Journal of Convex Analysis
Volume 25 (2018), No. 2, 571–594

# Metric Regularity, Polyhedrality and Complementarity

**Alexander D. Ioffe**
*Department of Mathematics, Technion, Haifa 32000, Israel*
*alexander.ioffe38@gmail.com*

*Dedicated to Antonino Maugeri on the occasion of his 70th birthday.*

Received: January 7, 2016
Revised manuscript received: August 16, 2016
Accepted: August 29, 2017

The paper is devoted to (metric) regularity theory of semi-linear mappings (set-valued mappings whose graphs are finite unions of convex polyhedral sets). The key results relate to mappings associated with certain "complementarity relations" on polyhedral sets that contain as particular case the standard complementarity of dual polyhedral cones. We also show how the well known results of Robinson and Dontchev-Rockafellar on variational inequalities over polyhedral sets follow from the results obtained in the paper.

*Keywords:* Variational analysis, semi-linear mapping, variational inequality, strong regularity, complementarity mapping.

*2010 Mathematics Subject Classification:* 49J40, 49J53, 52B11, 90C31

## 1. Introduction

Here we discuss metric regularity related problems for set-valued mappings whose graphs are unions of finitely many convex polyhedra. We call such sets *semi-linear* following tradition (cf. semi-algebraic, semi-analytic etc.) and in order to avoid possible terminological confusion.

Structurally, semi-linear sets are among the simplest. From the metric regularity point of view their most remarkable property is that there may be only finitely many different tangent and normal cones at points of any such set. This makes unnecessary any limiting procedures and typically reduces calculations to finitely many sufficiently elementary operations. As a result, the "standard" local metric regularity theory (centered around regularity criteria and estimates for regularity rates near a point of the graph) for general semi-linear mappings becomes rather simple.

ISSN 0944-6532 / $ 2.50 

On the other hand, a rich and sophisticated regularity theory was developed for linear variational inequalities

$$z \in Ax + N(C, x), \tag{1}$$

where $A$ is a linear operator in $I\!R^n$ and $C \subset I\!R^n$ is a convex polyhedron, closely connected with necessary optimality conditions in optimization. We refer to the papers of Robinson [12], Ralph [10] and Dontchev and Rockafellar [2] as well as to monographs by Facchinei – Pang [4] and Dontchev – Rockafellar [3] where the main results can be found. Most of them relate to the problem of global uniqueness and Lipschitz behavior of the solution mapping $S(z)$ of (1) that turned out to be highly nontrivial. The original and fairly complicated proofs given in [12, 10] and in some publications on which one of the principal results of [2] is based used mainly topological and algebraic machinery that typically does not appear in the variational analysis literature.

In an accompanying paper [8] we give much shorter proofs of the mentioned results fully based on elementary polyhedral geometry and some basic facts of metric regularity theory. We also give some additional geometric information about (1) and its solution mapping. The proofs are by no means elementary but they firmly place the results into the structures of variational analysis.

The core of this paper is an extension of the techniques and results of [8] to a more general class of semi-linear mappings associated with some “face complementarity relations” on convex polyhedral sets defined in § 3. In the nutshell, we replace normal cones in (1) by cones whose face structure is coordinated with the face structure of the set. This part of the paper (as well as [8]) is an elaboration upon a recent arXiv author’s article [7]. But we start with some results related to metric regularity of general semi-linear mappings proved at the end of the next section after a few statements containing necessary information about metric regularity and polyhedral geometry. In the third section we introduce and study the concept of face complementarity playing the central role in subsequent discussions. We also show that polyhedral cones that appear in complementarity relations are piecewise affine homeomorphic to normal cones to the set associated with the same faces. In the fourth section we introduce the main objects of our interest – complementarity mappings associated with complementarity relations on convex polyhedral sets. Finally, in Section 5 we prove three main results: equivalence of metric regularity and strong metric regularity for complementarity mappings, regularity criteria and a formula for the modulus of surjection/metric regularity. We conclude the paper with a brief discussion of interrelations between the results proved here and the corresponding results for linear variational inequalities.

**Some notation:**

$K$ stands for a cone (convex, usually polyhedral) and $L$ for a subspace;

$C^\circ = \{y : \langle y, x\rangle \leq 1, \ \forall\, x \in C\}$ is the *polar* of $C$, in particular,

$K^\circ = \{y : \langle y, x\rangle \le 0,\ \forall x \in K\}$ is the *polar cone* of $K$;

$L^\perp = \{y : \langle y, x\rangle = 0\}$ is the *orthogonal complement* to $L$;

$\operatorname{lin} K = K \cap (-K)$ is the *lineality* of $K$ (the maximal linear subspace contained in $K$);

$\operatorname{ri} C$ is the *relative interior* of a convex set $C$;

$L(C)$ is the *subspace* spanned by $C - C$;

$\operatorname{aff} C$ is the *affine hull* of $C$;

$d(x, C)$ is the *distance* from $x$ to $C$.

$T_C(x) = \{h : \liminf_{t\to+0} t^{-1} d(x + th, C) = 0\}$ is the *tangent (contingent) cone* to $C$ at $x \in C$;

$N_C(x) = \limsup_{x\to\overline{x}} (T_C(x))^\circ$ is the (limiting) *normal cone* to $C$ at $x$.

Recall that $N_C(x) = (T_C(x))^\circ$ if $C$ is convex.

## 2. Preliminaries

### 2.1. Regularity

We refer to a recent survey [6] for an account of an essential part of the metric regularity theory. Here we just mention a few facts needed for our purpose.

A set-valued mapping $\Phi\colon X \rightrightarrows Y$ is *(metrically) regular near* $(\bar{x}, \bar{y}) \in \operatorname{Graph}\Phi$ if there is a $\kappa > 0$ such that $d(x, \Phi^{-1}(y)) \le \kappa d(y, \Phi(x))$ for all $(x, y)$ close to $(\bar{x}, \bar{y})$. We say that $\Phi$ is *globally* metrically regular or just *regular* if the inequality holds for all $(x, y) \in I\!R^n \times I\!R^n$. The lower bound $\operatorname{reg}\Phi(\overline{x}|\overline{y})$ of such $\kappa$ is the *modulus* or *rate of metric regularity* of $\Phi$ near $(\bar{x}, \bar{y})$. If no such $\kappa$ exists, we set $\operatorname{reg}\Phi(\overline{x}|\overline{y}) = \infty$.

An equivalent property: $\Phi$ is *open at a linear rate near* $(\bar{x}, \bar{y})$ if there are $r > 0$ and $\varepsilon > 0$ such that $B(y, rt) \cap B(\overline{y}, \varepsilon) \subset \Phi(B(\overline{x}, t))$ for all $(x, y) \in \operatorname{Graph}\Phi$ sufficiently close to $(\bar{x}, \bar{y})$ and $t \in (0, \varepsilon)$. The upper bound $\operatorname{sur}\Phi(\overline{x}|\overline{y})$ of such $r$ is the *modulus* or *rate of surjection* of $\Phi$ near $(\bar{x}, \bar{y})$. If no such $r$ exists, we set $\operatorname{sur}\Phi(\overline{x}|\overline{y}) = 0$. The equality

$$\operatorname{reg}\Phi(\overline{x}|\overline{y}) \cdot \operatorname{sur}(\Phi(\overline{x}|\overline{y})) = 1$$

always holds, provided we adopt the convention that $0 \cdot \infty = 1$.

There is a third equivalent description of regularity using the *pseudo-Lipschitz* or *Aubin* property of the inverse map. We do not state it in full here, just mention that in case when $\Phi$ is regular near $(\bar{x}, \bar{y})$ and the restriction of the inverse mapping $\Phi^{-1}$ to a neighborhood of $(\bar{x}, \bar{y})$ is single-valued, it is necessarily Lipschitz.

The set-valued mapping $\Phi\colon X \rightrightarrows Y$ (both $X$ and $Y$ metric) is *strongly regular* near $(\bar{x}, \bar{y}) \in \operatorname{Graph}\Phi$ if it is metrically regular near $(\bar{x}, \bar{y})$ and there is a $\delta > 0$

such that $\Phi(x) \cap \Phi(u) \cap B(\overline{y}, \delta) = \emptyset$ for $x$, $u$ sufficiently close to $\overline{x}$ with $x \neq u$. If $\Phi$ is globally regular and $\Phi(x) \cap \Phi(u) = \emptyset$ for all $x \neq u$, we say that $\Phi$ is *globally* strongly regular.

The original (equivalent) definition by Robinson [11] is: $\Phi$ is strongly regular near $(\bar{x}, \bar{y}) \in \text{Graph } \Phi$ if for some $\varepsilon > 0$ the restriction of $\Phi^{-1}(y) \cap B(\overline{x}, \varepsilon)$ to a neighborhood of $\overline{y}$ is single-valued and Lipschitz.

A set-valued mapping $\Phi\colon X \rightrightarrows Y$ between linear spaces is *homogeneous* if its graph is a pointed cone. This means that $\Phi(\lambda x) = \lambda\Phi(x)$ and $\Phi^{-1}(\lambda y) = \lambda\Phi^{-1}(y)$ whenever $\lambda > 0$. The following theorem summarizes the most important properties of homogeneous set-valued mappings needed for our discussions.

**Theorem 2.1.** *Let $X$ and $Y$ be two Banach spaces, and let $\Phi\colon X \rightrightarrows Y$ be a homogeneous set-valued mapping.*

(a) *If $\Phi$ is (strongly) regular near* $(0, 0)$*, then it is globally (strongly) regular with the same rates.*

(b) *If* Graph $\Phi$ *is closed, then*

$$\text{sur}\Phi(0,0) = \inf_{((\bar{x}.\bar{y}))\in \text{Graph } \Phi} \inf\{\|x^*\| : (x^*, y^*) \in [T_{\text{Graph } \Phi}, (x, y))]^\circ,\ \|y^*\| = 1\}.$$

Another useful concept is subregularity: $\Phi$ is *subregular* at $(\bar{x}, \bar{y}) \in \text{Graph } \Phi$ if there is a $K > 0$ such that $d(x, \Phi^{-1}(\overline{y})) \leq Kd(\overline{y}, \Phi(x))$ for all $x$ of a neighborhood of $\overline{x}$. If moreover $\|x - \overline{x}\| \leq Kd(\overline{y}, \Phi(x))$ for $x$ close to $\overline{x}$, then we say that $\Phi$ is *strongly subregular* at $(\bar{x}, \bar{y})$. We refer to [3] for more information about strongly subregular mappings.

### 2.2. Polyhedral sets and mappings

The books [13] and [3] contain practically all necessary information about polyhedral sets needed in what follows. But there is some differences in accents as far as the problem we are going to consider here is concerned. We do not use the critical cone language as in [3] and instead pay more attention to the facial structure.

A set $C \subset I\!\!R^n$ is *polyhedral* or *convex polyhedral* if it is an intersection of finitely many closed linear half spaces, that is if

$$C = \{x \in I\!\!R^n :\ \langle y_i, x\rangle \leq \alpha_i,\ i = 1, \dots, k\} \tag{2}$$

for some nonzero $y_i \in I\!\!R^n$ and $\alpha_i \in I\!\!R$. Note that some of the inequalities may actually hold as equalities for elements of $C$, if $y_i = -y_j$ for some $i$ and $j$.

If all $\alpha_i$ are equal to zero, we get a definition of a *polyhedral cone.* A set-valued mapping $I\!\!R^n \rightrightarrows I\!\!R^m$ is *polyhedral* if its graph is polyhedral.

It is to be observed that in the literature the word "polyhedral" is sometimes used for finite unions of convex polyhedral sets. To avoid confusions, we shall call such unions *semi-linear sets.*

Given a closed convex set $C \subset I\!R^n$, a closed subset $F \subset C$ is a *face* of $C$, if for any line segment $[a, b]$

$$[a, b] \subset C \text{ and } x \in (\operatorname{ri}[a, b]) \cap F \;\Rightarrow\; [a, b] \subset F.$$

We denote by $\mathcal{F}_C$ the collection of all faces of $C$ and set for $x \in C$, $\overline{F} \in \mathcal{F}_C$

$$\mathcal{F}_C(x) = \{F \in \mathcal{F}_C :\ x \in F\}, \quad \mathcal{F}_C(\overline{F}) = \{F \in \mathcal{F}_C :\ \bar{F} \subset F\}.$$

A face $F \in \mathcal{F}_C$ is *exposed* if there is a nonzero $y \in I\!R^n$ such that

$$F = \{x \in C :\ \langle y, x\rangle = \max_{u \in C} \langle y, u\rangle\}$$

We shall say that $y$ *exposes* $F$ in $C$.

Clearly, given a representation of $C$ as in (2), every face of $C$ is a collection of elements of $C$ satisfying equations $\langle y_i, x\rangle = \alpha_i$ for some collection of indices. Therefore any face is itself a polyhedral set. The biggest face is the set itself. We shall call a face of $C$ *proper* if is different from $C$. The following facts about faces are well known (see [13], §§ 18,19) and geometrically obvious.

**Proposition 2.2.** *Let $C$ be a polyhedral set given by* (2)

(a) *for any $I \subset \{1, \ldots, k\}$ the set $F_I = \{x \in C :\ \langle y_i, x\rangle = \alpha_i,\ i \in I\}$, if nonempty, is a face, and for any $F \in \mathcal{F}_C$ there is an $I$ such that $F = F_I$. Hence $C$ has finitely many faces and intersection of two faces (if nonempty) is a face;*

(b) $\operatorname{ri} F_I = \{x \in F_I :\ \langle y_i, x\rangle < \alpha_i,\ \forall\, i \notin I\}$. *The family* $\{\operatorname{ri} F :\ F \in \mathcal{F}_C\}$ *is a partition of $C$;*

(c) *if $F \in \mathcal{F}_C$ and $F' \in \mathcal{F}_F$, then $F' \in \mathcal{F}_C$; if $F, F' \in \mathcal{F}_C$ and $F \cap \operatorname{ri} F' \neq \emptyset$, then $F' \subset F$;*

(d) $C \cap \operatorname{aff} F = F$ *for any $F \in \mathcal{F}_C$; if $x \in \operatorname{ri} F$, then every nonzero $y \in \operatorname{ri} N_C(x)$ exposes $F$;*

(e) *if $C$ is a cone, then any face of $C$ is a cone;* $\operatorname{lin} F = \operatorname{lin} C$ *for any $F \in \mathcal{F}_C$, that is, if $C$ is a cone, $\operatorname{lin} C$ is the minimal face of $C$ ;*

(f) *a linear image of a polyhedral set is also a polyhedral set.*

As $\mathcal{F}_C$ is a finite set, it follows from (a) that for any $x \in C$ there is a minimal face containing $x$. We shall denote it $F^C_{min}(x)$. But as a rule we shall omit the superscript and write simply $F_{min}(x)$ if it is clear from the context which set we are talking about.

Given a representation of $C$ by (2) and $x \in C$, let $I(x) = \{i \in \{1, \ldots, k\} : \langle y_i, x\rangle = \alpha_i\}$. Then

$$F_{min}(x) = \{u \in C :\ \langle y_i, u\rangle = \langle y_i, x\rangle,\ i \in I(x)\} = F_{I(x)}.$$

In view of (b), the following statement is now obvious.

**Proposition 2.3.** *Let $C$ be a polyhedral set. If $F \in \mathcal{F}_C$, then $x \in \operatorname{ri} F$ if and only if $F = F_{min}(x)$.*

Needless to say that the tangent cone to a polyhedral set at any point of the set is a polyhedral cone. We next consider some useful and simple properties of tangent cones to polyhedral sets.

**Proposition 2.4.** *Let $C$ be a polyhedral set and $F \in \mathcal{F}_C$. If $x, x' \in \mathrm{ri}\, F$, then $I(x) = I(x')$, $T_C(x) = T_C(x')$ and $N_C(x) = N_C(x')$.*

**Proof.** It is sufficient to show that any $h \in T_C(x)$ also belongs to $T_C(x')$. Indeed, as $x' \in$ ri $F$, we have $x' = \alpha x + (1-\alpha)w$ for some $\alpha \in (0,1)$ and $w \in F$. On the other hand, as $h \in T_C(x)$, we have $h = \lambda(u - x)$ for some $\lambda > 0$ and $u \in C$. Set $u' = \alpha u + (1-\alpha)w$. Then $u' \in C$ and $u' - x' = \alpha(u - x)$, that is $h = \alpha^{-1}(u' - x') \in T_C(x')$. □

Thus at points of a polyhedral set there may be only finitely many tangent and normal cones, as many as the number of different faces of the set. By the proposition, it is natural to use the notation $I(F)$, $T_C(F)$ and $N_C(F)$ for the common index sets, and tangent and normal cones to $C$ at elements of the relative interior of a face. It is natural to call the last two *tangent* and *normal cones to $C$ at $F$*.

**Proposition 2.5.** *Let $C \subset I\!R^n$ be a polyhedral set and $F \in \mathcal{F}_C$. Then*

$$T_C(F) = \mathrm{cone}(C - F).$$

*If furthermore $C$ is a cone, then $T_C(F) = C + L(F)$. Finally $K \in \mathcal{F}_{T_C(F)}$ if and only if there is a $F' \in \mathcal{F}_C(F)$ such that $K = \mathrm{cone}(F' - F)$.*

**Proof.** The equality in the displayed formula is an easy consequence of Proposition 2.6 below. The cone sign can be dropped in case $C$ is a cone as $C - F$ is a cone for any $F \in \mathcal{F}_C$ and the simple chain $C - F = (C + F) - F = C + L(F)$ proves the last equality.

Let finally $F' \in \mathcal{F}_C(F)$, the latter being the set of faces containing $F$. It is an easy matter to see that $y \neq 0$ exposes $F'$ in $C$, if and only if it exposes $\mathrm{cone}(F' - F)$ in $T_C(F)$. The last statement is now a consequence of Proposition 2.2(d). □

**Proposition 2.6 (local tangential representation).** *Let $C \subset I\!R^n$ be a polyhedral set and $\overline{x} \in C$. Then there is an $\varepsilon > 0$ such that*

$$C \cap B(\overline{x}, \varepsilon) = \overline{x} + T_C(\overline{x}) \cap (\varepsilon B).$$

**Proof.** Let $C = \{x :\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \dots, k\}$. It is an easy matter to see that

$$T_C(\overline{x}) = \{h :\ \langle x_i^*, h\rangle \le 0,\ i \in I(\overline{x})\}. \tag{3}$$

It follows that $\overline{x} + h \in C$ for an $h \in T_C(\overline{x})$ if the norm of $h$ is sufficiently small and $\langle x_i^*, \overline{x} + h\rangle < \alpha_i$ for all $i \notin I(\overline{x})$. Clearly, the latter holds if $\|h\| < \varepsilon$ with sufficiently small $\varepsilon > 0$. As $C \subset \overline{x} + T_C(\overline{x})$ (which is true for any closed convex set, not necessarily polyhedral), the result follows. □

The proposition says that the local geometry of a polyhedral set or even a semi-linear set near a certain point is fully determined by the geometry of its tangent cone at the point. (Indeed, it is immediate from the proposition that the tangent cone to a finite union of polyhedral sets a certain point is the union of the tangent cones to the sets at the point.) Thus, the regularity properties of a set-valued mapping with semi-linear graph are fully determined by the corresponding properties of the graphical derivative of the mapping at the point (the set-valued mapping whose graph is the tangent cone to the graph – see [14]) near $(0,0)$:

$$D\Phi(x,y)(h) = \{v :\ (h,v) \in T_{\text{Graph }\Phi}(x,y)\}.$$

This often allows to work with polyhedral or semi-linear cones rather than with general polyhedral or semi-linear sets.

The important point is that *there is a finite collection of homogeneous semi-linear mappings $R^n \rightrightarrows I\!R^m$ such that at any point $(x,y)$ of the graph of $\Phi$ the set $D\Phi(x,y)$ coincides with some element of the collection.* Indeed if $C$ is a polyhedral set, then, as follows from Proposition 2.4 there is a finite collection of convex polyhedral cones such that for each $x \in C$ the tangent cone to $C$ at $x$ coincides with some element of the collection.

## 2.3. Regularity properties of semi-linear mappings

**Proposition 2.7.** *A semi-linear mapping is subregular at any point of its graph.*

**Proof.** Consider first the case of a polyhedral set-valued mapping $F\colon I\!R^n \to I\!R^m$. Then

$$\text{Graph } F = \{(x,y) :\ \langle a_i, x\rangle + \langle b_i, y\rangle \le \alpha_i,\ i = 1,\dots,k\}.$$

so that $\Phi(x) = \{y :\ \langle b_i, y\rangle \le \alpha_i - \langle a_i, x\rangle,\ i = 1,\dots,k\}$. Thus, given a $(\bar{x},\bar{y}) \in \text{Graph } F$, we have by the Hoffmann error bound theorem (see e.g. [3]) that there is a $K > 0$ such that for any $x$ with $F(x) \neq \emptyset$ and any $y \in F(x)$

$$\begin{aligned} d(x, F^{-1}(\overline{y})) &\le K \textstyle\sum_i (\langle a_i, x\rangle + \langle b_i, \overline{y}\rangle - \alpha_i)^+ \\ &= K \textstyle\sum_i (\langle b_i, \overline{y} - y\rangle + (\langle a_i, x\rangle + \langle b_i, y\rangle - \alpha_i))^+ \\ &\le K \textstyle\sum_i \|b_i\| \|\overline{y} - y\| = K_1 \|\overline{y} - y\|. \end{aligned}$$

As this is true for any $y \in F(x)$, we conclude that $d(x, F^{-1}(\overline{y})) \le K_1 d(\overline{y}, F(x))$. Moreover, we see that $K_1$ is the same for all elements of Graph $F$.

Now let $\Phi$ be a semi-linear mapping. This means that there are finitely many mappings with polyhedral graphs $\Phi_i$ such that $\Phi(x) = \sum_i \Phi_i(x)$ for all $x$. Let again $(\bar{x},\bar{y}) \in$ Graph $\Phi$. Then for any $i$ such that $\overline{y} \in \Phi_i(\overline{x})$ and any $x$ such that $\Phi_i(x) \neq \emptyset$, we have

$$d(x, \Phi^{-1}(\overline{y})) \le d(x, \Phi_i^{-1}(\overline{y})) \le K_i d(\overline{y}, \Phi_i(x)).$$

If $\overline{y} \notin \Phi_i(\overline{x})$ for some $i$, that is $\langle b_i, \overline{y}\rangle > \alpha_i - \langle a_i, \overline{x}\rangle$ and we can find an $\varepsilon_i > 0$ such that $d(\overline{y}, \Phi_i(x)) \geq \varepsilon_i$ if $\|x - \overline{x}\| < \varepsilon_i$, so that $d(x, \Phi^{-1}(\overline{y})) \leq d(\overline{y}, \Phi_i(x))$ for such $x$. Taking $\varepsilon = \min \varepsilon_i$ and $K = \max\{1, \max K_i\}$, we see that $d(x, \Phi^{-1}(\overline{y})) \leq Kd(\overline{y}, \Phi_i(x))$for any $i$ and any $x$ such that $\|x - \overline{x}\| < \varepsilon$ and $\Phi(x) \neq \emptyset$. If, on the other hand, $\Phi(x) = \emptyset$, then $d(\overline{y}, \Phi(x)) = \infty$. This completes the proof. □

**Proposition 2.8.** *Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with semi-linear graph. It is (globally) regular if and only if it is locally open at every point of its graph, which means that for any $(x, y) \in$ Graph $\Phi$ the $\Phi$-image of any neighborhood of $x$ contains a neighborhood of $y$.*

**Proof.** We need only to prove the "if" part as the "only if" part of the proposition is obvious. So let us assume that $\Phi$ is open at every point of its graph. By Proposition 2.6 this means that, given a $(x, y) \in$ Graph $\Phi$, the graphical derivative of $\Phi$ at the point is open at zero, that is there is a $r(x, y) > 0$ such that

$$r(x, y)tB_{\mathbb{R}^m} \subset D\Phi(tB_{\mathbb{R}^n}), \quad \forall\, t \geq 0.$$

Applying Proposition 2.6 in the opposite direction, we conclude that there is an $\varepsilon = \varepsilon(x, y) > 0$ such that

$$B(y, r(x, y)t) \subset \Phi(B(x, t)), \quad \forall\, t \in (0, \varepsilon).$$

As we have mentioned there are finitely many different tangent cones at points of Graph $\Phi$, so there are finitely many different $r(x, y)$. Let $r$ stand for the minimal of them. Then $r > 0$ and the regularity of $\Phi$ follows.

We next note that the image of $\Phi$ is the whole of $\mathbb{R}^m$. Indeed, $\Phi(\mathbb{R}^n)$ is just the Cartesian projection of Graph $\Phi$ onto the range space. Hence, this is a closed set as Graph $\Phi$ is semi-linear. On the other hand, local openness implies that $\Phi(\mathbb{R}^n)$ is open.

Now fix an $(x, y) \in$ Graph $\Phi$ and let $v \in \mathbb{R}^m$. We claim that there is a $u \in \mathbb{R}^n$ such that $(u, v) \in$ Graph $\Phi$ and $\|u - x\| \leq r^{-1}\|y - v\|$. Indeed, if $t > 0$ is sufficiently small, then, as we have just seen, there is a $u_t$ such that $y + t(v - y) \in \Phi(u_t)$ and $\|u_t - x\| \leq r^{-1}t\|y - v\|$. We have to show that the upper bound of such $t$ is not smaller than 1. (Actually, it is $\infty$.) Indeed, let $\tau < \infty$ be the upper bound of such $t$. Then there is a sequence of $t$ converging to $\tau$ with the above property. They all belong to the ball of radius $\tau\|y - v\|$ around $x$, hence the collection of such $u_t$ is a precompact set. If $u$ is a limiting point of $(u_t)$ as $t \to \tau$ and $w = y + \tau(v - y)$, then necessarily $(u, w) \in$ Graph $\Phi$. Since $\Phi$ is regular near every point of the graph, we can move further along the direction $v - y$ keeping all the inequalities. This completes the proof. □

**Proposition 2.9.** *Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be semi-linear with* $\dim \text{Graph}\, \Phi \leq n$. *If $\Phi$ is regular near $(\bar{x}, \bar{y}) \in$ Graph $\Phi$, then it is strongly subregular at the point.*

This property is actually valid for a broader class of mappings, e.g. mappings with semi-algebraic graphs (see Theorem 13 in [5]). But the given below proof for the semi-linear case is much simpler.

**Proof.** We know already that $\Phi$ is subregular at $(\bar{x},\bar{y})$. So we only have to show that $\overline{y} \notin \Phi(x)$ for $x \neq \overline{x}$ sufficiently close to $\overline{x}$. Set $G = \text{Graph}\ \Phi$. Then $G = \cup G_i$, where $G_i,\ i = 1,\dots,k$ are convex polyhedral sets. As we are only interested in behavior of $\Phi$ near $(\bar{x},\bar{y})$ and all $G_i$ are closed, we may assume that $(\bar{x},\bar{y}) \in G_i$ for all $i$. We can view each $G_i$ as the graph of some $\Phi_i : I\!R^n \rightrightarrows I\!R^n$ and, moreover, we can assume furthermore that the dimensions of all images $\text{Im}\ \Phi_i = \Phi_i(I\!R^n)$ is exactly $n$. Indeed, if say $\dim(\text{Im}\ (\Phi_k)) = \ell < n$, and in any neighborhood of $(\bar{x},\bar{y})$ there were a point $(x,y) \in G_k$ not belonging to any other $G_i$, then the intersection of the $\Phi$-image of a certain neighborhood of $x$ with some neighborhood of $y$ would be contained in $\text{Im}\ (\Phi_i)$, hence $\Phi$ would not be regular contrary to our assumption.

The proposition will be proved if we show that $\overline{y}$ cannot belong to $\Phi_i(x)$ for $x \neq \overline{x}$. Clearly $\text{Im}\ (G_i)$ is the Cartesian projection of $G_i$ to the range space. As $G_i$ is convex and the dimensions of $G_i$ and $\text{Im}\ (G_i)$ coincide, the projection is a linear isomorphism between $G_i$ and $\text{Im}\ (G_i)$. It follows that for any $i$ and any $y$ there may be only one $x$ such that $y \in \Phi_i(x)$. $\square$

It is an easy matter to see that the proposition imples that $\Phi^{-1}(\overline{y})$ is a finite set if $\overline{y}$ is a regular value of $\Phi$, that is to say if $\Phi$ is metrically regular near every $(x,\overline{y}) \in \text{Graph}\ \Phi$. Moreover, as easily follows from the proof, the number of points in $\Phi^{-1}(\overline{y})$ in this case cannot exceed the number of convex polyhedral $G_i$ in any representation $\text{Graph}\ \Phi = \cup G_i$. The simple example of $\Phi(x) = \{\pm x\}$ $(x \in I\!R)$ shows that under the assumptions $\Phi$ may fail to be strongly regular.

## 3. Face complementarity and face isomorphisms

**Definition 3.1.** Let $C \subset I\!R^n$ be a polyhedral set. A *(face) complementarity relation on* $C$ is a set-valued mapping $\Lambda_C \colon \mathcal{F}_C \rightrightarrows I\!R^n$ with the following properties

(a) for any $F \in \mathcal{F}_C$ the set $\Lambda_C(F)$ is a polyhedral cone and
$$\dim F + \dim \Lambda_C(F) = n;$$
(b) if $F \in \mathcal{F}_C$ and $F'$ is a face of $F$, then $\Lambda_C(F)$ is a face of $\Lambda_C(F')$; if $F \cap F' \neq \emptyset$ and $F \neq F'$, then $(\text{ri}\ \Lambda_C(F)) \cap (\text{ri}\ \Lambda_C(F')) = \emptyset$.
(c) if $G$ is a face of $\Lambda_C(F)$ for some $F \in \mathcal{F}_C$, then there is an $F' \in \mathcal{F}(F)$ such that $G = \Lambda_C(F')$.

A complementarity relation is *non-singular* if $\dim(F + \Lambda_C(F)) = n$, that is to say, if $L(F)$ and $L(\Lambda_C(F))$ are complementary subspaces of $I\!R^n$ for all $F \in \mathcal{F}_C$.

Note that the requirement $F \cap F' \neq \emptyset$ in the second part of (b) is essential: we do not exclude the situation when $F \cap F' = \emptyset$ but $\Lambda_C(F) \cap \Lambda_C(F') \neq \emptyset$.

The simplest and best-known example of a complementarity relation is $F \to N_C(F)$, that is the correspondence associating with every face $F \in \mathcal{F}_C$ the normal cone to $C$ at $F$. We shall later discuss connections between various complementarity relations on the same set, in particular, between complementarity relations and $N_C(F)$. The following proposition is immediate from Definition 3.1 (taking into account that any two faces of a convex cone have a common point).

**Proposition 3.2.** *Let $K \subset I\!\!R^n$ be a polyhedral cone and $\Lambda_K$ a complementarity relation on $K$. Set $H = \Lambda_K(\{0\})$. Then $\Lambda_K$ is a one-to-one correspondence between $\mathcal{F}_K$ and $\mathcal{F}_H$ and $\Lambda_H := \Lambda_K^{-1}$ is a complementarity relation on $H$.*

The proposition, in turn, justifies the following definition.

**Definition 3.3.** Let $K$ and $H$ be two polyhedral cones in $I\!\!R^n$. We say that $K$ and $H$ are *face complementary* if there is a complementarity relation $\Lambda_K$ on $K$ such that $H = \Lambda_K(\{0\})$. We shall call $\Lambda_K$ a *complementarity correspondence* from $K$ to $H$ and set $\Lambda_H = \Lambda_K^{-1}$.

**Proposition 3.4.** *Let $C \subset I\!\!R^n$ be a polyhedral set and $\Lambda_C$ a complementarity relation on $C$. Then for any $F \in \mathcal{F}_C$ the cones $T_C(F)$ and $\Lambda_C(F)$ are face complementary. The associated complementarity correspondence connects the cones cone$(F' - F)$ and $\Lambda_C(F')$, where $F'$ ranges through $\mathcal{F}_C(F)$.*

We shall denote this correspondence by $\Lambda_F^C$ or simply $\Lambda_F$ if this does not lead to confusion. In what follows we shall call the triple $(T_C(F), \Lambda_C(F), \Lambda_F^C)$ the *tangential extension of $(C, \Lambda_C)$ along $F$.*

**Proof.** Set $K = T_C(F)$ and $H = \Lambda_C(F)$. By Proposition 2.5 $F' \to \text{cone}(F' - F)$ is a mapping from $\mathcal{F}_C(F)$ onto $\mathcal{F}_K$, clearly one-to-one. Likewise, the mapping $F' \to \Lambda_C(F')$ is a one-to-one mapping from $\mathcal{F}_C(F)$ onto $\mathcal{F}_H$ by Definition 3.1(b,d). Thus $\Lambda_F$ is a one-to-one correspondence between $\mathcal{F}_K$ and $\mathcal{F}_H$. It satisfies the properties (b) and (d) of Definition 3.1 if so does $\Lambda_C$, hence it is a complementarity correspondence. $\square$

The collection of face complementary pairs of polyhedral cones is rather rich. As a trivial example, we just mention that two polyhedral cones in $I\!\!R^3$ are face complementary if and only if they have the same number of extreme directions.

**Proposition 3.5.** *Let $K$ and $H$ be a pair of face complementary polyhedral cones with $\Lambda_K$ the corresponding face complementarity correspondence and $\Lambda_H = \Lambda_K^{-1}$. Then $\Lambda_K(F_{min}(x))$ is the maximal among faces $G \in \mathcal{F}_H$ such that $x \in \Lambda_H(G)$. Likewise $\Lambda_H(F_{min}(y))$ is the maximal element among $F \subset \mathcal{F}_K$ such that $y \in \Lambda_K(F)$. Moreover the relations*

$$y \in \Lambda_K(F_{min}(x)) \quad \text{and} \quad x \in \Lambda_H(F_{min})(y) \tag{4}$$

*are equivalent.*

**Proof.** The first part of the proposition is a direct consequence of the definitions. Let us prove the equivalence of the two relations in (4). Suppose $x \in F$ and $y \in \Lambda_K(F) \subset \Lambda_K(F_{min}(x))$. Then

$$x \in F = \Lambda_H(\Lambda_K(F)) = \Lambda_H(G) \subset \Lambda_H(F_{min}(y))).$$

This implies that $F \subset \Lambda_K^{-1}(F_{min})(y)$ and the left inclusion in (4) implies the right one. Reversing the roles of $x$ and $y$ and $K$ and $H$, we complete the proof. $\square$

In view of this result, it is natural to set

$$F^K_{max}(y) = \Lambda_H(F_{min}(y)); \qquad F^H_{max}(x) = \Lambda_K(F_{min}(x)).$$

We can view $F_{max}$ as set valued mappings from $I\!R^n$ into itself with domains $K$ and $H$ respectively. The proposition shows that *the mappings are mutually inverse:*

$$\left(F^K_{max}\right)^{-1} = F^H_{max}.$$

We shall next describe a complementarity preserving transformation, for non-singular complementarity relations. So suppose we are given a polyhedral set $C$ and a non-singular complementarity relation $\Lambda_C$ on $C$, and let $F \in \mathcal{F}_C$. Denote for brevity $M = L(\Lambda_C(F))$, the subspace spanned by $\Lambda_C(F)$. As $\Lambda_C$ is non-singular, $M$ and $L = L(F)$ are complementary subspaces of $I\!R^n$ and we can consider the projection $\pi_{ML}$ onto $M$ parallel to $L$.

Set $K_F = \text{cone}(\pi_{ML}(C)) = \pi_{ML}(T_C(F)) = M \cap T_C(F)$, $H_F = \Lambda_C(F)$. It is an easy matter to see that the image under $\pi_{ML}$ of a face of $T_C(F)$ is a face of $K_F$ and any face of $K_F$ can be obtained this way. Thus $G \in \mathcal{F}_{K_F}$ if and only if there is an $\bar{F} \in \mathcal{F}_C(F)$ such that $G = \text{cone}(\pi_{ML}(\bar{F} - F)) = (\text{cone}(\bar{F} - F)) \cap M$. Define a one-to-one correspondence $\Lambda_{K_F}$ between faces of $K_F$ and $H_F$ by setting for $\bar{F} \in \mathcal{F}_C(F)$ and $G = \Lambda_C(\bar{F}) \in \Lambda_C(F)$

$$\Lambda_{K_F}(\pi_{ML}(\text{cone}(\bar{F} - F)) = G,$$

**Proposition 3.6.** *Let $C$ be a polyhedral set in $I\!R^n$ and $\Lambda_C$ a non-singular complementarity relation on $C$. Then for any $F \in \mathcal{F}_C$ the cones $K_F$ and $H_F$ are face complementary in $M$ with $\Lambda_{K_F}$ a face complementarity correspondence between $K_F$ and $H_F$.*

**Proof.** Immediate consequence of Proposition 3.4.

We shall call the triple $(K_F, H_F, \Lambda_{K_F})$ the *factorization of* $(C, \Lambda_C)$ *along* $F$. Note that (if $\dim F > 0$) factorization is associated with a decrease of the dimension of the underlying space.

**Proposition 3.7.** *Let $C \subset I\!R^n$ be a polyhedral set and $\Lambda$ a complementarity relation on $C$. If $A$ is a linear isomorphism $I\!R^n \to I\!R^n$ and $D = A(C)$, then*

- $\Lambda'_C(F) = A(\Lambda_C(F))$ *is a complementarity relation on $C$;*
- $\Lambda_D(A(F)) := \Lambda_C(F)$ *is a complementarity relation on $D$.*

The proposition says that the complementarity property is stable with respect to linear isomorphisms. We omit the proof of the proposition which is fairly straightforward. The only remark that probably should be made is that in the second case the values of $A$ outside of $C$ are inessential for the result and it is enough to assume that $A$ is injective on $L(C)$.

**Definition 3.8.** Let $C$ and $C'$ be two convex polyhedra. We say that $C$ and $C'$ are *face isomorphic* if there is a one-to-one mapping $\tau$ from $\mathcal{F}_C$ onto $\mathcal{F}_{C'}$ such that for any $F \in \mathcal{F}_C$

- $\dim F = \dim \tau(F)$ and
- $F' \in \mathcal{F}_F$ implies that $\tau(F') \in \mathcal{F}_{\tau(F)}$.

We call $\tau$ *face isomorphism* between $C$ and $C'$.

It is immediate from the definition that, given two complementarity relations $\Lambda_C$ and $\Lambda'_C$, the cones $\Lambda_C(F)$ and $\Lambda'_C(F)$ are face isomorphic for every face of $C$. The question we are going to discuss is whether face isomorphism can be equivalently expressed by a usual point-to-point mapping in $I\!R^n$.

We start with building some special simplicial decomposition of a bounded convex polyhedron $C$. Given such a set, let us denote by $\operatorname{ex} C$ the collection of extreme points of $C$ and by $b_C$ the baricenter of $C$ that is $b_C = (1/m)(x_1 + \cdots + x_m)$ where $\operatorname{ex}C = \{x_1, \dots, x_m\}$. Suppose now that $\dim C = n$. We wish to construct for any $k = 0, 1, \dots, n$ a family $\Sigma_k$ of simplices of dimension $k$ in $C$ such that

(i) relative interiors of elements of $\cup\Sigma_k$ form a partition of $C$;

(ii) any $S \in \Sigma_k$ is contained in a certain $F \in \mathcal{F}_C$ with $\dim F = k$;

(iii) every $S \in \Sigma_k$ is the convex hull of the union of some $S' \in \Sigma_{k-1}$ and the baricenter $b_F$ of the $k$-face containing $S$.

To this end we set first $\Sigma_0 = \operatorname{ex} C$. Suppose we have already $\Sigma_{k-1}$ for some $k \geq 1$. To build $\Sigma_k$ we for every $F \in \mathcal{F}_C$ with $\dim F = k$ take all simplices $\operatorname{conv}\,(S' \cup \{b_F\})$, where $S' \in \Sigma_{k-1}$ lies in one of the faces of $F$.

**Proposition 3.9.** *Let $C$ and $C'$ be face isomorphic bounded convex polyhedra in $I\!R^n$, and let $\tau\colon \mathcal{F}_C \to \mathcal{F}_{C'}$ be a face isomorphism between them. Then there is a continuous piecewise affine homeomorphism $\omega\colon C \to C'$ such that $\omega(F) = \tau(F)$ for any $F \in \mathcal{F}_C$.*

**Proof.** Let $\Sigma_k,\ k = 0, \dots, n$ be the just described simplicial decomposition of $C$. We first set $\omega(x) = \tau(\{x\})$ for $x \in \Sigma_0 = \operatorname{ex}C$ and $\omega(b_F) = b_{\tau(F)}$ for all $F \in \mathcal{F}_C$. Suppose now that we have already defined $\omega$ on elements of $\Sigma_{k-1}$ for some $1 \leq k \leq n$. Then given a $S \in \Sigma_k$ and the $k$-face $F$ containing $S$ we have $S = \operatorname{conv}\,(S' \cup \{b_F\})$ for some $S' \in \Sigma_{k-1}$, so that every $x \in S$ is uniquely defined as $\alpha b_F + (1-\alpha)x'$ with $\alpha = \alpha(x) \in [0,1]$ and $x' \in S'$. Set $\omega(x) = \alpha b_{\tau(F)} + (1-\alpha)\omega(x')$. It is an easy matter to see that $\omega$ have the claimed properties on the union of elements of $\Sigma_k$ and that the union is precisely the union of all faces of $C$ whose dimensions do not exceed $k$. Continuing the process till we reach $k = n$, we complete the proof. □

Now what can we do if the sets are not bounded? We are specifically interested in the case of face isometric convex polyhedral cones $K$ and $K'$. We can assume both cones pointed. Indeed, if say $K$ is not pointed, then its minimal face is a subspace $L$. Let $\tau\colon \mathcal{F}_K \to \mathcal{F}_{K'}$ be the face isometry between the cones. Then necessarily $L' = \tau(L)$ is the minimal face of $K'$ and a linear subspace of the same dimension as $L$. If we now replace $K$ by $\tilde{K} = K \cap L^{\perp}$ and $K'$ by $\tilde{K}' = K' \cap (L')^{\perp}$, then (as it is easy to see) we shall again have a couple of face isomorphic pointed cones. Furthermore, as face isomorphisms are (obviously) preserved under congruent transfomations, we can harmlessly assume that the cones have the full dimension.

Now let $y$ and $y'$ be nonzero vectors exposing the origin in $K$ and $K'$ respectively and $e \in \operatorname{int} K$, $e' \in \operatorname{int} K'$. Consider the set $C = \{x \in K : \langle y, x\rangle = \langle y, e\rangle\}$ and the corresponding set $C'$ in $K'$. These sets are bounded convex polyhedra that are face isomorphic. Indeed, define a mapping $\tau_C : \mathcal{F}_C \to \mathcal{F}_{C'}$ as follows: if $F \in \mathcal{F}_K$, then $F \cap C$ is a face of $C$ (and any face of $C$ has this form!), then we set $\tau_C(F \cap C) = \tau(F) \cap C'$. By Proposition 3.9 there is a piecewise affine isomorphism $\omega_C : C \to C'$ that transforms faces of $C$ into faces of $C'$. Any nonzero $x \in K$ has a unique representation $x = \lambda u$ with $\lambda > 0$ and $u \in C$. Likewise, any nonzero $x' \in K'$ has a unique representation $x' = \lambda u'$ with $\lambda > 0$ and $u' \in C'$. Therefore $\omega_K(x) = \lambda\omega_C(u)$ realizes a piecewise affine isomorphism of $K$ and $K'$ and transforms any face $F$ of $K$ into the corresponding face of $K'$, namely $\tau(F)$. Thus we have proved the following result.

**Proposition 3.10.** *Let $K$ and $K'$ be face isomorphic convex polyhedral cones in $I\!R^n$, and let $\tau\colon \mathcal{F}_K \to \mathcal{F}_{K'}$ be a face isomorphism between them. Then there is a continuous and homogeneous piecewise affine homeomorphism $\omega\colon K \to K'$ such that $\omega(F) = \tau(F)$ for any $F \in \mathcal{F}_K$.*

## 4. The complementarity mapping

Let $C \subset I\!R^n$ be a polyhedral set and $\Lambda_C$ a complementarity relation on $C$. The *complementarity mapping associated with* $(C, \Lambda_C)$ is

$$\Phi_C(x) = \begin{cases} x + \Lambda_C(F_{min}(x)), & \text{if } x \in C; \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5}$$

In case we have two face complementary cones $K$ and $H$ with face complementarity correspondence $\Lambda_K\colon \mathcal{F}_K \to \mathcal{F}_H$, then $\Phi_K(x) = x + F^H_{max}(x)$ (for $x \in K$)

and we can consider the symmetric mapping $\Phi_H(y) = y + F^K_{max}(y)$ $(y \in H)$. By Proposition 3.5

$$x \in F^K_{max}(y) \Leftrightarrow\ y \in F^H_{max}(x)\ \Leftrightarrow\ z = x + y \in \Phi_K(x) \cap \Phi_H(y). \tag{6}$$

As a consequence of this observation, we get that the ranges of $\Phi_K$ and $\Phi_H$ coincide and, moreover, the following holds.

**Proposition 4.1.** *If $\Phi_K$ is (strongly) regular near $(\overline{x}, \overline{z}) \in$ Graph $\Phi_K$, then $\Phi_H$ is (strongly) regular near $(\overline{y}, \overline{z})$, where $\overline{y} = \overline{z} - \overline{x}$.*

**Proof.** Let $(x, z) \in$ Graph $\Phi_H$ be sufficiently close to $(\overline{x}, \overline{z})$. As $\Phi_K$ is regular near $(\overline{x}, \overline{z})$, for any $w$ sufficiently close to $z$ there is a $u$ such that $w \in \Phi_K(u)$ and $\|u-x\| \le r^{-1}\|z-w\|$, where $r$ is the rate of surjection of $\Phi$ near $(\overline{x}, \overline{z})$. We have $z = x + y,\ w = u + v$ for some $y \in \Lambda_K(x)$ and $v \in \Lambda_K(u)$ and therefore

$$\|y - v\| \le \|z - w\| + \|x - u\| \le k\|z - w\|$$

with $k$ not depending on $z, w, y, v$. But $z \in \Phi_H(y)$ and $w \in \Phi_H(v)$ by (6).

Now if $\Phi_K$ is strongly regular near $(\overline{x}, \overline{z})$, then there is a Lipschitz mapping $x(z)$ from a neighborhood $W$ of $\overline{z}$ into a neighborhood $U$ of $\overline{x}$ such that $\{(x, z) : x = x(z),\ z \in W\} = (U \times W) \cap$ Graph $\Phi_K$. Set $y(z) = z - x(z)$. This is a Lipschitz mapping and, obviously, there is no other solutions to $z \in \Phi_H(y)$ in a small neighborhood of $\overline{y}$ close to $\overline{z}$. □

Both $\Phi_K$ and $\Phi_H$ are homogeneous mappings. As a consequence we get, thanks to Theorem 2.1,

**Proposition 4.2.** *If $\Phi_K$ is regular near $(0, 0)$, then it is globally regular (in particular, regular near every point of its graph) with the same rate.*

The proposition allows to speak just about regularity of $\Phi_K$ and $\Phi_H$ without specifying whether it is global or local near $(0, 0)$ (but of course regularity near another point of the graph may not imply global regularity). This makes the complementarity maps with a pair of face complementary cones especially convenient to work with.

Let us return to the general model (5). Fix an $F \in \mathcal{F}_C$, and let $(T_C(F), \Lambda_C(F), \Lambda^C_F)$ be the tangential extension of $(C, \Lambda_C)$ along $F$. Set further $K = T_C(F)$, $H = \Lambda_C(F)$ and consider $\Phi_K(x) = x + \Lambda_C(F^K_{min}(x))$ (with dom $\Phi_K = K$).

**Proposition 4.3.** *The set-valued mapping $\Phi_K$ is (strongly) regular near $(0, \overline{z} - \overline{x})$, provided $\Phi_C$ is (strongly) regular near $(\overline{x}, \overline{z})$ and vice versa. In particular, if $\Phi_C$ is (strongly) regular near $(\overline{x}, \overline{x})$, then $\Phi_K$ is globally (strongly) regular.*

**Proof.** As we have seen in the proof of Proposition 3.4 for $(x, z)$ sufficiently close to $(0, \overline{z})$ the relation $z - \overline{x} \in \Phi_K(x)$ is equivalent to $z \in \Phi_C(x + \overline{x})$ This implies the first statement. The second follows from Proposition 2.1. □

Below are the main results of the section.

**Theorem 4.4.** *Let $C$ be a polyhedral set, and let $\Lambda_C$ be a complementarity relation on $C$. If the complementarity mapping $\Phi_C$ is regular near $(x,x)$ for some $x \in C$, then for $F = F_{\min}(x)$ the subspaces $L(F)$ and $L(\Lambda_C(F))$ are complementary subspaces of $I\!R^n$. Thus $\Lambda_C$ is a non-singular complementarity relation if $\Phi_C$ is regular in a neighborhood of every $(x,x)$, $x \in C$.*

**Proof.** It is sufficient to consider the case of two face complementary cones and $x = 0$. The general case reduces to that one with the help of Propositions 2.6 and 4.3 if we consider $K = T_C(F)$ and $H = \Lambda_C(F)$.

The minimal face of $K$ containing zero is obviously $L = \mathrm{lin}K$. So assuming that the statement is wrong, we have to conclude that $\dim(L + L(H)) < n$ in which case (recall that for any $F \in \mathcal{F}_K$ the sum of dimensions of $L(F)$ and $L(\Lambda_K(F))$ is $n$) we can find $x_i \in L(F)$ and $y_i \in H$, $i = 1, 2$ such that $0 \neq x_1 - x_2 = y_2 - y_1$. Adding, if necessary, to both $y_i$ some $y \in \mathrm{ri}\, H$, we may guarantee that $y_i \in \mathrm{ri}\, H$.

Consider $\Phi_K$, the same as above. By Propositions 4.3 and 4.2 $\Phi_K$ is regular, so there is an $r > 0$ such that $B(z, r\varepsilon) \subset \Phi_K(B(x,\varepsilon))$ for any $x \in K$, any $z \in \Phi_K(x)$ and all $\varepsilon > 0$. Set $z = x_1 + y_1 = x_2 + y_2$. Then for any $z_i' \in B(z,\varepsilon)$ there is a $x_i' \in K$ such that $z_i' \in \Phi_K(x_i')$ and $\|x_i' - x_i\| \le r^{-1}\|z_i' - z\|$. Set further $y_i' = z_i' - x_i' \in H$. We have $\|y_i' - y_i\| \le (1 + r^{-1})\|z_i' - z\| \le (1 + r^{-1})\varepsilon$. Let $\varepsilon$ be so small that $y_i' \in \mathrm{ri}\, H$. Then necessarily $x_i' \in L$ as $L$ is the smallest face of $K$ - see (b) in Definition 3.1 and Proposition 3.5. This means that $\Phi_K(B(x_i,\varepsilon)) \subset L + H$ and therefore

$$B(0, 2r\varepsilon) \subset \Phi_F(B(x_1,\varepsilon)) - \Phi_1(B(x_2,\varepsilon)) \subset L + L(H).$$

This contradicts to the assumption that the dimension of $L + L(H)$ is strictly smaller than $n$. $\square$

Combining this with Propositions 3.6 and 4.3 we get

**Corollary 4.5.** *If $\Phi_C$ is regular, then for any $F \in \mathcal{F}_C$ the complementarity mapping $\Phi_{K_F}$ associated with the factorization $(K_F, H_F, \Lambda_F)$ of $(C, \Lambda)$ along $F$ is also regular in $M$.*

**Proof.** We have $K_F = K \cap M$, $H_F = H$, where $(K, H, \Lambda_F)$ is the tangential extension of $C, \Lambda_C$ along $F$. So if $z \in \Phi_{K_F}(x)$, that is $z = x + y$ with both $x$ and $y$ in $M$ and $z' \in M$, then by regularity of $\Phi_{K_F}$ (see Proposition 4.3) there are $x' \in K_F$ and $y' \in H_F \subset M$ such that $z' = x' + y'$ and $\|x' - x\| \le \kappa\|z' - z\|$ with $\kappa$ being the modulus of regularity of $\Phi_{K_F}$. Then $x' \in M$ as both $z'$ and $y'$ are in $M$, that is $x' = \pi_{ML}(x)$ and $z' \in \Phi_{K_F}(x')$. $\square$

**Theorem 4.6.** *Let $K$ and $H$ be a pair of face complementary polyhedral cones. If $\Phi_K$ is regular, then*

$$K \cap H = \{0\}.$$

**Proof.** The theorem trivially holds if $n = 1$. Indeed, if $K = \{0\}$, then $H$ must be a one-face cone of dimension 1, that is $H = I\!R$. If $K$ is a half line then $H$ contains two faces, that is $H$ is also a half line. By regularity, $K$ and $H$ cannot coincide, hence zero is the only common point for $K$ and $H$.

Suppose now that the theorem is true for all dimensions up to $m-1$ with some $m \geq 2$ and consider the case $n = m$. Assume that there is a $0 \neq e \in H \cap K$. If $\dim K = k < n$, then by Theorem 4.4, $\Lambda(K)$, the minimal face of $\mathcal{F}_H$, must have dimension $n - k$. This may be only if it is a subspace, that is $\Lambda_K(K) = \text{lin}\, H$. Consider factorization of $(H, \Lambda_H)$ along $\bar{G} = \text{lin}\, H$. Then $K_{\overline{G}} = K$, $H'_{\bar{G}} = \pi_{ML}(H)$, where $M = L(K)$, $L = \text{lin} H$ and the complementarity correspondence $\Lambda'_{\overline{G}}\colon \mathcal{F}_K \leftrightarrow \mathcal{F}_{H'}$ is defined by $(\Lambda'_{\overline{G}})_{K'}(F) = \pi_{ML}(\Lambda_C(F))$. Applying consecutively Propositions 4.1 and 3.6, we conclude that $\Phi'_{K'} = x + (\Lambda'_{\overline{G}})_{K'}(F^{K'}_{min}(x))$ is a regular mapping in $M$. But $\dim M < m$, so by the induction hypothesis $K \cap H' = \{0\}$. As $\text{lin}\, H$ is a complementary subspace to $L(K)$, it follows that $K \cap (H'_{\overline{G}} + \text{lin}\, H) = \{0\}$ and therefore that $K \cap H = \{0\}$.

So we have to consider the case $\dim K = m$. Assume that there is a $0 \neq e \in K \cap H$. The inclusion $H \subset K$ would imply that $K + H = K \neq I\!R^m$ contrary to the assumption that $\Phi$ is regular. This means that $H$ contains an element not belonging to $K$. This cannot be $-e$ for in this case $\text{lin}\, H \neq \{0\}$ and therefore $\dim K = m - \dim(\text{lin}\, H) < m$. It follows (by convexity of $H$) that $K \cap H$ must contain a nonzero element not belonging to int $K$ and we may assume that $e$ is such an element.

Let $F$ be a proper face of $K$ containing $e$. The inclusion $e \in \Lambda_K(F)$ is impossible as $F$ and $\Lambda_K(F)$ lie in complementary subspaces of $I\!R^n$ by Theorem 4.4. Thus $e \notin \Lambda_K(F)$. Set $G = \Lambda_K(F)$ and consider the factorization $(H'_G, K'_G, \Lambda_G)$ of $(H, \Lambda_H)$ along $G$, that is $K'_G = F$, $M = L(F)$ is the subspace spanned by $F$ and $L = L(G)$, $\pi_{ML}$ is the projection onto $M$ parallel to $L$, $H'_G = \pi_{ML}(H)$ and $\Lambda'_G(F') = \pi_{ML}(\Lambda_K(F'))$ for $F' \in \mathcal{F}_K(F)$. As $e \in M$, it follows that $e = \pi_{ML}(e) \in H'_G$. Thus again we get $e \in K'_G \cap H'_G$ which, in view of Corollary 4.5 contradicts to the induction hypothesis. This completes the proof of the theorem. $\square$

## 5. Main results

Everywhere in this section $C$ is a polyhedral set, $\Lambda_C$ is a complementarity relation on $C$ and $\Phi_C(x) = x + \Lambda_C(F_{min}(x))$ is the corresponding complementarity mapping.

**Theorem 5.1 (regularity implies strong regularity).** *If $\Phi_C$ is regular then the inverse mapping is single-valued and Lipschitz on $I\!R^n$. Thus, regularity of $\Phi_C$ implies global strong regularity.*

**Proof.** We only need to show that $\Phi_C^{-1}$ is single-valued: the Lipschitz property will then automatically follow from regularity. To begin with we note that the equality $u = x + y$ for some $x, u \in C$ and $y \in \Lambda(F_{min}(x))$ can be valid only if $x = u$ and $y = 0$. Indeed, as $\Phi_C$ is regular, for any $x \in C$ the complementarity mapping $\Phi_F$ associated with the tangential extension $(K_F, H_F, \Lambda_F)$ of $(C, \Lambda_C)$ along $F = F_{min}(x)$ is also regular (Proposition 4.3). By Theorem 4.6 $K_F \cap H_F = \{0\}$. If we now assume that $u = x + y$ with $x, u, y$ as above, then $u \notin F = F_{min}(x)$ for otherwise $u - x = y$ and as $L(F)$ and $L(\Lambda_C(F))$ are complementary subspaces by Theorem 4.4, we get $x - u = y = 0$. But $C - x \subset T(C, x) = K_F$ and therefore for no $u \in C$ other than $x$ the difference $u - x$ may belong to $H_F = \Lambda_C(F)$.

Note that in case we have a couple of face complementary cones $K$ and $H$ with face correspondence $\Lambda$, and $\Phi_K$ is regular, then so is $\Phi_H$ (Proposition 4.1) and, likewise, the equality $y = v + u$ for $y, v \in H$ and $u \in \Lambda^{-1}(F^H_{min}(v))$ may hold only if $y = v$ and $u = 0$.

Assume now that $x + y = u + v$ for some $x, u \in C$, $x \neq u$, $y \in \Lambda_C(F_{min}(x))$ and $v \in \Lambda_C(F_{min}(u))$. We first show that this is impossible if in addition $x \in F^C_{\min}(u)$. Indeed, in this case $F = F^C_{\min}(x) \subset F^C_{\min}(u)$ and therefore $v \in \Lambda_C(F_{min}(x))$. Set $F = F_{min}(x)$, and let $(K'_F, H'_F, \Lambda'_F)$ be the factorization of $(C, \Lambda)$ along $F$. Then $y, v \in H'_F$, $u' = \pi_{ML}(u)$ (where as usual $\pi_{ML}$ is the projection onto $M = L(\Lambda_C(F))$ along $F$) and $y = u' + v$. But by what we have just proved this can be true only if $y = v$.

Thus we have to assume that neither of $x$, $u$ belongs to the minimal face of the other. Let $\kappa$ be the modulus of metric regularity of $\Phi$ or any bigger number. Choose $\varepsilon > 0$ so small that the ball of radius $(1 + \kappa)\varepsilon$ around $x$ does not meet any face $F \in \mathcal{F}_C$ such that $(\mathrm{ri}\, F^C_{min}(x)) \cap F = \emptyset$. This means that $x \in F_{min}(w)$ whenever $w \in C$ and $\|w - x\| \leq (1 + \kappa)\varepsilon$. Let further $m$ be an integer big enough to guarantee that $\delta = m^{-1}\|y\| < \varepsilon$. Regularity of $\Phi_C$ allows to construct recursively a finite sequence of pairs $(u_k, z_k)$, $k = 0, 1, \ldots, m$ such that $(u_0, z_0) = (u, z)$, $z_k \in \Lambda_C(u_k)$, $u_k + z_k = x + (1 - m^{-1}k)y$ and $\|u_k - u_{k-1}\| \leq \kappa\delta$.

Then $u_m + z_m = x$. As follows from the result obtained at the beginning of the proof, this can happen only if $u_m = x$. This in turn means, if $u_0 \neq x$, that for a certain $k$ we have $u_k \neq x$, $\|u_k - x\| \leq \kappa\delta < \kappa\varepsilon$. By the choice of $\varepsilon$ this implies that $x \in F_{min}(u_k)$. But in this case the result obtained at the preceding paragraph excludes the possibility of the equality $u_k + z_k = x + (1 - m^{-1}k)y$ unless $u_k = x$. So we again get a contradiction that completes the proof. $\square$

Our next goal is to characterize regularity of $\Phi_C$ in verifiable terms.

**Definition 5.2.** We say that $(C, \Lambda_C)$ satisfies the *face separation condition* if $\Lambda_C$ is non-singular and for any $F, F' \in \mathcal{F}_C$, $F \subset F'$, $\dim F' = \dim F + 1$, the hyperplane $\mathrm{aff}(F + \Lambda_C(F'))$ properly separates $F + \Lambda_C(F)$ and $F'$.

Proper separation means that none of the separated sets lies in the separating hyperplane. Note that $\mathrm{aff}(F + \Lambda_C(F'))$ is indeed a hyperplane of dimension $n - 1$ because by non-singularity $\dim(F + \Lambda_C(F')) = \dim F + \dim \Lambda_C(F')$ (as $\Lambda_C(F') \subset \Lambda_C(F)$) and $\dim \Lambda_C(F') = n - \dim F' = n - \dim F - 1$.

**Theorem 5.3 (characterization of regularity).** *The following properties are equivalent:*

(a) $\Phi_C$ *is regular;*
(b) $\Phi_C$ *is locally open at every point of its graph;*
(c) $\Lambda_C$ *is non-singular and* $\Phi_C$ *is onto (that is the range of* $\Phi$ *is the whole of* $I\!R^n$*);*
(d) *the face separation condition holds for* $(C, \Lambda_C)$*.*

**Proof.** The equivalence (a) $\Leftrightarrow$ (b) and the implication (b) $\Rightarrow$ (c) follow from Proposition 2.8 and its proof.

(c) $\Rightarrow$ (a). By non-singularity, for any $F \in \mathcal{F}_C$ the mapping $(x, y) \to x + y$ from $L(F) \times L(\Lambda_C(F))$ into $I\!R^n$ is a linear homeomorphism. This means that there are positive $r(F)$, $F \in \mathcal{F}_C$ such that for any $z \in I\!R^n$ there are uniquely defined $x \in L(F)$ and $y \in L(\Lambda_C(F))$ such that $z = x + y$ and $B(z, r(F)t) \subset B(x, t) \cap L(F) + B(y, t) \cap L(\Lambda_C(F))$.

Let $L = L(F) + F$ be the affine set parallel to $L(F)$ and containing $F$. Then for any $z \in I\!R^n$ there are uniquely defined $x \in L$ and $y \in L(\Lambda_C(F))$ such that $z = x + y$ and

$$B(z, r(F)t) \subset B(x, t) \cap L + B(y, t) \cap L(\Lambda_C(F)). \tag{7}$$

Now let $z \in \Phi_C(x)$ for some $x \in F$. We claim that

$$B(z, r(F)t) \cap \Phi_C(F) \subset B(x, t) \cap F + \Lambda_C(F). \tag{8}$$

Indeed, let $x \in F \in \mathcal{F}_C$ and $z \in \Phi_C(x)$. Then $z = x + y$ for some $y \in \Lambda_C(x)$. Let $w \in B(z, r(F)t) \cap \Phi_C(F)$. Then $w = u + v$ with $u \in F \subset L$, $v \in \Lambda_C(F) \subset L(\Lambda_C(F))$ and the sum of no other pair of elements of $L$ and $L(\Lambda_C(F))$ is can be equal to $w$. By (7) $u \in B(x, t)$ and (8) follows. On the other hand, $B(x, t) \cap F + \Lambda_C(F) = \Phi_C(B(x, t) \cap F)$, that is $B(z, r(F)t) \cap \Phi_C(F) \subset \Phi_C(B(x, t))$.

Setting $r = \min_{F \in \mathcal{F}_C} r(F)$, we get

$$B(z, rt) = \bigcup_F (B(z, rt) \cap \Phi_C(F)) \subset \Phi_C(B(x, t))$$

which is precisely the regularity of $\Phi_C$.

(b) $\Rightarrow$ (d). As $\Lambda_C$ is non-singular, all sets $Q(F) = F + \Lambda_C(F)$ have dimension $n$, that is $\operatorname{int} Q(F) \neq \emptyset$ for any $F \in \mathcal{F}_C$. Take certain $F, F' \in \mathcal{F}_C$, such that $F \in F'$ and $\dim F' = \dim F + 1$. Then $G = \Lambda_C(F')$ is a proper face of $\Lambda_C(F)$ with $\dim G = \dim \Lambda_C(F) - 1$ and $E = \operatorname{aff}(F + G)$ is a hyperplane of dimension $n - 1$. It is clear that $Q(F)$ and $Q(F')$ each belongs to one of the two closed half spaces defined by $E$. We have to show that the subspaces are different which amounts to saying that $E$ separates $Q(F)$ and $F'$.

Take an $\overline{x} \in$ ri $F$ and a $\overline{y} \in$ ri $G$. Then $\overline{z} = \overline{x} + \overline{y} \in Q(F) \cap Q(F')$ and $\overline{z} \in$ ri $(F + G)$. We claim that $x \in$ (ri $F) \cup$ (ri $F'$) if $(x, z) \in$ Graph $\Phi_C$ is sufficiently close to $(\overline{x}, \overline{z})$. Set $y = z - x$. Then $y \in \Lambda_C(F_{min}(x))$ (by definition of $\Phi_C$). As $\overline{x} \in$ ri $F$, any $x$ close to $\overline{x}$ is either in ri $F$ or in some face containing $F$. In the last case by the property (ii) of Definition 3.3 any $y \in \Lambda(F_{min}(x))$ belongs to $\Lambda_C(F)$.

If $y \in$ ri $\Lambda_C(F')$, then $x$ must be in $F'$. If now $x \notin$ (ri $F) \cup$ (ri $F'$), then $x \in [(\mathrm{bd}\, F) \cup (\mathrm{bd}\, F')] \backslash (\mathrm{ri}\ F)$. This is a closed set not containing $\overline{x}$, hence an $x$ sufficiently close to $\overline{x}$ cannot belong to this set and we have to conclude that $x \in$ (ri $F) \cup$ (ri $F'$).

If, on the other hand, $y \notin$ ri $\Lambda_C(F')$, then $y \notin \Lambda_C(F')$ (as $\overline{y} \in$ ri $\Lambda_C(F')$) and $y$ is close to $\overline{y}$). This means that $y$ belongs to a face of $\Lambda_C(F)$ containing $\Lambda_C(F')$, then (again by the property (ii) of Definition 3.3) $x$ belongs to some face of $C$ contained in $F'$ but different from the latter. But have seen, $F$ is the only such face to which $x$ may belong.

As $\Phi$ is locally open, it follows that $Q(F) \cup Q(F')$ covers a neighborhood of $\overline{z}$. As each of the set lies completely in one of the half spaces defined by $E$, this may happen only when the subspaces are different and (d) holds.

(d) $\Rightarrow$ (c). We need the following simple Lemma to proceed (see [8]).

**Lemma 5.4.** *Let $Q_1, \ldots, Q_k$ be polyhedral sets. Assume that $Q = \cup Q_i \neq \mathbb{R}^n$. Then there are a set $Q_j$, an $\overline{x}$ in the boundary of $Q_j$ and an $\varepsilon > 0$ such that $B(\overline{x}, \varepsilon) \cap Q_j = B(\overline{x}, \varepsilon) \cap Q$.*

Set $Q = \mathrm{Im}\ \Phi = \cup_{F \in \mathcal{F}_C} Q(F)$. If $Q \neq \mathbb{R}^n$, choose a $Q(F)$ and a $\overline{z} \in \mathrm{bd}\, Q(F)$ according to the lemma, that is such that the intersections of a neighborhood of $\overline{z}$ with $Q(F)$ and $Q$ coincide. In particular the boundaries of the sets near $\overline{z}$ are identical. The non-singularity condition guarantees that interiors of all sets $Q(F)$ are nonempty. Therefore, by moving $\overline{z}$ slightly, we can guarantee that $\overline{z}$ belongs to the relative interior of a face $\tilde{Q}$ of $Q(F)$ of dimension $n - 1$.

Set for brevity $G = \Lambda_C(F)$. Clearly, any face of $Q(F)$ is the sum of a face of $F$ and a face of $\Lambda_C(F)$.Thus $\tilde{Q} = \tilde{F} + \tilde{G}$, where $\tilde{F} \in \mathcal{F}_F$ and $\tilde{G} \in \mathcal{F}_G$. As $\dim \tilde{Q} = n - 1$, there are just two possibilities: $\tilde{F} = F$ and $\dim \tilde{G} = \dim G - 1$ and $\dim \tilde{F} = \dim F - 1$ and $\tilde{G} = G$.

In the first case $G = \Lambda(F')$ for some $F' \in \mathcal{F}_C$. As $G = \Lambda_C(F)$, we necessarily have $F \subset F'$.

By the assumption, the hyperplane $E(F) = L(F) + L(\Lambda_C(F'))$ properly separates $Q(F)$ and $F'$. This however contradicts to the fact that $\overline{z}$ is a boundary point of $Q(F)$. Indeed, $Q$ and $Q(F)$ coincide in a neighborhood of $\overline{z}$, that is the intersection of Im $\Phi_C$ with the neighborhood lies completely in one of the half-spaces determined by $E$. But the relative interior of $F'$, also a part of Im $\Phi$ lies in the interior of the opposite half-space.

In the second case the arguments are even simpler. As $G = \Lambda_C(F)$, we apply the separation property to $\tilde{F}$ and $F$ and come to the same contradiction. This completes the proof of the theorem. □

**Remark 5.5.** Note that for general semi-linear mappings properties (a) and (c) are not equivalent: see [15] (Example 2.3.2) for an example of even a piecewise affine (single-valued continuous semi-linear) mapping $\mathbb{R}^n \to \mathbb{R}^n$ which is non-singular and onto but not locally open.

The final theorem offers a formula for local regularity rates of $\Phi_C$. As this is going to be a quantitative result, it is more convenient to consider mappings of a somewhat more general, although formally equivalent, kind, namely

$$\Phi(x) = Tx + S(\Lambda_C(F_{min}(x)),$$

where $T$ and $S$ are linear operators in $\mathbb{R}^n$. We shall do this assuming that $C$ is a cone. This however does not affect the generality of the results as locally every polyhedral set coincides with a cone (Proposition 2.6).

**Theorem 5.6.** *Let $K$ be a polyhedral cone and $\Lambda_K$ a complementarity relation on $K$. For any $F_1, F_2 \in \mathcal{F}_K$ such that $F_1 \subset F_2$ set*

$$r(F_1, F_2) = \inf\{d(T^*z^*, (F_2 - F_1)^\circ) : \ \|z^*\| = 1, \ S^*z^* \in (\Lambda(F_1) - \Lambda(F_2))^\circ\},$$

*where $T^*$ and $S^*$ stand for the adjoint operators. Then*

$$\mathrm{sur}\Phi(0|0) = \min\{r(F_1, F_2) : \ F_1, \ F_2 \in \mathcal{F}_K, \ F_1 \subset F_2\}.$$

**Proof.** Set $H = \Lambda_K(\{0\})$. Then $\Lambda_K$ is a face complementarity correspondence between $K$ and $H$. We shall use Theorem 2.1(b) to compute the modulus of surjection of $\Phi$. To this end we first have to compute tangent cones to the graph of $\Phi$. Let $z \in \Phi(x)$, that is $z = Tx + Sy$ for some $y \in F^H_{max}(x) = \Lambda_K(F_{min}(x))$. Then $(h, w) \in T(\text{Graph } \Phi, (x, z))$ is the same as $w = Th + Sv$ and $(h, v) \in T(\text{Graph } F^H_{max}, (x, y))$. So all we need is to compute $T(\text{Graph } F^H_{max}, (x, y))$.

Take a pair $(x, y) \in \text{Graph } F^H_{max}$ and let $(h, v) \in T(\text{Graph } F^H_{max}, (x, y))$, that is $x + \lambda h \in F^H_{\max}(y + \lambda v)$. We claim that $x + \lambda h \in F^K_{max}(y)$ for small $\lambda$. Indeed, $F^H_{min}(y) \subset F^H_{min}(y + \lambda v)$ for small $\lambda$ and therefore

$$F^K_{max}(y + \lambda v) = \Lambda_H(F_{min}(y + \lambda v) \subset \lambda^H_K(y) = F^H_{\max}(y)$$

as claimed.

Denote for brevity $F_2 = F^K_{max}(y), \ F_1 = F^K_{min}(x)$. As $x$ is in the relative interior of $F_1$, it follows from Proposition 2.5 that $\text{cone}(F_2 - x) = \text{cone}(F_2 - F_1) =$

$F_2 - F_1$. On the other hand $\Lambda_K(F_2) = F^H_{min}(y)$ by definition, so that $y \in \operatorname{ri}\Lambda_K(F_2)$ and $\operatorname{cone}(\Lambda_K(F_2) - y) = \operatorname{cone}(\Lambda_K(F_2) - \Lambda_K(F_2)) = L(F_2)$. Likewise $\operatorname{cone}(F_1 - x) = L(F_1)$ and $\operatorname{cone}(\Lambda_K(F_1) - y) = \operatorname{cone}(\Lambda_K(F_1) - \Lambda_K(F_2))$. Thus, the tangent cone to Graph $F^K_{max}$ at $(x, y)$ contains

$$\{(h, v) :\ h \in L(F_1),\ v \in \Lambda_K(F_1) - \Lambda_K(F_2)\}$$

and

$$\{(h, v) :\ h \in F_2 - F_1,\ v \in L(\Lambda_K(F_2))\}.$$

The polar of the tangent cone is the same as the polar of its convex hull that contains the sum of the two cones above. It is an easy matter to see (taking consecutively $h = 0$ and $w = 0$) that the sum coincides with $(F_2 - F_1) \times (\Lambda_K(F_1) - \Lambda_K(F_2))$. We next notice that $F_1 \subset F \subset F_2$ for any face $F$ such that $x \in F$ and $y \in \Lambda_K(F)$. Therefore $T(F, x) = \operatorname{cone}(F - x) \subset F_2 - F_1$ and $T(\Lambda_K(F), y) = \operatorname{cone}(\Lambda_K(F) - y) \subset \Lambda_K(F_1) - \Lambda_K(F_2)$. It follows that

$$\operatorname{conv} T(\text{Graph } F^H_{max}(x, y)) = (F_2 - F_1) \times (\Lambda_K(F_1) - \Lambda_K(F_2)),$$

so that

$$\operatorname{conv} T(\text{Graph } \Phi, (x, z)) = \{(h, w) : h \in F_2 - F_1,\ w \in Th + S(\Lambda_K(F_1) - \Lambda_K(F_2))\}$$

Therefore the normal cone to Graph $\Phi$ at $(x, z)$ is the collection of pairs $(x^*, z^*)$ such that

$$\langle x^*, h\rangle + \langle z^*, Th + Sv\rangle \le 0, \quad \text{if } h \in F_2 - F_1,\ v \in \Lambda_K(F_1) - \Lambda_K(F_2),$$

that is

$$N(\text{Graph } \Phi, (x, z)) = \left\{ (x^*, z^*) :\ \begin{array}{c} x^* + T^* z^* \in (F_2 - F_1)^\circ, \\ S^* z^* \in (\Lambda_K(F_1) - \Lambda_K(F_2))^\circ \end{array} \right\}$$

It is an easy matter to see that

$$r(F_1, F_2) = \inf\{\|x^*\| : (x^*, z^*) \in N(\text{Graph } \Phi, (x, z)),\ \|z^*\| = 1\}.$$

To conclude the proof we only need to note that for any pair of faces $F_2,\ F_1 \in \mathcal{F}_K$ such that $F_1 \subset F_2$, we can choose $x$ and $y \in F^H_{max}(x)$ such that $F_2 = F_{max}(y)$, $F_1 = F^K_{min}(x)$, namely any $x \in \operatorname{ri} F_1$ and any $y \in \operatorname{ri}\Lambda_K(F_2)$ (see Proposition 2.3) and take into account that $\Lambda_K(F_2) \subset \Lambda_K(F_1)$). □

## 6. Concluding remarks

As we have mentioned, $N_C(x)$ is a complementarity relation on $C$. In particular, for any $x \in C$ and any $F \in \mathcal{F}_C$

$$N_{N_C(x)}(0) = T_C(x), \quad N_{N_C(F)}(0) = \operatorname{cone}(C - F).$$

Proofs of these facts are elementary and well known. Also, the right hand side of (1) can be easily rewritten as a complementarity mapping $\Phi_Q(x)$ with $Q = A(C)$ and $\Lambda_Q(Ax) = N_{A^{-1}(Q)}(x))$. Reformulating our main results to this specific case, we get almost all main results of [8].

The only exception (if we compare Theorem 5.3 with Theorem 5.4 of [8]) is the absence (among characterizations or refgularity in Theorem 5.4) of a condition that could be interpreted as a generalization of Robinson's coherent orientation condition.

The question of whether such a generalization exists remains open. The point is that in the proof of the corresponding part of Theorem 5.4 in [8] the fact that $\Lambda_C(x)$ is the normal cone, that is that the elements of the cone are orthogonal to $F_{min}(x)$ plays a crucial role. We have seen that there is a polyhedral homeomorphism between $\Lambda_C(x)$ and $N_C(x)$. If those homeomorphisms were linear for all $x$, an extension of the coherent orientation condition would be likely possible. But in general no reasonable way to such an extension is seen for the moment.

It is to be also observed that the proofs given in the preceding section are very similar to the corresponding proof for (1) given in [8], that is passing from face complementary mappings to linear variational inequalities does not result in noticeable simplification of arguments. Again, a tempting idea that might be suggested by this observation along with the homeomorphism property of Proposition 3.9 is that perhaps the results of the previous section are consequences of the correposnding results for linear variational inequalities. We cannot provide now a definite answer to that question but the feeling is that it is negative. Again, it might be possible if all the homeomorphisms were linear which is definitely not the case (e.g. a trapeze is not a linear image of a square).

Another point that should be addressed is the formula for the modulus of surjection/metric regularity for (1) obtained in [3] and the *critical face* condition for regularity [2] the closely connected with the formula. Below we show that both are direct consequences of Theorem 5.6.

**Definition 6.1.** Let $C$ and $A$ be as in (1). It is said that *critical face condition* is satisfied at $x \in C$ if

$$z \in \text{cone}(F_2 - F_1), \quad A^*z \in (F_2 - F_1)^\circ \;\Rightarrow\; z = 0.$$

for any $F_1, F_2 \in \mathcal{F}_C$, $x \in F_1 \subset F_2$.

**Proposition 6.2.** *Let $K$ be a polyhedral cone, and let $F, F_2 \in \mathcal{F}_K$, $F_1 \subset F_2$. Then*

$$(F_2 - F_1)^\circ = N(K, F_1) - N(K, F_2).$$

**Proof.** By Proposition 2.5 $F_2 - F_1 = F_2 + L(F_1)$. Therefore taking into account that $F_2 = K \cap L(F_2)$, we get

$$\begin{aligned}(F_2 - F_1)^\circ &= (F_2 + L(F_1))^\circ = F_2^\circ \cap (L(F_1))^\perp \\ &= (K^\circ + (L(F_2))^\perp) \cap (L(F_1))^\perp = N(K, F_1) + (L(F_2))^\perp.\end{aligned}$$

The last equality needs comments. First note that $K^\circ \cap L(F)^\perp = N(K, F)$ for any face $F$ of $K$. Thus

$$N(K, F_1) = K^\circ \cap L(F_1)^\perp \subset (K^\circ + (L(F_2))^\perp) \cap L(F_1)^\perp.$$

On the other hand,

$$(L(F_2))^\perp = (L(F_2))^\perp \cap (L(F_1))^\perp \subset (K^\circ + (L(F_2))^\perp) \cap L(F_1)^\perp$$

as well. However $(K^\circ + (L(F_2))^\perp) \cap L(F_1)$ is a convex cone. Therefore it contains also $N(K, F_1) + (L(F_2))^\perp$.

To prove the opposite inclusion, let $u \in K^\circ$, $v \in (L(F_2))^\perp$ and $x = u + v \in (L(F_1))^\perp$. Then $v \in (L(F_1))^\perp$ as $(L(F_2))^\perp \subset (L(F_1))^\perp$ and therefore $u \in (L(F_1))^\perp$, that is $u \in K^\circ \cap (L(F_1))^\perp$, so that $u + v \in N(K, F_1) + (L(F_2))^\perp$.

Furthermore, the equalities

$$(\mathrm{lin} K)^\circ = (K \cap (-K))^\circ = K^\circ + (-K^\circ) = L(K^\circ).$$

hold for any polyhedral cone $K$ and as follows from Proposition 2.5, $L(F) = \mathrm{lin}\, T(K, x)$ for an $x \in \mathrm{ri}\, F$. Thus

$$(F_2 - F_1)^\circ = N(K, F_1) + L(N(K, F_2)) = N(K, F_1) - N(K, F_2),$$

the last equality due to the fact that $N(K, F_2) \subset N(K, F_1)$. $\square$

It follows that

$$(N(C, F_1) - N(C, F_2))^\circ = \mathrm{cone}(F_2 - F_1)$$

if $C$ is a polyhedral set and $F_1, F_2 \in \mathcal{F}_C$, $F_1 \subset F_2$. Thus, the condition

$$T^* z^* \in (\mathrm{cone}(F_2 - F_1))^\circ, \quad S^* z^* \in (\Lambda(F_1) - \Lambda(F_2))^\circ \ \Rightarrow z^* = 0$$

with $T = A$, $S = Id$ and $\Lambda_C = N(C, \cdot)$ reduces precisely to the critical face condition.

In other words we have proved the following result.

**Theorem 6.3 ([2, 3]).** *Let $C \subset \mathbb{R}^n$ be a convex polyhedron, and let $A$ be a linear operator in $\mathbb{R}^n$. Set $\Phi(x) = Ax + N(C, x)$ (with $\Phi(x) = \emptyset$ if $x \notin C$). Set for any $F_1$, $F_2 \in \mathcal{F}_C$, $F_1 \subset F_2$*

$$r(F_1, F_2) = \inf\{d(A^* z, (\mathrm{cone}(F_2 - F_1)^\circ)) : \ \|z\| = 1, z \in \mathrm{cone}(F_2 - F_1)\}.$$

*Then for any $x \in C$*

$$\mathrm{sur}\Phi(x|x) = \min\{r(F_1, F_2) : \ F_1, F_2 \in \mathcal{F}_C, \ x \in F_1 \subset F_2\}.$$

*In particular $\Phi$ is regular near $(x, x)$ if and only if the critical face condition is satisfied at $x$.*

**Acknowledgement**. I wish to thank the reviewer for attentive reading and useful remarks.

## References

[1] A. V. Dmitruk, A. A. Milyutin, N. P. Osmolovskii: Ljusternik's theorem and the theory of extrema, Russian Math. Surveys 35(6) (1980) 11–51.

[2] A. L. Dontchev, R. T. Rockafellar: Characterizations of strong regularity for variational inequalities over polyhedral convex sets, SIAM J. Optimization 6 (1996) 1087–1105.

[3] A. L. Dontchev, R. T. Rockafellar: Implicit Functions and Solution Mappings, 2nd edition, Springer, Berlin (2014).

[4] F. Facchinei, J. S. Pang: Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer, New York (2003).

[5] A. D. Ioffe: Variational analysis and mathematical economics. 2: Nonsmooth regular economies, Adv. Math. Economics 14 (2011) 17–38.

[6] A. D. Ioffe: Metric regularity. Theory and applications – a survey, arXiv:1505.07920v2 [math.OC].

[7] A. D. Ioffe: Polyhedrality, complementarity and regularity, with applications to variational inequalities over polyhedral sets, arXiv:1508.06607v2 [math.OC].

[8] A. D. Ioffe: On variational inequalities over polyhedral sets, Math. Programming 168(1-2) (2018) 261-–278.

[9] D. Klatte, B. Kummer: Nonsmooth Equations in Optimization, Kluwer, Dordrecht (2002).

[10] D. Ralph: A new proof of Robinson's homeomorphism theorem for PL-normal maps, Linear Algebra Appl. 178 (1993) 249–260.

[11] S. M. Robinson: Strongly regular generalized equations, Math. Oper. Res. 5 (1980) 43–62.

[12] S. M. Robinson: Normal maps induced by linear transformations, Math. Oper. Res. 17 (1992) 691–714.

[13] R. T. Rockafellar: Convex Analysis, Princeton Univ. Press, Princeton (1970).

[14] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).

[15] S. Scholtes: Introduction to Piecewise Differential Equations, Springer, Berlin (2012).