

Minimization of Quadratic Functions on Convex Sets without Asymptotes

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Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

Received: January 2, 2016

Revised manuscript received: January 6, 2017

Accepted: January 8, 2017

The classical Frank and Wolfe theorem states that a quadratic function which is bounded below on a convex polyhedron P attains its infimum on P . We investigate whether more general classes of convex sets F can be identified which have this Frank-and-Wolfe property. We show that the intrinsic characterizations of Frank-and-Wolfe sets hinge on asymptotic properties of these sets.

Keywords: Quadratic optimization problem, asymptotes, conic asymptotes, Motzkin decomposition, Frank and Wolfe theorem, complementarity problem

2010 Mathematics Subject Classification: 90C20, 90C26

1. Introduction

The classical Frank and Wolfe theorem [4, 5] states that a quadratic function q which is bounded below on a convex polyhedron P attains its infimum on P . It is known that this result has consequences with regard to the existence of solutions to linear complementarity problems [6]. Here we investigate ways in which the Frank and Wolfe theorem can be extended.

A first line is to go beyond polyhedra and ask whether there are more general classes of *Frank-and-Wolfe sets*, that is, convex sets F with the property that every quadratic function q which is bounded below on F attains its infimum on

F . What one would like to obtain is an internal characterization of Frank-and-Wolfe sets via geometric properties, or likewise, verifiable sufficient conditions for the Frank-and-Wolfe property. In response we will characterize Frank-and-Wolfe sets as those convex sets which do not admit conic asymptotes in a sense to be made precise here.

A variant of the same question concerns the larger class of convex sets F with the property that every quadratic function q which is bounded below on F , and which is in addition convex or quasiconvex on F , attains its infimum on F . It turns out that this class has a nice internal characterization. It consists of those convex sets that do not have affine asymptotes in the sense of Klee [9].

A second idea to extend the Frank and Wolfe theorem would be to go beyond quadratics and look for more general classes of functions f attaining their finite infimum on polyhedra P . For instance, do higher degree polynomials f have this property? It turns out that without convexity this line has little hope for success, as shown by the quartic function $f(x) = x_1^2 + (1 - x_1x_2)^2$, which has infimum 0 on the plane, but does not attain its infimum there. Positive results can at best be expected for convex polynomial functions f . For instance, Rockafellar [12, Cor. 27.3.1] shows that a convex polynomial f which is bounded below on a polyhedron P attains its infimum on P . Other variations of this theme are for instance Perold [11], Hirsch and Hoffman [7], or Belousov and Klatte [2].

The structure of the paper is as follows. In Section 2 we define Frank-and-Wolfe sets and variants and obtain first basic properties. Section 3 establishes the link between the Frank-and-Wolfe property and the absence of f -asymptotes in the sense of Klee. In Section 4.7 we consider the Frank-and-Wolfe property within the class of Motzkin decomposable sets, where one expects key information to be provided by the recession cone. Section 5 characterizes Frank-and-Wolfe sets by the absence of q -asymptotes, a geometric notion we define in Section 5. In the final Section 6 we obtain an application to generalized complementarity problems.

Notations

We generally follow Rockafellar's book [12]. The *closure of a set* F is $\overline{F}$. The *Euclidean norm* in $\mathbb{R}^n$ is $\|\cdot\|$, and the *Euclidean distance* is $\text{dist}(x, y) = \|x - y\|$. For subsets M, N of $\mathbb{R}^n$ we write $\text{dist}(M, N) = \inf\{\|x - y\| : x \in M, y \in N\}$. A direction d with $x + td \in F$ for every $x \in F$ and every $t \geq 0$ is called a *direction of recession* of F , and the *cone of all directions of recession* is denoted as 0^+F .

A function $q(x) = \frac{1}{2}x^\top Ax + b^\top x + c$ with $A = A^\top \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ is called *quadratic*. The quadratic $q: \mathbb{R}^n \rightarrow \mathbb{R}$ is *quasiconvex* on a convex set $F \subset \mathbb{R}^n$ if the sublevel sets of $q|_F: F \rightarrow \mathbb{R}$ are convex. Similarly, q is convex on the set F if $q|_F$ is convex.

2. Frank and Wolfe sets

We call a convex set F in $\mathbb{R}^n$ *Frank-and-Wolfe* if every quadratic function $q : \mathbb{R}^n \rightarrow \mathbb{R}$ which is bounded below on F attains its infimum on F . For short we say that F is an *FW-set*. In the same vein we call the convex set F *quasi-Frank-and-Wolfe* if the property holds for every quadratic q which is in addition quasiconvex on F . For short, such sets are called *qFW-sets*.

Formally we may also consider convex sets F where the property holds for every quadratic q which is convex on F . We temporarily call those *cFW-sets*. Ultimately this class will turn out equivalent to quasi-Frank-and-Wolfe sets, i.e., $cFW = qFW$.

Clearly every bounded closed convex set is Frank-and-Wolfe, so the disquisition is only useful in studying unbounded convex sets. Trivially *FW-sets* are *qFW*, and *qFW-sets* are *cFW*. The classical theorem of Frank and Wolfe [5] says that every convex polyhedron P is a *FW-set*. Our first observation is the following.

Lemma 2.1. *Every cFW-set is closed, hence so are qFW- and FW-sets.*

Proof. Consider $x \in \overline{F}$, then $q(\cdot) = \|\cdot - x\|^2$ is quadratic convex and its infimum on F is 0. Since by hypothesis this infimum is attained, we must have $x \in F$. $\square$

Another useful property of Frank-and-Wolfe sets is the following.

Proposition 2.2. *Affine images of cFW-sets are cFW-sets. Similarly, affine images of qFW-sets are qFW, and affine images of FW-sets are FW. In particular, affine images of cFW-sets, qFW-sets and FW-sets are closed.*

Proof. Closedness of the affine image of a *cFW-set* F under an affine image follows from the first part of the statement in tandem with Lemma 2.1. To prove the first part let F be a *cFW-set* and $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ an affine operator. Let $\tilde{F} = T(F)$. We show that $\tilde{F}$ is *cFW*. Let $\tilde{q} : \mathbb{R}^m \rightarrow \mathbb{R}$ be quadratic and convex on $\tilde{F}$, and suppose it is bounded below on $\tilde{F}$ with infimum γ . Then $q = \tilde{q} \circ T$ is quadratic and convex on F , and bounded below on F with the same infimum γ . By the hypothesis on F the infimum γ of q is attained at $x_0 \in F$, and then clearly $\tilde{q}$ attains its infimum γ on $\tilde{F}$ at Tx_0 .

Similarly, if $\tilde{q}$ is quasiconvex on $\tilde{F}$, then q is quasiconvex on F . Therefore the other two statements follow as well. $\square$

Yet another elementary property of *FW-sets* is the following

Proposition 2.3. *Suppose F is a FW-set, and let F' be a closed convex set containing F such that $F' \setminus F$ is bounded. Then F' is FW. The analogous statement holds for qFW-sets.*

Proof. Suppose q is a quadratic function with finite infimum γ' on F' . Then q has also a finite infimum γ on F , where obviously $\gamma \geq \gamma'$. There are two

cases. If $\gamma = \gamma'$, then we choose $x \in F$ where γ is attained, and then γ' is also attained at x . On the other hand, if $\gamma' < \gamma$, then $\inf_{x \in F'} q(x) = \inf_{x' \in F' \setminus F} q(x')$. Since $F' \setminus F$ is bounded, there exists $x' \in \overline{F' \setminus F} \subset F'$ where the infimum γ' is attained. $\square$

3. f -asymptotes

Following Klee [9], an affine manifold M in $\mathbb{R}^n$ is called an f -asymptote of the closed convex set F if $F \cap M = \emptyset$ and $\text{dist}(F, M) = 0$. The link between f -asymptotes and the Frank-and-Wolfe property is given by the following

Theorem 3.1. *Let F be a convex set in $\mathbb{R}^n$. Then the following statements are equivalent:*

- (i) *Every quadratic function q which is quasiconvex on F and bounded below on F attains its infimum on F . That is, F is qFW.*
- (ii) *Every quadratic function q which is convex on F and bounded below on F attains its infimum on F . That is, F is cFW.*
- (iii) *F has no f -asymptotes.*

Proof. The implication (i) $\implies$ (ii) is clear. Consider (ii) $\implies$ (iii). We have to show that F has no f -asymptotes. Let M be an affine manifold with $\text{dist}(F, M) = 0$. We have to show that M is not an f -asymptote of F . Suppose $M = y + U$ for its direction space U and some $y \in U^\perp$. Let P be the orthogonal projection on $U^\perp$, then $P(M) = \{y\}$ and $M = P^{-1}(y)$. Since $\text{dist}(F, M) = 0$, there exist sequences $x_k \in F$, $z_k \in M$, such that $\text{dist}(x_k, z_k) \rightarrow 0$. Then $\text{dist}(Px_k, Pz_k) \leq \text{dist}(x_k, z_k) \rightarrow 0$, but $Pz_k = y$ for every k , hence $\text{dist}(Px_k, y) \rightarrow 0$, so the sequence Px_k converges to y . Now since F has property (ii), its affine image $P(F)$ is closed by Proposition 2.2, so $y \in P(F)$. Pick $x \in F$ with $y = Px$, then $x \in F \cap P^{-1}(y) = F \cap M$, so that $F \cap M \neq \emptyset$. This shows that F does not have f -asymptotes.

It remains to prove the implication (iii) $\implies$ (i). Note first that F is necessarily closed because every point $x \in \overline{F} \setminus F$ would be an f -asymptote, and F has none. We will prove this by induction on the dimension n of F . For dimension $n = 1$ the implication is clearly true, because any quadratic function $q: \mathbb{R} \rightarrow \mathbb{R}$ which is bounded below on a closed convex set $F \subset \mathbb{R}$ attains its infimum on F . Suppose therefore that the result is true for dimension $n - 1$, and consider a quadratic function $q: \mathbb{R}^n \rightarrow \mathbb{R}$ which is quasiconvex on F and bounded below on F . Assume without loss that the dimension of F is n , i.e., F has nonempty interior, as otherwise the result follows directly from the induction hypothesis. Let $\gamma = \inf\{q(x) : x \in F\} > -\infty$, and fix $\alpha > \gamma$. If the sublevel set $S_\alpha := \{x \in F : q(x) \leq \alpha\}$ is bounded, then by the Weierstrass extreme value theorem the infimum of q over S_α is attained. But this infimum is also the infimum of q over F , so in that case we are done. Assume therefore that S_α is unbounded. Since q is quasiconvex on F , the set S_α is closed convex, which

means S_α has a direction of recession d , that is, a direction with $x + td \in S_\alpha$ for every $t \geq 0$ and every $x \in S_\alpha$ (see e.g. [12, Theorem 8.4]). Fix $x \in S_\alpha$. Expanding q at $x + td \in S_\alpha$ gives

$$\gamma \leq q(x + td) = \frac{1}{2}x^\top Ax + b^\top x + c + td^\top(Ax + b) + \frac{1}{2}t^2d^\top Ad \leq \alpha$$

for every $t \geq 0$, and this implies $d^\top Ad = 0$. Substituting this back gives

$$\gamma \leq q(x + td) = \frac{1}{2}x^\top Ax + b^\top x + c + td^\top(Ax + b) \leq \alpha,$$

for every $t \geq 0$. That implies $d^\top(Ax + b) = 0$. But the argument is valid for every $x \in S_\alpha$. By assumption F has dimension n , so S_α has nonempty interior, meaning $x + \epsilon B \subset S_\alpha$ for some $\epsilon > 0$, with B the unit ball. That shows $Ad = 0$. Going back with this into $d^\top(Ax + b) = 0$ shows $d^\top b = 0$, too. Altogether we have shown

$$q(x + td) = q(x) \text{ for every } x \in S_\alpha \text{ and every } t \geq 0. \quad (1)$$

Since q is a quadratic function and S_α has nonempty interior, this implies $q(x + td) = q(x)$ for every $x \in \mathbb{R}^n$ and every $t \in \mathbb{R}$.

Now let P be the orthogonal projection onto the hyperplane $H = d^\perp$. Then $\tilde{q} := q|_H$ is quadratic on the $(n - 1)$ -dimensional space H and takes the same values as q due to (1). In particular, $\tilde{q} = q|_H$ is bounded below on the set $\tilde{F} = P(F)$. Since q is quasiconvex on F , $\tilde{q}$ is quasiconvex on $\tilde{F}$. Therefore $\tilde{q}$ attains its infimum on $\tilde{F}$. Namely, $\dim(\tilde{F}) = n - 1$, and since F does not have f -asymptotes, neither does $\tilde{F}$, so $\tilde{F}$ is qFW by the induction hypothesis. Since q has the same values as $\tilde{q}$, it also attains its infimum on F . $\square$

Remark 3.2. From the implication (iii) $\implies$ (i) it is clear that for a quadratic function q bounded below on F to attain its infimum on F , it is sufficient to have just one of its sublevel sets S_α with $\alpha > \gamma = \inf_{x \in F} q(x)$ convex, a condition which is weaker than quasiconvexity on F . An even weaker condition suffices, namely, the existence of a not necessarily convex sublevel set S_α and a direction $d \in \mathbb{R}^n$ with the following property: For every $x \in S_\alpha$ there exists $t_x \in \mathbb{R}$ such that $x + td \in S_\alpha$ for every $t \geq t_x$.

Remark 3.3. Yet another equivalent condition which we could add to the above list is

(iv) $P(F)$ is closed for every orthogonal projection P .

Indeed (ii) $\implies$ (iv) is Proposition 2.2, and (iv) $\implies$ (ii) is implicit in the proof of (ii) $\implies$ (iii) above. For the equivalence of (iii) and (iv) see also [9].

Corollary 3.4. *Frank-and-Wolfe sets have no f -asymptotes.* $\square$

We end this section by indicating that the converse of Corollary 3.4 is not true. Put differently, the absence of f -asymptotes does *not* characterize Frank-and-Wolfe sets. Or put again differently, there exist quasi-Frank-and-Wolfe sets, which are not Frank-and-Wolfe.

Example 3.5. We construct a closed convex set F without f -asymptotes, which is not Frank-and-Wolfe. We use Example 2 of [10], which we reproduce here for convenience. Consider the optimization program for $x \in \mathbb{R}^4$:

$$\begin{aligned} \text{minimize} \quad & q(x) = x_1^2 - 2x_1x_2 + x_3x_4 \\ \text{subject to} \quad & c_1(x) = x_1^2 - x_3 \leq 0 \\ & c_2(x) = x_2^2 - x_4 \leq 0 \end{aligned}$$

then as Lou and Zhang [10] show the constraint set $F = \{x \in \mathbb{R}^4 : c_1(x) \leq 0, c_2(x) \leq 0\}$ is closed convex, and the quadratic function q has infimum $\gamma = -1$ on F , but this infimum is not attained.

Let us show that F has no f -asymptotes. Note that $F = F_1 \times F_2$, where $F_1 = \{(x_1, x_3) \in \mathbb{R}^2 : x_1^2 - x_3 \leq 0\}$, $F_2 = \{(x_2, x_4) \in \mathbb{R}^2 : x_2^2 - x_4 \leq 0\}$. Observe that $F_1 \cong F_2$, and that F_1 does not have asymptotes, being a parabola. Therefore, F does not have f -asymptotes either. This can be seen from the following

Proposition 3.6. *Suppose F_1, F_2 do not have f -asymptotes. Then neither does $F_1 \times F_2$ have f -asymptotes.*

Proof. We write $F_1 \times F_2 = (F_1 \times \mathbb{R}^n) \cap (\mathbb{R}^n \times F_2)$. Suppose M is an f -asymptote of $F_1 \times F_2$, then by Klee [9, Theorem 4] the flat M contains either an f -asymptote N_1 of $F_1 \times \mathbb{R}^n$, or it contains an f -asymptote N_2 of $\mathbb{R}^n \times F_2$. Assume without loss that M contains N_1 . Let P be the projection on the first coordinate, then $P(N_1)$ is an affine manifold, and it is easy to see that it is an f -asymptote of F_1 . $\square$

Example 3.7. Let F be the epigraph of $f(x) = x^2 + \exp(-x^2)$ in $\mathbb{R}^2$. Then $q(x, y) = y - x^2$ is bounded below on F , but does not attain its infimum, so F is not FW . However, F has no f -asymptotes, so it is qFW . $\square$

4. Motzkin decomposable sets

The proof of the classical Frank-and-Wolfe theorem [5] exploits the fact that a polyhedron P can be decomposed as $P = C + D$, where C is a polytope, and D a convex polyhedral cone. This rises the question whether the Frank and Wolfe theorem may be extended to other classes of convex sets F with this type of decomposition. We recall the following

Definition 4.1. A nonempty closed convex set F in $\mathbb{R}^n$ is called *Motzkin decomposable* if there exists a compact convex set C and a closed convex cone D such that $F = C + D$. We call (C, D) a *Motzkin decomposition* of F . $\square$

We start with a disclaimer. Not all Motzkin decomposable sets are Frank-and-Wolfe.

Example 4.2. We put $D = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1x_2 - x_3^2 \geq 0\}$, then D is a closed convex cone, hence is trivially Motzkin decomposable. But D is not Frank-and-Wolfe. In fact, it is not even quasi-Frank-and-Wolfe, as we now show. Indeed, define $q : \mathbb{R}^3 \rightarrow \mathbb{R}$ by $q(x) = x_1^2 + (x_3 - 1)^2$, then q is quadratic convex and bounded below by 0. In fact, $\gamma = 0$ is the infimum of q on D , because $q\left(\frac{1}{k}, \frac{(k+1)^2}{k}, 1 + \frac{1}{k}\right) = \frac{2}{k^2} \rightarrow 0$, but 0 is not attained on D . In view of Theorem 3.1, the cone D must have f -asymptotes. $\square$

Example 4.3. In the same vein consider the quadratic function $q : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined as $q(x, y, z) := (x - 1)^2 - y + z$ and the ice-cream cone $F := \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$. Clearly $q \geq 0$ on F since $z \geq y$ for every $(x, y, z) \in F$. But the infimum of q on F is 0, since $(1, k, \sqrt{1 + k^2}) \in F$ and

$$q(1, k, \sqrt{1 + k^2}) = \sqrt{1 + k^2} - k \rightarrow 0,$$

and this infimum is not attained, as for $(x, y, z) \in F$, one has either $x \neq 1$ or $z \geq \sqrt{1 + y^2} > y$, which both imply $q(x, y, z) > 0$.

The orthogonal projection of F onto the hyperplane

$$H := \{(x, y, z) \in \mathbb{R}^3 : y + z = 0\}$$

is not closed. To see this, notice that the orthogonal projection P on H is given by $P(x, y, z) = (x, \frac{y-z}{2}, \frac{z-y}{2})$. Consider again $(1, k, \sqrt{1 + k^2}) \in F$, then $P(1, k, \sqrt{1 + k^2}) = (1, \frac{k - \sqrt{1 + k^2}}{2}, \frac{\sqrt{1 + k^2} - k}{2}) \in P(F)$, but its limit $(1, 0, 0)$ does not belong to $P(F)$, because $P^{-1}(1, 0, 0) = \{(x, y, z) \in \mathbb{R}^3 : x = 1, y = z\}$ does not intersect F . $\square$

These examples raise the question whether a Motzkin decomposable set F is Frank-and-Wolfe as soon as its recession cone 0^+F is Frank-and-Wolfe. A similar question can be asked for quasi-Frank-and-Wolfe sets. For the latter class things have been simplified due to Theorem 3.1, and we have the following answer.

Proposition 4.4. *Let F be a Motzkin decomposable set. Then F is quasi-Frank-and-Wolfe if and only if its recession cone 0^+F is quasi-Frank-and-Wolfe.*

Proof. 1) Suppose 0^+F is qFW . Assume contrary to what is claimed that F has an f -asymptote M . Write $M = y + U$ for the direction space U of M and $y \in U^\perp$. Let P be the orthogonal projection onto $U^\perp$. Then $M = P^{-1}(y)$. Observe that $P(F)$ is not closed. Indeed, there exist $x_k \in F$, $y_k \in M$, with $\text{dist}(x_k, y_k) \rightarrow 0$. Therefore $Px_k \rightarrow y$. But $y \notin P(F)$, because if $y = Px$ for some $x \in F$, then $x \in F \cap P^{-1}(y) = F \cap M$, which is impossible due to $F \cap M = \emptyset$.

Since F is Motzkin decomposable, there exist a compact convex C with $F = C + 0^+F$. Then $P(F) = P(C) + P(0^+F)$, while $\overline{P(F)} = P(C) + \overline{P(0^+F)}$. Since $P(F) \neq \overline{P(F)}$, we deduce that $P(0^+F)$ cannot be closed, and that means 0^+F has an f -asymptote parallel to U , contradicting the fact that 0^+F is a qFW -set.

2) Conversely, suppose F is qFW , but that 0^+F is not qFW . Then 0^+F must have an f -asymptote L by Theorem 3.1. Suppose $L = y + W$ with W the direction space of L and $y \in W^\perp$. Let P be the orthogonal projection on $W^\perp$, then again $P(0^+F)$ is not closed. Now by [8, Proposition 5] F has an f -asymptote parallel to W , and by Theorem 3.1 this contradicts the fact that F is qFW . $\square$

Remark 4.5. The statement of Proposition 4.4 is no longer correct if one drops the hypothesis that F is Motzkin decomposable. We take $F = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1\}$, then F , being a hyperbola, has f -asymptotes, but 0^+F is the positive orthant, which does not have f -asymptotes.

An immediate consequence of Proposition 4.4 is the following

Corollary 4.6. *Let F be a Motzkin decomposable qFW -set. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear operator. Then $T(0^+F)$ is closed in $\mathbb{R}^m$.* $\square$

Proposition 4.4 is a strong incentive to look for similar criteria for the Frank-and-Wolfe property in terms of 0^+F . A first partial answer is the following generalization of the classical Frank and Wolfe theorem.

Theorem 4.7. *Let F be a Motzkin decomposable convex set, and suppose its recession cone 0^+F is polyhedral. Then F is Frank-and-Wolfe.*

Proof. Write $F = C + 0^+F$ for C compact convex. Now consider a quadratic function $q(x) = \frac{1}{2}x^\top Ax + b^\top x$ bounded below by γ on F . Hence

$$\inf_{x \in F} q(x) = \inf_{y \in C} \inf_{z \in 0^+F} q(y + z) = \inf_{y \in C} \left(q(y) + \inf_{z \in 0^+F} y^\top Az + q(z) \right) \geq \gamma. \quad (2)$$

Now observe that for fixed $y \in C$ the function $q_y : z \mapsto y^\top Az + q(z)$ is bounded below on 0^+F by $\eta = \gamma - \max_{y' \in C} q(y')$. Indeed, for $z \in 0^+F$ we have

$$\begin{aligned} y^\top Az + q(z) &\geq \left(q(y) + \inf_{z' \in 0^+F} y^\top Az' + q(z') \right) - q(y) \\ &\geq \inf_{y' \in C} \left(q(y') + \inf_{z' \in 0^+F} y'^\top Az' + q(z') \right) - \max_{y' \in C} q(y') \\ &\geq \gamma - \max_{y' \in C} q(y') = \eta. \end{aligned}$$

Since q_y is a quadratic function bounded below on the polyhedral cone 0^+F , the inner infimum is attained at some $z = z(y)$. This is in fact the classical

Frank and Wolfe theorem on a polyhedral cone. In consequence the function $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ defined as

$$f(y) = \inf_{z \in 0^+ F} y^\top Az + q(z),$$

satisfies $f(y) = y^\top Az(y) + q(z(y)) > -\infty$ for $y \in C$, so the compact set C is contained in its domain. But now a stronger result holds, which one could call a parametric Frank and Wolfe theorem, and which we shall prove in Lemma 4.8 below. We show that f is continuous relative to its domain. Once this is proved, the infimum (2) can then be written as

$$\inf_{x \in F} q(x) = \inf_{y \in C} q(y) + f(y),$$

and this is now attained by the Weierstrass extreme value theorem due to the continuity of $q+f$ on the compact C . Continuity of f on C is now a consequence of the following $\square$

Lemma 4.8. *Let D be a polyhedral convex cone and define*

$$f(c) = \inf_{x \in D} c^\top x + \frac{1}{2} x^\top Gx,$$

where $G = G^\top$. Then $\text{dom}(f)$ is a polyhedral convex cone, and hence f is continuous relative to $\text{dom}(f)$.

Proof. If $x^\top Gx < 0$ for some $x \in D$, then $\text{dom}(f) = \emptyset$, so we may assume for the remainder of the proof that $x^\top Gx \geq 0$ for every $x \in D$. Clearly then,

$$\text{dom}(f) = \{c : c^\top x \geq 0 \text{ for every } x \in D \text{ such that } x^\top Gx = 0\}.$$

Now by the Farkas-Minkowski-Weyl theorem (cf. [12, Thm. 19.1] or [13, Cor. 7.1a]) the polyhedral cone D is the linear image of the positive orthant of a space $\mathbb{R}^p$ of appropriate dimension, i.e. $D = \{Zu : u \in \mathbb{R}^p, u \geq 0\}$. This implies

$$\text{dom}(f) = \{c : c^\top Zu \geq 0 \text{ for every } u \geq 0 \text{ such that } u^\top Z^\top GZu = 0\}.$$

Now observe that if $u \geq 0$ satisfies $u^\top Z^\top GZu = 0$, then it is a minimizer of the quadratic function $u^\top Z^\top GZu$ on the cone $u \geq 0$, hence $Z^\top GZu \geq 0$ by the Kuhn-Tucker conditions. Therefore we can write the set $P = \{u \in \mathbb{R}^p : u \geq 0, u^\top Z^\top GZu = 0\}$ as

$$P = \bigcup_{I \subset \{1, \dots, p\}} P_I,$$

where the P_I are the polyhedral convex cones

$$P_I = \{u \geq 0 : Z^\top GZu \geq 0, u_i = 0 \text{ for all } i \in I, (Z^\top GZu)_j = 0 \text{ for all } j \notin I\}.$$

For every $I \subset \{1, \dots, p\}$ choose m_I generators $u_{I1}, \dots, u_{Im_I}$ of P_I . Then,

$$\begin{aligned} \text{dom}(f) &= \{c : c^\top Z u \geq 0 \text{ for every } u \in P\} \\ &= \left\{ c : c^\top Z u \geq 0 \text{ for every } u \in \bigcup_{I \subset \{1, \dots, p\}} P_I \right\} \\ &= \bigcap_{I \subset \{1, \dots, p\}} \{c : c^\top Z u \geq 0 \text{ for every } u \in P_I\} \\ &= \bigcap_{I \subset \{1, \dots, p\}} \{c : c^\top Z u_{Ij} \geq 0 \text{ for all } j = 1, \dots, m_I\}. \end{aligned} \quad (3)$$

Since a finite intersection of polyhedral cones is polyhedral, this proves that $\text{dom}(f)$ is a polyhedral convex cone. To conclude, continuity of f relative to its domain now follows from [12, Thm. 10.2], since f is clearly concave and upper semicontinuous. $\square$

Remark 4.9. The proof includes the case when $x^\top G x > 0$ holds for every $x \in D \setminus \{0\}$. In that case one has $P_I = \{0\}$ for every $I \subset \{1, \dots, p\}$, and therefore $\{c : c^\top Z u \geq 0 \text{ for every } u \in P_I\} = \mathbb{R}^n$, so that the equality (3) still holds and reduces to $\text{dom}(f) = \mathbb{R}^n$.

Remark 4.10. We refer to Banks *et al.* [1, Thm. 5.5.1 (4)] or Best and Ding [3] for a related result in the case where $G \succeq 0$. For the indefinite case see also Tam [14].

Remark 4.11. The example in Remark 4.5 shows that Theorem 4.7 is no longer true if F is not Motzkin decomposable.

A second partial answer to the question whether the Frank-and-Wolfe property of 0^+F implies that of F is given in the following

Proposition 4.12. *Let F have a Motzkin decomposition of the form $F = P + 0^+F$ with P a polytope. If 0^+F is Frank-and-Wolfe, then so is F .*

Proof. Consider a quadratic q which is bounded below on F . Splitting the infimum according to (2), we see as in the proof of Theorem 4.7 that every $q_y : z \mapsto y^\top A z + q(z)$ is quadratic and bounded below on 0^+F , and since 0^+F is Frank-and-Wolfe by hypothesis, the inner infimum in (2) is attained at $z(y) \in 0^+F$. As in the proof of Theorem 4.7 define $f(y) = \inf_{z \in 0^+F} q_y(z) = q_y(z(y))$, then f is the infimum of the family of affine functions $y \mapsto y^\top A z + q(z)$ on the polytope P , hence is lower semi-continuous on P by [12, Theorem 10.2]. But then $y \mapsto q(y) + f(y)$ is lower semi-continuous on P , and by compactness of P the outer infimum $y \in P$ in (2) is therefore attained. $\square$

5. q -Asymptotes

The discussion in Section 3 shows that the absence of f -asymptotes is only a necessary condition for the Frank-and-Wolfe property. In this section we shall develop a related concept of asymptotes, where we replace affine (flat) surfaces by quadratic surfaces. We start with the following

Definition 5.1. A quadric in $\mathbb{R}^n$, also called a *quadratic surface* or a *conic*, is a set of the form $Q = \{x \in \mathbb{R}^n : \frac{1}{2}x^\top Ax + b^\top x + c = 0\}$ with $A = A^\top \neq 0$. $\square$

Definition 5.2. A nonempty closed set A is said to be *asymptotic* to the nonempty closed convex set F if $A \cap F = \emptyset$ and $\text{dist}(F, A) = 0$. $\square$

If A is an affine subspace of $\mathbb{R}^n$, then A is asymptotic to F iff it is an f -asymptote in the sense of Klee [9] and in the sense of Section 3. Now we can give the central definition of this section.

Definition 5.3. The quadric $Q = \{x \in \mathbb{R}^n : q(x) = \frac{1}{2}x^\top Ax + b^\top x + c = 0\}$ is a q -asymptote of the closed convex subset F of $\mathbb{R}^n$ if $F \cap Q = \emptyset$ and $\text{dist}(Q \times \{0\}, \{(x, q(x)) : x \in F\}) = 0$. $\square$

The condition means $F \cap Q = \emptyset$ and that there exist $x_k \in F$ and y_k with $q(y_k) = 0$ such that $x_k - y_k \rightarrow 0$ and $q(x_k) \rightarrow 0$. This shows that the notion of a q -asymptote is invariant under an affine change of coordinates in $\mathbb{R}^n$, hence is a concept of affine geometry. The condition could also be expressed as follows: The quadric $Q \times \{0\}$ in $\mathbb{R}^{n+1}$ is asymptotic to $\text{graph}_F(q) := \text{graph}(q) \cap (F \times \mathbb{R})$ in the sense of Definition 5.2.

Remark 5.4. If Q is a q -asymptote of F , then Q is clearly asymptotic to F , but the converse is not true in general. To see this consider the following example. Let $F = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ be the positive orthant, and let $q(x, y) = xy + 1$, then $Q = \{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\} = \{(x, y) : xy = -1\}$ is a hyperbola with $F \cap Q = \emptyset$. We have $\text{dist}(F, Q) = 0$, because $(-\frac{1}{n}, n) \in Q$ and $(0, n) \in F$, so Q is asymptotic to F in the sense of definition 5.2. But Q is not a q -asymptote of F , because the sets

$$Q \times \{0\} = \{(x, -\frac{1}{x}, 0) : x \neq 0\}, \quad \text{and} \\ \text{graph}(q) \cap (F \times \mathbb{R}) = \{(x, y, xy + 1) : x \geq 0, y \geq 0\}$$

cannot be close. $\square$

Remark 5.5. Consider the quadric

$$Q : q(x) = x_1^2 + \cdots + x_{n-1}^2 = 0,$$

then $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ is the x_n -axis. Suppose the x_n -axis is an f -asymptote of a closed convex set F . This is equivalent to Q being asymptotic to F . However, we argue that Q is then even a q -asymptote of F in the sense of definition 5.3. Namely, we have

$$Q \times \{0\} = \{(0_{n-1}, \xi, 0) : \xi \in \mathbb{R}\} \subset \mathbb{R}^{n+1}, \quad \text{and} \\ \text{graph}(q) \cap (F \times \mathbb{R}) = \{(x, q(x)) : x \in F\} = \left\{ \left(x, \sum_{i=1}^{n-1} x_i^2 \right) : x \in F \right\}.$$

Now given $\epsilon > 0$ choose $x \in F$ and $t \in \mathbb{R}$ such that $\|x - (0_{n-1}, t)\|^2 < \epsilon^2$, which is possible because $\text{dist}(Q, F) = 0$. (Naturally, we could take $t = x_n$). Then $q(x) = x_1^2 + \cdots + x_{n-1}^2 < \epsilon^2$ and $(x_n - t)^2 < \epsilon^2$. Therefore

$$\|(0, t, 0) - (x, q(x))\|^2 \leq \|x - (0_{n-1}, t)\|^2 + q(x)^2 \leq \epsilon^2 + q(x)^2 < \epsilon^2 + \epsilon^4.$$

This shows the claim. We can generalize this to a proof that any flat M which is an f -asymptote is also a q -asymptote when interpreted as a quadric:

Proposition 5.6. *Let F be closed convex in $\mathbb{R}^n$ and $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ be a quadric. Suppose Q is flat, that is, degenerates to an affine subspace. Then Q is a q -asymptote of F in the sense of Definition 5.3 if and only if it is an f -asymptote of F in the sense of [9]. Moreover, for any f -asymptote M of F there exists a quadric representation of M as $M = \{x \in \mathbb{R}^n : q(x) = 0\}$, and then M is also a q -asymptote of F .*

Proof. The fact that Q is an affine subspace of dimension $k \leq n - 1$ means that one can find affine coordinates in $\mathbb{R}^n$ such that Q has the form $Q = \{x \in \mathbb{R}^n : x_{k+1}^2 + \cdots + x_n^2 = 0\} = \mathbb{R}^k \times \{0_{n-k}\}$.

Since being a q -asymptote implies being asymptotic, and since for an affine subspace this coincides with being an f -asymptote, we have but to prove the opposite implication.

Assume therefore that Q is an f -asymptote of F , i.e., $F \cap Q = \emptyset$ and $\text{dist}(F, Q) = 0$. We have to show that $Q \times \{0\} = \{(x, 0) : x \in Q\}$ is asymptotic to $\text{graph}(q) \cap (F \times \mathbb{R}) = \{(y, q(y)) : y \in F\}$. Clearly the two sets are disjoint. Splitting $x = (x', x'')$, $y = (y', y'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we have $q(y) = y_{k+1}'^2 + \cdots + y_n'^2 = \|y''\|^2$. Now pick $x^r \in Q$, $y^r \in F$ with $\text{dist}(x^r, y^r) \rightarrow 0$ as $r \rightarrow \infty$. Then $x^r = (x'^r, x''^r) = (x'^r, 0)$ and $y^r = (y'^r, y''^r)$, hence $\|y''^r\|^2 \leq \|x^r - y^r\|^2 \rightarrow 0$, and this implies $q(y^r) = \|y''^r\|^2 \rightarrow 0$. Hence Q is a q -asymptote of F , because it now follows that $\|(x^r, 0) - (y^r, q(y^r))\| \rightarrow 0$. $\square$

This result shows that the notion of a q -asymptote is a natural extension of Klee's concept of f -asymptotes. We move from flat asymptotes to quadratic asymptotes. We are now ready to state the principal result of this section.

Theorem 5.7. *A convex set F is Frank-and-Wolfe if and only if it has no q -asymptotes.*

Proof. 1) Assume that there exists a quadratic function $q : \mathbb{R}^n \rightarrow \mathbb{R}$ which is bounded below on F , but does not attain its infimum on F . We have to show that F has a q -asymptote. Assume without loss that the infimum of q on F is 0. Since there exists $x \in F$ with $q(x) > 0$ and $y \notin F$ with $q(y) = 0$, the set $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ is a quadric in $\mathbb{R}^n$.

Note that if F is not qFW , then by Theorem 3.1 the set F has an f -asymptote, and then has also a q -asymptote by Proposition 5.6. So we can assume that F

is qFW , and by Proposition 2.2 we therefore know that orthogonal projections of F are closed.

We clearly have $F \cap Q = \emptyset$, so we have to show that $\text{dist}(\{(x, q(x)) : x \in F\}, Q \times \{0\}) = 0$. Since the statement we have to prove is invariant under an affine change of coordinates in $\mathbb{R}^n$, we may assume that the quadric Q is given by one of the following equations:

$$Q : \quad q(x) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^r x_i^2 + \gamma = 0, \quad (p < r \leq n) \quad (4)$$

where $\gamma \in \{0, 1\}$ if Q is a center quadric with 0 as its center, or

$$Q : \quad q(x) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^r x_i^2 + x_{r+1} = 0 \quad (p \leq r < n) \quad (5)$$

if Q is a paraboloid.

a) Let us first discuss the easier case of a paraboloid (5). Since q is a quadratic function, it satisfies a Łojasiewicz inequality at infinity. In other words, following [15, Theorem 2.1] there exist constants $\delta > 0$, $c > 0$ and a Łojasiewicz exponent $\alpha > 0$ at infinity such that for every $x \in \mathbb{R}^n$ with $|q(x)| < \delta$ we have

$$|q(x)| \geq c \text{dist}(x, \widehat{Q})^\alpha,$$

where $\widehat{Q} = Q \cup Q_1$ with

$$Q = \{x \in \mathbb{R}^n : q(x) = 0\}, \quad \text{and} \quad Q_1 = \left\{x \in \mathbb{R}^n : \frac{\partial}{\partial x_{r+1}} q(x) = 0\right\}.$$

This result uses the fact that q is a monic polynomial of degree $m = 1$ in the variable x_{r+1} . Since $\partial/\partial x_{r+1} q(x) = 1$, the set Q_1 is empty, hence we obtain

$$|q(x)| \geq c \text{dist}(x, Q)^\alpha$$

for $|q(x)| < \delta$. Now choose a sequence $x_k \in F$ with $q(x_k) \rightarrow 0$. Then from some k onward, $q(x_k) \geq c \text{dist}(x_k, Q)^\alpha \rightarrow 0$, which proves $\text{dist}(\{(x, q(x)) : x \in F\}, Q \times \{0\}) = 0$. This settles the case where Q is a paraboloid.

b) Let us next consider the more complicated case where Q is a center quadric. Choose a sequence $x_k \in F$ with $q(x_k) \rightarrow 0$. We want to show $\text{dist}(x_k, Q) \rightarrow 0$, at least for a subsequence. Assume on the contrary that $\text{dist}(x_k, Q) > d > 0$ for every k . Write $x_k = (\xi_1^k, \dots, \xi_n^k)$, and note that $\|x_k\| \rightarrow \infty$. We now have two principal cases.

Case I is when $(\xi_1^k, \dots, \xi_r^k) \rightarrow 0$, while the part $(\xi_{r+1}^k, \dots, \xi_n^k)$ on which q given by (4) does not depend satisfies $\|(\xi_{r+1}^k, \dots, \xi_n^k)\| \rightarrow \infty$. In this case we necessarily have $r < n$.

Case II is when there exists $i \in \{1, \dots, r\}$ such that $\xi_i^k \rightarrow \xi_i \neq 0$ for a subsequence $k \in \mathcal{K}$, including the possibilities $\xi_i = \pm\infty$.

We start by discussing case II. Suppose ξ_i is finite and the signature of i is negative (i.e. $i \in \{p+1, \dots, r\}$). Then there must also exist another index with positive signature $j \in \{1, \dots, p\}$ say, for which $\xi_j^k \rightarrow \xi_j \neq 0$. (This is because in (4) the $-\xi_i^2$ and ξ_j^2 have to sum to $\gamma \geq 0$. Therefore if there is a non-vanishing contribution from an index i with negative signature, there is necessarily also one from an index j with positive signature.) We may without loss assume that this contribution with positive signatures comes from $j = 1$. A similar argument applies when $\xi_i = \pm\infty$. We now have two subcases. Case II.1 is when $\xi_1^k \rightarrow \xi_1 \in (0, +\infty]$, case II. 2 is when $\xi_1^k \rightarrow \xi_1 \in [-\infty, 0)$.

Let us discuss case II.1. Shrinking d if need be, we assume $\xi_1 - d > 0$, and then also $\xi_1^k > d$ for all k large enough. (This works also for $\xi_1 = +\infty$). Now define $f_k(t) = q(t, \xi_2^k, \dots, \xi_n^k)$, then $f_k(t) = t^2 + r(\xi_2^k, \dots, \xi_n^k)$. We have $f_k(t) > 0$ for every $t \in I_k := [\xi_1^k - d, \xi_1^k + d]$, because $d < \text{dist}(x_k, Q)$. Moreover, $f'_k(t) = 2t \geq 2(\xi_1^k - d) > 0$ for $t \in I_k$. So f_k is positive and increasing on I_k . Therefore

$$\max_{t \in [\xi_1^k - d, \xi_1^k]} f_k(t) = f_k(\xi_1^k) = q(x_k) \rightarrow 0.$$

Now define $g_k(t) = f_k(t + \xi_1^k)$, then

$$\max_{s \in [-d, 0]} g_k(s) = q(x_k) \rightarrow 0.$$

Therefore the sequence g_k converges to 0 in the space of quadratic polynomials in the variable t . But that implies its coefficients tend to 0, a contradiction with $g_k(t) = (t + \xi_1^k)^2 + r(\xi_2^k, \dots, \xi_n^k)$, because the coefficient of t^2 is 1 and does not tend to 0. That is a contradiction in case II. 1, and therefore settles that case.

Now consider case II. 2. Here we arrange $\xi_1 + d < 0$, and then also $\xi_1^k + d < 0$ for k sufficiently large, and that works also for $\xi_1 = -\infty$. So here f_k is positive and decreasing on I_k . We use an analogous argument, and get a similar contradiction. That settles case II.

c) It remains to discuss case I. Note that here we must have $\gamma = 0$, so Q is a cone (in the sense of quadric theory). Suppose $r > 0$, then the sublevel set $\{x \in F : q(x) \leq r\}$ is nonempty and unbounded. Fix x in this set, then $q(x) = q(x + td)$ for every d of the form $d = (0, \dots, 0, \xi_{r+1}, \dots, \xi_n)$, because q does not depend on the coordinates $\xi_{r+1}, \dots, \xi_n$. Now let P be the orthogonal projection on $d^\perp$, then $P(F)$ is convex and, in addition, closed by what was observed at the beginning of the proof. But the infimum of q on $P(F)$ is still 0, and it is not attained. With regard to the form (4) we have therefore reduced the dimension n by 1, but the quadric is still of the form (4) with the same r . Continuing in this way, we end up with the case where $r = n$ in (4). But then we are in case II, because remember that case I can only occur when $r < n$. That settles case I and therefore completes the first part of the proof.

2) Let us now prove that if F has a q -asymptote Q , then it is not Frank-and-Wolfe. From the definition of a q -asymptote we have $F \cap Q = \emptyset$. We may therefore assume without loss that $F \subset \{x \in \mathbb{R}^n : q(x) > 0\}$, because F is connected and q is continuous. Now there exists a sequence $x_k \in F$ and a sequence $y_k \in Q$ such that $\text{dist}((x_k, q(x_k)), (y_k, 0)) \rightarrow 0$. That means 0 is the infimum of q on F , and it is not attained. $\square$

Remark 5.8. One might be tempted to guess that F is Frank-and-Wolfe iff there is no quadratic Q which is asymptotic to F . The example of the positive orthant in remark 5.4 shows that this guess is incorrect. The corresponding condition is too strong.

Remark 5.9. The crucial point in the first part of the proof of Theorem 5.7 is that $\text{dist}(Q, F) = 0$ is inferred from the fact that 0 is in the closure of the value set $\{q(x) : x \in F\}$. This is obviously strongly linked to the fact that q is quadratic, and we now indicate why there is little hope for an extension of the argument to higher degree polynomials. Consider for instance $f(x, y, z) = (y^2 + (xy - 1)^2)z$, then $Q = \{(x, y, z) : f(x, y, z) = 0\} = \{z = 0\}$ is an affine manifold. We have $f(k, \frac{1}{k}, 1) \rightarrow 0$, but $\text{dist}((k, \frac{1}{k}, 1), Q) \rightarrow \frac{1}{\sqrt{2}}$ as $k \rightarrow \infty$. Putting $F = \{(x, y, 1) \in \mathbb{R}^3 : xy \geq 1\}$, we see that f does not attain its infimum 0 on F , yet the affine manifold $Q = \{f = 0\}$ is not asymptotic to F . What is missing is indeed an argument which allows to infer from $f(x_k) \rightarrow 0$ for $x_k \in F$ that also $\text{dist}(x_k, Q) \rightarrow 0$, and for higher order polynomials such an argument may not exist.

Example 5.10. To illustrate Theorem 5.7 we consider the set $F = \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}$ and claim that it is Frank-and-Wolfe. We check this by showing that F has no q -asymptotes. Suppose $Q = \{q = 0\}$ is a q -asymptote of F . If Q is a hyperbola or consists of two lines, then F itself has lines as asymptotes, which is impossible, because F is a parabola. It is equally impossible that Q is an ellipse, so Q must be a parabola, too. But it is intuitively clear that no other parabola can be a q -asymptote of $y = x^2$.

To prove this rigorously, suppose $q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. By the definition of a q -asymptote there exist $(x_k, y_k) \in Q$ and $(x_k, x_k^2) \in F$, such that $\|(x_k, x_k^2) - (x_k, y_k)\| \rightarrow 0$ and $q(x_k, x_k^2) \rightarrow 0$. Picking a subsequence, we may without loss assume $x_k \rightarrow +\infty$. Then $q(x_k, x_k^2) = ax_k^2 + bx_k^3 + cx_k^4 + dx_k + ex_k^2 + f \rightarrow 0$ implies successively $c = 0$, then $b = 0$, then $a = -e$, then $d = 0$ and $f = 0$, and finally $a \neq 0$. Hence $Q = \{(x, y) : a(x^2 - y) = 0\}$, but this is the boundary curve of F , which contradicts $F \cap Q = \emptyset$. $\square$

Remark 5.4 suggests an equivalent geometric characterization of q -asymptotes, which we now develop. Let $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ be a quadric and consider the associated one-parameter family $\mathcal{Q} = \{Q_\alpha\}_{\alpha \in \mathbb{R}}$ of quadrics $Q_\alpha = \{x \in \mathbb{R}^n : q(x) - \alpha = 0\}$. Note that $\mathcal{Q}$ is a geometric object, as an affine change of coordinates leads to the same family of sets. Informally, we intend to show

that $Q \in \mathcal{Q}$ is a q -asymptote of the closed convex set F if and only if Q, F are asymptotic, and no other element Q' of the bundle $\mathcal{Q}$ can be squeezed in between F and Q .

Definition 5.11. Let F, Q be closed sets with $F \cap Q = \emptyset$ and $\text{dist}(F, Q) = 0$. We say that the closed set Q' is *squeezed* in between F and Q if $F \cap Q' = \emptyset = Q \cap Q'$ and if every segment $[x, y]$ with $x \in F$ and $y \in Q$ contains a point $z \in Q'$, i.e., $[x, y] \cap Q' \neq \emptyset$. $\square$

We now have the following

Proposition 5.12. *Let F be closed convex and let $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ be a quadric. Then Q is a q -asymptote of F if and only if Q is asymptotic to F and no other member Q' of the bundle $\mathcal{Q}$ can be squeezed in between F and Q . In other words, Q is a tight quadric asymptote to F .*

Proof. 1) Suppose Q is a q -asymptote of F . Then there exist $x_k \in F, y_k \in Q$ such that $x_k - y_k \rightarrow 0$ and $q(x_k) \rightarrow 0$. Clearly Q is asymptotic to F . Since $F \cap Q = \emptyset$ and F is connected, we either have $F \subset \{x : q(x) > 0\}$ or $F \subset \{x : q(x) < 0\}$. Assume without loss that $F \subset \{x : q(x) > 0\}$. Suppose $Q' = \{x : q(x) = \alpha\}$ can be squeezed in between Q and F . Since $Q \cap Q' = \emptyset$, we have $\alpha \neq 0$. There are two cases to be discussed.

Suppose first that $\alpha < 0$. Then we find a point z_k in the open segment (x_k, y_k) such that $q(z_k) = \alpha < 0$. But $q(x_k) > 0$, hence by the mean value theorem there exists another point v_k in the open segment (x_k, z_k) with $q(v_k) = 0$. Now we repeat the argument on $[x_k, v_k]$, which must also contain a point with value $q = \alpha$. That leads to a contradiction, because we thereby find a third root of $q - \alpha$ on the segment $[x_k, y_k]$, which is impossible as q is quadratic. In consequence the squeezing value must be $\alpha > 0$.

Suppose therefore that the quadric Q' which may be squeezed in between F and Q has $\alpha > 0$. Then we have the following situation on the segment $[x_k, y_k]$. There exists $z_k \in (x_k, y_k)$ with $q(z_k) = \alpha > 0$, while $q(y_k) = 0$ and $q(x_k) \rightarrow 0$, $0 < q(x_k) \ll \alpha$. Let L_k be the line generated by $[x_k, y_k]$. Since q is a quadratic function on L_k , there exists a point $v_k \in L_k$ preceding x_k where $q(v_k) = 0$. Here preceding means that $x_k \in [v_k, y_k]$. Since $F \subset \{q > 0\}$, we have $v_k \notin F$. In particular, $F \cap L_k$ is contained in the segment $[v_k, x_k]$. But $v_k \in Q$, $x_k \in F$, hence the segment $[v_k, x_k]$ must also contain an element w_k of Q' , i.e., with $q(w_k) = \alpha$, and that is impossible because q is quadratic. Namely, the arrangement on the line L_k is now $v_k < w_k < x_k < z_k < y_k$ with $q(y_k) = 0$, $q(z_k) = \alpha > 0$, $q(x_k) \ll \alpha$, $q(w_k) = \alpha$, $q(v_k) = 0$. But $q|_{L_k}$ is concave, so this is impossible. This proves that $Q' \in \mathcal{Q}$ could not possibly be squeezed in between F and Q .

2) Conversely, suppose Q is asymptotic to F and is tight in the sense that no other member Q' of the bundle $\mathcal{Q}$ can be squeezed in between F and Q . Since

$F \cap Q = \emptyset$, we may assume $F \subset \{x : q(x) > 0\}$. Let $\gamma := \inf_{x \in F} q(x)$. We claim that $\gamma = 0$. For suppose we had $\gamma > 0$ then on choosing $0 < \alpha < \gamma$ we find that $Q' = \{x : q(x) - \alpha = 0\}$ is squeezed in between F and Q , which is impossible. Hence $\gamma = 0$. Now pick $x_k \in F$ with $q(x_k) \rightarrow 0$ and $y_k \in Q$. Using the argument of part 1) of the proof of Theorem 5.3, it follows that $y_k - x_k \rightarrow 0$. Hence $(x_k, q(x_k)) - (y_k, 0) \rightarrow 0$. That shows $\text{dist}(Q \times \{0\}, \text{graph}_F(q)) = 0$, hence Q is a q -asymptote of F . $\square$

Remark 5.13. In view of this new characterization of q -asymptotes we have the following description of Frank-and-Wolfe sets. Whenever Q is a quadric asymptote of a Frank and Wolfe set F , then there exists another quadric Q' in the bundle $\mathcal{Q}$ associated with Q that can be squeezed in between F and Q . We could say that Q' is a tighter asymptote than Q . As this argument can be repeated, the FW -set F has no tightest asymptote among the quadrics in $\mathcal{Q}$.

Remark 5.14. It is instructive to give a direct argument for the fact that an f -asymptote in the sense of Klee is tight in the sense of the previous remark, hence is a q -asymptote. To see this, suppose $M = \{x \in \mathbb{R}^n : Ax - b = 0\}$ is an f -asymptote of F and represent M as the quadric $M = \{x : q(x) = \|Ax - b\|^2 = 0\} = Q$. Consider the associated bundle $\mathcal{Q} = \{Q_\alpha\}$ and suppose some Q_α with $\alpha \neq 0$ can be squeezed in between $Q = M$ and F . Clearly this means $\alpha > 0$, as the $Q_{\alpha'}$ with $\alpha' < 0$ are empty. But q is convex, hence $F \subset \{x : q(x) = \|Ax - b\|^2 > \alpha\}$. Indeed, suppose there exists $x \in F$ with $q(x) < \alpha$. Pick $y \in M$, then $q(y) = 0$. Since $\{z : q(z) < \alpha\}$ is a convex set containing x, y , it also contains the segment, $[x, y] \subset \{z : q(z) < \alpha\}$. But since Q_α is by assumption squeezed in between F and M , there exists $z \in [x, y]$ with $q(z) = \alpha$, a contradiction. So we have shown $F \subset \{x : q(x) > \alpha\}$. But now we arrive at a contradiction with the fact that $\text{dist}(M, F) = 0$, as the latter implies $\inf_{x \in F} \|Ax - b\| = 0$.

6. Generalized linear complementarity problem

Let F be a closed convex cone in $\mathbb{R}^n$, let $A = A^\top \in \mathbb{R}^{n \times n}$, and $b \in \mathbb{R}^n$. Then we consider the following generalized linear complementarity problem on F with data (A, b) :

$$\text{Find } x^* \in F \text{ with } (Ax^* + b)^\top x \geq 0 \text{ for every } x \in F, \text{ and } Ax^* + b \perp x^*. \quad (6)$$

Every $x^* \in F$ satisfying (6) is called a solution of the problem. We say that problem (6) is feasible if $\gamma = \inf_{x \in F} (Ax + b)^\top x > -\infty$ and if there exists $x_0 \in F$ such that $(Ax_0 + b)^\top x \geq 0$ for every $x \in F$.

Theorem 6.1. *Suppose problem (6) is feasible. If F is a Frank-and-Wolfe cone, then (6) has a solution x^* .*

Proof. Let x_0 be a feasible solution, then $(Ax_0 + b)^\top x \geq 0$ for every $x \in F$. Since F is a cone we have $2x_0 \in F$, and $(2Ax_0 + 2b)^\top x \geq 0$ for every $x \in F$.

Due to feasibility the quadratic function $q(x) = (Ax + 2b)^\top x$ is now bounded below by 2γ on F , and since F is Frank-and-Wolfe, there exists $x^* \in F$ such that

$$(Ax + 2b)^\top x \geq (Ax^* + 2b)^\top x^* \quad (7)$$

for every $x \in F$. For $x \in F$ and $0 < t \leq 1$ we have $\tilde{x} = x^* + t(x - x^*) \in F$, hence on substituting $\tilde{x}$ in (7) and expanding, we get

$$t(Ax^* + 2b)^\top (x - x^*) + t(x - x^*)^\top Ax^* + t^2(x - x^*)^\top A(x - x^*) \geq 0.$$

Dividing by t and letting $t \rightarrow 0$ gives $2(Ax^* + b)^\top (x - x^*) \geq 0$, hence

$$(Ax^* + b)^\top (x - x^*) \geq 0 \quad \text{for every } x \in F.$$

Putting $x = 0 \in F$ we get $(Ax^* + b)^\top x^* \leq 0$, while putting $x = 2x^* \in F$ gives $(Ax^* + b)^\top x^* \geq 0$, so together we get the complementarity $Ax^* + b \perp x^*$. From that follows $(Ax^* + b)^\top (x - x^*) = (Ax^* + b)^\top x \geq 0$ for all $x \in F$, hence x^* is a solution of (6). $\square$

For sufficient conditions guaranteeing $\inf_{x \in F} (Ax + b)^\top x > -\infty$ we refer to [6] and the references given there. Links with the linear complementarity problem can already be found in the original work [5].

Acknowledgments. J. E. Martínez-Legaz was partially supported by the program MINECO of Spain, Grant MTM2014-59179-C2-2-P, the Severo Ochoa Programme for Centres of Excellence in R&D [SEV-2015-0563], and under the Australian Research Council's Discovery Projects funding scheme (project number DP140103213). He is affiliated with MOVE (Markets, Organizations and Votes in Economics). Wilfredo Sosa was partially supported by CNPq Grants 302074/2012-0 and 471168/2013-0. Part of this research was carried out during Wilfredo's visit to UAB (partially supported by the above mentioned MINECO grant) and during Juan Enrique's visit to UCB (as 'Professor visitante especial' in the program 'Ciência sem fronteiras'). D. Noll was supported by Fondation Mathématiques Jacques Hadamard (FMJH) under Programme Gaspard Monge pour l'Optimisation et la Recherche Opérationnelle (PGMO).

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