}_{sk} \qquad \forall i = 1,\dots,I, \forall s = 1,\dots,S.$$

- $r_{isg}^{(v)}$ be the amount of unsold product i that retailer s gives back to manufacturer g, and let us assume $r_{isg}^{(v)}$ is a function of Q_1 and Q_4:
$$r_{isg}^{(v)} = r_{isg}^{(v)}(Q_1, Q_4);$$
specifically the following estimate holds:
$$r_{isg}^{(v)} = \sum_{v=1}^{V+1}\sum_{e=1}^{2} q_{gise}^{v} - \sum_{k=1}^{K} q_{sk}^{gi}$$
$\forall i = 1, \dots, I,\ \forall s = 1, \dots, S,\ \forall g = 1, \dots, G,$
which means that $r_{isg}^{(v)}$ is given by the difference between the shipment that retailers purchase from manufacturers and the shipment from retailers to all demand markets;
- $\hat{c}_{isg}^{(v)}$ be the transaction cost of product i from retailer s to manufacturer g, via the chosen carrier v, and let us assume $\hat{c}_{isg}^{(v)}$ is a function of $r_{isg}^{(v)}$:
$$\hat{c}_{isg}^{(v)} = \hat{c}_{isg}^{(v)}(r_{isg}^{(v)}) = \hat{c}_{isg}^{(v)}(Q_1, Q_4)$$
$$\forall i = 1, \dots, I,\ \forall s = 1, \dots, S,\ \forall g = 1, \dots, G;$$
- $\hat{p}_{gi}$ be the unit reworking cost of the unsold product i of manufacturer g in order to get a new product;
- γ_i be the transformation rate of unsold product i needed to produce one unit of a new product;
- β_i be the portion of unsold product i which can be used to produce a new product;
- $\hat{q}_{gj}$ be the amount of the new product sold by manufacturer g to retailer j and let us assume that the shipment of new products sold by g to all retailers is the same as the quantity of new product obtained by the goods given back by all retailers:
$$\sum_{j=1}^{J} \hat{q}_{gj} = \sum_{i=1}^{I} \gamma_i\beta_i \sum_{s=1}^{S} r_{isg}^{(v)} = \sum_{i=1}^{I} \gamma_i\beta_i \Big(\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2} q_{gise}^{v} - \sum_{s=1}^{S}\sum_{k=1}^{K} q_{sk}^{gi} \Big);$$
hence, $\hat{q}_{gj}$ is a combination of Q_1 and Q_4.
We remark that, in order to obtain the actual amount of a new product, we have to sum the return and the scrap (caused by a wrong cut, a wrong leavening, a wrong cooking, ...). Since, in our case, the scrap is in a small amount, it will be neglected in this discussion.
- $\hat{\rho}_{1gj}^{*}$ be the price charged to the new product by g in the transaction with j;
- $\hat{c}_{gj}$ be the transaction cost of the new product from manufacturer g to retailer j and let us assume $\hat{c}_{gj}$ is a function of $\hat{q}_{gj}$, that is Q_1 and Q_4:
$$\hat{c}_{gj} = \hat{c}_{gj}(Q_1, Q_4)\ \forall g = 1, \dots, G,\ \forall j = 1, \dots, J;$$

- ρ be the unit waste cost, so the total waste cost is given by:

$$\rho\left\{\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}[(1-\beta_{li})q_{hlgi}]+\sum_{i=1}^{I}\sum_{s=1}^{S}\left[(1-\beta_i)r_{isg}^{(v)}\right]\right\}.$$

We assume there are no demand excesses or deficiencies, therefore the demand at demand market k related to the product i of g is:

$$\delta_k^{gi}=\sum_{s=1}^{S}q_{sk}^{gi}+q_{gik} \qquad \forall k=1,\dots,K,\ \forall i=1,\dots,I.$$

Moreover, the total demand of product i of g is less than or equal to the produced amount:

$$\begin{aligned}\sum_{k=1}^{K}\delta_k^{gi}&=\sum_{k=1}^{K}\left[\sum_{s=1}^{S}q_{sk}^{gi}+q_{gik}\right]=\sum_{k=1}^{K}\sum_{s=1}^{S}q_{sk}^{gi}+\sum_{k=1}^{K}q_{gik}\\&\leq\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}q_{gise}^{v}+\sum_{k=1}^{K}q_{gik}, \quad \forall i=1,\dots,I.\end{aligned}$$

Since the total demand is not actually determined, we denote by $[\lambda^{gi},\mu^{gi}]$ $\forall i=1,\dots,I$, the range of demand forecasting.

The aim of manufacturer g is to maximize his own profits. Therefore, g's optimality conditions are given by the following problem:

$$\begin{aligned}\max\Bigg\{&\sum_{i=1}^{I}\Bigg[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}\rho_{1gise}^{v*}q_{gise}^{v}+\sum_{k=1}^{K}\rho_{1gik}^{*}q_{gik}-C^i-c_g^i(Q_0,Q_1,Q_2)-\\&-\sum_{h=1}^{H}\sum_{l=1}^{L}\rho_{0hlgi}^{*}q_{hlgi}-\sum_{h=1}^{H}\sum_{l=1}^{L}c_{hlgi}(q_{hg})-p_{gi}(Q_1,Q_2)-\\&-\tau\Bigg[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}q_{gise}^{v}+\sum_{k=1}^{K}q_{gik}\Bigg]-\rho\sum_{l=1}^{L}\sum_{h=1}^{H}(1-\beta_{li})q_{hlgi}-\\&-\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}c_{gise}^{v}(q_{gise}^{v})-\sum_{k=1}^{K}c_{gik}(q_{gik})\Bigg]+\sum_{j=1}^{J}\hat{\rho}_{1gj}^{*}\hat{q}_{gj}(Q_1,Q_4)-\\&-\sum_{i=1}^{I}\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}\rho_{1gise}^{v*}r_{isg}^{(v)}(Q_1,Q_4)-\sum_{j=1}^{J}\hat{c}_{gj}(Q_1,Q_4)-\sum_{i=1}^{i}\hat{p}_{gi}r_{isg}^{(v)}(Q_1,Q_4)-\\&-\sum_{i=1}^{I}\sum_{s=1}^{S}\hat{c}_{isg}^{(v)}(Q_1,Q_4)-\rho\sum_{i=1}^{I}\sum_{s=1}^{S}(1-\beta_i)r_{isg}^{(v)}(Q_1,Q_4)\Bigg\}\end{aligned} \tag{1}$$

$$\sum_{h=1}^{H} \frac{\beta_{li}}{\alpha_{hl}^{gi}} q_{hlgi} = \sum_{s=1}^{S} \sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} + \sum_{k=1}^{K} q_{gik}, \ \forall l = 1, \ldots, L; \forall i = 1, \ldots, I$$

$$\lambda^{gi} \leq \sum_{k=1}^{K} q_{gik} + \sum_{s=1}^{S} \sum_{k=1}^{K} q_{sk}^{gi} \leq \mu^{gi} \qquad \forall i = 1, \ldots, I$$

$$\sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} \geq \sum_{k=1}^{K} q_{sk}^{gi} \qquad \forall i = 1, \ldots, I, \ \ \forall s = 1, \ldots, S$$

$$q_{hlgi}, \, q_{gise}^{v}, \, q_{gik}, \, q_{sk}^{gi} \geq 0 \quad \forall h = 1, \ldots, H, \ \forall l = 1, \ldots, L, \ \forall s = 1, \ldots, S, \\ \forall v = 1, \ldots, V+1, \ \forall e = 1, 2, \ \forall k = 1, \ldots, K, \ \forall i = 1, \ldots, I.$$

We assume that the objective function is continuously differentiable and concave and this is guaranteed assuming that all the cost functions

$$c_g^i(Q_0, Q_1, Q_2), \ c_{hlgi}(q_{hg}), \ p_{gi}(Q_1, Q_2), \ c_{gise}^{v}(q_{gise}^{v}) \text{ and } c_{gik}(q_{gik})$$

are convex functions. Therefore, since the objective function is concave and the feasible set is convex, it is easy to verify that the optimality conditions for all manufacturers simultaneously are characterized by the following variational inequality (for the proof, see [1], Theorem 5):

Find $(Q_0^*, Q_1^*, Q_2^*, Q_4^*) \in \mathbb{K}$:

$$\begin{aligned}
&\sum_{h=1}^{H} \sum_{l=1}^{L} \sum_{i=1}^{I} \left[\frac{\partial c_g^i(Q_0^*, Q_1^*, Q_2^*)}{\partial q_{hlgi}} + \rho_{0hlg}^* + \frac{\partial c_{hlgi}(q_{hg}^*)}{\partial q_{hlgi}} + \rho(1 - \beta_{li}) \right] \left(q_{hlgi} - q_{hlgi}^* \right) \\
&+ \sum_{i=1}^{I} \sum_{s=1}^{S} \sum_{v=1}^{V+1} \sum_{e=1}^{2} \left[\frac{\partial c_g^i(Q_0^*, Q_1^*, Q_2^*)}{\partial q_{gise}^{v}} - \rho_{1gise}^{v*} + \frac{\partial p_{gi}(Q_1^*, Q_2^*)}{\partial q_{gise}^{v}} + \tau + \right. \\
&\qquad + \frac{\partial c_{gise}^{v}(q_{gise}^{v*})}{\partial q_{gise}^{v}} - \sum_{j=1}^{J} \hat{\rho}_{1gj}^* \frac{\partial \hat{q}_{gj}(Q_1^*, Q_4^*)}{\partial q_{gise}^{v}} + \rho_{1gise}^{v*} \frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^{v}} + \\
&\qquad + \sum_{j=1}^{J} \frac{\partial \hat{c}_{gj}(Q_1^*, Q_4^*)}{\partial q_{gise}^{v}} + \hat{p}_{gi} \frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^{v}} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^{v}} + \\
&\qquad \left. + \rho(1 - \beta_i) \frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^{v}} \right] \left(q_{gise}^{v} - q_{gise}^{v*} \right) + \\
&+ \sum_{i=1}^{I} \sum_{k=1}^{K} \left[\frac{\partial c_g^i(Q_0^*, Q_1^*, Q_2^*)}{\partial q_{gik}} - \rho_{1gik}^* + \frac{\partial p_{gi}(Q_1^*, Q_2^*)}{\partial q_{gik}} + \tau + \right. \\
&\qquad \left. + \frac{\partial c_{gik}(q_{gik}^*)}{\partial q_{gik}} \right] \left(q_{gik} - q_{gik}^* \right) + \qquad (2)
\end{aligned}$$

$$+\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{i=1}^{I}\Bigg[-\sum_{j=1}^{J}\hat{\rho}^*_{1gj}\frac{\partial \hat{q}_{gj}(Q_1^*,Q_4^*)}{\partial q^{gi}_{sk}}+\sum_{v=1}^{V+1}\rho^{v*}_{1gis}\frac{\partial r^{(v)}_{isg}(Q_1^*,Q_4^*)}{\partial q^{gi}_{sk}}+$$

$$+\sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*,Q_4^*)}{\partial q^{gi}_{sk}}+\hat{p}_{gi}\frac{\partial r^{(v)}_{isg}(Q_1^*,Q_4^*)}{\partial q^{gi}_{sk}}+\frac{\partial \hat{c}^{(v)}_{isg}(Q_1^*,Q_4^*)}{\partial q^{gi}_{sk}}+$$

$$+\rho(1-\beta_i)\frac{\partial r^{(v)}_{isg}(Q_1^*,Q_4^*)}{\partial q^{gi}_{sk}}\Bigg]\left(q^{gi}_{sk}-q^{gi*}_{sk}\right)\geq 0 \quad \forall\,(Q_0,Q_1,Q_2,Q_4)\in\mathbb{K}$$

where $$\mathbb{K}=\Bigg\{(Q_0,Q_1,Q_2,Q_4)\in\mathbb{R}_+^{HLGI+GIS2(V+1)+GIK+SGKI}:$$

$$\sum_{h=1}^{H}\frac{\beta_{li}}{\alpha^{gi}_{hl}}q_{hlgi}=\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}q^{v}_{gise}+\sum_{k=1}^{K}q_{gik},\ \forall l=1,\dots,L;\ \forall i=1,\dots,I\Bigg\}.$$

The variational inequality (2) represents the optimality conditions of manufacturer g for all products simultaneously. The solution of (2) gives the optimal amount of raw materials manufacturer g has to purchase from all suppliers (Q_0^*), the optimal amount of final product g has to sell to all retailers (Q_1^*) and to all demand markets (Q_2^*) and the optimal amount of products sold by all retailers to all demand markets, in equilibrium.

We now provide an existence result for a solution to variational inequality (2).

Theorem 2.1. *A solution* $(Q_0^*,Q_1^*,Q_2^*,Q_4^*)\in\mathbb{K}$ *to variational inequality* (2) *is guaranteed to exist.*

Proof. The result follows from the classical theory of variational inequalities (see [6]), since the feasible set is compact and the function that enters the variational inequality is continuous. □

3. A case study

3.1. The company

We have applied our model to a well-known agribusiness company, the Cooperativa Agricola "*Valle del Dittaino*", which is located in the heart of Sicily among huge expanses of wheat called Valle del Dittaino (Enna, Italy). It is a cooperative of wheat producers founded in 1976 with the aim of enhancing the precious raw material and of verticalizing the entire production process, from the storage up to the grinding and bakery. The company is widely distributed in the Sicilian (and not only) market with more than 900 stores and thirteen types of bread, bread crumbs (plain and flavored), pastries and snacks branded *Pandittaino* (see www.pandittaino.it). The cooperative offers a highly competitive product from the point of view of quality and food safety and these two powers are at the base of their marketing strategy. The plant contains the storage silos as well as the mill for the production of flour which is directly connected to the bakery.

3.2. The model

The Cooperativa Agricola "*Valle del Dittaino*" has the topology of the network as shown in the previous sections, with four tiers of decision-makers: suppliers of raw materials, the company, retailers, demand markets (and also carriers).

We consider the production of the newest and developing line: the soft wheat bread. From this production line you can get seven different types of goods, which will be distinguished by shape (round or elongated) or duration (fresh or long-life). It is our goal to determine, with the help of the model presented in Section 2.1, the optimal amount of raw material that the manufacturer has to buy from all suppliers (Q_0^*), the optimal amount of product that the manufacturer must sell to all retailers (Q_1^*) and to all demand markets (Q_2^*) and the optimal amount of goods that all retailers sell to all demand markets (Q_4^*), in equilibrium, in order to maximize the profits (minimizing the amount of unsold product i that retailer s gives back to manufacturer g).

Please note that, since ecommerce has not yet been introduced in the company, we will only consider the physical network.

The topology of the network is shown in Fig. 3.1.

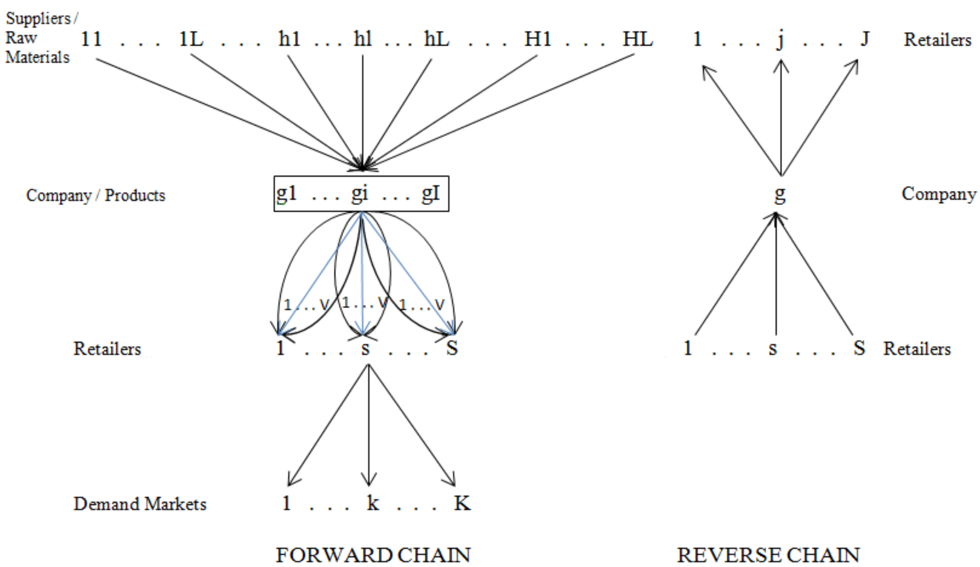

Figure 3.1: Pandittaino's network

Since the production takes place daily, in a continuous fashion, it is necessary to estimate the optimal amounts in advance and, therefore, we consider a seven-day interval. In Section 3.5 we shall determine the demand forecast of every good. After obtaining the estimates, for the long-life goods, we shall make use of the well-known (see [14]) Wagner-Whitin algorithm shown in Section 3.6 for inventory management, in order to determine the production periods (days), which satisfy the expected demand. Later, we shall apply our model so to determine the optimal amounts we are looking for.

3.3. Additional constraints

3.3.1. Worsening of the objective function

In our case study, an additional constraint is given by the maximum capacity of every single tray and by the finite number of trays which can be used following one another. We can distinguish two different types of sandwiches (type I and type II), according to their format (round or elongated). Let:

- I be the number of products;
- x_i be the amount of the packages to produce, $\forall i = 1, \dots, j$ containing type I sandwiches and $\forall i = j+1, \dots, I$ type II sandwiches; we have:
$$x_i = \sum_{s=1}^{S} \sum_{v=1}^{V+1} q_{gis}^v;$$
in the general case, we have: $x_i = \sum_{s=1}^{S} \sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^v + \sum_{k=1}^{K} q_{gik};$
- n_i be the number of sandwiches in package i;
- t_I be the number of trays for type I sandwiches;
- t_{II} be the number of trays for type II sandwiches;
- p_I the maximum number of type I sandwiches in every tray;
- p_{II} the maximum number of type II sandwiches in every tray.

Once the trays used for each type of product are over, we need $\frac{h}{4}$ minutes to experience the *format change*, during which there is a worsening of the objective function given by the consumption of energy, labor and other components, waiting for the time it takes to restart the production process. The number of format changes turns out to be:

$$C = \left\lceil \frac{\sum_{i=1}^{j} n_i x_i}{p_I \cdot t_I} \right\rceil + \left\lceil \frac{\sum_{i=j+1}^{I} n_i x_i}{p_{II} \cdot t_{II}} \right\rceil - 1.$$

We remark that the number of format changes is a function of x_i and, hence, of Q_1. Therefeore, we get: $C = C(Q_1)$. As a consequence, we have to add a new term to the objective function, namely:

$$-C(Q_1) \cdot E,$$

where E is the price of all components of every format change.

3.3.2. Time

In the real situation, an important role is played by the production time, which provides an additional constraint, since it is bounded. Let:

- N_i be the number of sandwiches i produced in one hour;
- T be the total available hours (including also the lead-time).

Therefore, the new constraint is:

$$\sum_{i=1}^{I} \frac{n_i x_i}{N_i} + C \cdot \frac{h}{4} \leq T;$$

namely the time needed to produce all sandwiches plus the time for format changes must be less than or equal to the total available time (in hours).

3.4. Optimality conditions and Variational formulation

The company aims at maximizing its profit. Hence, according with the model in Section 2.1 and the additional constraints, the optimality conditions for the company are given by the following problem:

$$\begin{aligned}
\max \Bigg\{ & \sum_{i=1}^{I} \Bigg[\sum_{s=1}^{S} \sum_{v=1}^{V+1} \rho_{1gis}^{v*} q_{gis}^{v} - C^i - c_g^i(Q_0, Q_1) - \sum_{h=1}^{H} \sum_{l=1}^{L} \rho_{0hlg}^{*} q_{hlgi} \\
& - \sum_{h=1}^{H} \sum_{l=1}^{L} c_{hlgi}(q_{hg}) - p_{gi}(Q_1) - \tau \left[\sum_{s=1}^{S} \sum_{v=1}^{V+1} q_{gis}^{v} \right] \\
& - \rho \sum_{l=1}^{L} \sum_{h=1}^{H} [(1-\beta_{li}) q_{hlgi}] - \sum_{s=1}^{S} \sum_{v=1}^{V+1} c_{gis}^{v}(q_{gis}^{v}) \Bigg] \\
& + \sum_{j=1}^{J} \hat{\rho}_{1gj}^{*} \hat{q}_{gj}(Q_1, Q_4) - \sum_{i=1}^{I} \sum_{s=1}^{S} \sum_{v=1}^{V+1} \rho_{1gis}^{v*} r_{isg}^{(v)}(Q_1, Q_4) \\
& - \sum_{j=1}^{J} \hat{c}_{gj}(Q_1, Q_4) - \sum_{i=1}^{I} \hat{p}_{gi} r_{isg}^{(v)}(Q_1, Q_4) - \sum_{i=1}^{I} \sum_{s=1}^{S} \hat{c}_{isg}^{(v)}(Q_1, Q_4) \\
& - \rho \sum_{i=1}^{I} \sum_{s=1}^{S} \left[(1-\beta_i) r_{isg}^{(v)}(Q_1, Q_4) \right] - C(Q_1) E \Bigg\} \qquad (3)
\end{aligned}$$

$$\sum_{h=1}^{H} \frac{\beta_{li}}{\alpha_{hl}^{gi}} q_{hlgi} = \sum_{s=1}^{S} \sum_{v=1}^{V+1} q_{gis}^{v} + \sum_{k=1}^{K} q_{gik}, \quad \forall\, l = 1, \ldots, L, \quad \forall\, i = 1, \ldots, I$$

$$\sum_{i=1}^{I} \frac{n_i x_i(Q_1)}{N_i} + C(Q_1) \frac{h}{4} \leq T; \quad \lambda^{gi} \leq \sum_{s=1}^{S} \sum_{k=1}^{K} q_{sk}^{gi} \leq \mu^{gi} \quad \forall\, i = 1, \ldots, I$$

$$\sum_{v=1}^{V+1} q_{gis}^v \geq \sum_{k=1}^{K} q_{sk}^{gi} \quad \forall\, i=1,\dots,I, \quad \forall\, s=1,\dots,S$$

$$q_{hlgi},\, q_{gis}^v,\, q_{gik},\, q_{sk}^{gi} \geq 0 \quad \forall\, h=1,\dots,H, \quad \forall\, l=1,\dots,L, \quad \forall\, s=1,\dots,S,$$
$$\forall\, v=1,\dots,V+1,\ \forall\, k=1,\dots,K,\ \forall\, i=1,\dots,I.$$

We assume that the objective function is continuously differentiable and concave and this is guaranteed assuming that all the cost functions

$$c_g^i(Q_0,Q_1),\ c_{hlgi}(q_{hg}),\ p_{gi}(Q_1),\ c_{gis}^v(q_{gis}^v) \text{ and } C(Q_1)E$$

are convex functions. Therefore, since the objective function is concave and the feasible set is convex, it is easy to verify that the optimality conditions for the manufacturer are characterized by the following variational inequality (where δ^{gi} does not appear):

Find $\quad (Q_0^*, Q_1^*, Q_4^*) \in \mathbb{K} \quad$ such that:

$$\sum_{h=1}^{H}\sum_{l=1}^{L}\sum_{i=1}^{I}\left[\frac{\partial c_g^i(Q_0^*,Q_1^*)}{\partial q_{hlgi}} + \rho_{0hlg}^* + \frac{\partial c_{hlgi}(q_{hg}^*)}{\partial q_{hlgi}} + \rho(1-\beta_{li})\right]\left(q_{hlgi} - q_{hlgi}^*\right) +$$
$$+\sum_{i=1}^{I}\sum_{s=1}^{S}\sum_{v=1}^{V+1}\left[\frac{\partial c_g^i(Q_0^*,Q_1^*)}{\partial q_{gis}^v} - \rho_{1gis}^{v*} + \frac{\partial p_{gi}(Q_1^*)}{\partial q_{gis}^v} + \tau + \frac{\partial c_{gis}^v(q_{gis}^{v*})}{\partial q_{gis}^v} -\right.$$
$$-\sum_{j=1}^{J}\hat{\rho}_{1gj}^*\frac{\partial \hat{q}_{gj}(Q_1^*,Q_4^*)}{\partial q_{gis}^v} + \rho_{1gis}^{v*}\frac{\partial r_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{gis}^v} + \sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*,Q_4^*)}{\partial q_{gis}^v} +$$
$$+\hat{p}_{gi}\frac{\partial r_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{gis}^v} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{gis}^v} + \rho(1-\beta_i)\frac{\partial r_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{gis}^v} +$$
$$\left.+E\frac{\partial C(Q_1^*)}{\partial q_{gis}^v}\right]\left(q_{gis}^v - q_{gis}^{v*}\right) +$$
$$+\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{i=1}^{I}\left[-\sum_{j=1}^{J}\hat{\rho}_{1gj}^*\frac{\partial \hat{q}_{gj}(Q_1^*,Q_4^*)}{\partial q_{sk}^{gi}} + \sum_{v=1}^{V+1}\rho_{1gis}^{v*}\frac{\partial r_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{sk}^{gi}} +\right.$$
$$+\sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*,Q_4^*)}{\partial q_{sk}^{gi}} + \hat{p}_{gi}\frac{\partial r_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{sk}^{gi}} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{sk}^{gi}} +$$
$$\left.+\rho(1-\beta_i)\frac{\partial r_{isg}^{(v)}(Q_1^*,Q_4^*)}{\partial q_{sk}^{gi}}\right]\left(q_{sk}^{gi} - q_{sk}^{gi*}\right) \geq 0, \quad \forall\,(Q_0,Q_1,Q_4)\in\mathbb{K}, \tag{4}$$

where:

$$\mathbb{K} = \left\{(Q_0,Q_1,Q_4)\in\mathbb{R}_+^{HLGI+GIS2(V+1)+SKGI} :\ \forall\, l=1,\dots,L,\ \forall\, i=1,\dots,I,\right.$$
$$\left.\sum_{h=1}^{H}\frac{\beta_{li}}{\alpha_{hl}^{gi}}\,q_{hlgi} = \sum_{s=1}^{S}\sum_{v=1}^{V+1}q_{gis}^v + \sum_{k=1}^{K}q_{gik} \ \text{ and } \ \sum_{i=1}^{I}\frac{n_i x_i(Q_1)}{N_i} + C(Q_1)\frac{h}{4} \leq T\right\}.$$

The variational inequality (4) represents the optimality conditions for the company g related to all commodities simultaneously. The solution to (4) gives the optimal amount of raw material that g has to buy from all suppliers (Q_0^*), the optimal amount of product sold by g to all retailers (Q_1^*) and the optimal amount of products sold by all retailers to all demand markets (Q_4^*), in equilibrium.

3.5. Achieved goals

In this section, we determine the demand forecast of every good, because it is necessary to estimate the amounts in advance (in the problem there is the total demand of product i and the range of demand forecasting) and, since the production takes place daily in a continuous fashion, we consider a seven-day interval.

Therefore, in this section, we discuss the time series analysis. Its main aim is to identify an appropriate model that has trajectories that fit to the data, to be able to predict and determine the ranges. The company software also allowed us to achieve the daily data of sales, starting from the production till today, required for forecasting and production planning.

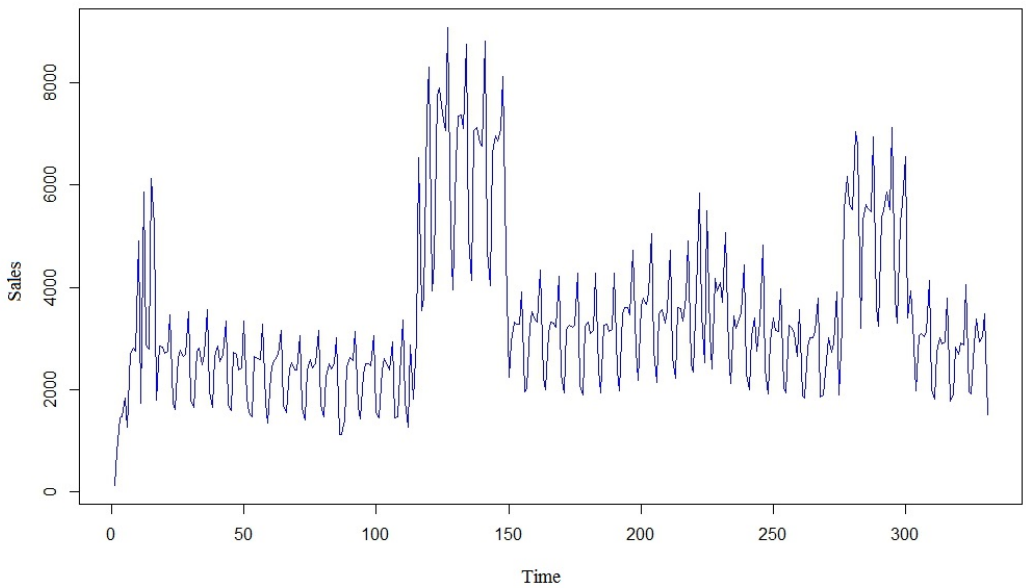

Figure 3.2: Time series

In this paper we make use of the statistical software **R**, with a plurality of commands and functions very useful in the time series analysis. Before the analysis, we examined the raw data and we made some adjustments to purify the data by discontinuities or effects of different interval duration or time periods considered; in particular we have eliminated cases of *additives outlier*, observations that lie an abnormal distance from other values, by replacing them with appropriate values. Therefore, we show the graph of the series (Figure 3.2) which describes an increase in the sales in some weeks of the year and especially a regularity at periodic intervals.

This periodicity is confirmed by the correlogram (Figure 3.3) which represents the autocorrelation of the series as a function of the lag.

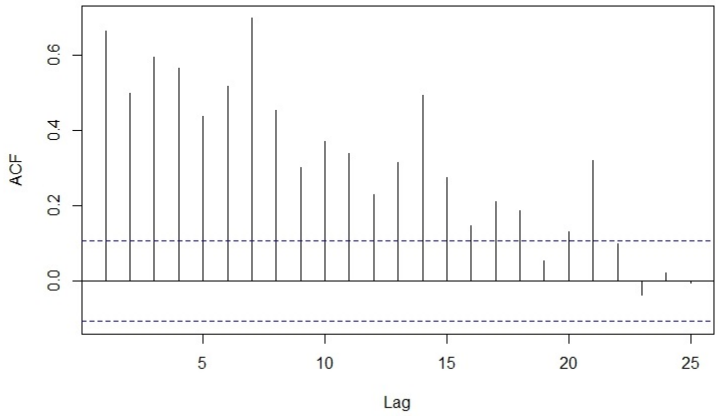

Figure 3.3: Correlogram

The result, shown in Figure 3.4, clearly proves how in all weeks the trend remains similar: we notice a maximum on the first day which decreases, until a minimum on the third day, and after that it has a slight increase.

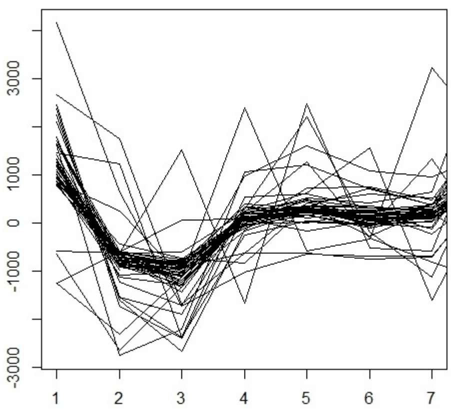

Figure 3.4: Profile

In order to analyze the series, we have to make it stationary and therefore we have implemented the detrending through the moving average, although we loose some initial and final values (the latter being very useful for the prediction). So we used the *Holt-Winters model* (exponential smoothing), which is denoted in red in Figure 3.5, thus resulting in an initial sales forecast in the next week:

```
> serief<-ts(serie1,frequency=7)
> serie.hw<-HoltWinters(serief,seasonal="additive")
```

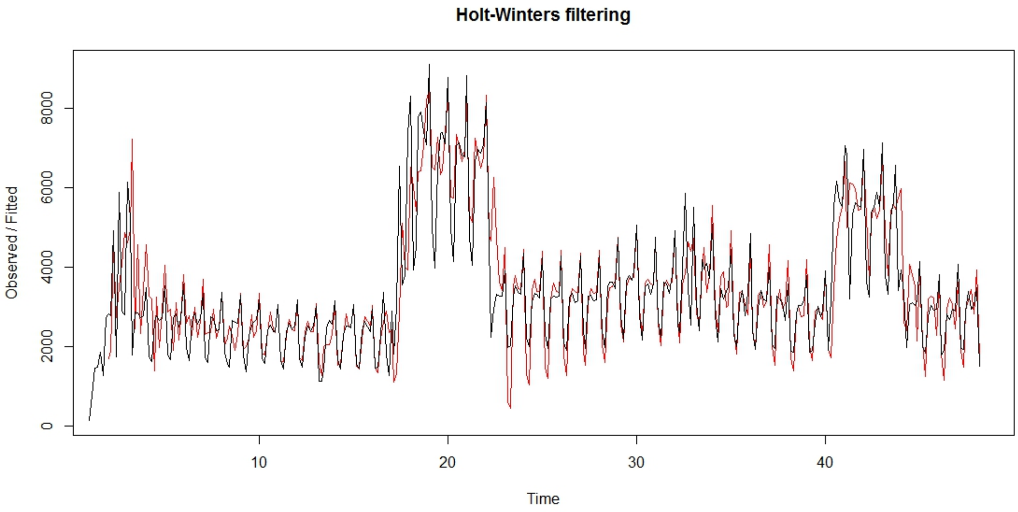

Figure 3.5: Holt-Winters

```
> serie.hw
Holt-Winters exponential smoothing with trend and additive seasonal
component.
Call:
HoltWinters(x = serief, seasonal = "additive")
Smoothing parameters:
alpha:  0.4344439
beta :  0.04848514
gamma:  0.4232419
Coefficients:
                [,1]
a       2989.25514
b       -36.94766
s1       -1703.41649
s2       -532.79118
s3       -199.48552
s4       -169.15905
s5       -364.33260
s6       514.17507
s7       -1425.35193
> plot(serie.hw)
> prev<-predict(serie.hw,n.ahead=7)
> prev
Time Series:
Start = c(48, 3)
End = c(49, 2)
Frequency = 7
                fit
```

```
[1,]     1248.891
[2,]     2382.569
[3,]     2678.927
[4,]     2672.305
[5,]     2440.184
[6,]     3281.744
[7,]     1305.270
```

A second prediction is obtained by the *decomposition and estimation of the components* with the method *loess* which consists in estimating a locally weighted polynomial regression at the point t_k (with k fixed) using points of its neighborhood. We show the graph of this decomposition into seasonal, trend and irregular components (Figure 3.6).

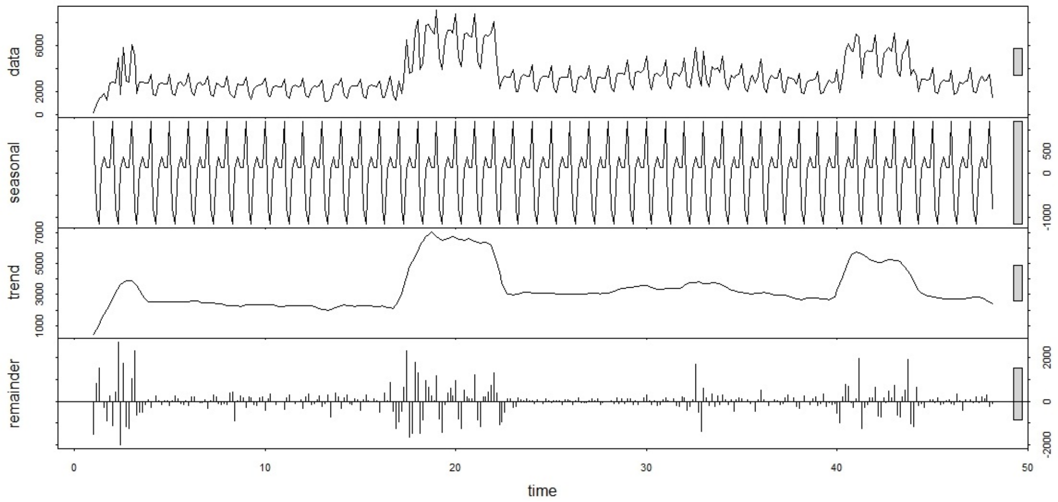

Figure 3.6: Decomposition in components

Now we calculate forecasts and ranges of the three components:

> prev.stag<-predict(stag.stl,7)
> prev.stag

	Point Forecast	Lo 80	Hi 80	Lo 95	Hi 95
48.28571	-1142.75224	-1142.75224	-1142.75224	-1142.75224	-1142.75224
48.42857	89.56213	89.56213	89.56213	89.56213	89.56213
48.57143	378.56325	378.56325	378.56325	378.56325	378.56325
48.71429	131.13198	131.13198	131.13198	131.13198	131.13198
48.85714	147.16995	147.16995	147.16995	147.16995	147.16995
49.00000	1193.92431	1193.92431	1193.92431	1193.92431	1193.92431
49.14286	-797.59933	-797.59933	-797.59933	-797.59933	-797.59933

> prev.trend<-predict(trend.stl,7)
> prev.trend

	Point Forecast	Lo 80	Hi 80	Lo 95	Hi 95
48.28571	2313.735	2255.400	2372.070	2224.5195	2402.951
48.42857	2220.072	2046.462	2393.683	1954.5575	2485.587
48.57143	2131.398	1859.787	2403.009	1716.0049	2546.791
48.71429	2047.446	1677.720	2417.172	1481.9992	2612.893
48.85714	1967.966	1498.356	2437.575	1249.7603	2686.171
49.00000	1892.718	1321.512	2463.924	1019.1340	2766.302
49.14286	1821.478	1147.356	2495.600	790.4974	2852.459

```
> prev.res<-predict(res.stl,7)
> prev.res
```

	Point Forecast	Lo 80	Hi 80	Lo 95	Hi 95
48.28571	2.236474	-712.1711	716.6441	-1090.356	1094.829
48.42857	2.236474	-712.1712	716.6441	-1090.356	1094.829
48.57143	2.236474	-712.1712	716.6441	-1090.356	1094.829
48.71429	2.236474	-712.1712	716.6441	-1090.356	1094.829
48.85714	2.236474	-712.1712	716.6441	-1090.356	1094.829
49.00000	2.236474	-712.1712	716.6441	-1090.356	1094.829
49.14286	2.236474	-712.1712	716.6441	-1090.356	1094.829

Therefore, we get the following values of prediction:

	Point Forecast
48.28571	1173,219234
48.42857	2311,870604
48.57143	2512,197724
48.71429	2180,814454
48.85714	2117,372424
49.00000	3088,878784
49.14286	1026,115144

After estimating the seasonal component, it is necessary to prove that the seasonality over the months shows no trend; in this case we call it trend-season. A simple and immediate way that we use is to refer to the function `seaplot()`.

In our case, the seasonality is constant in every week, in fact, Figure 3.7 shows the weekly sub-series of seasonality; the order is from left to right and from bottom to top (thus the first diagram at the bottom left is referred to the seasonality observed on the first day of the week, and so on).

A final prediction is obtained by the *ARIMA model.* We show the graph (Figure 3.8) and the results of the forecast:

```
> plot(previsione)
> previsione
```

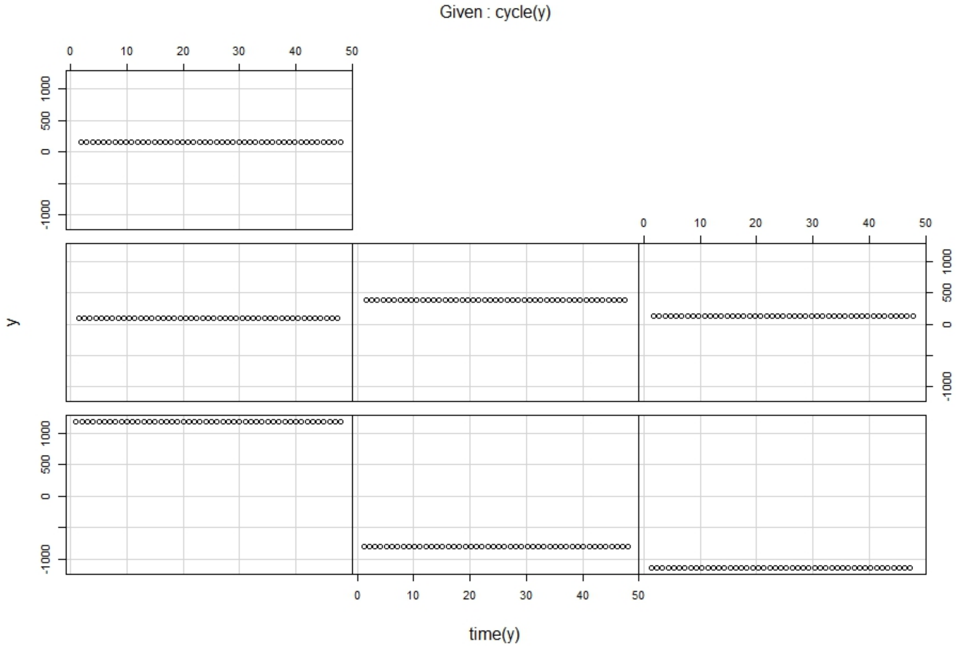

Figure 3.7: Seaplot.stl

	Point Forecast	Lo 80	Hi 80	Lo 95	Hi 95
48.28571	1423.474	572.2470	2274.700	121.6348	2725.313
48.42857	2568.165	1497.7902	3638.539	931.1682	4205.162
48.57143	2819.999	1679.1242	3960.875	1075.1814	4564.818
48.71429	2653.697	1414.4843	3892.909	758.4850	4548.908
48.85714	2535.439	1187.3477	3883.531	473.7111	4597.167
49.00000	3433.762	1978.2263	4889.297	1207.7124	5659.811
49.14286	1421.351	-101.5853	2944.287	-907.7789	3750.480
49.28571	1128.699	-545.8531	2803.251	-1432.3073	3689.705
49.42857	2399.166	581.1618	4217.169	-381.2312	5179.562
49.57143	2591.403	677.6506	4505.156	-335.4290	5518.236
49.71429	2486.855	486.8052	4486.905	-571.9573	5545.668
49.85714	2304.635	203.0068	4406.264	-909.5281	5518.799
50.00000	3264.413	1059.9267	5468.900	-107.0581	6635.885
50.14286	1282.890	-999.7158	3565.496	-2208.0542	4773.834

After estimating the model, we must verify how well the data fit a statistical model. An important element for such a verification is the *coefficient of determination* R^2; it measures the rate of variability explained by the model compared to the variability of Y:

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS}$$

where:

- $ESS = \sum_{t=1}^{T}(\hat{Y}_t - \overline{Y})^2$ is the regression sum of squares, also called the explained sum of squares;

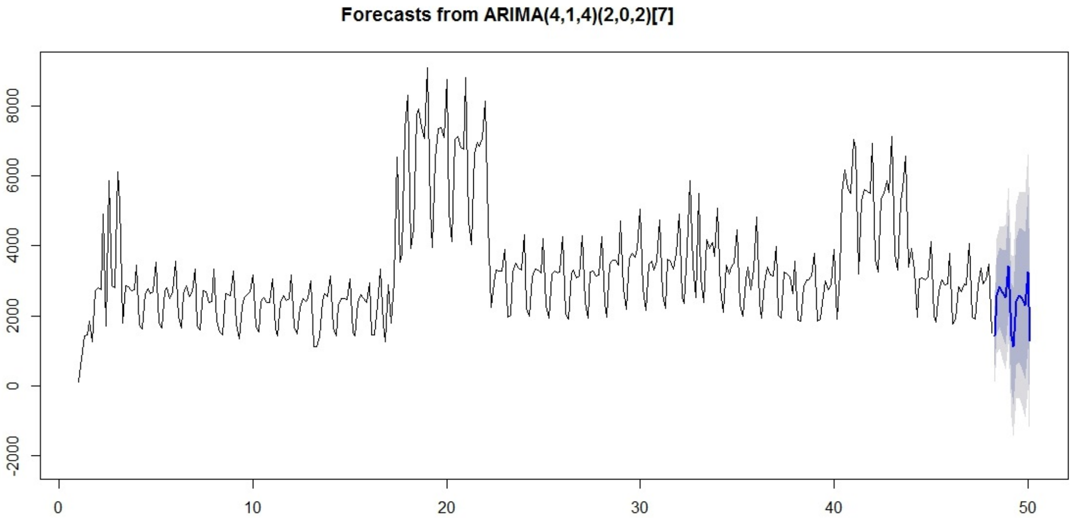

Figure 3.8: ARIMA

- $TSS = \sum_{t=1}^{T}(Y_t - \overline{Y})^2$ is the total sum of squares (proportional to the variance of the data);
- $RSS = \sum_{t=1}^{T} R_t^2 = \sum_{t=1}^{T}(Y_t - \hat{Y}_y)^2$ is the sum of squares of residuals, also called the residual sum of squares.

When the coefficient R^2 is equal to 1, it means that the regression line perfectly fits the data, while when R^2 is equal to 0, it means that the line does not fit the data at all. Therefore, we calculate the coefficient for the Holt-Winters, the decomposition and the ARIMA models getting the following values: `0.8368674`, `0.8781064` and `0.7328934`, respectively. Since we obtain results very close to 1, we can conclude that the models could usefully be used for predictive purposes.

For the three obtained forecasts, we verified the validity of the assumptions (test specification):

- average waste equal to zero (t-test);
- errors normality (Shapiro-Wilk and Jarque-Bera);
- homoscedasticity (Breusch-Pagan);
- correlation between residues (Box-Pierce and Ljung-Box).

Only models passing all tests and having a good determination coefficient R^2 can be accepted. So, since for the examined product, in the *Holt-Winters model* and in the *ARIMA model* residuals are not normally distributed (and therefore the second assumption is not verified), only the *decomposition model* is acceptable and, hence, it is the one we will use for our optimization problem. With a similar procedure we obtained estimates of all 7 goods produced by the considered production line, but, as a matter of corporate privacy, will not present them.

3.6. Inventory management

In this section, we study a model of inventory management in order to determine the production periods (days), which satisfy the demand forecast obtained in the previous section, while minimizing the production and the storing costs. Later, we shall apply our model so to determine the optimal amounts we are looking for.

The Wagner-Whitin model of inventory management (1958) is part of the Lot sizing problems, characterized by the fact that the demand and the production and storage costs may vary over time. The aim is to determine the production $x_1, x_2, \ldots, x_N$ and the storage $s_1, s_2, \ldots, s_N$ in each period in such a way that the demands $d_1, d_2, \ldots, d_N$ (obtained in the previous section), have, in stock, an initial value $s_0 = 0$. Of course we should try to minimize production and storage costs. Grafically we have a situation shown in Figure 3.9.

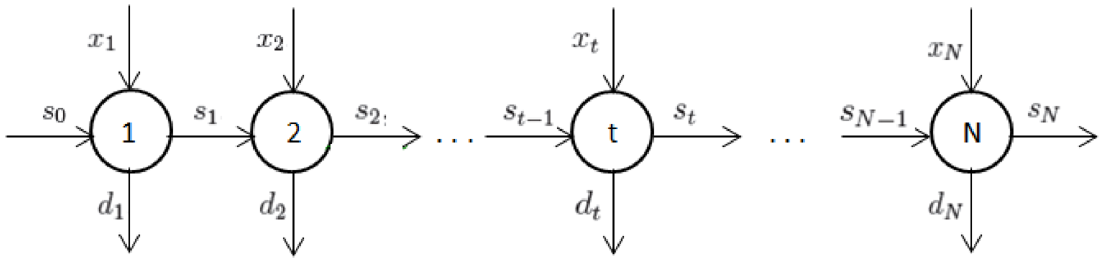

Figure 3.9: Inventory management

Since the production and the stotage costs may vary over time, we have the following estimates:

- total production cost at time t: $C_t(x_t) = A_t\eta(x_t) + c_t(x_t)$,
- total storage cost at time t: $H_t(s_t) = \pi_t\eta(s_t) + h_t(s_t)$,

where A_t and π_t are the fixed production cost and the fixed storage cost respectively, c_t and h_t denote the production and the storage costs and

$$\eta(x_t) = \begin{cases} 1 & \text{if } x_t > 0, \\ 0 & \text{if } x_t \leq 0. \end{cases}$$

We have to solve the following optimization problem:

$$\begin{cases} \min f(x,s) = \sum_{t=1}^{N} \big(C_t(x_t) + H_t(s_t)\big) \\ x_t + s_{t-1} - s_t = d_t \quad t = 1, \ldots, N \\ x_t, s_t \geq 0 \qquad\qquad\quad t = 1, \ldots, N. \end{cases}$$

If (x, s) is a solution to the previous problem, then in any period $t \in \{1, \ldots, N\}$ one of the following cases holds:

- if $x_t > 0$ and $s_{t-1} = 0$, then we have a positive production and no inventory;
- if $x_t = 0$ and $s_{t-1} > 0$, then we have no production and a positive inventory.

Definition 3.1. If $x_t > 0$, then time $t \in \{1, \dots, N\}$ is called *productive time.*

The demand in a nonproductive time $h \in \{1, \dots, N\}$ is satisfied by the production in the last productive time $k \in \{1, \dots, N\}$, where $k < h$.

Definition 3.2. The set of nonproductive times when the demand is satisfied in a special productive time is called *production interval.*

Let j be the productive time satisfying the demand of nonproductive times $\{j+1, \dots, k\}$, then these times are subsequent. After establishing the production interval $\{j, j+1, \dots, k\}$, it is possible to evaluate the production and the storage:

$$\bar{x}_j = \sum_{r=j}^{k} d_r \text{ and } \bar{s}_t = \sum_{r=t+1}^{k} d_r, \quad t \in \{j, j+1, \dots, k\}.$$

Further, we have:

- $\bar{x}_t = 0$ for $t = j+1, \dots, k$;
- $\bar{s}_t > 0$ for $t = j, \dots, k-1$;
- $\bar{s}_{j-1} = \bar{s}_k = 0$.

A solution $(\bar{x}, \bar{s})$ depends on the set of productive times $\bar{J} = \{j_1, \dots, j_q\}$. Indeed, from this set we can deduce the production and the storage at each time. In any productive interval $\{j, \dots, k\}$, the total cost, given by the sum of production and storage costs, is as follows:

$$M(j,k) = C_j(\bar{x}_j) + \sum_{t=j}^{k-1} H_t(\bar{s}_t) = C_j\Big(\sum_{r=j}^{k} d_r\Big) + \sum_{t=j}^{k-1} H_t\Big(\sum_{r=t+1}^{k} d_r\Big).$$

Now we introduce F_k, namely the minimum production and storage cost required in order to satisfy the demand in the interval $\{1, \dots, k\}$. Hence, F_N represents the optimal value. We assume that the productive times form the set $J = \{j_1, \dots, j_q\}$; the total production and storage cost is given by:

$$Z(J) = \sum_{s=1}^{q-1} M(j_s, j_{s+1} - 1) + M(j_q, N).$$

In consequence, we have to find the set $J^* \subseteq \{1, \dots, N\}$ such that $Z(J^*) = F_N$.

3.6.1. Computational procedure

We find F_k recursively. Assume we have to satisfy the demand in the time horizon $\{1, \dots, k\}$ and we know the optimal solution $J_k = \{j_1, \dots, j_q\}$. Then, the minimal cost in such a time interval is:

$$F_k = M(j_1, j_2 - 1) + M(j_2, j_3 - 1) + \ldots + M(j_{q-1}, j_q - 1) + M(j_q, k).$$

But, if the horizon is $\{j_1, \ldots, j_q - 1\}$, then we have:

$$F_{j_q - 1} = M(j_1, j_2 - 1) + M(j_2, j_3 - 1) + \ldots + M(j_{q-1}, j_q - 1),$$

therefore:

$$F_k = F_{j_q - 1} + M(j_q, k).$$

Time j_q represents the last productive time of the optimal solution associated with the time interval $\{1, \ldots, k\}$. If $j \in \{1, \ldots, k\}$ is the last productive time, then the minimal production and storage cost is given by $F_{j-1} + M(j, k)$. As a consequence, $F_k \leq F_{j-1} + M(j, k)\ \forall j \in \{1, \ldots, k\}$. Hence, we can conclude:

$$F_k = \min_{1 \leq j \leq k} \{F_{j-1} + M(j, k)\}.$$

Such a result is a recursive estimate of F_k depending on the values of F_j con $j = 1, \ldots, k-1$. If we set $F_0 = 0$, we can calculate all the values $F_1, F_2, \ldots, F_N$, and the last one allows us to solve the problem. We implemented the algorithm in `C++` code.

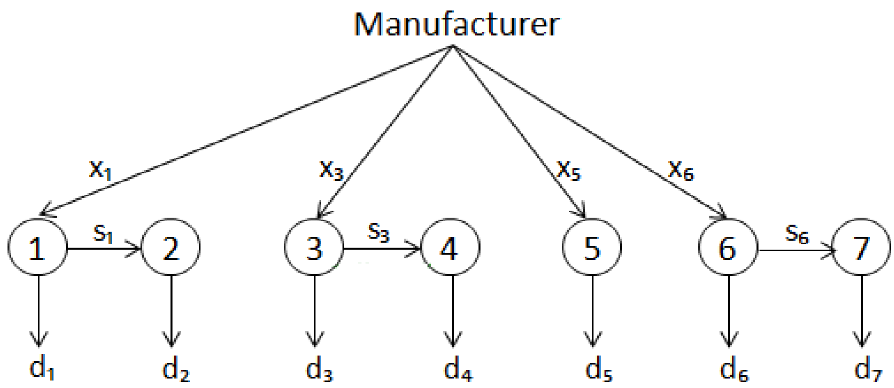

Figure 3.10: Inventory management: product 807

3.7. Results

In this section we present the results of the optimization model. We will consider as the discrete time interval the first 7 next days. For each of the long-life products, we get the $x_1, \ldots, x_7$ quantities to be produced and the $s_1, \ldots, s_7$ quantities to be stored in each period, so that the expected demands $d_1, \ldots, d_7$ (obtained in Section 3.5) are met. We use the algorithm shown in Section 3.6, developed in `C++` code, to determine the periods of production, while minimizing the production and storing costs. By analyzing product 807, as shown in Fig. 3.10, it turns out that the optimal production periods are the days 1, 3, 5 and 6; in the remaining days, however, the demands have to be met from stocks obtained from previous productive days prior.

By applying the same algorithm to products 808, 809 and 810, it turns out that only the demand in day 7 must be satisified by using stocks (Fig. 3.11).

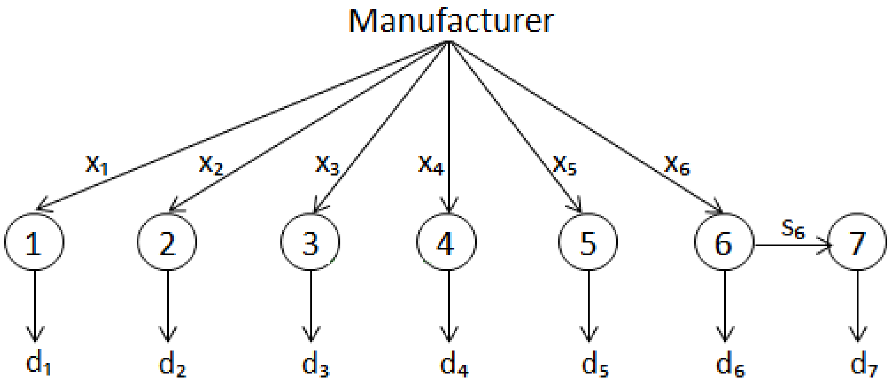

Figure 3.11: Inventory management: products 808-809-810

Finally, using the optimal production periods previously obtained, we calculate, for every day, the optimal amounts Q_1^*, by solving our model, that is equivalent to the Variational inequality (4), through the mathematical software `Maple` (and also the software `Lingo`).

The following table shows the results:

	Products						
	302	303	807	808	809	810	811
Day 1	1073	526	0	0	0	0	0
Day 2	2211	502	0	140	592	580	406
Day 3	2012	752	2664	137	494	6341	1008
Day 4	2080	672	0	255	672	663	1069
Day 5	2017	632	3970	580	290	1783	750
Day 6	3000	825	2384	260	568	5892	95
Day 7	926	335	0	0	0	0	150

We remark that the “0’s” in Day 1 are due to the demand forecast of the market because that day coincides with Sunday; the others, namely those related to product 807 and Day 7, are determined by the inventory management.

The model also gives us the optimal amount of product that all retailers must sell to all demand markets (Q_4^*) and, as expected, they coincide with the produced amount, so that there is the maximization of the objective function. Therefore, we note that the amount of unsold product i that retailer s gives back to manufacturer g is equal to zero:

$$r_{isg}^{(v)} = \sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} - \sum_{k=1}^{K} q_{sk}^{gi} = 0,$$

$\forall\, i = 1, \ldots, I,\ \forall\, s = 1, \ldots, S,\ \forall\, g = 1, \ldots, G.$

Once we have the quantities to be produced, by means of the constraint:

$$\sum_{h=1}^{H} \frac{\beta_{li}}{\alpha_{hl}^{gi}} q_{hlgi} = \sum_{s=1}^{S} \sum_{v=1}^{V+1} q_{gis}^{v} + \sum_{k=1}^{K} q_{gik},\ \forall\, l = 1, \ldots, L;\ \forall\, i = 1, \ldots, I,$$

we can establish, for every day, the amount of raw materials needed for the products (Q_0^*). For example, in the table below, we show a portion of the bill of material of Day 3:

Day 3	Products							Total
Raw material	302	303	807	808	809	810	811	
Label 66105	2012	752	0	0	0	0	0	2764
Envelope 66287	2012	0	0	0	0	0	0	2012
Farina FAR3	577	105	484	15	90	1034	113	2418 kg
Salt 90101	13.8	2.5	10.7	0.3	2.0	22.8	2.5	54.6 kg
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Water 64110	312	54	252	8	47	538	59	1270 lt

4. Conclusions

In this paper we initially studied a multilevel supply chain network to which the variation of the different products of the company was added with respect to those produced by competing companies. We also analyzed the chain in a context where both ecommerce and production excesses are present, thus introducing the reverse chain and presenting the optimality conditions and the variational formulation of the single manufacturer. Therefore, we have built we have built the supernetwork structure for a business management with electronic commerce in which suppliers of raw materials provide their commodities to manufacturers who produce i different products and sell their goods to retailers through V different carriers and directly to demand markets. Moreover, we have assumed that the products unsold by retailers can be given back to manufacturers who, after reworking such products, are now able to produce new commodities which will be shipped to new retailers (some of them could coincide with the previous ones). So we have extended the model including a forward and a reverse chain network. We have obtained the optimality conditions of the manufacturer which have been characterized by a variational inequality. Existence results for solutions follow from the classical theory.

Then, we improved the problem by applying it to a company, the Valle del Dittaino, considering a more complete model in which we not only the production excess and reverse logistics were examined, but we also added additional constraints arising from concrete and production limits.

Finally, we have applied such models to a real company, Valle del Dittaino, obtaining the optimal production and storage in a 7 days period. In the case study, we have included some additional constraints related to production and time.

Since every company aspiring to become a leader in the market has to rely on an internal organization capable of managing its work, trying to reduce costs and to get maximum profits, in order to study the best production planning it is necessary to analyze all of the organizational, managerial and strategic activities that govern the company's flow of materials and related information. Therefore, through the historical business series, we have also dealt with the demand forecast analysis needed to solve the maximization problem and a stock management model by implementing an algorithm that enables us to obtain excellent production times, in addition to the optimal amount of goods to be manufactured and sold and raw materials to be purchased.

Given the importance and relevance of the optimization and network models nowadays and the need to continuously create new and innovative ways to exploit for achieving business goals, the most important result of this paper is the complete and concrete model, which can be applied to many companies in order to improve their production and minimize their waste or can be used in other fields that maintain a multilevel structure such as the described one.

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