Journal of Convex Analysis
Volume 25 (2018), No. 2, 515–527

Generalized Variational Inequality and General Equilibrium Problem

Maria B. Donato
Department of Mathematics and Computer Science, University of Messina,
Via Ferdinando Stagno D'Alcontres, 31, 98166 Messina
mbdonato@unime.it

Monica Milasi
Department of Mathematics and Computer Science, University of Messina,
Via Ferdinando Stagno D'Alcontres, 31, 98166 Messina
mmilasi@unime.it

Carmela Vitanza
Department of Mathematics and Computer Science, University of Messina,
Via Ferdinando Stagno D'Alcontres, 31, 98166 Messina
vitanzac@unime.it

Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

Received: October 5, 2015
Revised manuscript received: February 17, 2017
Accepted: April 26, 2017

We present a study of the general economic equilibrium problem in the framework of the variational inequality approach. We describe the main aspects characterizing the model and the equilibrium definition. The utility functions, which represent the consumers' preferences over goods, are taken to be generalized concave and non-differentiable. Our aim is to study the equilibrium by means of a suitable quasi-variational inequality.
As is well-known, the difficulty to study a quasi-variational inequality lies in the dependence of the convex constraint set on the solution. A crucial issue of our study is proving the equivalence between our quasi-variational inequality and suitable variational inequalities. More precisely, we introduce a new and useful methodology, which allows us to solve associated variational inequalities (where the convex sets do not depend on solution) instead of a quasi-variational inequality. Finally, our main result is achieved by using arguments of set-valued analysis.

Keywords: Generalized quasi-variational inequalities, generalized concavity, competitive equilibrium, production economy.

2010 Mathematics Subject Classification: 49J53, 91B42, 91B50

ISSN 0944-6532 / $ 2.50

1. Introduction

The variational inequality theory was developed in the early 1960s by Fichera and Stampacchia in connection with partial differential equations and it was soon recognized its power in the study of several realistic equilibrium problems (see e.g. [21, 29]). We refer the reader to the early results [14, 16, 26] and to other important contributions [6, 7, 13, 19, 22, 27, 28] (with the references therein), where several equilibrium problems are studied by means of variational inequality arguments in the static and in the time-dependent case. In fact, it is recognized that variational inequality methods represent an important tool in the optimization theory. A well-known fact is that if the objective function of an optimization problem is concave and continuously differentiable, then the Stampacchia variational inequality, involving the gradient operator of the objective function, provides a necessary and sufficient optimality condition for a point to be a solution to the optimization problem. However, in many applied problems the concavity appears to be a restrictive condition. This lead to the introduction of the class of quasiconcave functions which have powerful properties and which are very used in mathematical economics (see e.g. [3] and references therein, and [9, 10, 24]). In this general setting of the quasiconcavity, the operator involved in the variational problem is replaced by a suitable set-valued map which uses the concept of normal cone. The idea of using the normal operator was given by Borde and Crouzeix [11] and subsequently new operators involving this concept was introduced (see, for example, the interesting application to the Nash equilibrium problem given in [4]).

The aim of this article is to improve the results obtained in [8, 20]. More precisely, the general equilibrium is studied in a production economy in which the utility functions are quasiconcave and non-differentiable. In literature, the general equilibrium problem was extensively studied by using a variety of approaches: the fixed point theory (see for instance [2]), the differential topology approach (see for instance [30]) and the variational approach (see for instance [18, 23]). For this economic model, in the present paper a new formulation in terms of a generalized quasi-variational inequality is obtained. We would like to underline that the method used to solve our variational problem represents a powerful methodology to provide existence results. More precisely, we are able to achieve the existence of solutions of the generalized quasi-variational inequality establishing the existence of solutions of associated variational inequalities where the convex constraint sets do not depend on the solution (see Remark 1 in Section 3).

The paper is organized as follows. The Section 2 is devoted to recall basic definitions and results, in particular to show the connection between maximization problems and suitable variational inequality problems under different assumptions on the objective function. The Section 3 is devoted to the study of the general equilibrium problem for a model of consumption, production and exchange. Firstly, we describe main aspects characterizing the model and we give the equilibrium definition. Subsequently, we reformulate the problem in terms of a generalized quasi-variational inequality under semistrict quasiconcavity and continuity assumptions on utility functions. Finally, we investigate on the existence of equilibrium points.

2. Preliminaries

In this Section for the reader's convenience, we recall some basic definitions of the set-valued analysis and some definitions and known existence results on variational problems that will be useful in the sequel, with particular attention to the connection between a maximization problem and a suitable variational inequality. For the proofs and further details, we refer to [1, 4, 5, 12, 25, 29] and the references therein.

Definition 2.1. (see [25]) Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. We define the *domain* and the *graph* of F, respectively, by:

$$Dom\, F := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\};\ \ Graph(F) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\}\,.$$

Let F be such that $Dom\, F \neq \emptyset$. F is:

(a) *upper semicontinuous* at $x \in \mathbb{R}^n$ iff for each open set $V \subset \mathbb{R}^m$, where $F(x) \subset V$, there exists a neighborhood $U \subset \mathbb{R}^n$ of x such that for all $x' \in U : F(x') \subset V$;

(b) *lower semicontinuous* at $x \in \mathbb{R}^n$ iff for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$, $x_n \to x$, and for any $y \in F(x)$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$, with $y_n \in F(x_n)\ \forall n$ and $y_n \to y$.

(c) *closed* iff for any sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$, $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$, if $x_n \to x$ and $y_n \in F(x_n)$, $y_n \to y$ then $y \in F(x)$.

F is upper semicontinuous and lower semicontinuous if it is at all $x \in \mathbb{R}^n$.

Definition 2.2. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and let $S\colon C \rightrightarrows \mathbb{R}^n$ and $\Phi\colon C \rightrightarrows \mathbb{R}^n$ be set-valued maps. A *Generalized Quasi-Variational Inequality* associated with C, S, Φ, denoted by *GQVI*, is the following problem:

$$\text{Find}\ \ \overline{u} \in S(\overline{u})\ \ \text{s.t.}\ \ \exists \varphi \in \Phi(\overline{u})\ \ \text{with}\ \ \langle \varphi, u - \overline{u}\rangle \geq 0\ \ \forall u \in S(\overline{u}). \tag{1}$$

In particular, when $S(u) = C$ for all $u \in C$, (1) is a *Generalized Variational Inequality, GVI*; when Φ is single-valued, (1) reduces to the *Quasi-Variational Inequality, QVI.* When both $\Phi(u)$ is singleton and $S(u) = C$, for all $u \in C$, we have the classical Stampacchia *Variational Inequality, VI.*

Theorem 2.3. (see [29]) *If C is a compact, convex and nonempty set and the function $\Phi : C \to \mathbb{R}^n$ is continuous, then the VI admits a solution.*

Theorem 2.4. (see [12]) *If C is a compact set and the set-valued map $\Phi\colon C \rightrightarrows \mathbb{R}^n$ is upper semicontinuous and with compact and convex values, then there exists $\overline{u} \in C$ solution to GVI.*

Let $u\colon X \to \mathbb{R}$ be a function, with $X \subseteq \mathbb{R}^l$, let $K \subseteq X$ be a closed, convex and nonempty set. Let us consider the following maximization problem:

$$\text{Find } \ \overline{x} \in K \ \text{ such that } \ u(\overline{x}) = \max_{x \in K} u(x)\,. \tag{2}$$

It is well known that, when the objective function u is *continuous differentiable*, if $\overline{x}$ is a solution to problem (2), then $\overline{x}$ is solution to the following VI:

$$\langle -\nabla u(\overline{x}), x - \overline{x} \rangle \geq 0 \qquad \forall x \in K. \tag{3}$$

When u is also *concave*, VI (3) represents a necessary and sufficient optimality condition to be $\overline{x}$ a solution to (2). Whereas, under more general assumptions, if the function u is *concave and continuous*, without differentiability assumption, in (3) the operator $-\nabla u$ is replaced by a supergradient of u in $\overline{x}$. Hence, the problem (2) is equivalent to the following GVI:

$$\text{Find } \ \overline{x} \in K \ \text{ such that } \ \exists\, h \in \partial u(\overline{x}) \ \text{ with } \ \langle -h, x - \overline{x} \rangle \geq 0 \ \ \forall x \in K,$$

where $\partial u(x) = \{l \in \mathbb{R}^l : \ u(y) \leq u(x) + \langle l, y - x \rangle \ \ \forall\, y \in \mathbb{R}^l\}$ is the superdifferential of the concave function u and $h \in \partial u(\overline{x})$ is the supergradient of u in $\overline{x}$. In this paper our aim is to deal with quasiconcave and continuous objective functions. We recall some concepts of quasiconcavity.

Definition 2.5. The function $u\colon X \subseteq \mathbb{R}^l \to \mathbb{R}$ is said to be

(i) *quasiconcave* iff for any $x, y \in X$ and $\lambda \in [0,1]$ one has

$$u(\lambda x + (1-\lambda) y) \geq \min\{u(x), u(y)\}\,;$$

(ii) *semistrictly quasiconcave* iff for any $x, y \in X$ with $u(x) \neq u(y)$, one has

$$u(\lambda x + (1-\lambda) y) > \min\{u(x), u(y)\}, \quad \forall \lambda \in (0,1)\,.$$

Let us denote, for any $\alpha \in \mathbb{R}$, by $U_\alpha(u)$ and $U_\alpha^>(u)$ the upper level set and the strict upper level set, respectively, associated with u and α:

$$U_\alpha(u) := \{x \in X : u(x) \geq \alpha\}, \qquad U_\alpha^>(u) := \{x \in X : u(x) > \alpha\}.$$

If u is semistrictly quasiconcave, one has $U_{u(x)} = \overline{U}^>_{u(x)} \ \ \forall x \in X \setminus argmax_X u$. Furthermore, the following characterization holds: a function u is quasiconcave if and only if, for any $\alpha \in \mathbb{R}$, upper level set $U_\alpha(u)$ (strict upper level set $U_\alpha^>(u)$) is a convex set. When the function u is quasiconcave, let us set $N(x)$ and $N^>(x)$ the normal cones, respectively, to sets $U_{u(x)}$ and $U^>_{u(x)}$:

$$N(x) := \{h \in \mathbb{R}^l : \ \langle h, z - x \rangle \leq 0 \quad \forall\, z \in U_{u(x)}\},$$
$$N^>(x) := \{h \in \mathbb{R}^l : \ \langle h, z - x \rangle \leq 0 \quad \forall\, z \in U^>_{u(x)}\}.$$

It is clear that $N(x) \subseteq N^{>}(x)$ for all $x \in X$, and when u is semistricly quasiconcave one has $N(x) = N^{>}(x)$ for all $x \in X \setminus argmax_X u$. To relate the maximization of quasiconcave and continuous functions to a variational problem, we will apply the results obtained in the last years about the optimization theory in the framework of the generalized convexity. To this aim, we observe that

$$\max u = -\min(-u)\,, \qquad U_\alpha(u) = S_\alpha(-u)\,,$$

where $S_\alpha(-u)$ stands for the classical sublevel set of the function $-u$ at level α. Hence, we can use in particular the results of [1, 4, 5], by replacing the sublevel set of the quasiconvex function $-u$ with the upper level set of the quasiconcave function u. Following [4], we adapt the definition of operator $G\colon X \rightrightarrows \mathbb{R}^l$ such that

$$G(x) := \begin{cases} \overline{B}(0,1) & \textit{if } x \in argmax_X u \\ conv\Big(N^{>}(x) \cap S(0,1)\Big) & \textit{if } x \in X \setminus argmax_X u \end{cases}$$

where $\overline{B}(0,1) = \{x \in \mathbb{R}^l : \|x\| \le 1\}$, $S(0,1) = \{x \in \mathbb{R}^l : \|x\| = 1\}$ are, respectively, the closed unit ball and the unit sphere of $\mathbb{R}^l$. The following results hold.

Theorem 2.6. (see [4]) *Let u be continuous and quasiconcave. Then the set-valued map G is with convex and compact values, and upper semicontinuous.*

Proof. Clearly, from definition, for all $x \in X$, $G(x)$ is a convex and compact set. The proof of the closedness of the set-valued map G can be done using the same arguments as in the proof of Theorem 4.2 of [4] and the upper semicontinuity of G can thus be deduced from the fact that the values of G are all included into the closed unit ball. $\square$

Let us consider the following GVI:

$$\text{Find } \overline{x} \in K \text{ s.t. } \exists\, g \in G(\overline{x}) \text{ with } \langle g, x - \overline{x}\rangle \ge 0 \;\; \forall x \in K. \tag{4}$$

Problem (4) leads to a necessary and sufficient optimality condition.

Theorem 2.7. (see [1]) *Let $u\colon X \subseteq \mathbb{R}^l \to \mathbb{R}^n$ be continuous and let K be a convex set. One has:*

(i) Sufficient condition*: If u is quasiconcave in X and $K^\perp = \{0\}$, then any solution $\overline{x} \in K$ to GVI* (4) *is a solution to maximization problem* (2)*.*

(ii) Necessary and sufficient condition*: If u is semistrictly quasiconcave, then $\overline{x} \in K$ is a solution to maximization problem* (2) *if and only if $\overline{x}$ is a solution to GVI* (4)*.*

Let $K\colon U \rightrightarrows \mathbb{R}^l$ be a set-valued map and U a normed vector space; let us consider the following parametric generalized variational inequality:

$$\text{Find } \overline{x} \in K(\mu) \text{ s.t. } \exists\, g \in G(\overline{x}) \text{ with } \langle g, x - \overline{x}\rangle \ge 0 \;\; \forall x \in K(\mu). \tag{5}$$

Let us define the solution map of (5) as $S\colon U \rightrightarrows \mathbb{R}^l$ such that

$$S(\mu) = \{\overline{x} \in X : \ \overline{x} \text{ is } \ \text{a solution to GVI (5)}\}.$$

Proposition 2.8. (see [5]) *Let us suppose that $K\colon U \rightrightarrows \mathbb{R}^l$ is bounded, closed, lower semicontinuous and with convex values and u is quasiconcave. Then:*

(i) *the solution map S is closed, upper semicontinuous with nonempty compact values;*

(ii) *if, additionally, u is semistrictly quasiconcave, one has that S is with convex values.*

Proof. From Theorem 2.6, the set-valued map G is upper semicontinuous with convex and compact values. Then, from Theorem 2.4, $S(\mu) \neq \emptyset$ for all $\mu \in U$. Now, combining with Proposition 4.1 of [5], the solution map S is closed and thus, using the boundedness the map K, S is also upper semicontinuous with compact values.
Finally, S has convex values. Indeed, for all $\mu \in U$ we consider $\overline{x}_1, \overline{x}_2 \in S(\mu)$. From the sufficient condition in Theorem 2.7 one has

$$u(\overline{x}_1) = u(\overline{x}_2) = \max_{x \in K(\mu)} u(x)\,.$$

Since $K(\mu)$ is convex, $z = \lambda\overline{x}_1 + (1-\lambda)\overline{x}_2 \in K(\mu)$ for all $\lambda \in [0,1]$. Moreover, u being quasiconcave, it results $u(z) \geq u(\overline{x}_1) = u(\overline{x}_2)$, that is, $u(z) = \max_{x\in K(\mu)} u(x)$. Hence, for the semistrict quasiconcavity of u, from the necessary condition of Theorem 2.7, it results $z \in S(\mu)$. $\square$

3. A general economic equilibrium problem

We consider a marketplace consisting of a finite number l of completely homogeneous commodities, indexed by h and of two types of agents: n consumers (or householders), indexed by i, and m producers (or firms), indexed by j. We denote by $I = \{1, ..., n\}$, $J = \{1, ..., m\}$ and $H = \{1, ..., l\}$, respectively, sets of consumers, producers and commodities. Each consumer chooses a consumption plan $x_i = (x_i^1, ..., x_i^l)$ over the nonempty set $X_i \subseteq \mathbb{R}^l_+$ called consumption set, where x_i^h represents the quantity of commodity h consumed by i and $x = (x_1, ..., x_n)^T$ is the total consumption of market. As usual in the classical theory, sets X_i are convex, closed and bounded from below. Moreover, each consumer has an initial endowment $e_i = (e_i^1, ..., e_i^l) \in \mathbb{R}^l_+$ of different types of commodities, where e_i^h represents the amount of commodity h held by i. We assume that each consumer is endowed with each commodity h: $e_i^h > 0$ for all $h \in H$. Each producer uses goods, called inputs, to produce other goods, called outputs. We denote by y_j^h the quantity of commodity h produced by producer j and we remark that the quantity y_j^h can also assume negative values: the positive component y_j^h represents the output of commodity h, the negative component y_j^h represents the input of commodity h. The vector $y_j = (y_j^1, ..., y_j^l) \in \mathbb{R}^l$

and the matrix $y = (y_1, ..., y_m) \in \mathbb{R}^{l \times m}$ represent, respectively, the production plan of producer j and the total production of the market. The set of all feasible production vectors for producer j is called the production set Y_j. As is standard in economics, Y_j is assumed to be a closed, bounded and convex set of $\mathbb{R}^l$ with $0_{\mathbb{R}^l} \in Y_j$. The boundedness of Y_j means that producers cannot have an infinite production and $0_{\mathbb{R}^l} \in Y_j$ includes the possibility that there is no activity for producer j. We indicate with $Y = \prod_{j \in J} Y_j$ the total market production. The total production $\sum_{j \in J} y_j^h$ of commodity h is shared between consumers: each consumer i receives the given fraction $\sum_{j \in J} \theta_{ij} y_j^h$, determined by a system of fixed weights $\theta_{ij} \geq 0$ such that $\sum_{i \in I} \theta_{ij} = 1$ for all $j \in J$. Hence, each consumer i, relative to commodity h, has at command the quantity $e_i^h + \sum_{j \in J} \theta_{ij} y_j^h$. We note that, since y_j^h can assume negative values, it is possible that, the holdings of consumer i, $e_i^h + \sum_{j \in J} \theta_{ij} y_j^h$ is negative. To each commodity $h \in H$ a nonnegative price p^h is associated; we denote by $p = (p^1, ..., p^l) \in \mathbb{R}_+^l$ the price vector and we suppose that prices belong to the set

$$P = \left\{ p \in \mathbb{R}_+^l : \sum_{h \in H} p^h = 1 \right\}.$$

Given a price vector p, the inner product $\langle p, x_i \rangle$ represents the value of the consumption plan x_i, $\langle p, y_j \rangle$ represents the value of the production plan y_j and $\langle p, e_i \rangle + \max \left\{ 0, \langle p, \sum_{j \in J} \theta_{ij} y_j \rangle \right\}$ represents the wealth of consumer i at the going prices. We assume that consumers and producers are price takers, i.e., they behave as if they could not influence prices with their behavior. Thus, they live in a competitive production economy. Producers act to maximize their profit $\langle p, y_b \rangle$, while consumers act to maximize their preferences described by a utility function $u_i \colon X_i \to \mathbb{R}$, subject to the budget constraints set $M_i(y, p)$: at the going prices, the value of the consumption plan of consumer i cannot exceed his total wealth given by the value of his initial endowment and of the shared production. Let us set the set-valued map $M_i \colon Y \times P \rightrightarrows \mathbb{R}_+^l$ defined as

$$M_i(y, p) := \left\{ x_i \in X_i \ : \langle p, x_i \rangle \leq \langle p, e_i \rangle + \max \left\{ 0, \langle p, \sum_{j \in J} \theta_{ij} y_j \rangle \right\} \right\}. \tag{6}$$

We suppose that each utility function is continuous and semistrictly quasiconcave on X_i and the following *non-satiation* assumption holds: for any $x_i \in X_i$, there is an $x_i' \in X_i$ such that $u_i(x_i') > u_i(x_i)$. In summary, an economy with production is described by

$$\varepsilon = \Big((X_i, u_i, e_i), (\theta_{ij}), (Y_j) \Big).$$

We summarize below the assumptions assumed for an economy in this paper:

1. For all $j \in J$:
   (a) Y_j is a compact and convex subset of R^l;
   (b) $0 \in Y_j$;
2. For all $i \in I$:
   (a) X_i is a closed, convex subset of R^l_+ and bounded from below;
   (b) $e_i^h > 0$ for all $h \in H$;
   (c) u_i is semistrictly quasiconcave on X_i;
   (d) u_i is continuous on X_i;
   (e) for any $x_i \in X_i$, there is an $x_i' \in X_i$ such that $u_i(x_i') > u_i(x_i)$.

Concerning the aspects characterizing the model and the equilibrium definition we follow the pioneering works [2] and [15]. More detailed information from an economic viewpoint can be found in [30]. Remembering that an *allocation* in a production economy is an element $(x, y) \in X \times Y$, we can introduce the definition of equilibrium.

Definition 3.1. A vector $(\overline{x}, \overline{y}, \overline{p}) \in \prod_{i\in I} X_i \times Y \times P$ is an *allocation-price equilibrium for the production economy* ε if and only if

$$\text{for all } j \in J \quad \langle \overline{p}, \overline{y}_j\rangle = \max_{y_j\in Y_j} \langle \overline{p}, y_j\rangle \tag{7}$$

$$\text{for all } i \in I \quad u_i(\overline{x}_i) = \max_{M_i(\overline{y},\overline{p})} u_i(x_i) \tag{8}$$

$$\text{for all } h \in H \quad \sum_{i\in I}(\overline{x}_i^h - e_i^h) - \sum_{j\in J} \overline{y}_j^h \le 0, \quad \langle \overline{p}, \sum_{i\in I}(\overline{x}_i - e_i) - \sum_{j\in J} \overline{y}_j\rangle = 0. \tag{9}$$

Our aim, now, is to obtain a new formulation of the equilibrium by means of a suitable variational problem. We consider a suitable operator introduced in [4] to study a generalized Nash equilibrium problem by means of a variational approach. For all $i \in I$, let $G_i \colon X_i \rightrightarrows \mathbb{R}^l$ be the following set-valued map:

$$G_i(x_i) := \begin{cases} \overline{B}(0,1) & \text{if } x_i \in argmax_{X_i} u_i \\ conv\big(N^{>}(x_i) \cap S(0,1)\big) & \text{if } x_i \in X_i \setminus argmax_{X_i} u_i \end{cases}$$

We define the set

$$K(\overline{y}, \overline{p}) = \prod_{i\in I} K_i(\overline{y}, \overline{p}), \text{ where } K_i(\overline{y}, \overline{p}) = M_i(\overline{y}, \overline{p}) \cap \prod_{h\in H} \Big[0, \sum_{i\in I} e_i^h + M\Big]$$

with $M > 0$ and $\left|\sum_{j\in J} y_j^h\right| < M \;\; \forall h \in H$ (the existence of M is ensured by boundedness of each Y_j). We underline that the bounded set $K(y, p)$ is introduced particularly for reasons of mathematical convenience. We introduce the following GQVI:

$$\text{Find } (\overline{x},\overline{y},\overline{p}) \in K(\overline{y},\overline{p}) \times Y \times P \text{ s.t. } \exists\, g = \{g_i\}_{i\in I},\ g_i \in G_i(\overline{x}_i)\ \forall i \in I \text{ with}$$
$$\sum_{j\in J}\langle -\overline{p}, y_j - \overline{y}_j\rangle + \sum_{i\in I}\langle g_i, x_i - \overline{x}_i\rangle + \langle \sum_{i\in I}(e_i - \overline{x}_i) + \sum_{j\in J}\overline{y}_j, p - \overline{p}\rangle \geq 0$$
$$\forall (x,y,p) \in K(\overline{y},\overline{p}) \times Y \times P\,. \tag{10}$$

Remark 3.2. The vector $(\overline{x},\overline{y},\overline{p})$ is a solution to GQVI (3) if and only if

$$\text{for all } j \in J \qquad \langle -\overline{p}, y_j - \overline{y}_j\rangle \geq 0 \quad \forall y_j \in Y_j\,, \tag{11}$$
$$\text{for all } i \in I \qquad \langle g_i, x_i - \overline{x}_i\rangle \geq 0 \quad \forall x_i \in K_i(\overline{y},\overline{p}), \text{ and} \tag{12}$$
$$\langle \sum_{i\in I}(e_i - \overline{x}_i) + \sum_{j\in J}\overline{y}_j, p - \overline{p}\rangle \geq 0 \quad \forall p \in P\,. \tag{13}$$

This is easily seen by testing equation (3), respectively, with:

- $(x,\overline{y},\overline{p})$ for $x \in K(\overline{y},\overline{p})$ such that
$$x_i = \begin{cases} \overline{x}_k & if\ k \neq i, \\ x_i & \forall\, x_i \in K_i(\overline{y},\overline{p}), \end{cases}$$
- $(\overline{x},y,\overline{p})$ for all $y \in Y$ such that
$$y_j = \begin{cases} \overline{y}_k & if\ k \neq j, \\ y_j & \forall\, y_j \in Y_j, \end{cases}$$
- $(\overline{x},\overline{y},p)$ for all $p \in P$.

This remark is a crucial issue for our study, indeed it allows us to characterize the equilibrium and to achieve the existence of the solution to (3), establishing the existence of solutions to variational inequalities (11), (12) and (13).

Lemma 3.3. *For all $i \in I$, $p \in P$, and $y \in Y$ one has $(K_i(y,p))^{\perp} = \{0\}$.*

Proof. The thesis is achieved with similar techniques of Lemma 3.2 of [20]. □

Theorem 3.4. *Let Assumptions* 1. *and* 2. *be true. Then $(\overline{x},\overline{y},\overline{p})$ is a solution to GQVI* (3) *if and only if it is an allocation-price equilibrium for a production economy.*

Proof. To achieve our thesis we use Remark 3.2. Trivially the maximization problem (7) is equivalent to the VI (11) and from Theorem 2.7 and Lemma 3.3 the maximization problem

$$u_i(\overline{x}_i) = \max_{K_i(\overline{y},\overline{p})} u_i(x_i) \tag{14}$$

is equivalent to GVI (12).

Sufficient condition: Let $(\overline{x}, \overline{y}, \overline{p})$ be a solution to GQVI (3). We observe that, since $\overline{y}_j$ is a solution to maximization problem (7) and $0 \in Y_j$ and one has:

$$\langle \sum_{i\in I}(e_i - \overline{x}_i) + \sum_{j\in J} \overline{y}_j, \overline{p}\rangle \geq 0\,.$$

Replacing in (13) $p = (0, ..., 0, 1, 0, ..., 0)$, with 1 at the h-th position, one has:

$$\sum_{i\in I}(e_i^h - \overline{x}_i^h) + \sum_{j\in J} \overline{y}_j^h = \langle \sum_{i\in I}(e_i - \overline{x}_i) + \sum_{j\in J} \overline{y}_j, p - \overline{p}\rangle + \langle \sum_{i\in I}(e_i - \overline{x}_i) + \sum_{j\in J} \overline{y}_j, \overline{p}\rangle \geq 0\,.$$

Then, we get the first equilibrium condition of (9). Moreover, for all $h \in H$ and for all $i \in I$, one has $\overline{x}_i^h < \sum_{i\in I} e_i^h + M$, that is $\overline{x}_i \in ri_{X_i} \prod_{h\in H}\left[0, \sum_{i\in I} e_i^h + M\right]$:

$$\exists\delta > 0 :\ B(\overline{x}, \delta) \cap X_i \subseteq \prod_{h\in H}\left[0, \sum_{i\in I} e_i^h + M\right].$$

Now, we prove that there exists $g_i \in G_i(\overline{x}_i)$ such that

$$\langle g_i, x_i - \overline{x}_i\rangle \geq 0 \quad \forall x_i \in M_i(\overline{y}, \overline{p}). \tag{15}$$

We suppose that, for all $\widetilde{g}_i \in G_i(\overline{x}_i)$ there exists $x_i' \in M_i(\overline{y}, \overline{p})$ such that $\langle \widetilde{g}_i, x_i' - \overline{x}_i\rangle < 0$. Since $M_i(\overline{y}, \overline{p})$ is a convex set, we have $\widetilde{x}_i = \lambda x_i' + (1-\lambda)\overline{x}_i \in M_i(\overline{y}, \overline{p})$ for all $\lambda \in [0, 1]$ and $\langle \widetilde{g}_i, \widetilde{x}_i - \overline{x}_i\rangle < 0$. We can choose λ such that $\widetilde{x} \in B(\overline{x}, \delta)$:

$$\|\widetilde{x}_i - \overline{x}_i\| = \lambda\|x_i' - \overline{x}_i\| < \delta \quad \Leftrightarrow \quad \lambda < \frac{\delta}{\|x_i' - \overline{x}_i\|}\,.$$

Hence, for all $0 < \lambda < \frac{\delta}{\|x_i' - \overline{x}_i\|}$ one has that $\widetilde{x}_i \in K_i(\overline{y}, \overline{p})$ and $\langle \widetilde{g}_i, \widetilde{x}_i - \overline{x}_i\rangle < 0$ for all $\widetilde{g}_i \in G_i(\overline{x}_i)$; but this contradicts the fact that $\overline{x}_i$ is a solution to (12). Hence, it follows that $\overline{x}_i$ is a solution to GVI (15). From Theorem 2.7, one has:

$$u_i(\overline{x}_i) = \max_{x_i \in M_i(\overline{y}, \overline{p})} u_i(x_i)\,.$$

Finally, since u_i is semistrictly quasiconcave and Assumption 2.(e) holds, one has: $\langle \overline{p}, \sum_{i\in I}(\overline{x}_i - e_i) - \sum_{j\in J} \overline{y}_j\rangle = 0$ (see e.g. Section 1.4.2 of [2]).

Hence, we can conclude that $(\overline{x}, \overline{y}, \overline{p})$ is an allocation-price equilibrium for a production economy.

Necessary condition: Follows from Theorem 2.7. □

Lemma 3.5. *For all $i \in I$, the set-valued map $K_i\colon Y \times P \rightrightarrows X_i$ is closed, lower semicontinuous, and with convex values.*

Proof. Proved in Theorem 9 of [8]. □

Theorem 3.6. *Let Assumptions* 1. *and* 2. *hold. Then there exists a solution to GQVI* (3)*.*

Proof. Fix $p \in P$ and $j \in J$ and let us consider the parametric VI

$$\text{Find } \ \overline{y}_j \in Y_j \ \text{ s.t. } \ \langle -p, y_j - \overline{y}_j \rangle \geq 0 \quad \forall y_j \in Y_j. \tag{16}$$

We define the solution map $\mathcal{Y}_j \colon P \rightrightarrows Y_j$, such that for all $p \in P$

$$\mathcal{Y}_j(p) := \{\overline{y}_j \in Y_j : \ \overline{y}_j \text{ is a solution to (16)}\}.$$

One has that $\mathcal{Y}_j(p) \neq \emptyset$ for all $p \in P$, $\mathcal{Y}_j$ is upper semicontinuous, with compact and convex values. Now, we fix $i \in I$, $p \in P$ and $y \in Y$ and we consider the parametric GVI:

$$\text{Find } \overline{x}_i \in K_i(y,p) \text{ s.t. } \exists\, g_i \in G_i(\overline{x}_i) \text{ with } \langle g_i, x_i - \overline{x}_i \rangle \geq 0 \ \forall x_i \in K_i(y,p). \tag{17}$$

We define the solution map $\mathcal{X}_i \colon Y \times P \rightrightarrows \mathbb{R}^l$ such that for all $(y,p) \in Y \times P$

$$\mathcal{X}_i(y,p) := \{\overline{x}_i \in X_i : \ \overline{x}_i \text{ is a solution to (17)}\}.$$

From Proposition 2.8 and Theorem 2.6, one has that, $\mathcal{X}_i(y,p) \neq \emptyset$ for all $(y,p) \in Y \times P$ and $\mathcal{X}_i$ is upper semicontinuous, with compact and convex values. Finally, we define the set-valued map $F \colon P \rightrightarrows \mathbb{R}^l$ as

$$F(p) := \left\{ \sum_{i \in I} (e_i - \overline{x}_i) + \sum_{j \in J} \overline{y}_j : \ \overline{y} = (\overline{y}_j)_{j \in J}, \overline{y}_j \in \mathcal{Y}_j(p), \ \overline{x}_i \in \mathcal{X}_i(p, \overline{y}) \right\}.$$

The set-valued map F is the composition of a linear and continuous single valued map $f \colon \mathbb{R}^{n \times l} \times \mathbb{R}^{m \times l} \to \mathbb{R}^l$ defined as

$$f(x,y) = \sum_{i \in I} (e_i - x_i) + \sum_{j \in J} y_j$$

with the set-valued maps $\mathcal{X} \colon Y \times P \rightrightarrows \mathbb{R}^{n \times l}$, $\mathcal{X}(y,p) = \prod_{i \in I} \mathcal{X}_i(p,y)$ and $\mathcal{Y} \colon P \rightrightarrows \mathbb{R}^{m \times l}$, $\mathcal{Y}(p) = \prod_{j \in J} \mathcal{Y}_j(p)$. The set-valued maps $\mathcal{X}$ and $\mathcal{Y}$ are upper semicontinuous with compact and convex values since they are finite product of upper semicontinuous with compact and convex values set-valued maps $\mathcal{X}_i$, $i \in I$ and $\mathcal{Y}_j$, $j \in J$, respectively. Hence the set-valued map F is upper semicontinuous with compact and convex values.

Let us consider the GVI:

$$\text{Find } \ \overline{p} \in P \ \text{ s.t. } \ \langle \sum_{i \in I} (e_i - \overline{x}_i) + \sum_{j \in J} \overline{y}_j, p - \overline{p} \rangle \geq 0 \ \ \forall p \in P. \tag{18}$$

Taking into account of properties of set-valued map F and the compactness of the set P, from Theorem 2.3, the GVI (18) admits at least one solution $\overline{p}$.

Thus, let $(\overline{x}, \overline{y}, \overline{p})$ be such that $\overline{x} \in \mathcal{X}(\overline{p})$, $\overline{y} \in \mathcal{Y}(\overline{p})$ and $\overline{p}$ is a solution to (18), one has that $(\overline{x}, \overline{y}, \overline{p})$ is a solution to (3). $\square$

Hence, from Theorems 3.4 and 3.6, it follows that, under Assumptions 1. and 2. there exists at least one allocation-price equilibrium for the production economy ε.

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