Journal of Convex Analysis
Volume 25 (2018), No. 2, 545–569

Identification in Variational and Quasi-Variational Inequalities

Joachim Gwinner
Institute of Mathematics, Department of Aerospace Engineering, Universität der Bundeswehr, Werner-Heisenberg-Weg 39, 85577 Neubiberg/Munich, Germany
joachim.gwinner@unibw-muenchen.de

Baasansuren Jadamba[*], **Akhtar A. Khan**[†]
Center for Applied and Computational Mathematics, School of Mathematical Sciences, Rochester Inst. of Technology, 85 Lomb Memorial Drive, Rochester, NY 14623, USA
bxjsma@rit.edu, aaksma@rit.edu

Miguel Sama[‡]
Departamento de Matemática Aplicada, Universidad Nacional de Educación a Distancia, Calle Juan del Rosal, 12, 28040 Madrid, Spain

Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

Received: December 30, 2015
Accepted: August 4, 2016

Our objective is to investigate the inverse problem of identifying variable parameters in certain variational and quasi-variational inequalities. To this end we extend a trilinear form based optimization framework that has been used quite effectively for parameter identification in variational equations emerging from partial differential equations. An abstract nonsmooth regularization approach is developed that encompasses the total variation regularization and permits the identification of discontinuous parameters. We investigate the inverse problem in an optimization setting using the output least-squares formulation. We give existence and convergence results for the optimization problem. We also penalize the variational inequality and arrive at an optimization problem for which the constraint variational inequality is replaced by the penalized equation. For this case, the smoothness of the parameter-to-solution map is studied and convergence analysis and optimality conditions are given. We also discretize the identification problem for quasi-variational inequalities and give the convergence analysis for the discrete problems. Examples are given to justify the theoretical framework.

Keywords: Inverse problems, ill-posed problems, regularization, total variation, parameter identification, output least-squares, variational inequalities, quasi-variational inequalities, penalization, finite elements.

2010 Mathematics Subject Classification: 49J40, 49N45, 90C26.

[*] Supported by RIT's COS FEAD grant for 2016-2017.
[†] Supported by a grant from the Simons Foundation (#210443) and RIT's COS FEAD grant for 2016-2017.
[‡] Partially supported by Ministerio de Economìa y Competitividad (Spain), project MTM2015-68103-P.

ISSN 0944-6532 / $ 2.50

1. Introduction

Numerous models in applied sciences can conveniently be written as variational problems involving certain variable coefficients. These coefficients are known and they often characterize some physical properties of the underlying model. The direct problem in this context is to solve the variational problem. By contrast, an inverse problem asks for the identification of variable coefficients when a certain measurement of a solution of the variational problem is available. In recent years, the field of inverse problems emerged as one of the most vibrant and expanding branches of applied and industrial mathematics. Certainly the primary reason behind this is the ever-growing number of real-world situations that are being modeled and studied in a unified framework of inverse problems. However, the theoretical aspects of inverse problems are also challenging and require a fine blending of various branches of mathematics.

In this paper, we study the inverse problem of identifying variable parameters in certain variational and quasi-variational inequalities. The theory of variational and quasi-variational inequalities has now established itself as one of the most promising areas of applied mathematics. This field offers us a powerful mathematical tool for investigating a wide range of problems arising in diverse disciplines. To be specific, applications of quasi-variational inequalities can be found in economics growth models [32], elastohydrodynamics [28], energy production management [9], equilibrium problems [4, 13, 17, 39], frictional elastostatic contact [12, 15, 41], frictionless quasistatic contact with history-dependent stiffness [47], heat convection with temperature dependent velocity constraint [16], image processing [38], Nash game equilibrium [24, 48] and multi-objective elliptic control [8, 14, 26], reaction diffusion [42], sandpiles formation [6], semiconductor and transistor design [37, 43, 46], shape optimization [1], superconductivity models [5, 45], and numerous others.

We begin by describing the general setting. The parameter space in this work is denoted by B which is assumed to be a Banach space. The set of all admissible coefficients is denoted by A which we assume to be a nonempty, closed, and convex subset of B. To prove the existence of some derivatives, we will also assume that A has a nonempty interior. The variational and the quasi-variational inequality will be posed in a Hilbert space V which is identified with its topological dual V^*. The measured data will be taken in another Hilbert space Z such that V is continuously embedded in Z. (For simplicity, we take the embedding constant to be 1). By $\|\cdot\|_X$, we denote the norm in any space X as well as the associated norm in its dual. The strong convergence and the weak/weak* convergence will be denoted by notations $\to$ and $\rightharpoonup$, respectively. Let K be a nonempty, closed, and convex subset of V with $0 \in K$. Let $F\colon V \to V^*$ and $\Phi\colon V \times V \to \mathbb{R}$ be given maps to be specified later, and let $m \in V^*$ be fixed. Let $T\colon B \times V \times V \to \mathbb{R}$ be a trilinear form with $T(a, u, v)$ symmetric in u, v. Although most result given below hold without the symmetry of T, this property will play a crucial role in some of the results. Assume there are constants $\alpha > 0$ and $\beta > 0$ such that

$$T(a, u, v) \leq \beta \|a\|_B \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V, \ a \in B, \tag{1}$$

$$T(a, u, u) \geq \alpha \|u\|_V^2, \ \text{for all } u \in V, \ a \in A. \tag{2}$$

Therefore, for every $(a, u) \in A \times V$, the Riesz map associated to $T(a, u, \cdot)$ is well-defined and will be denoted by $T(a, u)$. That is, $\langle T(a, u), v\rangle = T(a, u, v)$, for every $a \in A$, $u, v \in V$.

Consider the quasi-variational inequality: Given $a \in A$, find $u = u(a) \in K$ such that

$$\langle T(a, u) + F(u) - m, v - u\rangle \geq \Phi(u, u) - \Phi(u, v), \quad \text{for every } v \in K. \tag{3}$$

Further, consider the variational inequality: Given $a \in A$, find $u = u(a) \in K$ such that

$$T(a, u, v - u) \geq \langle m, v - u\rangle, \quad \text{for every } v \in K. \tag{4}$$

Quasi-variational inequality (3) and variational inequality (4) constitute the direct problems in this study. Our focus, however, is on identifying the parameter a from a measurement z of $u(a)$ in these problems. In other words, we are interested in the parameter a such that $u(a) = z$. However, due to the lack of regularity in the measured data, it is unrealistic to expect such a coefficient. Consequently, the inverse problem of parameter identification will be posed as an optimization problem which aims to minimize the gap between the solution computed via (3) (or (4)) and the measured data z. This approach has been precisely the case with simpler inverse problems. To the best of our knowledge, this is the first work on inverse problem of parameter identification in quasi-variational inequalities. However, there is an abundance of literature on parameter identification in partial differential equations, see the recent survey [31] and the cited references therein. There are also some works on parameter identification in variational inequalities (see [7, 20, 25, 27]). We note that K. H. Hoffmann and J. Sprekels [27] were among the first ones to study parameter identification in variational inequalities. However, in [27], in contrast to the most papers on inverse problems where an optimization framework is the preferred choice, the authors developed an iterative scheme that is based on the construction of certain regularized time-dependent problems containing the original problem as asymptotic steady state. We also note that in [2] an optimal control of quasi-variational inequalities was studied.

For identification in partial differential equations, a convenient abstract framework has been developed in [19] for variational problems defined using trilinear forms. This work aims to study the inverse problem in variational and quasi-variational inequalities by exploring the advantages of a trilinear-based approach.

Most work on inverse problems appeared for the boundary value problem

$$\text{(BVP)} \qquad -\nabla\cdot(q\nabla u) = f \ \text{ in } \Omega, \ u = 0 \ \text{ on } \partial\Omega, \tag{5}$$

where Ω is a suitable domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\partial\Omega$ is its boundary. In (5), $u = u(x)$ may represent the steady-state temperature at a given point x of a body; then q would be a variable thermal conductivity coefficient, and f the external heat source. The system (5) also models underground steady state aquifers in which the parameter q is the aquifer transmissivity coefficient, u is the hydraulic head, and f is the recharge. The inverse problem in the context of (5) is to estimate q from a measurement z of the solution u.

A number of approaches to the aforementioned inverse problem have been proposed in the literature; most involve either regarding (5) as a hyperbolic PDE in q or posing an optimization problem whose solution is an estimate of q. Furthermore, the approach of reformulating (5) as an optimization problem is divided into two possibilities, namely either formulating the problem as an unconstrained optimization problem or treating it as a constrained optimization problem, in which the PDE itself is the constraint. Among the optimization-based techniques the output least-squares method is among the most widely investigated methods. The output least-squares (OLS) approach minimizes the functional

$$q \to \|u(q) - z\|^2, \tag{6}$$

where z is the data and $u(q)$ solves the variational form of (5) given by

$$\int_\Omega q\nabla u\cdot\nabla v = \int_\Omega fv, \quad \text{for all } v \in H_0^1(\Omega). \tag{7}$$

There are other functionals that have been used for the above inverse problem. For example, the equation error method (see [33]), consists of minimizing the functional

$$q \to \|\nabla\cdot(q\nabla z) + f\|^2_{H^{-1}(\Omega)}$$

where $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$ and z is the data. Ito and Kunisch [29] and Chen and Zou [11] developed an augmented Lagrangian algorithm to solve the OLS problem by treating the PDE as an explicit constraint.

Knowles [36] proposed using a coefficient-dependent norm in the OLS setting

$$q \to \int q\nabla(u(q) - z)\cdot\nabla(u(q) - z), \tag{8}$$

where z is the data (the measurement of u) and $u(a)$ solves (7). Knowles [36] established that the above functional is convex. In [19], a new modified output least-squares (MOLS) was proposed to extend (8) and its convexity was proved in an abstract setting. An extension of the MOLS for elasticity imaging inverse problem was recently given in [30].

The contents of this paper are organized into five sections. Section 2 deals with parameter identification in quasi-variational inequalities. Section 3 gives numerous results on parameter identification in variational inequalities. In Section 4, a finite-dimensional approximation scheme for the inverse problem is given. Section 5 discusses future goals and open questions.

2. Parameter identification in quasi-variational inequalities

We will study the inverse problem of parameter identification in the output least-squares framework by minimizing

$$a \to \frac{1}{2}\|u(a) - z\|_Z^2,$$

where $u(a)$ is a solution of (3) and $z \in Z$ is the data. Since inverse problems are highly ill-posed, the above OLS functional will be regularized shortly.

The following existence result confirms that the OLS formulation is well defined:

Theorem 2.1. *In addition to* (1) *and* (2), *assume that F is hemi-continuous*[1], $F(0) = 0$, $\Phi(u, \cdot)\colon V \to \mathbb{R}$ *is convex and lower-semicontinuous for any $u \in K$, and $\Phi(0, \cdot) = 0$. Assume that there are constants $\delta > 0$ and $\gamma > 0$ with $\tau := \alpha - \gamma - \delta > 0$ such that for all $u, v, u_1, v_1 \in K$, we have*

$$\langle F(u) - F(v), u - v\rangle \geq -\gamma\|u - v\|_V^2, \tag{9}$$

$$|\Phi(u, v_1) + \Phi(u_1, v) - \Phi(u, v) - \Phi(u_1, v_1)| \leq \delta\|u - u_1\|_V\|v - v_1\|_V. \tag{10}$$

Then, for $a \in A$, (3) *has a unique solution $u(a)$. Moreover, for any $a, b \in A$, we have*

$$\|u(a)\|_V \leq \tau^{-1}\|m\|_{V^*}, \tag{11}$$

$$\|u(a) - u(b)\|_V \leq \beta\tau^{-2}\|m\|_{V^*}\|a - b\|_B. \tag{12}$$

Proof. For a fixed $w \in K$, we consider the following parametric variational inequality: Given $\bar{a} \in A$, find $Sw \in K$ such that

$$\langle T(\bar{a}, Sw) + F(Sw) - m, v - Sw\rangle \geq \Phi(w, Sw) - \Phi(w, v), \text{ for every } v \in K. \tag{13}$$

For every $a \in A$, the map $(F + T(a, \cdot))$ is strongly monotone and consequently for a fixed $w \in K$, (13) is uniquely solvable. This permits us to define the variational selection $S : K \to K$ such that for every $w \in K$, the element Sw is the unique solution of (13). Note that any fixed point of the variational selection S solves (3).

[1] That is, the map $t \in [0, 1] \to \langle F(tu + (1 - t)v), u - v\rangle$ is continuous.

Let w_1, w_2 be two elements in K and let Sw_1 and Sw_2 be the corresponding solutions of (13). That is, the following two parametric variational inequalities hold:

$$\langle T(\bar{a}, Sw_1) + F(Sw_1) - m, v - Sw_1 \rangle \geq \Phi(w_1, Sw_1) - \Phi(w_1, v) \text{ for every } v \in K,$$
$$\langle T(\bar{a}, Sw_2) + F(Sw_2) - m, v - Sw_2 \rangle \geq \Phi(w_2, Sw_2) - \Phi(w_2, v) \text{ for every } v \in K.$$

We rearrange these inequalities with $v = Sw_2$ in the first and $v = Sw_1$ in the second to get

$$\begin{aligned}(\alpha-\gamma)\|Sw_1 - Sw_2\|_V^2 &\leq \langle T(\bar{a}, Sw_1) + F(Sw_1) - T(\bar{a}, Sw_2) - F(Sw_2), Sw_1 - Sw_2 \rangle \\ &\leq \Phi(w_1, Sw_2) + \Phi(w_2, Sw_1) - \Phi(w_1, Sw_1) - \Phi(w_2, Sw_2) \\ &\leq \delta \|w_1 - w_2\|_V \|Sw_1 - Sw_2\|_V,\end{aligned}$$

which implies

$$\|Sw_1 - Sw_2\|_V \leq \frac{\delta}{\alpha - \gamma} \|w_1 - w_2\|_V,$$

and hence the map S is a contraction. The Banach fixed point theorem ensures that there is a unique fixed point of S and hence quasi-variational inequality (3) is uniquely solvable.

For (12), let $a, b \in A$ be arbitrary and let u and w be the associated solutions, that is,

$$\begin{aligned}T(a, u, v - u) + \langle F(u) - m, v - u \rangle &\geq \Phi(u, u) - \Phi(u, v), \text{ for every } v \in K, \\ T(b, w, v - w) + \langle F(w) - m, v - w \rangle &\geq \Phi(w, w) - \Phi(w, v), \text{ for every } v \in K.\end{aligned}$$

A rearrangement of the above, with $v = w$ in the first and $v = u$ in the second, yields

$$\begin{aligned}T(a - b, w, w - u) + \Phi(u, w) + \Phi(w, u) - \Phi(u, u) - \Phi(w, w) \geq \\ \geq \langle Fu - Fw, u - w \rangle + T(a, u - w, u - w),\end{aligned}$$

which further implies that

$$(\alpha - \gamma - \delta)\|u - w\|_V \leq \beta \|a - b\|_B \|w\|_V. \tag{14}$$

To show (12), we need to prove (11). For this, we set $v = 0 \in K$ in (3), to get

$$\langle T(a, u) + Fu - m, -u \rangle \geq \Phi(u, u) - \Phi(u, 0)$$

which, by using the assumptions that $\Phi(0, \cdot) = 0$ and $F(0) = 0$, yields

$$\begin{aligned}(\alpha - \gamma)\|u\|_V^2 &\leq \|m\|_{V^*} \|u\|_V + \Phi(u, 0) - \Phi(u, u) + \Phi(0, u) - \Phi(0, 0) \\ &\leq \|m\|_{V^*} \|u\|_V + \delta \|u\|_V^2,\end{aligned}$$

and hence $(\alpha - \gamma - \delta)\|u\|_V \leq \|m\|_{V^*}$, which proves (11). This inequality, in conjunction with (14), gives (12). The proof is complete. □

We shall now focus on the inverse problem posed as an optimization problem. To counter the adverse affects of the ill-posed nature of the inverse problem, we need to incorporate some sort of regularization. For this we tailor a general regularization framework by assuming:

1. The Banach space B is continuously embedded in a Banach space L. There is another Banach space $\widehat{B}$ that is compactly embedded in L. The set A is a subset of $B \cap \widehat{B}$, closed and bounded in B and also closed in L.

2. For any $\{b_k\} \subset B$ with $b_k \to 0$ in L, any bounded $\{u_k\} \subset V$, and fixed $v \in V$, we have

$$T(b_k, u_k, v) \to 0. \tag{15}$$

3. $R\colon \widehat{B} \to \mathbb{R}$ is positive, convex, and lower-semicontinuous with respect to $\|\cdot\|_L$ such that

$$R(a) \geq \tau_1 \|a\|_{\widehat{B}} - \tau_2, \quad \text{for every } a \in A, \text{ for some } \tau_1 > 0, \ \tau_2 > 0. \tag{16}$$

To illustrate the spectrum of the regularization framework and to justify the imposed conditions on the trilinear map T, we consider the inverse problem of identifying a in the quasi-variational inequality of finding $u \in K$ such that for every $v \in K$, we have

$$T(a, u, v - u) \geq \langle m, v - u \rangle + \Phi(u, u) - \Phi(u, v),$$

with $T(a, u, v) = \int_\Omega a \nabla u \cdot \nabla v$ and $V = H_0^1(\Omega)$, where Ω is an open and bounded domain in $\mathbb{R}^N$. The set K and the maps m and Φ do not play much role in this discussion and can be chosen as above.

To justify the above framework, recall that the *total variation* of $f \in L^1(\Omega)$ is given by

$$\mathrm{TV}(f) = \sup\left\{ \int_\Omega f\,(\nabla \cdot g) \ : \ g \in \left(C_0^1(\Omega)\right)^N, \ |g(x)| \leq 1 \text{ for all } x \in \Omega \right\}$$

where $|\cdot|$ represents the Euclidean norm. Clearly, if $f \in W^{1,1}(\Omega)$, then $\mathrm{TV}(f) = \int_\Omega |\nabla f|$.

If $f \in L^1(\Omega)$ satisfies $\mathrm{TV}(f) < \infty$, then f is said to have *bounded variation*, and $\mathrm{BV}(\Omega)$ is defined by $\mathrm{BV}(\Omega) = \{f \in L^1(\Omega) : \mathrm{TV}(f) < \infty\}$ with norm $\|f\|_{\mathrm{BV}(\Omega)} = \|f\|_{L^1(\Omega)} + \mathrm{TV}(f)$. The functional $\mathrm{TV}(\cdot)$ is a seminorm on $\mathrm{BV}(\Omega)$ and is often called the BV-*seminorm*.

We set $V = H_0^1(\Omega)$, $B = L^\infty(\Omega)$, $L = L^1(\Omega)$, $\widehat{B} = \mathrm{BV}(\Omega)$, and $R(a) = TV(a)$, and define

$$A = \{a \in L^\infty(\Omega) |\ 0 < c_1 \leq a(x) \leq c_2, \ \text{ a.e. in } \Omega, \ \mathrm{TV}(a) \leq c_3 < \infty\}, \tag{17}$$

where c_1, c_2, and c_3 are positive constants. Clearly, A is bounded in $\widehat{B}$ and compact in L. It is known that $L^\infty(\Omega)$ is continuously embedded in $L^1(\Omega)$, $\mathrm{BV}(\Omega)$ is compactly embedded in $L^1(\Omega)$, and $TV(\cdot)$ is convex and lower-semicontinuous in $L^1(\Omega)$-norm, see Attouch, Buttazzo, and Michaille [3] and Ziemer [49]. Thus in this setting, assumptions 1. and 3. are satisfied.

Let $u, v \in H_0^1(\Omega)$ be fixed and let $\{q_k\}$ be a bounded sequence in $L^\infty(\Omega)$. If $q_k \to 0$ in the $L^1(\Omega)$ norm, then, by the Lebegue dominated convergence theorem, $\int_\Omega q_k \nabla u \cdot \nabla v \to 0$.

Moreover, for any $q \in L^\infty(\Omega)$ and $u, v \in H_0^1(\Omega)$, we can bound $\left|\int_\Omega q \nabla u \cdot \nabla v\right|$ as follows:

$$\begin{aligned}\left|\int_\Omega q \nabla u \cdot \nabla v\right| &\leq \int_\Omega |q \nabla u \cdot \nabla v| = \int_\Omega \left(|q|^{1/2} \nabla u\right) \cdot \left(|q|^{1/2} \nabla v\right) \\ &\leq \left[\int_\Omega |q| \nabla u \cdot \nabla u\right]^{1/2} \left[\int_\Omega |q| \nabla v \cdot \nabla v\right]^{1/2}.\end{aligned}$$

The above two observations motivate the following assumptions:

1. If $\{a_k\}$ is a bounded sequence in B, $a_k \to 0$ in L, and $u, v \in V$ fixed, then $T(a_k, u, v) \to 0$.

2. Assume that L is a space of real-valued functions and that

$$|T(a, u, v)| \leq [T(|a|, u, u)]^{1/2} [T(|a|, v, v)]^{1/2} \text{ for all } u, v \in V, a \in B. \tag{18}$$

3. For both $\|\cdot\|_B$ and $\|\cdot\|_L$: $\quad \||a|\| \leq \|a\| \quad$ for all a. (19)

The following result shows the relevance of the above three assumptions in proving (15):

Lemma 2.2. *Let $\{a_k\} \subset A$ be bounded in B with $a_k \to a$ in $\|\cdot\|_L$ and let $\{v_k\}$ be bounded in V. Then (15) holds, that is, for all $w \in V$, we have $T(a_k - a, v_k, w) \to 0$.*

Proof. We have $|T(a_k - a, v_k, w)| \leq [T(|a_k - a|, v_k, v_k)]^{1/2} [T(|a_k - a|, w, w)]^{1/2}$. Since $|a_k - a| \to 0$ in L, we have $[T(|a_k - a|, w, w)]^{1/2} \to 0$, while the boundedness of $\{v_k\}$ in V and $\{a_k\}$ in B implies that $[T(|a_k - a|, v_k, v_k)]^{1/2}$ is uniformly bounded in k. This shows that $|T(a_k - a, v_k, w)| \to 0$. □

Remark 2.3. The regularization framework developed above can be simplified by assuming that the set A belongs to a Hilbert space H that is compactly embedded in the space B. An example for this setting is $B = L^\infty(\Omega)$ and $H = H^2(\Omega)$, for a suitable domain Ω. In this case, the properties (1) and (2) provide simpler proofs of all of our results.

We return to the regularized OLS based optimization problem: Find $a \in A$ by solving

$$\min_{a \in A} J_\kappa(a) = \frac{1}{2}\|u(a) - z\|_Z^2 + \kappa R(a), \tag{20}$$

where $\kappa > 0$ is a regularization parameter, R is the regularization map, $u(a)$ is the unique solution of (3), and z is the measured data.

For the solvability of (20), we need the following additional condition on Φ originally introduced in [41]: For any $v \in K$ and for any $\{z_n\} \subset V$ with $z_n \rightharpoonup z \in V$, we have

$$\limsup_{n\to\infty} [\Phi(z_n, v) - \Phi(z_n, z_n)] \leq \Phi(z, v) - \Phi(z, z). \tag{21}$$

With the above preparation, we give the following existence result:

Theorem 2.4. *For any $\kappa > 0$, problem* (20) *has a nonempty solution set.*

Proof. By definition of the functional J_κ, for every $a \in A$, we have $J_\kappa(a) \geq 0$. Therefore, there exists a minimizing sequence $\{a_n\}$ in A such that

$$\lim_{n\to\infty} J_\kappa(a_n) = \inf\{J_\kappa(a) |\, a \in A\},$$

which implies that $\{J_\kappa(a_n)\}$ is bounded from above, and by the definition of J_κ, there exists a constant $c > 0$ such that $R(a_n) \leq c$. From (16), $\{a_n\}$ is bounded in $\|\cdot\|_{\widehat{B}}$ as well. Due to the compact embedding of $\widehat{B}$ into L, $\{a_n\}$ has a subsequence which converges strongly in $\|\cdot\|_L$. Keeping the same notation for subsequences as well, let $\{a_n\}$ be the subsequence converging in $\|\cdot\|_L$ to some $\bar{a} \in A$.

Let $\{u_n\}$ be the sequence of solutions of (3) corresponding to $\{a_n\}$, that is,

$$\langle T(a_n, u_n) + Fu_n - m, v - u_n\rangle \geq \Phi(u_n, u_n) - \Phi(u_n, v), \text{ for every } v \in K. \tag{22}$$

By (11), $\{u_n\}$ is bounded and hence there exists a subsequence $\{u_n\}$ that converges weakly to some $\bar{u} \in V$. We shall show that $\bar{u} = \bar{u}(\bar{a})$.

Inequality (22), which holds for the subsequence, can be rearranged as follows

$$\begin{aligned}\langle T(\bar{a}, v) + Fv - m, v - u_n\rangle + T(a_n - \bar{a}, v, v - u_n) \geq \\ \geq T(a_n, u_n - v, u_n - v) + \langle Fu_n - Fv, u_n - v\rangle + \Phi(u_n, u_n) - \Phi(u_n, v) \\ \geq \Phi(u_n, u_n) - \Phi(u_n, v),\end{aligned}$$

where we used the monotonicity of $T(a_n, \cdot) + F(\cdot)$. Consequently

$$\langle Fv - m, v - u_n\rangle + T(\bar{a}, v, v - u_n) + T(a_n - \bar{a}, v, v - u_n) \geq \Phi(u_n, u_n) - \Phi(u_n, v),$$

which, in view of (15) and (21), when passed to the limit $n \to \infty$, implies that

$$\langle Fv - m, v - \bar{u}\rangle + T(\bar{a}, v, v - \bar{u}) \geq \Phi(\bar{u}, \bar{u}) - \Phi(\bar{u}, v), \quad \text{for every } v \in K.$$

We set $v = \bar{u} + t(v - \bar{u})$, $t \in (0, 1)$, in the above inequality to get

$$t\langle F(\bar{u}+t(v-\bar{u}))-m, v-\bar{u}\rangle+tT(\bar{a}, \bar{u}+t(v-\bar{u}), v-\bar{u}) \geq \Phi(\bar{u}, \bar{u})-\Phi(\bar{u}, \bar{u}+t(v-\bar{u})),$$

and use the convexity of $\Phi(\bar{u},\cdot)$ to obtain

$$t\langle F(\bar{u}+t(v-\bar{u}))-m,v-\bar{u}\rangle + tT(\bar{a},\bar{u},v-\bar{u}) + t^2T(\bar{a},v-\bar{u},v-\bar{u}) \geq$$
$$\geq t[\Phi(\bar{u},\bar{u})-\Phi(\bar{u},v)]$$

which implies that

$$\langle F(\bar{u}+t(v-\bar{u}))-m,v-\bar{u}\rangle + T(\bar{a},\bar{u},v-\bar{u}) + tT(\bar{a},v-\bar{u},v-\bar{u}) \geq \Phi(\bar{u},\bar{u})-\Phi(\bar{u},v)$$

We pass the above inequality to limit $t \to 0$ and use the hemi-continuity of F to get

$$\langle F(\bar{u})-m,v-\bar{u}\rangle + T(\bar{a},\bar{u},v-\bar{u}) \geq \Phi(\bar{u},\bar{u})-\Phi(\bar{u},v),$$

which, in view of the fact that $v \in K$ was chosen arbitrarily, confirms that $\bar{u}$ solves (3). However, since quasi-variational inequality (3) is uniquely solvable, we get that $\bar{u} = u(\bar{a})$.

The weak lower-semicontinuity of a norm and lower-semicontinuity of R in $\|\cdot\|_L$ yield

$$\begin{aligned} J_\kappa(\bar{a}) &= \frac{1}{2}\|\bar{u}-z\|_Z^2 + \kappa R(\bar{a}) \leq \liminf_{n\to\infty} \frac{1}{2}\|u(a_n)-z\|_Z^2 + \liminf_{n\to\infty} \kappa R(a_n) \\ &\leq \liminf_{n\to\infty} \left\{ \frac{1}{2}\|u(a_n)-z\|_Z^2 + \kappa R(a_n) \right\} \leq \lim_{n\to\infty} J_\kappa(a_n) = \inf\{J_\kappa(a) |\, a \in A\}, \end{aligned}$$

which confirms that $\bar{a}$ is a solution of (20). The proof is complete. $\square$

Remark 2.5. In fact, $\{u_n\}$ in the above proof converges strongly to $\bar{u}$. By (22), we get

$$\langle T(a_n,u_n) + Fu_n - m, \bar{u}-u_n\rangle \geq \Phi(u_n,u_n) - \Phi(u_n,\bar{u}),$$

which can be rearranged as follows

$$\langle T(a_n,u_n) + Fu_n - T(a_n,\bar{u}) - F(\bar{u}), u_n-\bar{u}\rangle \leq$$
$$\leq \langle T(a_n,\bar{u}) + F(\bar{u}) - m, \bar{u}-u_n\rangle + \Phi(u_n,\bar{u}) - \Phi(u_n,u_n),$$

and the strong monotonicity of $(T(a_n,\cdot)+F)$, (15), and (21) yield $u_n \to \bar{u}$ as $n \to \infty$.

3. Parameter identification in variational inequalities

Our focus now is on the following variational inequality: Given $a \in A$, find $u(a) = u \in K$ such that

$$T(a,u,v-u) \geq \langle m, v-u\rangle, \quad \text{for every } v \in K. \tag{23}$$

In view of the coercivity and the continuity of the trilinear form T, for each $a \in A$, the above variational inequality has a unique solution $u(a)$ (see [40]).

Furthermore, by using the arguments given in the proof of Theorem 2.1, it can be shown that for any $a, b \in A$, we have

$$\|u(a)\|_V \leq \alpha^{-1}\|m\|_{V^*}, \tag{24}$$

$$\|u(a) - u(b)\|_V \leq \frac{\beta\|m\|_{V^*}}{\alpha^2}\|a - b\|_B. \tag{25}$$

Consider the regularized OLS based optimization problem: Find $a \in A$ by solving

$$\min_{a\in A} J_\kappa(a) = \frac{1}{2}\|u(a) - z\|_Z^2 + \kappa R(a), \tag{26}$$

where for $a \in A$, the element $u(a)$ solves (23), $R : A \to \mathbb{R}$ is a regularizer, $\kappa > 0$ is the regularization parameter, and $z \in Z$ is the measured data.

We have the following existence result:

Theorem 3.1. *For every $\kappa > 0$, problem* (26) *has a nonempty solution set.*

Proof. The proof is quite similar to the proof of Theorem 2.4 and hence omitted. □

Our goal is to replace the constraint variational inequality for the optimization problem by a variational equation. For this we define a familiar notion of a penalty map. A bounded, hemicontinuous, monotone map $P\colon V \to V^*$ is said to be the *penalty map of K* provided

$$K = \{v \in V |\ P(v) = 0\}. \tag{27}$$

A typical example of the penalty map is $P = (I - P_K)$, where I is the identity map and P_K is the projection map defined from V onto K (see [21]). In any Banach space X with its topological dual X^*, this example takes the form $P = J(I - P_K)$, where $J : X \to X^*$ is the normalized duality map and P_K is the projection in the best approximation sense.

Given a penalty parameter $\varepsilon > 0$ and the penalty map P, we consider the following *penalized variational equation*: Find $u_\varepsilon \in V$ such that

$$T(a, u_\varepsilon, v) + \frac{1}{\varepsilon}\langle P(u_\varepsilon), v\rangle = m(v), \quad \text{for every } v \in V. \tag{28}$$

We have the following existence and convergence result.

Theorem 3.2. *For any $\varepsilon > 0$ and for any $a \in A$,* (28) *has a unique solution $u_\varepsilon(a)$.*

Proof. The proof follows from the ellipticity of the trilinear map T and the monotonicity of P, ensuring that the map $T(a, \cdot) + \varepsilon^{-1}P$ is strictly monotone and coercive. □

We now consider an analogue of (26) where the underlying variational inequality has been replaced by a variational equation: Find $a \in A$ by solving the optimization problem:

$$\min_{a \in A} J_\varepsilon(a) = \frac{1}{2} \|u_\varepsilon(a) - z\|_Z^2 + \kappa R(a), \tag{29}$$

where for $a \in A$, the element $u_\varepsilon(a)$ solves the penalized variational equation (28). We give the following existence and convergence result:

Theorem 3.3. *For $\varepsilon > 0$, (29) has a solution $\bar{a}_\varepsilon$. There is a subsequence $\{(\bar{a}_\varepsilon, \bar{u}_\varepsilon)\}$, where $\bar{u}_\varepsilon = \bar{u}_\varepsilon(\bar{a}_\varepsilon)$ is the unique solution of (28), such that for $\varepsilon \to 0$, we have $\bar{a}_\varepsilon \to \tilde{a}$ in L, $\bar{u}_\varepsilon \to \tilde{u}$ in V where $\tilde{a}$ is a solution of (26) and $\tilde{u} = u(\tilde{a})$ is the unique solution of (23).*

Proof. For a fixed $\varepsilon > 0$, there exists a solution $\bar{a}_\varepsilon$ by arguments similar to those used in the proof of Theorem 2.4. For the convergence result, we note that the sequence $\{\bar{a}_\varepsilon\} \subset A$ of solutions of (29) is bounded in $\|\cdot\|_{\widehat{B}}$ and the bound does not depend on ε. Therefore, due to the compact embedding of $\widehat{B}$ into L, there exists a subsequence, denoted by the same notation, that converges strongly in $\|\cdot\|_L$ to some $\tilde{a} \in A$. Let $\bar{u}_\varepsilon$ be the corresponding sequence of the penalized solutions, that is, a sequence of solutions of (28) corresponding to $\bar{a}_\varepsilon$. Clearly, the sequence $\{\bar{u}_\varepsilon\}$ remains bounded as well. In fact, we take $v = \bar{u}_\varepsilon$ in (28) and by ellipticity of the trilinear form, monotonicity of P, and $0 \in K$, we obtain

$$\alpha \|\bar{u}_\varepsilon\|_V^2 \leq T(\bar{a}_\varepsilon, \bar{u}_\varepsilon, \bar{u}_\varepsilon) + \frac{1}{\varepsilon} \langle P(\bar{u}_\varepsilon), \bar{u}_\varepsilon \rangle = \langle m, \bar{u}_\varepsilon \rangle \leq \|m\|_{V^*} \|u_\varepsilon\|_V ,$$

and the boundedness of $\{\bar{u}_\varepsilon\}$ follows. Consequently, there exists a subsequence, again keeping the same notation, which converges weakly to some $\tilde{u} \in V$, that is, $\bar{u}_\varepsilon \rightharpoonup \tilde{u}$ as $\varepsilon \to 0$. We claim that $\tilde{u} \in K$. Using (28), for every $v \in K$, we have

$$\langle P(\bar{u}_\varepsilon), v \rangle = \varepsilon \left(\langle m, v \rangle - T(\bar{a}_\varepsilon, \bar{u}_\varepsilon, v) \right), \tag{30}$$

which, due to the continuity of m and the trilinear form T, ensures that the expression $\|P(\bar{u}_\varepsilon)\|_V \to 0$ as $\varepsilon \to 0$. Moreover, by the monotonicity of the penalty map P, for every $v \in V$, we have

$$0 \leq \lim_{\varepsilon \to 0} \langle P(v) - P(\bar{u}_\varepsilon), v - \bar{u}_\varepsilon \rangle = \langle P(v), v - \tilde{u} \rangle. \tag{31}$$

Now by setting $v = \tilde{u} + tz$ where $t > 0$, and $z \in V$ is an arbitrarily chosen element, we get that $\langle P(\tilde{u} + tz), z \rangle \geq 0$, and, by passing $t \to 0$, it follows from the hemicontinuity of P that for every $z \in V$, we have $\langle P(\tilde{u}), z \rangle \geq 0$. Since $z \in V$ is arbitrary, we have $P(\tilde{u}) = 0$ which confirms that $\tilde{u} \in K$.

We rearrange (28) so that for every $v \in K$, we have

$$T(\bar{a}_\varepsilon, \bar{u}_\varepsilon, v - \bar{u}_\varepsilon) - \frac{1}{\varepsilon} \langle P(v) - P(\bar{u}_\varepsilon), v - \bar{u}_\varepsilon \rangle + \frac{1}{\varepsilon} \langle P(v), v - \bar{u}_\varepsilon \rangle = \langle m, v - \bar{u}_\varepsilon \rangle,$$

and by using the monotonicity of P and the fact that $P(v) = 0$, for any $v \in K$, we deduce

$$T(\bar{a}_\varepsilon, \bar{u}_\varepsilon, v - \bar{u}_\varepsilon) \geq \langle m, v - \bar{u}_\varepsilon \rangle, \quad \text{for every } v \in K. \tag{32}$$

By using the ellipticity of T, it follows from (32) that

$$T(\bar{a}_\varepsilon, v, v - \bar{u}_\varepsilon) \geq T(\bar{a}_\varepsilon, v - \bar{u}_\varepsilon, v - \bar{u}_\varepsilon) + \langle m, v - \bar{u}_\varepsilon \rangle \geq \langle m, v - \bar{u}_\varepsilon \rangle,$$

which further implies that

$$T(\tilde{a}, v, v - \bar{u}_\varepsilon) + T(\bar{a}_\varepsilon - \tilde{a}, v, v - \bar{u}_\varepsilon) \geq \langle m, v - \bar{u}_\varepsilon \rangle,$$

and by passing to the limit $\varepsilon \to 0$ and using (15) and the symmetry of T, we obtain

$$T(\tilde{a}, v, v - \tilde{u}) \geq \langle m, v - \tilde{u} \rangle,$$

for every $v \in K$. To obtain (23), we set $v = \tilde{u} + t(v - \tilde{u})$, where $t \in (0, 1]$, and obtain

$$T(\tilde{a}, \tilde{u}, v - \tilde{u}) + tT(\tilde{a}, v - \tilde{u}, v - \tilde{u}) \geq \langle m, v - \tilde{u} \rangle,$$

and by taking $t \to 0$, we get

$$T(\tilde{a}, \tilde{u}, v - \tilde{u}) \geq \langle m, v - \tilde{u} \rangle, \quad \text{for every } v \in K, \tag{33}$$

verifying that $\tilde{u} = u(\tilde{a})$.

By now we know that $\{\bar{u}_\varepsilon\}$ converges weakly to $\tilde{u}$. We next show that, in fact, $\{\bar{u}_\varepsilon\}$ converges strongly to $\tilde{u}$. Note that by the definition of $\bar{u}_\varepsilon$, we have

$$T(\bar{a}_\varepsilon, \bar{u}_\varepsilon, \tilde{u} - \bar{u}_\varepsilon) + \frac{1}{\varepsilon} \langle P(\bar{u}_\varepsilon), \tilde{u} - \bar{u}_\varepsilon \rangle = \langle m, \tilde{u} - \bar{u}_\varepsilon \rangle,$$

and since this equality can be written as follows

$$T(\bar{a}_\varepsilon, \bar{u}_\varepsilon - \tilde{u}, \tilde{u} - \bar{u}_\varepsilon) + T(\bar{a}_\varepsilon, \tilde{u}, \tilde{u} - \bar{u}_\varepsilon) + \frac{1}{\varepsilon} \langle P(\bar{u}_\varepsilon) - P(\tilde{u}), \tilde{u} - \bar{u}_\varepsilon \rangle = \langle m, \tilde{u} - \bar{u}_\varepsilon \rangle,$$

we obtain by the monotonicity of P and the ellipticity of T that

$$\begin{aligned} \alpha \|\bar{u}_\varepsilon - \tilde{u}\|_V^2 &\leq T(\bar{a}_\varepsilon, \bar{u}_\varepsilon - \tilde{u}, \bar{u}_\varepsilon - \tilde{u}) + \frac{1}{\varepsilon} \langle P(\bar{u}_\varepsilon) - P(\tilde{u}), \bar{u}_\varepsilon - \tilde{u} \rangle \\ &\leq T(\bar{a}_\varepsilon - \tilde{a}, \tilde{u}, \tilde{u} - \bar{u}_\varepsilon) + T(\tilde{a}, \tilde{u}, \tilde{u} - \bar{u}_\varepsilon) + \langle m, \tilde{u} - \bar{u}_\varepsilon \rangle, \end{aligned}$$

and, by passing the above inequality to the limit for $\varepsilon \to 0$, using property (15) of T, symmetry and the fact that $\bar{a}_\varepsilon \rightharpoonup \tilde{a}$ strongly converges in L, we deduce that $\|\bar{u}_\varepsilon - \tilde{u}\|_V \to 0$.

Let $a_0 \in A$ and let u_0 be the corresponding solution of variational inequality (23). For a_0, let $\bar{u}_\varepsilon(a_0)$ be the unique solution of the penalized equation: Find $u \in V$ such that

$$T(a_0, u, v) + \frac{1}{\varepsilon} \langle P(u), v \rangle = \langle m, v \rangle, \quad \text{for every } v \in V.$$

Then, firstly, $\bar{u}_\varepsilon(a_0) \to u_0$ as $\varepsilon \to 0$, and secondly, $(a_0, \bar{u}_\varepsilon(a_0))$ is feasible for (29). Hence,

$$\begin{aligned} J_\kappa(\tilde{a}) &= \frac{1}{2}\|\tilde{u} - z\|_Z^2 + \kappa R(\tilde{a}) \le \liminf_{\varepsilon\to 0} \frac{1}{2}\|\bar{u}_\varepsilon - z\|_Z^2 + \kappa \liminf_{\varepsilon\to 0} R(\bar{a}_\varepsilon) \\ &\le \liminf_{\varepsilon\to 0} \left\{\frac{1}{2}\|\bar{u}_\varepsilon - z\|_Z^2 + \kappa R(\bar{a}_\varepsilon)\right\} \le \liminf_{\varepsilon\to 0} \left\{\frac{1}{2}\|\bar{u}_\varepsilon(a_0) - z\|_Z^2 + \kappa R(a_0)\right\} \\ &= J_\kappa(a_0), \end{aligned}$$

and consequently $\tilde{a} \in A$ solves (26). The proof is complete. □

The optimization problem (26) typically has multiple solutions. However, if we have some a priori information about a solution of such a problem , can it be incorporated in the regularized problem? Relevant to this question, we recall a known fact that for the linear inverse problems if some a priori information about a point being close to a solution is available, then the regularizer can be modified so that the regularized solutions converge to the point closest to the a priori chosen element. Following this idea, we consider the perturbed regularized problem where the newly added term aims to incorporate some a priori information.

Let $(\bar{a}_0, \bar{u}_0)$, where $\bar{u}_0 = u(\bar{a}_0)$, be a solution of the optimization problem (26). We consider the following perturbed regularized optimization problem: Find $\bar{a}_\varepsilon \in A$ by solving

$$\min_{a\in A} \widetilde{J}_\varepsilon(a) = \frac{1}{2}\|u_\varepsilon(a) - z\|_Z^2 + \frac{1}{2}\|a - \bar{a}_0\|_L^2 + \kappa R(a), \tag{34}$$

where for $a \in A$, the element $u_\varepsilon(a)$ solves (28).

The following result shows the convergence steering towards the chosen minimizer:

Theorem 3.4. *For every $\varepsilon > 0$, (34) has a minimizer. There exists a sequence $\{(\bar{a}_\varepsilon, \bar{u}_\varepsilon)\}$, where $\bar{u}_\varepsilon = \bar{u}_\varepsilon(\bar{a}_\varepsilon)$ solves (28) and for $\varepsilon \to 0$, $\bar{a}_\varepsilon \to \bar{a}_0$ in L and $\bar{u}_\varepsilon \to \bar{u}_0$ in V.*

Proof. We skip the first part of the proof which is identical to that of Theorem 3.3. Let $\{\bar{a}_\varepsilon\}$ be a sequence of solutions of (34) converging in $\|\cdot\|_L$ to $\tilde{a} \in A$. Let $\bar{u}_\varepsilon$ be the corresponding sequence converging strongly to some $\tilde{u} = u(\tilde{a}) \in V$.

Given $\bar{a}_0$, let $\bar{u}_\varepsilon(\bar{a}_0)$ be the unique solution of the equation: Find $u \in V$ such that

$$T(\bar{a}_0, u, v) + \frac{1}{\varepsilon}\langle P(u), v\rangle = \langle m, v\rangle, \quad \text{for every } v \in V.$$

By using the arguments given in Theorem 3.3, it can be shown that $\bar{u}_\varepsilon(\bar{a}_0) \to \bar{u}_0$ as $\varepsilon \to 0$. Moreover, $(\bar{a}_0, \bar{u}_\varepsilon(\bar{a}_0))$ is feasible for (29), and we have, in fact, $\widetilde{J}_\varepsilon(\bar{a}_0) = \frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_0) - z\|_Z^2 + \kappa R(\bar{a}_0)$. Because of the minimality of $\bar{a}_\varepsilon$, for $(\bar{a}_\varepsilon, \bar{u}_\varepsilon)$, where $\bar{u}_\varepsilon$ solves (28), we have $\widetilde{J}_\varepsilon(\bar{a}_\varepsilon) \le \widetilde{J}_\varepsilon(\bar{a}_0)$, or

$$\widetilde{J}_\varepsilon(\bar{a}_\varepsilon) = \frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_\varepsilon) - z\|_Z^2 + \kappa R(\bar{a}_\varepsilon) + \frac{1}{2}\|\bar{a}_\varepsilon - \bar{a}_0\|_L^2 \le \frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_0) - z\|_Z^2 + \kappa R(\bar{a}_0),$$

which further implies that

$$\begin{aligned} 0 &\le \frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_\varepsilon) - z\|_Z^2 + \kappa R(\bar{a}_\varepsilon) = \widetilde{J}_\varepsilon(\bar{a}_\varepsilon) - \frac{1}{2}\|\bar{a}_\varepsilon - \bar{a}_0\|_L^2 \\ &\le \frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_0) - z\|_Z^2 + \kappa R(\bar{a}_0) - \frac{1}{2}\|\bar{a}_\varepsilon - \bar{a}_0\|_L^2, \end{aligned} \tag{35}$$

which, when passed to the limit $\varepsilon \to 0$, yields

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0}\left\{\frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_0) - z\|_Z^2 + \kappa R(\bar{a}_0) - \frac{1}{2}\|\bar{a}_\varepsilon - \bar{a}_0\|_L^2\right\} \\ &= \frac{1}{2}\|\bar{u}_0 - z\|_Z^2 + \kappa R(\bar{a}_0) - \frac{1}{2}\left\|\tilde{a} - \bar{a}_0\right\|_L^2 = J_\kappa(\bar{a}_0) - \frac{1}{2}\left\|\tilde{a} - \bar{a}_0\right\|_L^2. \end{aligned} \tag{36}$$

However, due to the minimality of $\bar{a}_0$, we also have $J_\kappa(\bar{a}_0) \le J_\kappa(\tilde{a})$ and hence

$$\begin{aligned} J_\kappa(\bar{a}_0) &\le \frac{1}{2}\|\tilde{u} - z\|_Z^2 + \kappa R(\tilde{a}) \le \liminf_{\varepsilon\to 0}\frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_\varepsilon) - z\|_Z^2 + \liminf_{\varepsilon\to 0}\kappa R(\bar{a}_\varepsilon) \\ &\le \liminf_{\varepsilon\to 0}\left\{\frac{1}{2}\|\bar{u}_\varepsilon(\bar{a}_\varepsilon) - z\|_Z^2 + \kappa R(\bar{a}_\varepsilon)\right\} \le J_\kappa(\bar{a}_0) - \frac{1}{2}\left\|\tilde{a} - \bar{a}_0\right\|_L^2, \end{aligned}$$

by (35) and (36). Therefore, $\frac{1}{2}\left\|\tilde{a} - \bar{a}_0\right\|_L^2 \le 0$, and hence $\tilde{a} = \bar{a}_0$ and also $\tilde{u} = \bar{u}_0$. □

An advantage of replacing the variational inequality by the penalized equation is that for the latter, the parameter-to-solution map is smooth, given that the penalty map is smooth. Therefore, we shall replace P by its smooth approximation. For simplicity, we take $P(u) = (I - P_K)(u)$, and for $\varepsilon \ge 0$, define a family of its smooth approximation $P_\varepsilon \colon V \to V$ in a Hilbert space V, satisfying the following conditions which are partly inspired by [7]:

PC1. For $\varepsilon > 0$, P_ε is monotone, $N(P_\varepsilon) = K$, and for $v \in V$, $P_\varepsilon(v) \to P(v)$ as $\varepsilon \to 0$.

PC2. For $\varepsilon > 0$, the derivative P'_ε of P_ε exists at every point and satisfies the following:

$$\langle P'_\varepsilon(u)v, v\rangle \ge 0, \quad \text{for every } u, v \in V. \tag{37}$$

$$\langle P'_\varepsilon(u)v, P_K(u)\rangle = 0, \quad \text{for every } u, v \in V. \tag{38}$$

Given a penalty parameter $\varepsilon > 0$ and a family of smooth penalty maps $\{P_\varepsilon\}$, we now consider the following penalized variational equation: Find $u_\varepsilon \in V$ such that

$$T(a, u_\varepsilon, v) + \frac{1}{\varepsilon} \langle P_\varepsilon u_\varepsilon, v \rangle = \langle m, v \rangle, \quad \text{for every } v \in V. \tag{39}$$

For any $\varepsilon > 0$, let $u_\varepsilon(a)$ be the unique solution of (39) (which exists due to Theorem 3.2).

The following result shows the smoothness of the parameter-to-solution map:

Theorem 3.5. *For $\varepsilon > 0$, the map $a \to u_\varepsilon(a)$ is differentiable at any point a in the interior of A. For any direction $\delta a \in B$, the derivative $\delta u_\varepsilon = Du_\varepsilon(a)(\delta a)$ is the unique solution of the following variational equation:*

$$T(a, \delta u_\varepsilon, v) + \frac{1}{\varepsilon} \langle P'_\varepsilon(u_\varepsilon)\delta u_\varepsilon, v \rangle = -T(\delta a, u_\varepsilon, v), \quad \textit{for every } v \in V. \tag{40}$$

Proof. The differentiability follows by applying the implicit function theorem to the map $G\colon A \times V \to V$ mapping $(a, u) \to T(a, u, \cdot) + \frac{1}{\varepsilon} \langle P_\varepsilon(u), \cdot \rangle$, where $T(a, u, \cdot)$, $\langle P_\varepsilon(u), \cdot \rangle$ are the associated dual element given by the Riesz representation theorem. The derivative $D_u G(a, u)\colon A \times V \to V$ is given by $D_u G(a, u)(\delta u) = T(a, \delta u, \cdot) + \frac{1}{\varepsilon} \langle P'_\varepsilon(u)(\delta u), \cdot \rangle$. For any $w \in V$, the coercivity of $T(a, \cdot) + \frac{1}{\varepsilon} P'_\varepsilon(u)(\cdot)$ (see (2) and (37)) ensures that the equation

$$T(a, \delta u, \cdot) + \frac{1}{\varepsilon} \langle P'_\varepsilon(u)(\delta u), \cdot \rangle = \langle w, \cdot \rangle$$

is uniquely solvable. Therefore $D_u G(a, \cdot)(u)\colon V \to V$ is a linear homeomorphism and the differentiability follows from the implicit function theorem. Finally, from (28) or (39), for every $v \in V$, we have

$$T(\delta a, u_\varepsilon, v) + T(a, \delta u_\varepsilon, v) + \frac{1}{\varepsilon} \langle P'_\varepsilon(u_\varepsilon)\delta u_\varepsilon, v \rangle = 0,$$

and (40) follows. The proof is complete. □

We consider the following regularized optimization problem: Find $\bar{a}_\varepsilon \in A$ by solving

$$\min_{a \in A} J_\varepsilon(a) = \frac{1}{2} \| u_\varepsilon(a) - z \|_Z^2 + \kappa R(a), \tag{41}$$

where for $a \in A$, the element $u_\varepsilon(a)$ solves the penalized variational equation (39).

In the following optimality condition for (41), $P'_\varepsilon(\bar{u}_\varepsilon)^*$ is the adjoint of $P'_\varepsilon(\bar{u}_\varepsilon)$:

Theorem 3.6. *For each $\varepsilon > 0$, problem (41) has a minimizer. Moreover, for any solution $\bar{a}_\varepsilon \in A$ of (41), there exists $\bar{p}_\varepsilon \in V$, uniformly bounded in V, such that*

$$T(\bar{a}_\varepsilon, \bar{p}_\varepsilon, v) + \frac{1}{\varepsilon} \langle P_\varepsilon'(\bar{u}_\varepsilon)^* \bar{p}_\varepsilon, v \rangle = \langle \bar{u}_\varepsilon - z, v \rangle_Z, \quad \textit{for every } v \in V. \tag{42}$$

$$T(\bar{a}_\varepsilon - a, \bar{u}_\varepsilon, \bar{p}_\varepsilon) \geq \kappa \left(R(\bar{a}_\varepsilon) - R(a) \right), \quad \textit{for every } a \in A. \tag{43}$$

Proof. Let $\varepsilon > 0$ be fixed. Let $\bar{a}_\varepsilon \in A$ be a solution of (41). Then the following mixed variational inequality holds as a necessary optimality condition:

$$D\widehat{J}_\varepsilon(\bar{a}_\varepsilon)(a - \bar{a}_\varepsilon) \geq \kappa \left(R(\bar{a}_\varepsilon) - R(a) \right), \quad \text{for every } a \in A, \tag{44}$$

where $\widehat{J}_\varepsilon(\bar{a}_\varepsilon) := \frac{1}{2} \|u_\varepsilon(\bar{a}_\varepsilon) - z\|_Z^2 = \frac{1}{2} \|\bar{u}_\varepsilon - z\|_Z^2$. Clearly, for any $a \in A$, we have

$$D\widehat{J}_\varepsilon(\bar{a}_\varepsilon)(a - \bar{a}_\varepsilon) = \langle \delta\bar{u}_\varepsilon, \bar{u}_\varepsilon - z \rangle_Z,$$

where $\delta\bar{u}_\varepsilon = Du\left(\bar{a}_\varepsilon\right)(a - \bar{a}_\varepsilon)$.

We now define the adjoint equation (associated to (39)): Find $p_\varepsilon \in V$ such that

$$T(\bar{a}_\varepsilon, p_\varepsilon, v) + \frac{1}{\varepsilon} \langle P_\varepsilon'(\bar{u}_\varepsilon)^* p_\varepsilon, v \rangle = \langle \bar{u}_\varepsilon - z, v \rangle_Z, \quad \text{for every } v \in V. \tag{45}$$

Clearly (45) is uniquely solvable. Denoting by $\bar{p}_\varepsilon$ the unique solution to (45), we have

$$\begin{aligned}
\langle \bar{u}_\varepsilon - z, \delta\bar{u}_\varepsilon \rangle_Z &= T(\bar{a}_\varepsilon, \bar{p}_\varepsilon, \delta\bar{u}_\varepsilon) + \frac{1}{\varepsilon} \langle P_\varepsilon'(\bar{u}_\varepsilon)^* \bar{p}_\varepsilon, \delta\bar{u}_\varepsilon \rangle \\
&= T(\bar{a}_\varepsilon, \delta\bar{u}_\varepsilon, \bar{p}_\varepsilon) + \frac{1}{\varepsilon} \langle P_\varepsilon'(\bar{u}_\varepsilon) \delta\bar{u}_\varepsilon, \bar{p}_\varepsilon \rangle \\
&= -T(a - \bar{a}_\varepsilon, \bar{u}_\varepsilon, \bar{p}_\varepsilon) = T(\bar{a}_\varepsilon - a, \bar{u}_\varepsilon, \bar{p}_\varepsilon),
\end{aligned}$$

where we used the symmetry and linearity of T and derivative formula (40). Therefore,

$$D\widehat{J}_\varepsilon(\bar{a}_\varepsilon)(a - \bar{a}_\varepsilon) = T(\bar{a}_\varepsilon - a, \bar{u}_\varepsilon, \bar{p}_\varepsilon),$$

and (43) follows directly by substituting this expression in (44).

We still need to show that $\{\bar{p}_\varepsilon\}$ is uniformly bounded. For this, we take $v = \bar{p}_\varepsilon$ in the adjoint equation (45) and by using ellipticity of T and (37), we obtain

$$\alpha \|\bar{p}_\varepsilon\|^2 \leq T(\bar{a}_\varepsilon, \bar{p}_\varepsilon, \bar{p}_\varepsilon) + \frac{1}{\varepsilon} \langle P_\varepsilon'(\bar{u}_\varepsilon)^* \bar{p}_\varepsilon, \bar{p}_\varepsilon \rangle = \langle \bar{u}_\varepsilon - z, \bar{p}_\varepsilon \rangle_Z \leq C_1 \|\bar{p}_\varepsilon\|_V$$

where we also used the fact that $\{\bar{u}_\varepsilon\}$ is bounded. The proof is complete. □

We are now ready to give the following optimality conditions:

Theorem 3.7. *There exists a minimizer* $\bar{a}_0$ *of* (26) *and* $\bar{u}_0 \in V$, $\bar{p}_0 \in V$, $\lambda_0 \in V^*$ *with*

$$T(\bar{a}_0, \bar{u}_0, v - \bar{u}_0) \geq \langle m, v - \bar{u}_0 \rangle, \quad \text{for every } v \in K, \tag{46}$$

$$T(\bar{a}_0, \bar{p}_0, v) + \lambda_0^*(v) = \langle \bar{u}_0 - z, v \rangle_Z, \quad \text{for every } v \in V, \tag{47}$$

$$T(\bar{a}_0 - a, \bar{u}_0, \bar{p}_0) \geq \kappa \left(R(\bar{a}_0) - R(a) \right), \quad \text{for every } a \in A, \tag{48}$$

$$\lambda_0^*(\bar{u}_0) = 0, \tag{49}$$

$$\|\bar{p}_0\|_V \leq \alpha^{-1} \|\bar{u}_0 - z\|_Z. \tag{50}$$

Proof. For $\varepsilon > 0$, let $\{\bar{a}_\varepsilon\}$ be a sequence of solutions of (41), let $\bar{u}_\varepsilon$ be the corresponding solutions of the penalized equation (39), and let $\bar{p}_\varepsilon$ be the sequence of adjoint solutions defined in (45). Following the line of arguments used in Theorem 3.3, it can be shown that there exists a subsequence that converges in $\|\cdot\|_L$ to some $\bar{a}_0 \in A$. We need to keep in mind that the corresponding solutions $\bar{u}_\varepsilon$ solve the penalized equations that involve the family of smooth penalty maps and hence the arguments need to be slightly modified. In fact, it suffices to notice that (30) holds for P_ε, however, due to the assumption $P_\varepsilon(u) \to P(u)$ for $\varepsilon \to 0$, the conclusion of (31) remains valid and the rest of the arguments go through without any change. Moreover, the proof of the strong convergence of $\bar{u}_\varepsilon$ to $\bar{u}_0 = \bar{u}_0(\bar{a}_0)$ holds without any change and hence (46) holds.

We have shown that the sequence $\{\bar{p}_\varepsilon\} \subset V$ is bounded and consequently there exists a weakly convergent subsequence. By keeping the same notation for subsequences as well, we assume that $\bar{p}_\varepsilon \rightharpoonup \bar{p}_0 \in V$. We define two functionals λ_ε^*, $\lambda_0^*\colon V \to \mathbb{R}$ by

$$\lambda_\varepsilon^*(v) := \frac{1}{\varepsilon} \langle P'_\varepsilon(\bar{u}_\varepsilon)^* \bar{p}_\varepsilon, v \rangle = \langle \bar{u}_\varepsilon - z, v \rangle_Z - T(\bar{a}_\varepsilon, \bar{p}_\varepsilon, v), \tag{51}$$

$$\lambda_0^*(v) := \langle \bar{u}_0 - z, v \rangle_Z - T(\bar{a}_0, \bar{p}_0, v). \tag{52}$$

The above functionals are well defined, linear, and continuous, and hence λ_ε^*, $\lambda_0^* \in V^*$. Furthermore, due to the convergence $\bar{a}_\varepsilon \to \bar{a}_0$ in $\|\cdot\|_L$ and $\bar{u}_\varepsilon \to \bar{u}_0$ in $\|\cdot\|_V$, we have

$$\lambda_\varepsilon^*(v) = \langle \bar{u}_\varepsilon - z, v \rangle_Z - T(\bar{a}_\varepsilon, \bar{p}_\varepsilon, v) \to \langle \bar{u}_0 - z, v \rangle_Z - T(\bar{a}_0, \bar{p}_0, v) = \lambda_0^*(v)$$

as $\varepsilon \to 0$. Since this convergence is true for any $v \in V$, we deduce that the sequence $\{\lambda_\varepsilon^*\}$ converges weakly to $\{\lambda_0^*\}$ in V^* topology, that is, $\lambda_\varepsilon^* \underset{V^*}{\rightharpoonup} \lambda_0^*$. Taking $v = P_K(\bar{u}_\varepsilon)$ in (51) and using (38), we get

$$\lambda_\varepsilon^*(P_K(\bar{u}_\varepsilon)) = \frac{1}{\varepsilon} \langle P'_\varepsilon(\bar{u}_\varepsilon)^* \bar{p}_\varepsilon, P_K(\bar{u}_\varepsilon) \rangle = 0.$$

By using the continuity of the projection map, we get $0 = \lambda_\varepsilon^*(P_K(\bar{u}_\varepsilon)) \to \lambda_0^*(\bar{u}_0)$ for $\varepsilon \to 0$, and consequently, $\lambda_0^*(\bar{u}_0) = 0$. Furthermore, by property (37), we have

$$\lambda_\varepsilon^*(\bar{p}_\varepsilon) = \frac{1}{\varepsilon} \langle P_\varepsilon'(\bar{u}_\varepsilon)^* \bar{p}_\varepsilon, \bar{p}_\varepsilon \rangle \geq 0, \tag{53}$$

however, we also have

$$\lambda_\varepsilon^*(\bar{p}_\varepsilon) = \langle \bar{u}_\varepsilon - z, \bar{p}_\varepsilon \rangle_Z - T(\bar{a}_\varepsilon, \bar{p}_\varepsilon, \bar{p}_\varepsilon) \leq \|\bar{u}_\varepsilon - z\|_Z \|\bar{p}_\varepsilon\|_V - \alpha \|\bar{p}_\varepsilon\|_V^2,$$

which in conjunction with (53), implies that

$$\|\bar{p}_\varepsilon\|_V \leq \alpha^{-1} \|\bar{u}_\varepsilon - z\|_Z,$$

which, when passed to the limit $\varepsilon \to 0$, confirms that

$$\|\bar{p}_0\|_V \leq \liminf_{\varepsilon \to 0} \|\bar{p}_\varepsilon\|_V \leq \alpha^{-1} \|\bar{u}_0 - z\|_Z,$$

proving (50). For (48), it suffices to take limits in (43). In fact, since $\bar{a}_\varepsilon \underset{L}{\to} \bar{a}_0$, $\bar{u}_\varepsilon \underset{V}{\to} \bar{u}_0$, and $\bar{p}_\varepsilon \underset{V}{\rightharpoonup} \bar{p}_0$, by using the properties of T, and by taking the limit $\varepsilon \to 0$ in

$$T(\bar{a}_\varepsilon - a, \bar{u}_\varepsilon, \bar{p}_\varepsilon) \geq \kappa \left(R(\bar{a}_\varepsilon) - R(a) \right)$$

we get

$$T(\bar{a}_0 - a, \bar{u}_0, \bar{p}_0) \geq \kappa \left(R(\bar{a}_0) - R(a) \right),$$

which completes the proof. $\square$

4. Finite dimensional approximation of quasi-variational inequalities

Our objective now is to discretize the parameter identification problem. For simplicity, we drop F and consider the following quasi-variational inequality: Given $a \in A$, find $u = u(a) \in K$ such that

$$\langle T(a, u) - m, v - u \rangle \geq \Phi(u, u) - \Phi(u, v), \quad \text{for every } v \in K. \tag{54}$$

We consider the following regularized optimization problem: Find $a \in A$ by solving

$$\min_{a \in A} J_\kappa(a) = \frac{1}{2} \|u(a) - z\|_Z^2 + \kappa R(a), \tag{55}$$

where $\kappa > 0$ is a regularization parameter, R is the regularization map, $u(a)$ is the unique solution of (54), and z is the measured data.

Assume that we have a parameter h converging to 0 and a family $\{V_h\}$ of finite dimensional subspaces of V. We assume that $\{B_h\}$ is a family of finite-dimensional subspaces of B such that $\overline{\cup \{B_h\}} = B$. We set $A_h = B_h \cap A$ and assume that $\cap_h A_h \neq \emptyset$. Let $\{K_h\}$ be a family of closed and convex sets with $K_h \subset V_h$ approximating K in the following sense:

1. For every $z \in K$, there exists $z_h \in K_h$ such that $z_h \to z$ as $h \to 0$.
2. For every $z_h \in K_h$ such that $z_h \rightharpoonup z$ as $h \to 0$, we have $z \in K$.

We denote by R_h the discrete analogue of R and assume the conditions:

1. For any $a \in A$, there exists a sequence $\{a_h\}$ with $a_h \in A_h$ such that $a_h \to a$ in L and
$$\limsup_{h\to 0} R_h(a_h) \leq R(a). \tag{56}$$

2. For every $\{a_h\}$ with $a_h \in A_h$ and $a_h \to a$ in L, we have
$$R(a) \leq \liminf_{h\to 0} R_h(a_h). \tag{57}$$

We approximate Φ by the discrete function $\Phi_h \colon V_h \times V_h \to \mathbb{R}$ and assume:

1. For $v_h \in V_h$, the map $\Phi_h(v_h, \cdot) : V \to \mathbb{R}$ is convex, and lower-semicontinuous, $\Phi_h(0, \cdot) = 0$, and there exists $0 < \delta \leq \alpha$ such that for all $u, v, u_1, v_1 \in K_h$, we have
$$|\Phi_h(u, v_1) + \Phi_h(u_1, v) - \Phi_h(u, v) - \Phi_h(u_1, v_1)| \leq \delta \|u - u_1\|_V \|v - v_1\|_V. \tag{58}$$

2. For any sequence $\{z_h\}$ with $z_h \in K_h$ and $z_h \to z$ in V as $h \to 0$ and for any sequence $\{u_h\} \subset V$ with $u_h \rightharpoonup u \in V$, as $h \to 0$, we have
$$\limsup_{h\to 0} [\Phi_h(u_h, z_h) - \Phi_h(u_h, u_h)] \leq \Phi(u, z) - \Phi(u, u). \tag{59}$$

The discrete analogue of (54) then reads: Given $a_h \in A_h$, find $u_h \in K_h$ such that
$$\langle T(a_h, u_h) - m, v_h - u_h\rangle \geq \Phi_h(u_h, u_h) - \Phi_h(u_h, v_h), \text{ for every } v_h \in K_h. \tag{60}$$

Consider the following finite-dimensional minimization problem: Find $a_h^* \in A_h$ by solving
$$\min_{a_h \in A_h} J_\kappa^h(a_h) = \frac{1}{2}\|u_h - z\|_Z^2 + \kappa R_h(a_h), \tag{61}$$
where u_h is the unique solution of (60). Evidently, the data z could be replaced by discrete data z_h such that $z_h \to z$ as $h \to 0$.

The following result ensures the convergence of the finite-dimensional (61):

Theorem 4.1. *For each $h > 0$, discrete problem (61) has a minimizer $a_h^* \in A_h$. Moreover, there is a subsequence of $\{a_h^*\}$ that converges in $\|\cdot\|_L$ to a solution of (55).*

Proof. For every h, there exists a constant C such that $J_\kappa^h(a_h^*) \leq C$, and hence $\{a_h^*\}$ is bounded in $\|\cdot\|_{\widehat{B}}$. Due to the compact embedding, there is a subsequence that converges strongly, in $\|\cdot\|_L$, to an element of A. We assume that $\{a_h^*\}$

converges, in $\|\cdot\|_L$, to some $a^* \in A$. Let u^* be the unique solution of quasi-variational inequality (54) for the parameter a^* and let u_h^* be the unique solution of quasi-variational inequality (60) for the parameter a_h^*, respectively. It can be shown that the sequence $\{u_h^*\}$ is bounded, independently of h. Therefore, there exists a subsequence, still denoted by $\{u_h^*\}$, such that u_h^* converges weakly to some $u^* \in K$ as $h \to 0$. We will show that $u^* = u^*(a^*)$.

For every $v_h \in K_h$, we have

$$\langle T(a_h^*, u_h^*) - m, v_h - u_h\rangle \geq \Phi_h(u_h, u_h) - \Phi_h(u_h, v_h).$$

Let $z \in K$ be arbitrary and let $\{z_h\}$ be the sequence with $z_h \in K_h$ such that $z_h \to z$ in V. We set $v_h = z_h$ in the above inequality, and after a rearrangement, obtain,

$$\begin{aligned}\langle T(a_h^*, z_h) - m, z_h - u_h^*\rangle &\geq T(a_h^*, u_h^* - z_h, u_h^* - z_h) + \Phi_h(u_h^*, u_h^*) - \Phi_h(u_h^*, z_h)\\ &\geq \Phi_h(u_h^*, u_h^*) - \Phi_h(u_h^*, z_h),\end{aligned}$$

and consequently,

$$\langle T(a^*, z_h) - m, z_h - u_h^*\rangle + T(a_h^* - a^*, z_h, z_h - u_h^*) \geq \Phi_h(u_h^*, u_h^*) - \Phi_h(u_h^*, z_h),$$

which, in view of (15) and (21), when passed to the limit $h \to 0$, implies that

$$\langle T(a^*, z) - m, z - u^*\rangle \geq \Phi(u^*, u^*) - \Phi(u^*, z), \quad \text{for every } z \in K$$

and by proceeding as before, we can ensure that $u^* = u^*(a^*)$.

This observation, in view of (57), yields

$$\begin{aligned}J_\kappa(a^*) &= \frac{1}{2}\|u^*(a^*) - z\|_Z^2 + \kappa R(a^*)\\ &\leq \lim_{h\to 0} \frac{1}{2}\|u_h^*(a_h^*) - z\|_Z^2 + \liminf_{h\to 0} \kappa R_h(a_h^*)\\ &\leq \liminf_{h\to 0} \left\{\frac{1}{2}\|u_h^*(a_h^*) - z\|_Z^2 + \kappa R_h(a_h^*)\right\}.\end{aligned}$$

Let $a \in A$ be arbitrary. Then there exists a sequence $\{a_h\} \subset A_h$ such that $a_h \to a$ in L and $R_h(a_h) \to R(a)$ as $h \to 0$ (see (56) and (57)). Hence, by the above inequality, we get

$$\begin{aligned}J_\kappa(a^*) &\leq \liminf_{h\to 0} \left\{\frac{1}{2}\|u_h^*(a_h^*) - z\|_Z^2 + \kappa R_h(a_h^*)\right\}\\ &\leq \liminf_{h\to 0} \left\{\frac{1}{2}\|u_h(a_h) - z\|_Z^2 + \kappa R_h(a_h)\right\}\\ &= \frac{1}{2}\|u(a) - z\|_Z^2 + \kappa R(a) = J_\kappa(a),\end{aligned}$$

and hence a^* is a solution of (55). This completes the proof. □

Conditions on the discrete approximate Φ_h are motivated by some related verifiable conditions used in [10]. Moreover, the construction of the sequence a_h in (56) (and condition (57)) can be done by following the arguments given in [11] for the total variation regularizer.

5. Concluding remarks

We have studied the inverse problem of parameter identification in certain variational and quasi-variational inequalities. In this work, we have only employed the output least-squares formulation and it is of interest to introduce analogues of the modified output least-squares and equation error approach which have been used extensively for identification problems in simpler variational problems. Aspects such as stability with respect to data perturbation also need to be addressed. Moreover, it is also natural to consider more special quasi-variational inequalities emerging from different applications (see [18, 35]). Another important question is whether the study of inverse problems can be extended to noncoercive/semicoercive variational problems (see [22]). We would also like to study inverse problems for parabolic quasi-variational inequalities (see [34]). Finally, in this work, we used a smooth family of penalty maps to approximate the nonsmooth penalty map. We note that for the variational inequalities of second kind, smoothing techniques recently developed in [23, 44] seem quite relevant to develop an alternative smooth approximation scheme.

References

[1] J. Abouchabaka, R. Aboulaïch, A. Nachaoui, A. Souissi: Quasi-variational inequality and shape optimization for solution of a free boundary problem, COMPEL 18(2) (1999) 143–164.

[2] S. Adly, M. Bergounioux, M. Ait Mansour: Optimal control of a quasi-variational obstacle problem, J. Global Optim. 47(3) (2010) 421–435.

[3] H. Attouch, G. Buttazzo, G. Michaille: Variational Analysis in Sobolev and BV Spaces, 2nd edition, MOS-SIAM Series on Optimization, Society for Industrial and Applied Mathematics, Philadelphia (2014).

[4] A. Barbagallo: Regularity results for evolutionary nonlinear variational and quasi-variational inequalities with applications to dynamic equilibrium problems, J. Global Optim. 40(1-3) (2008) 29–39.

[5] J. W. Barrett, L. Prigozhin: A quasi-variational inequality problem in superconductivity, Math. Models Methods Appl. Sci. 20(5) (2010) 679–706.

[6] J. W. Barrett, L. Prigozhin: A quasi-variational inequality problem arising in the modeling of growing sandpiles, ESAIM Math. Model. Numer. Anal. 47(4) (2013) 1133–1165.

[7] G. Bayada, M. El Alaoui Talibi: An application of the control by coefficients in a variational inequality for hydrodynamic lubrication, Nonlinear Anal. Real World Appl. 1(2) (2000) 315–328.

[8] A. Borzì, C. Kanzow: Formulation and numerical solution of Nash equilibrium multiobjective elliptic control problems, SIAM J. Control Optim. 51(1) (2013) 718–744.

[9] S. Boulaaras, M. A. Bencheikh le Hocine, M. Haiour: The finite element approximation in a system of parabolic quasi-variational inequalities related to management of energy production with mixed boundary condition, Comput. Math. Model. 25(4) (2014) 530–543.

[10] R. A. Capatina, M. Cocu: Internal approximation of quasi-variational inequalities, Numer. Math. 59(4) (1991) 385–398.

[11] Z. Chen, J. Zou: An augmented Lagrangian method for identifying discontinuous parameters in elliptic systems, SIAM J. Control Optim. 37(3) (1999) 892–910.

[12] M. Cocu: Existence of solutions of Signorini problems with friction, Int. J. Eng. Sci. 22 (1984) 567–575.

[13] P. Daniele: A remark on a dynamic model of a quasi-variational inequality, Rend. Circ. Mat. Palermo (2) Suppl. 48 (1997) 91–100.

[14] A. Dreves, J. Gwinner: Jointly convex generalized Nash equilibria and elliptic multiobjective optimal control, J. Optim. Theory Appl. 168(3) 1065–1086.

[15] C. Eck, J. Jarusek: Solvability of static contact problems with Coulomb friction for orthotropic material, J. Elasticity 93(1) (2008) 93–104.

[16] T. Fukao, N. Kenmochi: Quasi-variational inequality approach to heat convection problems with temperature dependent velocity constraint, Discrete Contin. Dyn. Syst. 35(6) (2015) 2523–2538.

[17] F. Giannessi: Separation of sets and gap functions for quasi-variational inequalities, in: Variational inequalities and network equilibrium problems, Erice 1994, Plenum Publishers, New York (1995) 101–121.

[18] F. Giannessi, A. A. Khan: Regularization of non-coercive quasi variational inequalities, Control Cybernet. 29(1) (2000) 91–110.

[19] M. S. Gockenbach, A. A. Khan: An abstract framework for elliptic inverse problems. I: An output least-squares approach, Math. Mech. Solids 12(3) (2007) 259–276.

[20] G. A. González: Theoretical framework of an identification problem for an elliptic variational inequality with bilateral restrictions, J. Comput. Appl. Math. 197(1) (2006) 245–252.

[21] J. Gwinner: On the penalty method for constrained variational inequalities, in: Optimization: Theory and Algorithms, Confolant 1981, Lecture Notes Pure Appl. Math. 86 (1983) 197–211.

[22] J. Gwinner: A discretization theory for monotone semicoercive problems and finite element convergence for p-harmonic Signorini problems, Z. Angew. Math. Mech. 74(9) (1994) 417–427.

[23] J. Gwinner, N. Ovcharova: From solvability and approximation of variational inequalities to solution of nondifferentiable optimization problems in contact mechanics, Optimization 64(8) (2015) 1683–1702.

[24] P. T. Harker: Generalized Nash games and quasi-variational inequalities, Eur. J. Oper. Res. 54(1) (1991) 81–94.

[25] M. Hintermüller: Inverse coefficient problems for variational inequalities: optimality conditions and numerical realization, M2AN Math. Model. Numer. Anal. 35(1) (2001) 129–152.

[26] M. Hintermüller, T. Surowiec: A PDE-constrained generalized Nash equilibrium problem with pointwise control and state constraints, Pac. J. Optim. 9(2) (2013) 251–273.

[27] K. H. Hoffmann, J. Sprekels: On the identification of parameters in general variational inequalities by asymptotic regularization, SIAM J. Math. Anal. 17(5) (1986) 1198–1217.

[28] B. Hu: A quasi-variational inequality arising in elastohydrodynamics, SIAM J. Math. Anal. 21(1) (1990) 18–36.

[29] K. Ito, K. Kunisch: The augmented Lagrangian method for parameter estimation in elliptic systems, SIAM J. Control Optim. 28(1) (1990) 113–136.

[30] B. Jadamba, A. A. Khan, G. Rus, M. Sama, B. Winkler: A new convex inversion framework for parameter identification in saddle point problems with an application to the elasticity imaging inverse problem of predicting tumor location, SIAM J. Appl. Math. 74(5) (2014) 1486–1510.

[31] B. Jadamba, A. A. Khan, M. Sama: Inverse problems of parameter identification in partial differential equations, in: Mathematics in Science and Technology, World Sci. Publ., Hackensack (2011) 228–258.

[32] A. Kadoya, N. Kenmochi, M. Niezgódka: Quasi-variational inequalities in economic growth models with technological development, Adv. Math. Sci. Appl. 24(1) (2014) 185–214.

[33] T. Kaerkkaeinen: An equation error method to recover diffusion from the distributed observation, Inverse Problems 13(4) (1997) 1033–1051.

[34] A. A. Khan, D. Motreanu: Existence theorems for elliptic and evolutionary variational and quasi-variational inequalities, J. Optim. Theory Appl. 167(3) (2015) 1136–1161.

[35] A. A. Khan, C. Tammer, C. Zalinescu: Regularization of quasi-variational inequalities, Optimization 64(8) (2015) 1703–1724.

[36] I. Knowles: Parameter identification for elliptic problems, J. Comput. Appl. Math. 131(1-2) (2001) 175–194.

[37] F. Lefèvre, N. Nassif: A quasi-variational inequality for the simulation of a MODFET transistor, COMPEL 18(2) (1999) 175–186.

[38] F. Lenzen, F. Becker, J. Lellmann, S. Petra, C. Schnörr: A class of quasi-variational inequalities for adaptive image denoising and decomposition, Comput. Optim. Appl. 54(2) (2013) 371–398.

[39] A. Maugeri: Variational and quasi-variational inequalities in network flow models. Recent developments in theory and algorithms, in: Variational Inequalities and Network Equilibrium Problems, Erice 1994, Plenum Publishers, New York (1995) 195–211.

[40] A. Maugeri, F. Raciti: On existence theorems for monotone and nonmonotone variational inequalities, J. Convex Analysis 16(3-4) (2009) 899–911.

[41] D. Motreanu, M. Sofonea: Quasivariational inequalities and applications in frictional contact problems with normal compliance, Adv. Math. Sci. Appl. 10(1) (2000) 103–118.

[42] K. Naito: A system of quasivariational inequalities and its application to reaction diffusion equations, Publ. Res. Inst. Math. Sci. 21(4) (1985) 677–698.

[43] N. R. Nassif: On the existence and uniqueness of solutions for a class of quasivariational inequalities to solve reverse-biased semiconductor devices, Ann. Mat. Pura Appl. (4) 144 (1986) 95–112.

[44] N. Ovcharova, J. Gwinner: A study of regularization techniques of nondifferentiable optimization in view of application to hemivariational inequalities, J. Optim. Theory Appl. 162(3) (2014) 754–778.

[45] L. Prigozhin: Variational inequalities in critical-state problems, Physica D 197(3-4) (2004) 197–210.

[46] J.-F. Rodrigues: On a quasi-variational inequality arising in semiconductor theory, Rev. Mat. Univ. Complut. Madrid 5(1) (1992) 137–151.

[47] M. Sofonea, A. Matei: History-dependent quasi-variational inequalities arising in contact mechanics, European J. Appl. Math. 22(5) (2011) 471–491.

[48] B. A. Ton: Time-dependent quasi-variational inequalities and the Nash equilibrium, Nonlinear Anal. 41(7-8, Ser. A: Theory Methods) (2000) 1057–1081.

[49] W. P. Ziemer: Weakly Differentiable Functions, Graduate Texts in Mathematics 120, Springer-Verlag, New York (1989).