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Radial Solutions and Free Boundary of the Elastic-Plastic Torsion Problem

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Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

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The paper is concerned with radial solutions to the elastic-plastic torsion problem, assuming the free term to belong to $L^p(\Omega)$. In particular, we obtain a necessary and sufficient condition in order that the plastic region exists and we characterize the free boundary. Moreover, we find the explicit radial solution $u \in W^{2,p}(\Omega)$ and the Lagrange multiplier $\overline{\mu} \in L^p(\Omega)$.

Keywords: Elastic-plastic torsion, radial solutions, Lagrange multipliers.

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1. Introduction

In this paper we are concerned with radial solutions to the elastic-plastic torsion problem.

According to Von Mises (see [15]): *the elastic-plastic torsion problem of a cylindrical bar with cross section Ω is to find a function $u(x)$ which vanishes on the boundary $\partial\Omega$ and, together with its first derivatives, is continuous on Ω; nowhere on Ω the gradient of u must have an absolute value (modulus) less than or equal to a given positive constant τ; whenever in Ω the strict inequality holds, the function u must satisfy the differential equation $\Delta u = -2\mu\theta$, where*

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the positive constants μ and θ denote the shearing modulus and the angle of twist per unit length, respectively.

This problem is a free boundary one and a suitable tool for studying this kind of problems is the variational inequality theory. To this end, let us consider the following variational inequality:

$$
\begin{aligned}
&\text{Find } \ u \in K = \left\{ v \in H_0^{1,\infty}(\Omega) : |Dv| = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \leq 1 \text{ a.e. on } \Omega \right\}, \\
&\text{such that } \quad \int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \geq \int_\Omega F(v-u)dx \quad \forall v \in K,
\end{aligned}
\tag{1}
$$

with $\Omega \subset \mathbb{R}^n$ open bounded convex set with Lipschitz boundary $\partial\Omega$, $F \in L^p(\Omega)$, $p > 1$. As it is well known, the variational inequality (1) admits a unique solution $u \in W^{2,p}(\Omega) \cap K$ (see [2], [3]).

Moreover in [8, 9] the authors prove that a Lagrange multiplier, as a positive Radon measure, always exists for the variational inequality (1), namely, if u is the solution to variational inequality (1), there exists $\overline{\mu} \in (L^\infty(\Omega))^*$, such that

$$
\begin{cases}
\langle \overline{\mu}, y \rangle \geq 0 \quad \forall y \in L^\infty(\Omega), \ y \geq 0 \quad \text{a.e. in } \Omega; \\
\langle \overline{\mu}, \left(\sum_{i=1}^{n} \left(\dfrac{\partial u}{\partial x_i} \right)^2 - 1 \right) \rangle = 0; \\
\displaystyle\int_\Omega \left\{ \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} - F\varphi \right\} dx = \langle \overline{\mu}, -2 \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \rangle \quad \forall \varphi \in H_0^{1,\infty}(\Omega).
\end{cases}
\tag{2}
$$

Conversely, if $u \in K$ and there exists μ satisfying (2), it is easily proved that u is the solution to (1).

In virtue of this equivalence the variational inequality (1) is the elastic-plastic torsion problem formulated by Von Mises.

In the papers [4, 5, 6] the authors introduced a necessary and sufficient condition, called Assumption S (from Separation), for strong duality. Assumption S has revealed to be very effective in applications, for example in the study of the existence of the Lagrange multipliers in infinite dimensional case, when the classical theory cannot be applied, since it requires in general that the interior of the ordering cone, which defines the sign constraints, is nonempty.

If Assumption S is verified, the previous existence result of a Lagrange multiplier may be regularized (see [7]), namely, if u is the solution of variational inequality (1), then there exists $\overline{\mu} \in L^p(\Omega)$, such that

$$\begin{cases} \overline{\mu} \geq 0 \quad \text{a.e. in } \Omega; \\ \overline{\mu} \left(1 - \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right) = 0 \quad \text{a.e. in } \Omega; \\ -\Delta u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\overline{\mu} \frac{\partial u}{\partial x_i} \right) = F \quad \text{in the sense of distributions.} \end{cases} \tag{3}$$

In this paper we are aiming at the investigation of radial solutions of (3) and finding the free boundary.

From the Von Mises formulation it follows that the cross section Ω is divided into two regions: an elastic region $E = \{x \in \Omega : |Du(x)| < 1\}$ and a plastic region $P = \{x \in \Omega : |Du(x)| = 1\}$.

Studying the radial solutions to (3), assuming $F \in L^p(\Omega)$, $p > n$, of radial type, namely $F(x) = f(|x|) = f(\rho)$, with $|x| = \rho$, we prove that an elastic region $[0, \overline{\rho})$ and a plastic region $(\overline{\rho}, 1]$ are determined, under the assumption

$$\text{there exists } \; \overline{\rho} \in (0,1) \; : \; \frac{\int_{C_{\overline{\rho}}(0)} F(x)dx}{|\partial C_{\overline{\rho}}(0)|} = 1, \tag{4}$$

where $C_{\overline{\rho}}(0)$ is the closed ball of radius $\overline{\rho}$ centered at the origin. Moreover, we find the explicit radial solution of the elastic-plastic problem and the free boundary $\partial C_{\overline{\rho}}(0)$.

The problem is studied for $n = 2$; the case $n > 2$ will be studied in following papers.

In the literature, in the planar case the existence and the properties of a smooth solution of the elastic-plastic torsion problem have been studied by Ting ([12], [13], [14]), whereas multidimensional case has been studied by Brezis in [1], who proved the existence of a Lagrange multiplier for variational inequality (1), assuming $F = const > 0$. In this case, the solution to elastic-plastic torsion problem coincides with the solution to obstacle problem and is nonnegative.

In [11] the author considers the same problem as Brezis, but in the case of radial symmetry, and finds the formal explicit solution of the elastic-plastic problem. This result is generalized in the present paper to the case $F \in L^p(\Omega)$, $p > n$.

In [8, 9, 10] the result by Brezis has been improved to linear operators and nonlinear monotone operators.

The paper is organized in the following way: in Section 2 we state the assumptions and the main results of the paper, in Section 3 we determine, under condition (4), an elastic region $[0, \overline{\rho}]$ and a plastic region $[\overline{\rho}, 1]$ and we verify that the solutions $u(x)$ of (3) belong to $W^{2,p}(\Omega)$. Moreover, we show that condition (4) is a necessary and sufficient condition in order that a plastic region exists. In Section 4 we study the cases in which condition (4) is not satisfied and one has only elastic torsion. In Section 5 we provide some examples in the different situations and the explicit solutions are determined and, finally, in Section 6 we summarize our results and future work.

2. Assumptions and main results

Now, let us assume that Ω is the ball of $\mathbb{R}^n$ of radius 1 centered at the origin, and $F \in L^p(\Omega)$, $p > n$, is of radial type, namely $F(x) = f(|x|) = f(\rho)$, with $|x| = \rho$. Under the following assumption:

$$\text{there exists } \overline{\rho} \in (0,1) \text{ such that } \frac{\int_{C_{\overline{\rho}}(0)} F(x)dx}{|\partial C_{\overline{\rho}}(0)|} = 1, \tag{5}$$

where $C_{\overline{\rho}}(0)$ is the closed ball of radius $\overline{\rho}$ centered at the origin, namely

$$\int_0^{\overline{\rho}} \rho f(\rho) d\rho = \overline{\rho}, \tag{6}$$

we search for solutions to (3) such that $u(x) \in W^{2,p}(\Omega)$ and $\overline{\mu}(x) \in L^p(\Omega)$ are of radial type, namely $\overline{\mu}(x) = \mu(|x|) = \mu(\rho)$, $u(x) = \varphi(|x|) = \varphi(\rho)$.

In this case, since $u(x) = \varphi(\rho)$, $\frac{\partial u}{\partial x_i} = \varphi'(\rho)\frac{x_i}{\rho}$, $\Delta u = \varphi'' + \frac{n-1}{\rho}\varphi'(\rho)$, and $|Du| = \varphi'(\rho)$, bearing in mind that $u \in K$, conditions (3) become

$$\begin{cases} |\varphi'(\rho)| = 1;\ \mu(\rho) \geq 0 \quad \text{a.e. in } [0,1]; \\ \mu(\rho)\left(1 - |\varphi'(\rho)|\right) = 0 \quad \text{a.e. in } [0,1]; \\ -\varphi''(\rho) - \dfrac{n-1}{\rho}\varphi'(\rho) - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\mu \frac{\partial u}{\partial x_i}\right) = f(\rho). \end{cases} \tag{7}$$

In what follows we assume that

$$\rho f(\rho) \geq 0 \text{ is a nondecreasing function in } [0,1]. \tag{8}$$

We are able to prove the following result.

Theorem 2.1. *Under the conditions* (4) *and* (8)*, the region* $[0, \overline{\rho}\,]$ *is an elastic region and the region* $[\overline{\rho}, 1]$ *is a plastic region. Moreover, the solution* φ *to* (7) *is*

$$\varphi(\rho) = \begin{cases} 1 - \overline{\rho} + \displaystyle\int_\rho^{\overline{\rho}} \frac{1}{t} \int_0^t \sigma f(\sigma) d\sigma dt & \rho \in [0, \overline{\rho}\,] \\ 1 - \rho & \rho \in (\overline{\rho}, 1], \end{cases}$$

and we have $\varphi(\rho) \in W^{2,p}([0,1])$ *and* $\mu(\rho) \in L^p([0,1])$.

If equation (6) does not admit any solution $\overline{\rho} \in (0,1)$, namely $\forall \rho \in (0,1)$

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma < 1 \quad \text{or} \quad \frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma > 1,$$

the plastic region does not exist. In Section 4 we show that the case

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma > 1 \quad \forall \rho \in (0,1)$$

is not admissible, since it implies $\varphi'(\rho) < -1$ for all $\rho \in (0,1)$.

Then, we are able to prove the following result.

Theorem 2.2. *Under condition* (8)*, if*

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma < 1 \quad \forall \rho \in (0,1),$$

then, $[0,1]$ *is an elastic region. Moreover, the solution* φ *to* (7) *is*

$$\varphi(\rho) = \int_\rho^1 \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt \quad \forall \rho \in [0,1],$$

and we have $\varphi(\rho) \in W^{2,p}([0,1])$.

3. Proof of Theorem 2.1

3.1. Elastic region $[0,\overline{\rho}]$, plastic region $[\overline{\rho},1]$

In order to investigate the nature of the torsion and when the transition happens, let us start by studying if it is possible to have an elastic region $[0,\overline{\rho}\,]$ and a plastic region $[\overline{\rho},1]$, namely, in $[0,\overline{\rho}\,]$, $-1 < \varphi'(\rho) < 1$ and $\mu(\rho) = 0$, whereas, in $[\overline{\rho},1]$, $|\varphi'(\rho)| = 1$.

Then, in $[\overline{\rho},1]$, we may have, $\varphi(\rho) = \rho + c$ or $\varphi(\rho) = -\rho + c$. We search for solutions $\varphi(\rho) \geq 0$ in $[0,1]$ (indeed, when $f = const. > 0$, $\varphi(\rho)$ is nonnegative). Taking into account that $\varphi(1) = 0$, then, in $[\overline{\rho},1]$, $\varphi(\rho) = 1 - \rho$. Furthermore, since we are searching for $\varphi(\rho)$ such that $\varphi'(\rho)$ is continuous, then $\varphi'(\overline{\rho}) = -1$.

In $[0,\overline{\rho}\,]$ $\mu(\rho) = 0$, then, from (7) we get

$$-\varphi''(\rho) - \frac{n-1}{\rho}\varphi'(\rho) = f(\rho),$$

that is, for $n = 2$,

$$-\varphi''(\rho) - \frac{\varphi'(\rho)}{\rho} = f(\rho), \quad \text{and} \quad D(\rho\varphi'(\rho)) = -\rho f(\rho),$$

and by integration between ρ and $\overline{\rho}$,

$$\varphi'(\rho) = \frac{1}{\rho}\int_\rho^{\overline{\rho}} \sigma f(\sigma)d\sigma - \frac{\overline{\rho}}{\rho} \quad \forall \rho \in (0,\overline{\rho}\,]. \tag{9}$$

Let us observe that, from the assumption $F(x) \in L^p(\Omega)$, $p > 2$, namely

$$\int_\Omega |F(x)|^p dx = 2\pi \int_0^1 [f(\rho)]^p \ \rho \ d\rho < \infty,$$

it follows that $\rho f(\rho) \leq 1$ in a right neighborhood of 0 and

$$\lim_{\rho \to 0^+} \rho f(\rho) = 0. \tag{10}$$

In consequence we obtain

$$\lim_{\rho \to 0^+} \varphi'(\rho) = \lim_{\rho \to 0^+} \frac{\int_\rho^{\overline{\rho}} \sigma f(\sigma) d\sigma - \overline{\rho}}{\rho}. \tag{11}$$

Let us also show that, from condition (6), the limit in (11) is equal to 0 and $\varphi'(\rho) \in L^p(0, \overline{\rho})$. Indeed, we have, in virtue of (6),

$$\begin{aligned} \varphi'(\rho) &= \frac{\int_\rho^{\overline{\rho}} \sigma f(\sigma) d\sigma - \overline{\rho}}{\rho} = \frac{\int_0^{\overline{\rho}} \sigma f(\sigma) d\sigma - \int_0^\rho \sigma f(\sigma) d\sigma - \overline{\rho}}{\rho} \\ &= -\frac{\int_0^\rho \sigma f(\sigma) d\sigma}{\rho} \ . \end{aligned} \tag{12}$$

Then, since $\varphi'(\rho)$ is continuous in $(0, \overline{\rho}]$, $\varphi'(\rho) \leq 0$ in $(0, \overline{\rho}]$, and from (8) and (10) we conclude

$$\lim_{\rho \to 0^+} -\frac{\int_0^\rho \sigma f(\sigma) d\sigma}{\rho} = -\inf_{[0,\overline{\rho}]} \sigma f(\sigma) = 0. \tag{13}$$

Since $[0, \overline{\rho})$ is an elastic region, we have to verify that

$$\varphi'(\rho) = -\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma \geq -1 \quad \forall \rho \in (0, \overline{\rho}], \tag{14}$$

or equivalently

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma \leq 1 \quad \forall \rho \in (0, \overline{\rho}]. \tag{15}$$

The assumption (8) implies that the function $\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma$ is also a nondecreasing function in $(0, \overline{\rho}]$. In fact, it results

$$D \left(\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma \right) = \frac{1}{\rho^2} \left(\rho^2 f(\rho) - \int_0^\rho \sigma f(\sigma) d\sigma \right),$$

and, since $\rho f(\rho)$ is a nondecreasing function in $(0, 1)$, we get

$$\int_0^\rho \sigma f(\sigma) d\sigma \leq \int_0^\rho \rho f(\rho) d\sigma = \rho^2 f(\rho),$$

and therefore

$$D\left(\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma\right) \geq 0\,.$$

Then, in virtue of (4),

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma \leq \frac{1}{\overline{\rho}}\int_0^{\overline{\rho}} \sigma f(\sigma)d\sigma = 1 \quad \forall \rho \in (0,\overline{\rho}\,],$$

and (15) is achieved. From (12) we get $\forall \rho \in (0,\overline{\rho}\,]$

$$\varphi(\rho) = -\int_0^\rho \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt + c. \tag{16}$$

Since

$$\varphi(\overline{\rho}) = 1-\overline{\rho} = -\int_0^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt + c, \tag{17}$$

we have

$$c = 1-\overline{\rho} - \int_{\overline{\rho}}^0 \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt$$

and hence in $(0,\overline{\rho}\,]$

$$\varphi(\rho) = 1-\overline{\rho} + \int_\rho^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt. \tag{18}$$

Obviously, $\varphi(\rho) \geq 0$ in $(0,\overline{\rho}\,]$, and φ is nonincreasing in $(0,\overline{\rho}\,]$. We now study the continuity of φ at $\rho = 0$. Since

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma = \lim_{t\to 0^+} t f(t) = 0,$$

the function

$$\varphi'(t) = -\frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma$$

is continuous in $[0,\overline{\rho}\,]$, assuming that $\varphi'(0) = 0$. Then, we obtain

$$\lim_{\rho\to 0^+}\int_\rho^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt = \int_0^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt < +\infty,$$

and $\varphi(\rho)$ is continuous in $[0,\overline{\rho}\,]$. Moreover, we have already seen that $\varphi'(\rho)$ is continuous in $[0,\overline{\rho}\,]$. Finally,

$$\varphi''(\rho) = \frac{1}{\rho^2}\int_0^\rho \sigma f(\sigma)d\sigma - f(\rho) \quad \text{a.e. in } [0,\overline{\rho}\,].$$

Let us observe that, in virtue of (8),

$$\varphi''(\rho) = \frac{1}{\rho^2}\int_0^\rho \sigma f(\sigma)d\sigma - f(\rho) \leq \frac{1}{\rho^2}\int_0^\rho \rho f(\rho)d\sigma - f(\rho) = 0 \quad \text{a.e. in } [0,\overline{\rho}\,].$$

Let us now investigate the plastic region $[\overline{\rho},1]$. We have already observed that, in $[\overline{\rho},1]$, $\varphi(\rho)=1-\rho$, that is $\varphi(\rho)$ is the distance from the boundary 1.

In the interval $[\overline{\rho},1]$, since $\varphi'(\rho)=-1$, $\varphi''(\rho)=0$, $\frac{\partial u}{\partial x_i}=\varphi'(\rho)\frac{x_i}{\rho}$, we get

$$\frac{n-1}{\rho}+\frac{1}{\rho^{n-1}}D(\rho^{n-1}\mu(\rho))=f(\rho)\quad \text{a.e. in } [\overline{\rho},1]$$

If we assume $n=2$, we get

$$\frac{1}{\rho}+\frac{1}{\rho}D(\rho\mu(\rho))=f(\rho)\quad \text{and}\quad D(\rho\mu(\rho))=\rho f(\rho)-1\quad \text{a.e. in } [\overline{\rho},1].$$

Integrating, since $\mu(\overline{\rho})=0$, we get

$$\mu(\rho)=\frac{1}{\rho}\int_{\overline{\rho}}^{\rho}(\sigma f(\sigma)-1)\,d\sigma\quad \text{a.e. in } [\overline{\rho},1],$$

and hence

$$\mu(\rho)=\frac{1}{\rho}\int_{\overline{\rho}}^{\rho}\sigma f(\sigma)\,d\sigma-1+\frac{\overline{\rho}}{\rho}\quad \text{a.e. in } [\overline{\rho},1]. \tag{19}$$

In virtue of (4), we get from (19)

$$\mu(\rho)=\frac{1}{\rho}\int_{0}^{\rho}\sigma f(\sigma)\,d\sigma-1\quad \text{a.e. in } [\overline{\rho},1]. \tag{20}$$

In order that $\mu(\rho)\geq 0$, we must have

$$\frac{1}{\rho}\int_{0}^{\rho}\sigma f(\sigma)\,d\sigma\geq 1. \tag{21}$$

Repeating the same arguments as before, we may prove that the function $\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)\,d\sigma$ is nondecreasing in $[\,\overline{\rho}\,,1]$, that is, in virtue of (4),

$$\frac{1}{\rho}\int_{0}^{\rho}\sigma f(\sigma)d\sigma\geq\frac{1}{\overline{\rho}}\int_{0}^{\overline{\rho}}\sigma f(\sigma)\,d\sigma=1\quad \forall\rho\in[\,\overline{\rho}\,,1],$$

namely the desidered condition (21).

Remark 3.1. We have already proved that $\varphi(\rho)$ and $\varphi'(\rho)$ are continuous in $[0,1]$, hence also $u(x)$ and $Du(x)$ are continuous in $\overline{\Omega}$.

Concerning the second derivatives, we have

$$\begin{aligned}\int_\Omega|D^2u|^p dx&=\int_0^1\left|\sum_{i,j=1}^{2}\varphi''(\rho)\frac{x_ix_j}{\rho^2}+\sum_{i,j=1}^{2}\varphi'(\rho)\ \frac{\delta_{ij}\rho^2-x_ix_j}{\rho^3}\right|^p\rho\,d\rho\\&\leq c\left[\int_0^{\overline{\rho}}\rho\left[|\varphi''(\rho)|^p+\left|\frac{\varphi'(\rho)}{\rho}\right|^p\right]d\rho+\int_{\overline{\rho}}^1\rho^{1-p}d\rho\right],\end{aligned} \tag{22}$$

since $\sum_{i,j=1}^2 x_ix_j\leq 2\rho^2$ and $\sum_{i,j=1}^2\delta_{ij}\rho^2-x_ix_j\leq\rho^2$.

Analyzing the first two terms, in virtue of assumption (8) and since $F \in L^p(\Omega)$,

$$\begin{aligned}\int_0^{\overline{\rho}} \rho\,|\varphi''(\rho)|^p\,d\rho &= \int_0^{\overline{\rho}} \rho \left|\frac{1}{\rho^2}\int_0^{\rho} \sigma f(\sigma)d\sigma - f(\rho)\right|^p d\rho \\ &\le c\int_0^{\overline{\rho}} \rho \left|\frac{1}{\rho^2}\int_0^{\rho} \sigma f(\sigma)d\sigma\right|^p d\rho + c\int_0^{\overline{\rho}} \rho|f(\rho)|^p d\rho \\ &\le c\int_0^{\overline{\rho}} \rho \left|\frac{1}{\rho^2}\int_0^{\rho} \rho f(\rho)d\sigma\right|^p d\rho + c\int_0^{\overline{\rho}} \rho|f(\rho)|^p d\rho \\ &= 2c\int_0^{\overline{\rho}} \rho|f(\rho)|^p d\rho \;<\; +\infty,\end{aligned}$$

and, analogously,

$$\begin{aligned}\int_0^{\overline{\rho}} \rho\left|\frac{\varphi'(\rho)}{\rho}\right|^p d\rho &= \int_0^{\overline{\rho}} \rho\left|\frac{\int_0^{\rho}\sigma f(\sigma)d\sigma}{\rho^2}\right|^p d\rho \le \int_0^{\overline{\rho}} \rho\left|\frac{\int_0^{\rho}\rho f(\rho)d\sigma}{\rho^2}\right|^p d\rho \\ &= \int_0^{\overline{\rho}} \rho|f(\rho)|^p d\rho \;<\; +\infty.\end{aligned}$$

This implies $u \in W^{2,p}(\Omega)$.

3.2. Plastic region $[0,\overline{\rho}]$

Let us investigate if it is possible to have a plastic region $[0,\overline{\rho}]$. As observed in Section 3.1, we have $\varphi(\rho) = 1-\rho$ in the plastic region. Then, from (7) we get

$$\frac{n-1}{\rho} + \frac{1}{\rho^{n-1}}D(\rho^{n-1}\mu(\rho)) = f(\rho) \quad \text{a.e. in } (0,\overline{\rho}].$$

If we assume $n=2$, we get

$$\frac{1}{\rho} + \frac{1}{\rho}D(\rho\mu(\rho)) = f(\rho) \quad \text{a.e. in } (0,\overline{\rho}]$$

and

$$D(\rho\mu(\rho)) = \rho f(\rho) - 1 \quad \text{a.e. in } (0,\overline{\rho}]. \tag{23}$$

Integrating (23) in $[\rho,\overline{\rho}]$, $\rho \in [0,\overline{\rho}]$, and assuming $\mu(\overline{\rho}) = 0$, since $\mu(\rho) = 0$ in $[\overline{\rho}, 1]$, we have

$$\begin{aligned}\mu(\rho) &= -\frac{1}{\rho}\int_{\rho}^{\overline{\rho}} (\sigma f(\sigma) - 1)d\sigma = \\ &= -\frac{1}{\rho}\left[\int_0^{\overline{\rho}} (\sigma f(\sigma)-1)d\sigma - \int_0^{\rho}(\sigma f(\sigma)-1)d\sigma\right].\end{aligned} \tag{24}$$

$\mu(\rho)$ does not belong to $L^p(0,\overline{\rho})$, because the first term

$$\frac{1}{\rho}\int_0^{\overline{\rho}}(\sigma f(\sigma)-1)\,d\sigma$$

does not belong to $L^p(0,\overline{\rho})$, while for the second one, in virtue of (8), we have

$$\int_0^{\overline{\rho}}\frac{1}{\rho^p}\left(\int_0^{\rho}(1-\sigma f(\sigma))d\sigma\right)^p d\rho<\left(1-\inf_{(0,\overline{\rho})}\sigma f(\sigma)\right)^p\,\overline{\rho}=\overline{\rho}.$$

In conclusion, a plastic region of type $[0,\overline{\rho}\,]$ cannot exist.

Remark 3.2. If we assume that condition (4) holds, then we obtain

$$\mu(\rho)=-\frac{1}{\rho}\int_0^{\overline{\rho}}(\sigma f(\sigma)-1)\,d\sigma\leq 0.$$

In particular, the conditions of (7) are not verified and $[0,\overline{\rho}\,]$ cannot be a plastic region.

4. Proof of Theorem 2.2

As a consequence of the investigation of the free boundary in Section 3, we obtain that the plastic region does not exist, if equation (4) does not admit any solution $\overline{\rho}\in(0,1]$, namely $\forall\rho\in(0,1]$

$$\frac{1}{\rho}\int_0^{\rho}\sigma f(\sigma)\,d\sigma<1\quad\text{or}\quad\frac{1}{\rho}\int_0^{\rho}\sigma f(\sigma)\,d\sigma>1. \tag{25}$$

Hence, one has only the elastic torsion, that is the solution $\varphi(\rho)$ to (7) verifies, $\forall\,0<\rho\leq 1$,

$$-\varphi''(\rho)-\frac{n-1}{\rho}\varphi'(\rho)=f(\rho);$$

in particular, for $n=2$,

$$-\varphi''(\rho)-\frac{\varphi'(\rho)}{\rho}=f(\rho)\quad\text{and}\quad D(\rho\varphi'(\rho))=-\rho f(\rho).$$

By integration between ρ and 1, we get

$$\varphi'(\rho)=\frac{1}{\rho}\int_\rho^1\sigma f(\sigma)\,d\sigma+\frac{\varphi'(1)}{\rho}\quad\forall\rho\in(0,1]. \tag{26}$$

This implies

$$\lim_{\rho\to 0^+}\varphi'(\rho)=\lim_{\rho\to 0^+}\frac{\int_\rho^1\sigma f(\sigma)d\sigma+\varphi'(1)}{\rho}. \tag{27}$$

Let us show that, if we require

$$\lim_{\rho\to 0^+}\int_\rho^1 \sigma f(\sigma)d\sigma = \int_0^1 \sigma f(\sigma)d\sigma = -\varphi'(1)\,, \tag{28}$$

the limit in (27) is equal to 0 and $\varphi'(\rho) \in L^p([0,1])$. Indeed, we have, in virtue of (28),

$$\begin{aligned}\lim_{\rho\to 0^+}\varphi'(\rho) &= \lim_{\rho\to 0^+}\frac{\int_0^1 \sigma f(\sigma)d\sigma - \int_0^\rho \sigma f(\sigma)d\sigma + \varphi'(1)}{\rho}\\ &= \lim_{\rho\to 0^+}\frac{-\int_0^\rho \sigma f(\sigma)d\sigma}{\rho} = 0.\end{aligned}$$

Then

$$\varphi'(\rho) = -\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)\,d\sigma \quad \forall\rho\in(0,1] \tag{29}$$

is continuous in $[0,1]$, assuming $\varphi'(0) = 0$. Moreover, it is always nonnegative.

Let us now study the two cases in assumption (25). If we suppose that

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)\,d\sigma < 1 \quad \forall\rho\in(0,1], \tag{30}$$

this implies

$$\varphi'(\rho) > -1 \quad \forall\rho\in(0,1]. \tag{31}$$

Then, from (29) and (31), we get

$$|\varphi'(\rho)| < 1 \quad \forall\rho\in[0,1],$$

that is, $[0,1]$ is an elastic region. The other case

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma > 1 \quad \forall\rho\in(0,1] \tag{32}$$

is not possible, since it implies

$$\varphi'(\rho) < -1 \quad \forall\rho\in[0,1]. \tag{33}$$

Therefore, from now on, we suppose that (30) holds. From (29) and since $\varphi(1) = 0$, we get $\forall\rho\in(0,1]$

$$\varphi(\rho) = \int_\rho^1 \frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma\,dt, \tag{34}$$

$\varphi(\rho) \geq 0$ in $(0,1]$ and φ is nonincreasing in $(0,1]$. We have to study the continuity of φ at $\rho = 0$. Since

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma = \lim_{t\to 0^+} t f(t) = 0,$$

the function

$$\varphi'(t) = -\frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma$$

is continuous in $[0,1]$, assuming that $\varphi'(0) = 0$. Then

$$\lim_{\rho\to 0^+} \int_\rho^1 \frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma\,dt = \int_0^1 \frac{1}{t}\int_0^t \sigma f(\sigma)\,d\sigma\,dt < +\infty$$

and $\varphi(\rho)$ is continuous in $[0,1]$. Finally,

$$\varphi''(\rho) = \frac{1}{\rho^2}\int_0^\rho \sigma f(\sigma)\,d\sigma - f(\rho) \quad \text{a.e. in } [0,1].$$

Let us observe that, in virtue of (8),

$$\varphi''(\rho) = \frac{1}{\rho^2}\int_0^\rho \sigma f(\sigma)\,d\sigma - f(\rho) \leq \frac{1}{\rho^2}\int_0^\rho \rho f(\rho)\,d\sigma - f(\rho) = 0 \quad \text{a.e. in } [0,1].$$

As a conclusion, if condition (30) is verified, then the complete interval [0,1] is an elastic region. In an analogous way as in Remark 3.1 we may also prove that $u(x) = \varphi(\rho) \in W^{2,p}(\Omega)$.

5. Examples

Let us consider a first example, namely $f = const = k > 0$. In this case we obtain the same results as in [11], p. 15.

Obviously, the function $\rho f(\rho) = k\rho$ is a nondecreasing function in $[0,1]$, so that condition (8) is verified.

If we consider a first case, $f = k > 2$, a plastic region exists, since

$$\lim_{\rho\to 0^+} \int_\rho^{\overline{\rho}} k\sigma\,d\sigma = \lim_{\rho\to 0^+} \frac{k}{2}\,(\overline{\rho}^2 - \rho^2) = \frac{k}{2}\,\overline{\rho}^2.$$

In particular, $\overline{\rho} = \frac{2}{k} < 1$ is the solution to (6).

Then, by (18) we get the continuous function

$$\varphi(\rho) = \begin{cases} 1 - \dfrac{2}{k} + \displaystyle\int_\rho^{\frac{2}{k}} \frac{1}{t}\int_0^t k\sigma\,d\sigma\,dt = \frac{k}{4}\left[(1-\rho^2) - (1-\frac{2}{k})^2\right] & \text{in } E = [0, \frac{2}{k}) \\ 1 - \rho & \text{in } P = [\frac{2}{k}, 1]. \end{cases}$$

It is easily seen that $u, Du \in L^p(\Omega)$ and

$$\int_\Omega |D^2u|^p \, dx = \int_0^1 \left| \sum_{i,j=1}^{2} \varphi''(\rho)\frac{x_i x_j}{\rho^2} + \sum_{i,j=1}^{2} \varphi'(\rho) \, \frac{\delta_{ij}\rho^2 - x_i x_j}{\rho^3} \right|^p \rho \, d\rho$$
$$\leq c(p) \left[\int_0^{\overline{\rho}} k^p \rho \, d\rho + \int_{\overline{\rho}}^1 \rho^{1-p} d\rho \right] < \infty \, .$$

Finally, the Lagrange multiplier $\mu(\rho)$ exists and belongs to $L^p([0,1])$:

$$\mu(\rho) = \begin{cases} \dfrac{1}{\rho} \displaystyle\int_0^\rho \sigma f(\sigma) \, d\sigma \; - \; 1 = k \, \dfrac{\rho}{2} \; - \; 1 \geq 0 & \text{in } P = [\frac{2}{k}, 1] \\ 0 & \text{in } E = [0, \frac{2}{k}). \end{cases}$$

If we consider the other case $f = const = k$, $0 < k \leq 2$, the plastic region does not exist, since $\overline{\rho} = \frac{2}{k} \geq 1$ is the solution to (6).

Therefore, the torsion is all elastic. In fact, from (29) we get for all $\rho \in [0,1]$

$$\varphi'(\rho) = -\frac{k}{2} \, \rho, \quad \text{hence} \quad -1 \leq \varphi'(\rho) \leq 1,$$

and (28) is verified. Then, by (34) we get the continuous function

$$\varphi(\rho) = \frac{k}{4}(1 - \rho^2). \tag{35}$$

This function $\varphi(\rho)$ and $\mu = 0$ verify the conditions of (7) in $[0,1]$.

Moreover $u \in W^{2,p}(\Omega)$. In fact, it is easily seen that u, $Du \in L^p(\Omega)$ and

$$\int_\Omega |D^2u|^p \, dx = \int_0^1 \left| \sum_{i,j=1}^{2} \varphi''(\rho)\frac{x_i x_j}{\rho^2} + \sum_{i,j=1}^{2} \varphi'(\rho) \, \frac{\delta_{ij}\rho^2 - x_i x_j}{\rho^3} \right|^p \rho \, d\rho$$
$$\leq c \int_0^1 \rho \, d\rho < \infty.$$

Let us now consider problem (7) with $f(\rho) = \dfrac{k}{\rho^\alpha}$, $0 < \alpha < 1$.

The condition $\alpha < 1$ ensures that $F(x) \in L^p(\Omega)$, $2 = n < p < \frac{2}{\alpha}$. Moreover, condition (8) is verified.

If we consider the case $k > 2 - \alpha$, the plastic region exists, since

$$\lim_{\rho \to 0^+} \int_\rho^{\overline{\rho}} \sigma f(\sigma) \, d\sigma = \lim_{\rho \to 0^+} \int_\rho^{\overline{\rho}} k \, \sigma^{1-\alpha} \, d\sigma = \frac{k}{2 - \alpha} \, \overline{\rho}^{2-\alpha}.$$

In particular, $\overline{\rho} = \left(\frac{2-\alpha}{k}\right)^{\frac{1}{1-\alpha}} < 1$ is the solution to (6). Then, by (18) we get the continuous function

$$\varphi(\rho) = \begin{cases} 1 - \left(\frac{2-\alpha}{k}\right)^{\frac{1}{1-\alpha}} + \frac{k}{(2-\alpha)^2}\left(\frac{2-\alpha}{k}\right)^{\frac{2-\alpha}{1-\alpha}} - \frac{k}{(2-\alpha)^2}\,\rho^{2-\alpha} & \text{in } E = [0,\overline{\rho}) \\ 1-\rho & \text{in } P = [\overline{\rho},1]. \end{cases}$$

It is easily seen that u, $Du \in L^p(\Omega)$ and

$$\begin{aligned} \int_\Omega |D^2u|^p\,dx &= \int_0^1 \left| \sum_{i,j=1}^2 \varphi''(\rho)\frac{x_ix_j}{\rho^2} + \sum_{i,j=1}^2 \varphi'(\rho)\,\frac{\delta_{ij}\rho^2 - x_ix_j}{\rho^3} \right|^p \rho\,d\rho \\ &\le c\int_0^{\overline{\rho}} \rho^{1-\alpha p}d\rho < \infty\,. \end{aligned}$$

Moreover, the Lagrange multiplier $\mu(\rho)$ exists and belongs to $L^p([0,1])$:

$$\mu(\rho) = \begin{cases} \frac{1}{\rho}\int_0^\rho \sigma f(\sigma)\,d\sigma \;-\; 1 = \frac{k}{2-\alpha}\rho^{1-\alpha} \;-\; 1 \ge 0 & \text{in } P = (\overline{\rho},1] \\ 0 & \text{in } E = [0,\overline{\rho}\,]. \end{cases}$$

Finally, if we consider the other case $0 < k \le 2-\alpha$, the plastic region does not exist, since $\overline{\rho} = \left(\frac{2-\alpha}{k}\right)^{\frac{1}{1-\alpha}} \ge 1$ is the solution to (6). Consequently the torsion is all elastic. In fact, from (29) we get for all $\rho \in [0,1]$

$$\varphi'(\rho) = -\frac{k}{2-\alpha}\rho^{1-\alpha}, \quad \text{and then} \quad -1 \le \varphi'(\rho) \le 1,$$

and (28) is verified. Then, by (34) we get the continuous function

$$\varphi(\rho) = \frac{k}{(2-\alpha)^2}\,(1-\rho^{2-\alpha}) \quad \forall \rho \in [0,1]. \tag{36}$$

This function $\varphi(\rho)$ and $\mu = 0$ verify the conditions of (7) in $[0,1]$. Moreover, $u \in W^{2,p}(\Omega)$. In fact, it is easily seen that $u, Du \in L^p(\Omega)$ and, since $p < \frac{1}{\alpha}$,

$$\begin{aligned} \int_\Omega |D^2u|^p\,dx &= \int_0^1 \left| \sum_{i,j=1}^2 \varphi''(\rho)\frac{x_ix_j}{\rho^2} + \sum_{i,j=1}^2 \varphi'(\rho)\,\frac{\delta_{ij}\rho^2 - x_ix_j}{\rho^3} \right|^p \rho\,d\rho \\ &\le c\int_0^1 \rho^{1-\alpha p}\,d\rho < \infty\,. \end{aligned}$$

6. Conclusions

In this paper we studied the elastic-plastic torsion problem in a ball of R^n, when the free term belongs to $L^p(\Omega)$. We searched for radial solutions and characterized the free boundary. In particular, for $n = 2$, we found the explicit solution in the two possible cases, namely, when the elastic and the plastic regions both exist and when the torsion is only elastic. Finally, we provided some examples.
Future work will be devoted to the study of radial solutions in R^n, $n > 2$, and of solutions with axial symmetry.

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