Journal of Convex Analysis
Volume 25 (2018), No. 2, 675–690

New Challenges on the Regularity of Minimizers of Functionals

Maria Alessandra Ragusa
Dipartimento di Matematica e Informatica, Universitá di Catania,
Viale A. Doria 6, 95125 Catania, Italy
maragusa@dmi.unict.it

Atsushi Tachikawa
Department of Mathematics, Faculty of Science and Technology,
Tokyo University of Science, Noda, Chiba, 278-8510, Japan
tachikawa_atsushi@ma.noda.tus.ac.jp

Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

Received: November 3, 2015
Revised manuscript received: December 21, 2015
Accepted: December 21, 2015

Our aim is to obtain regularity of minimizers of functionals under the nonstandard growth conditions. Starting point is the study made by the authors finalized to investigate the regularity of the minimizers of quadratic functionals, whose integrands have vanishing mean oscillation coefficients, using some majorizations for the functionals, rather than the well known Euler's equation associated with it.

Keywords: Regularity, minimizers, non-stardard growth.

2000 Mathematics Subject Classification: Primary 49N60, 35J50; Secondary 58E20, 35J62, 35J20, 35P30.

1. Introduction and notations

The authors investigate the regularity of the minimizers of quadratic functionals, whose integrands have vanishing mean oscillation coefficients, using some majorizations for the functionals, rather than the well known Euler-Lagrange equation associated with it.

In the study of partial regularity of minimizers of quadratic functionals with discontinuous coefficients have been obtained by the authors estimates in Morrey spaces.

In the last decades, it was of big attractiveness to study problems related to $p(x)$-growth, and we focus our attention in this regard. It is of increasing interest, for instance, in physical problems. Zhikov in 1997, in [38], treated a

ISSN 0944-6532 / \$ 2.50

$p(x)$-growth functional related to the thermistor problem, and got some results on the higher integrability of the gradient of minimizers. Rajagopal and Ružička in 2001, see [33], (and also [34]) proposed a model of electrorheological fluid, using equations with $p(x)$-growth term. Moreover, Acerbi and Mingione in [3] obtain some regularity results for stationary electrorheological fluid and Ragusa, Tachikawa and Takabayashi in [30] studied non-standard growth conditions of $p(x)$-type.

Let Ω be an open bounded set in $\mathbb{R}^m$, $n \geq 2$ with $\partial\Omega$ sufficiently smooth, $0 < \lambda < m,\ 1 < p < \infty$.

We recall the following classical *Sobolev spaces*, where k is an integer,

$$W^{k,p}(\Omega) = \{f(x)\,;D^\alpha f \in L^p(\Omega),\ |\alpha| \leq k\}\,,$$

equipped with the norm $\|f\|_{W^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|D^k f\|_{L^p(\Omega)}$.

The closure of the space $C_0^\infty(\Omega)$ with respect to the norm in $W^{k,p}(\Omega)$ will be denoted, as usual, by $W_0^{k,p}(\Omega)$.

We say that a function $f \in L^1_{loc}(\Omega)$ belongs to the *Morrey Space* $L^{p,\lambda}(\Omega)$, if it is finite

$$\|f\|^p_{L^{p,\lambda}(\Omega)} \equiv \sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)\cap\Omega} |f(y)|^p dy\,,$$

where $B_\rho(x)$ is a ball of radius ρ centered at the point x (see e.g. [22], [26]).

We set $W^{k,p,\lambda}(\Omega)$ for the *Banach space* of functions belonging to $W^{k,p}(\Omega)$ and having k-th order derivative lying in the Morrey space $L^{p,\lambda}(\Omega)$. A natural norm in this space is

$$\|f\|_{W^{k,p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + \|D^k f\|_{L^{p,\lambda}(\Omega)}.$$

In the following, for a function $f\colon \Omega \to \mathbb{R}^k$ and a bounded measurable set $D \subset \Omega$, we write

$$⨍_D f(x)dx := \frac{1}{|D|}\int_D f(x)dx,$$

where $|D|$ denotes the Lebesgue measure of D. For a ball $B(y,r) := \{x \in \mathbb{R}^m;$ $|x-y| < 0\}$, we write

$$f_{y,r} := ⨍_{B(y,r)\cap\Omega} f(x)dx.$$

When there is no doubt of confusion, we omit the center y and set $f_r := f_{y,r}$.

For a function $p\colon \Omega \to [1,+\infty)$, we define $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ as follows:

$$L^{p(x)} := \{u \in L^1(\Omega)\,; \int_\Omega |u|^{p(x)}dx < +\infty\}.$$

$$W^{1,p(x)} := \{u \in L^{p(x)} \cap W^{1,1}(\Omega)\,; Du \in L^{p(x)}(\Omega)\}.$$

We define $L^{p(x)}_{\mathrm{loc}}(\Omega)$ and $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ similarly.

As mentioned in [8], if $p(x)$ is uniformly continuous and $\partial\Omega$ satisfies uniform cone property, then

$$W^{1,p(x)}(\Omega) = \{u \in W^{1,1}(\Omega)\,;\, Du \in L^{p(x)}(\Omega)\}.$$

In any case, if $p(x)$ is continuous in Ω, we have

$$W^{1,p(x)}_{\rm loc}(\Omega) = \{u \in W^{1,1}_{\rm loc}(\Omega)\,;\, |Du|^{p(x)} \in L^1_{\rm loc}(\Omega)\}.$$

We also define

$$W^{1,p(x)}_0(\Omega) := \{u \in W^{1,1}_0(\Omega)\,;\, \int_\Omega |Du|^{p(x)} dx < \infty\},$$

and for a given map φ

$$\varphi + W^{1,p(x)}_0(\Omega) := \{u \in W^{1,p(x)}(\Omega)\,;\; u - \varphi \in W^{1,p(x)}_0(\Omega)\}.$$

Throughout this paper we use the following notation: for $x_0 = (x_0^1, ..., x_0^{m-1}, 0)$ and $r > 0$, we put

$$\begin{aligned} B^+(x_0, r) &:= \{x \in \mathbb{R}^m\ ;\ |x - x_0| < r,\ x^m > 0\},\\ \Gamma(x_0, r) &:= \{x \in \mathbb{R}^m\ ;\ |x - x_0| < r,\ x^m = 0\},\\ \partial^+ B^+(x_0, r) &:= \partial B^+(x_0, r) \setminus \Gamma(x_0, r). \end{aligned}$$

When $x_0 = 0$, we omit the center $x_0 = 0$ and write simply

$$B^+(r) := B^+(0, r),\quad \Gamma(r) := \Gamma(0, r),\quad \partial^+ B^+(r) := \partial^+ B^+(0, r).$$

For $x \in B^+(x_0, R)$ and $r < \operatorname{dist}(x, \partial B(x_0, R)) = R - |x - x_0|$, we put

$$\begin{aligned} &\Omega(x, r) := B(x, r) \cap B^+(x_0, R),\\ &p_1(x, r) := \inf_{\Omega(x,r)} p(y),\quad p_2(x, r) := \sup_{\Omega(x,r)} p(y). \end{aligned} \tag{1}$$

For p_1 and p_2, when the center $x = x_0$ is clearly understood, we abbreviate as

$$p_1(r) := p_1(x, r),\quad p_2(r) := p_2(x, r).$$

When we consider the behavior of a solution near the boundary point $x_0 \in \partial\Omega$, we flatten the $\partial\Omega$ so that $x_0 = (0, ..., 0)$, $B(x_0, R_1) \cap \Omega = B^+(0, R_1)$ for some $R_1 > 0$ and $\partial\Omega \cap B(x_0, R_1) = \Gamma(0, R_1)$.

We use c without subscript as generic constants, which may change from line to line, but does not depend on the crucial quantities.

Let $\omega_1\colon [0,+\infty) \to [0,+\infty)$ be a nondecreasing continuous function with $\omega_1(0) = 0$ which represents the modulus of continuity, then ω_1 satisfies

$$|p(x) - p(y)| \le \omega_1(|x-y|). \tag{2}$$

Let us consider the following condition on ω_1:

$$\lim_{r\to 0} \omega_1(r) \log\left(\frac{1}{r}\right) = \mu_0 < +\infty\,. \tag{3}$$

The above condition implies

$$(1/t)^{\omega_1(t)} = \exp\big(-\log t\omega_1(t)\big) \to e^{\mu_0} \quad \text{as} \quad t \to 0^+. \tag{4}$$

When ω_1 satisfies (3), we say that $p(x)$ is *logarithmic continuous*. We mention also that if $p(x)$ is Hölder continuous, then condition (3) is fulfilled.

For a continuous function $p(x) > 1$ on Ω satisfying (2) and (3), let $f_{p(x)}(x,u,\xi)$ be a Carathéodory function on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$, which satisfies the growth condition (9).

We say that a function f belongs to the *John-Nirenberg space BMO*, or that f has *bounded mean oscillation*, if

$$\|f\|_* \equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int\limits_B |f(x) - f_B|dx < \infty\,,$$

where f_B is the integral average $\frac{1}{|B|}\int\limits_B f(x)dx$ of the function $f(x)$ over the set B, and B belongs to the class of balls of $\mathbb{R}^n$. This space was firstly considered in the paper [17] by John and Nirenberg in 1961.

Let, for a function $f \in BMO$

$$\eta(r) = \sup_{x\in\mathbb{R}^n,\ \rho\le r} \frac{1}{|B_\rho|} \int\limits_{B_\rho} |f(x) - f_{B_\rho}|dx\,.$$

A function $f \in BMO$ belongs to the class VMO, or f has *vanishing mean oscillation*, if

$$\lim_{r\to 0^+} \eta(r) = 0\,.$$

This space, first defined by Sarason 1975, see [35], was used by many authors (see e.g. [6], [19], [20], [24], [25], [32]), and characterized in [27].

The authors intend to continue and deepen this research; the challenge is to get better regularity to the boundary.

2. Preliminary tools and results

Let us start the section considering Morrey regularity results for minimizers of quadratic functionals having discontinuous coefficients.

The functional is

$$\mathcal{A}(u,\Omega) \;=\; \int_\Omega \Big\{ A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j + g(x,u,Du) \Big\}\, dx,$$

where $\Omega \subset \mathbb{R}^n$, $n \ge 3$, is a bounded open set, $u\colon \Omega \to \mathbb{R}^N$, $N > 1$, $u(x) = (u^1(x),\dots,u^N(x))$, $Du = (D_\alpha u^i)$, $D_\alpha = \partial/\partial x_\alpha$, $\alpha = 1,\dots,n,\ i = 1,\dots,N$.

We recall that a function $u \in H^{1,2}(\Omega,\mathbb{R}^N)$ is called a *minimizer* of the functional $\mathcal{A}(u,\Omega)$ iff

$$\mathcal{A}\,(u,\Omega)\; \le \mathcal{A}\,(v,\Omega),\ \forall v \in H^{1,2}(\Omega,\mathbb{R}^N),$$

with $u - v \in H_0^{1,2}(\Omega,\mathbb{R}^N)$.

Let us suppose that $A_{ij}^{\alpha\beta}$ are bounded functions on $\Omega \times \mathbb{R}^N$ and satisfy the following conditions.

1) $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.
2) For every $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot,u) \in VMO(\Omega)$.
3) For every $x \in \Omega$ and $u, v \in \mathbb{R}^N$,

$$\big|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)\big| \le \omega(|u-v|^2)$$

   for some monotone increasing concave function ω with $\omega(0) = 0$.
4) There exists a positive constant ν such that

$$\nu|\xi|^2 \le A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j$$

   for a.e. $x \in \Omega$, all $u \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$.

We should mention that since C^0 is a proper subset of VMO, the continuity of $A_{ij}^{\alpha\beta}(x,u)$ with respect to x is not assumed.

We suppose that the function g is a Charathéodory function and has growth less than quadratic.

Theorem 2.1. *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a minimum of the functional $\mathcal{A}(u,\Omega)$ defined above. Suppose that the above-mentioned assumptions on $A_{ij}^{\alpha\beta}(x,u)$ and $g(x,u,Du)$ are satisfied.*

Then, for $\lambda = n(1-\frac{2}{p})$, we have $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^{nN})$, where

$$\Omega_0 = \{x \in \Omega\colon\ \liminf_{R\to 0} \frac{1}{R^{n-2}} \int_{B(x,R)} |Du(y)|^2 dy \;=\; 0\}.$$

As a consequence, for $\alpha \in (0,1)$: $u \in C^{0,\alpha}(\Omega_0,\mathbb{R}^N)$.

For linear systems regularity results assuming $A_{ij}^{\alpha\beta}$ constants or in $C^0(\overline{\Omega})$, have been obtained by Campanato in [5]. Without assuming continuity of coefficients we mention the note [4] by Acquistapace, where he refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class neither containing nor contained in $C^0(\overline{\Omega})$, hence in general discontinuous. Also, regularity results for solutions of linear systems with discontinuous coefficients are contained in [23].

Moreover we recall the study made by Huang in 1996 where he shows regularity results of weak solutions of linear elliptic systems with coefficients in the class *VMO*. Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in VMO$, even for nonlinear cases.

Daněček and Viszus in [9] treated the regularity of minimizer for the functional

$$\int_{\Omega} \left\{ A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx,$$

where $g(x, u, Du)$ is a lower order term which satisfies

$$|g(x, u, z)| \leq f(x) + L|z|^\gamma,$$

where $f \in L^p(\Omega)$, $2 < p \leq \infty$, $f \geq 0$ a.e. on Ω, L is a nonnegative constant and $0 \leq \gamma < 2$. They obtain Hölder regularity of the minimizer assuming that $A_{ij}^{\alpha\beta}(x) \in VMO$.

The authors have extended both the results by Huang and Daněček and Viszus ([28], [29]) because they treat the functional whose integrand contains the term $g(x, u, Du)$ and has coefficients $A_{ij}^{\alpha\beta}$ dependent not only on x but also on u. Let us now give an idea of the proof. Basic tool is the following Lemma for constant coefficients by Campanato.

Lemma 2.2. *Let $u \in W^{1,2}(B(x_0, R), \mathbb{R}^n)$ be a weak solution of the system*

$$D_\alpha(g_{ij}^{\alpha\beta}\, D_\beta u^j) \;=\; 0 \quad \text{for } i = 1, ..., N,$$

where $g_{ij}^{\alpha\beta}$ are constant coefficients such that strong ellipticity holds. Then, for any $t \in [0, 1]$ we have

$$\int_{B(x_0, t\,R)} |\, Du\,|^2\, dx \;\leq\; c\, t^n \int_{B(x_0,\, R)} |\, Du\,|^2\, dx.$$

Proof of Theorem 2.1. Let $R > 0$, $x_0 \in \Omega$ such that $B(x_0, R) \subset \Omega$. Let us consider now the case $g = 0$, because the general case easily follows.

Let $v \in W^{1,2}(B(x_0, \frac{R}{2}), \mathbb{R}^N)$ be the minimum of the "frozen" functional $\mathcal{A}^0$:

$$\mathcal{A}^0(v, B(x_0, \frac{R}{2})) = \int_{B(x_0, \frac{R}{2})} A_{ij}^{\alpha\beta}(x_0, u_R) D_\alpha v^i D_\beta v^j dx\,,$$

where $v - u \in H_0^{1,2}(B(x_0, \frac{R}{2}))$.

Having constant coefficients $A_{ij}^{\alpha\beta}(x_0, u_R)$, we get from Campanato's Lemma:

$$\int_{B(x_0,\frac{tR}{2})} |Dv|^2\,dx \le c\cdot t^n \int_{B(x_0,\frac{R}{2})} |Dv|^2\,dx$$

for every $t \in [0,1]$. Let us set $w = u - v$. Then,

$$\int_{B(x_0,\frac{tR}{2})} |Du|^2 dx \le c\cdot \left\{ t^n \int_{B(x_0,\frac{R}{2})} |Du|^2 dx + \int_{B(x_0,\frac{R}{2})} |Dw|^2 dx \right\}.$$

Let us estimate the term $\displaystyle\int_{B(x_0,\frac{R}{2})} |Dw|^2\,dx$. We can write

$$\nu \int_{B(x_0,\frac{R}{2})} |Dw|^2 dx \le c\left\{ \mathcal{A}^0(u,\, B(x_0,\frac{R}{2})) - \mathcal{A}^0(v,\, B(x_0,\frac{R}{2})) \right\};$$

adding and subtracting the terms

$$A_{ij}^{\alpha\beta}(x,u_R) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha\beta}(x,u_R) D_\alpha v^i D_\beta v^j$$

$$A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j\,, \quad A_{ij}^{\alpha\beta}(x,v)\, D_\alpha v^i D_\beta v^j$$

we obtain two kind of integrals. The first is:

$$\int_{B(x_0,\frac{R}{2})} |A_{ij}^{\alpha\beta}(x_0,u_R) - A_{ij}^{\alpha\beta}(x,u_R)| \cdot |Du|^2\,dx \le$$

$$\le c\left\{\eta(A(\cdot,u_R);R)\right\}^{\frac{1}{r'}} \int_{B(x_0,R)} |\,Du\,|^2\,dx\,.$$

The second:

$$\int_{B(x_0,\frac{R}{2})} |A_{ij}^{\alpha\beta}(x,u_R) - A_{ij}^{\alpha\beta}(x,u)|\, D_\alpha u^i D_\beta u^j\ dx \le$$

$$\le c\left(\int_{B(x_0,R)} |Du|^2 dx\right)\left(⨍_{B(x_0,\frac{R}{2})} \omega(|u_R - u|^2) dx\right)^{1-\frac{2}{p}} \le$$

$$\le \left(\int_{B(x_0,R)} |Du|^2 dx\right)\omega\left(⨍_{B(x_0,\frac{R}{2})} |Du|^2\,dx\right).$$

Then if $\rho = tR$

$$\int_{B(x_0,\rho)} |Du|^2 dx \le c\left\{\left(\frac{\rho}{R}\right)^n + \omega\left(R^{2-n}\int_{B(x_0,\frac{R}{2})} |D\,u|^2\,dx\right)^{1-\frac{2}{q}} + \right.$$

$$\left. + \Big(\eta(A(\cdot,u_R),R)\Big)^{1-\frac{2}{q}}\right\}\cdot\left(\int_{B(x_0,\frac{R}{2})} |D\,u|^2 dx\right).$$

We obtain the conclusion by using a Lemma of Giaquinta [14], and letting R be sufficiently small: $\displaystyle\int_{B(x_0,\rho)} |Du|^2 dx \leq c \cdot \rho^\lambda$. □

The study of the minimizers for functionals under non-standard growth conditions have been a new envelope during the last decades with the introductions of Lebesgue spaces having variable coefficients. Our aim is to continue this line of research and to prove regularity for minimizers of $p(x)$-energy functionals.

Let us now consider maps $u\colon \Omega \to \mathbb{R}^n$ and the $p(x)$-*energy functional* defined as

$$\mathcal{E}(u;\Omega) := \int_\Omega \Big(g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x)\Big)^{p(x)/2} dx. \tag{5}$$

We suppose that $(G_{ij}(u))$ and $(g^{\alpha\beta}(x))$ are symmetric positive definite matrices whose entries are continuous functions defined on Ω and $\mathbb{R}^n$ respectively. The function $p(x) \geq 2$ is supposed continuous on Ω. Greek indices $\alpha, \beta, ...$ are to be summed from 1 to m, and Latin indices $i, j, ...$ from 1 to n. The Einstein summation convention is used. In the sequel we consider as integrand of (5) the function

$$e(u)(x) := g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x). \tag{6}$$

We are interested in the study the regularity of the minimizers of the $p(x)$-energy functionals.

The above functional $\mathcal{E}$ is a particular case of the more general one

$$\mathcal{Z}(u;\Omega) = \int_\Omega z(x,u,Du)dx, \tag{7}$$

where $z\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ is a *Carathéodory function* satisfying the following so-called (p,q)-*growth* condition: there exist constants $\Lambda \geq \lambda > 0$, $q \geq p \geq 1$ such that

$$\lambda|\xi|^p \leq z(x,u,\xi) \leq \Lambda\big(1 + |\xi|^q\big) \tag{8}$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$. We call $\mathcal{Z}$ a functional with *standard growth* if $p = q$, and with *non-standard growth* if $q > p$. If the integrand $z = z_{p(x)}$ satisfies

$$\lambda|\xi|^{p(x)} \leq z_{p(x)}(x,u,\xi) \leq \Lambda\big(1 + |\xi|^{p(x)}\big), \tag{9}$$

for all $(x,u,\xi) \in \Omega \times R^n \times \mathbb{R}^{mn}$, then

$$\mathcal{Z}_{p(x)}(u;\Omega) := \int_\Omega z_{p(x)}(x,u,Du)dx \tag{10}$$

is called a functional with $p(x)$-*growth.* A map $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ is called to be a *local minimizer* of $\mathcal{Z}_{p(x)}$ if it satisfies

$$\mathcal{Z}_{p(x)}(u;\mathrm{supp}\varphi) \leq \mathcal{Z}_{p(x)}(u+\varphi;\mathrm{supp}\varphi),$$

for any $\varphi \in W^{1,p(x)}_0(\Omega)$ with compact support in Ω.

The $p(x)$-energy functional $\mathcal{E}$ is a $p(x)$-growth functional with a special structure.

Non-standard growth conditions have aroused considerable curiosity and stimulated much interest since Marcellini treated them in the pioneeristic paper [21]. In 1995, Zhikov [37] and later in 1997 in [38], studied Lavrentiev phenomenon for the functional

$$\mathcal{W}_{p(x)}(u) := \int_\Omega |Du|^{p(x)} dx. \tag{11}$$

On the regularity of minimizers of $\mathcal{W}_{p(x)}$, a fundamental result was established by Coscia and Mingione [7] in 1999. The study of interior partial regularity results for general $p(x)$-growth functionals, continue in [1, 2, 3, 11, 12, 13].

For $p(x)$-energy $\mathcal{E}$, Ragusa, Tachikawa and Takabayashi [30] obtained interior partial regularity of minimizers, precisely showing that the singular set S_u of a minimizer u can have Hausdorff dimension $\dim^{\mathbb{H}}(S_u)$ at most $m - \inf p(x)$. In [31] the interior everywhere regularity was shown under the *one-sided condition*. In [36], assuming the boundedness of a minimizer u, the second author improved the estimate on the Hausdorff dimension of S_u as $\dim_{\mathbb{H}}(S_u) \leq m - [\inf p(x)] - 1$.

We continue this study of regularity of minimizers for $p(x)$-energy $\mathcal{E}$. In this area we recall that, for standard growth case, Jost and Meier [18] showed that a minimizers for some quadratic functionals can not have singular points on the boundary and later Duzaar, Grotowski and Kronz [10] generalized this result to general p-energy functionals

$$\int_\Omega \left(g^{\alpha\beta}(x) G_{ij}(x,u) D_\alpha u^i(x) D_\beta u^j(x) \right)^{p/2} dx,$$

for $p > 1$. We suppose that the following assumptions are satisfied.

(H) There exist positive constants $\lambda_g, \Lambda_g, \lambda_G, \Lambda_G$ such that

$$\lambda_g |\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_g|\zeta|^2, \quad \lambda_G|\eta|^2 \leq G_{ij}(u)\eta^i\eta^j \leq \Lambda_G|\eta|^2 \tag{12}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^m$ and $u, \eta \in \mathbb{R}^n$.

(I) The the coefficients $g^{\alpha\beta}(x)$, $G_{ij}(u)$ and $p(x)$ are Hölder continuous functions; there exist positive constants $\tau, \tau' < 1$, $\sigma \leq 1$, L_p, L_g, and L_h such that

$$|p(x) - p(y)| \leq L_p|x-y|^\sigma/2 =: \omega_p(|x-y|/2) \quad \text{for all } x, y \in \Omega, \tag{13}$$

$$|g^{\alpha\beta}(x) - g^{\alpha\beta}(y)| \leq L_g|x-y|^\tau =: \omega_g(|x-y|) \quad \text{for all } x, y \in \Omega, \tag{14}$$

$$|G_{ij}(u) - G_{ij}(v)| \leq L_h|u-v|^{\tau'} =: \omega_G(|u-v|^2) \quad \text{for all } u, v \in \mathbb{R}^n. \tag{15}$$

(J) The function $p(x)$ is such that

$$2 \leq \gamma_1 := \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) =: \gamma_2 < +\infty. \tag{16}$$

We state, see [30], the following lemma on higher integrability.

Proposition 2.3. *We suppose that the function $p(x) > 1$ has modulus of continuity ω_1 and satisfies* (3). *Let $p_1 := \inf_{B+(T)} p(x)$ and $p_2 := \sup_{B+(T)} p(x)$, and assume that*

$$(p_2)_* = \frac{mp_2}{m+p_2} < p_1, \quad \text{or} \quad p_1^* = \frac{mp_1}{m-p_1} > p_2. \tag{17}$$

For some $\varepsilon > 0$, let h be a given map in the class $W^{1,(1+\varepsilon)p(x)}(B^+(T))$. Let v be a local minimizer of $\mathcal{Z}_{p(x)}$ in the set

$$\{w \in W^{1,p(x)}(B^+(T), \mathbb{R}^n) \ ; \ w = h \ \text{ on } \Gamma(T)\}.$$

Then, there exists a positive constant $\hat{\delta} < \varepsilon$ such that for any $\delta \in (0, \hat{\delta})$ the local minimizer v satisfies $v \in W^{1,(1+\delta)p(x)}(B^+(T'))$ for any $T' < T$. Moreover, if $x_0 \in B^+(T') \cup \Gamma(T')$ and $r < T - T'$, we have

$$\Big(\fint_{\Omega(x_0,r/2)} (1+|Dv|^2)^{(1+\delta)p(x)/2} dx\Big)^{1/(1+\delta)} \leq \tag{18}$$
$$\leq c_0 \fint_{\Omega(x_0,r)} (1+|Dv|^2)^{p(x)/2} dx + c_1 \Big(\fint_{\Omega(x_0,r)} (1+|Dh|^2)^{(1+\delta)p(x)/2} dx\Big)^{1/(1+\delta)},$$

where we put $\Omega(y, \rho) := B(y, \rho) \cap B^+(T)$.

By virtue of Theorem 3.1 in [11] and Proposition 2.3, we have the following estimate of the minimizer in [30].

Proposition 2.4. *We consider a bounded domain $D \subset \mathbb{R}^m$ with smooth boundary ∂D and $S > 0$, a positive number, which satisfies the following hypotheses*

(i) *$p_1 = p_1(x, 4S)$ and $p_2 = p_2(x, 4S)$ satisfy* (17),

(ii) *there is a diffeomorphism $\psi \colon B(y, 4S) \to B(T)$ which satisfies*

$$\psi(B(y, 4S) \cap D) \subset B^+(T) \quad \text{and} \quad \psi(B(y, 4S) \cap \partial D) = \Gamma(T).$$

Let us suppose that $p(x)$ and $h(x)$ verify the assumptions of Proposition 2.3 *and that v is a minimizer of $\mathcal{Z}_{p(x)}(\cdot, D)$ in the class*

$$\{w \in W^{1,p(x)}(D, \mathbb{R}^n) \ ; \ w - h \in W_0^{1,1}(D; \mathbb{R}^n)\}.$$

Then, there exists a constant $\hat{\delta} \in (0, \varepsilon)$ such that for any $\delta \in (0, \hat{\delta}]$, we have that $v \in W^{p(x)(1+\delta)}(D, \mathbb{R}^n)$ and that

$$\int_D (1+|Dv|^2)^{(1+\delta)p(x)/2} dx \leq c_2 \big(1 + |D|^\delta S^{-m\delta}\big) \int_D (1+|Dh|^2)^{(1+\delta)p(x)/2} dx, \tag{19}$$

where c_2 is dependent only on m, λ, Λ, $p(x)$, and $\mathcal{Z}_{p(x)}(h, D)$.

Let us fix, from now on, a constant $\delta \in (0,1)$ in such a way the previous properties hold and such that

$$m\delta < \sigma\,. \tag{20}$$

We now study the boundary regularity of minimizers of the following functional

$$\mathcal{S}_{p(x)}(w, D) := \int_D |Dw|^{p(x)} dx. \tag{21}$$

Theorem 2.5. *Let us suppose that $p(x)$ satisfies (13), and that $R > 0$ is sufficiently small such that*

$$\left(1 + \frac{\delta}{2}\right) p_2(2R) \le (1+\delta) p_1(2R). \tag{22}$$

Also, let $v \in W^{1,p(x)}(B^+(R), \mathbb{R}^n)$ be a local minimizer of $\mathcal{S}_{p(x)}$ in the set

$$\{w \in W^{1,p(x)}\ ;\ w = h \ \text{ on } \Gamma(R)\},$$

where h is a given boundary data in the class of functions $W^{1,s}(B^+(R), \mathbb{R}^n)$, $s > (1+\delta)p_2$. Moreover, we suppose that $\mathcal{S}_{p(x)}(v) \le K$ for some positive constant K. Then, for any $\varepsilon \in (0, mp_2(2R)/s)$, we have

$$\int_{B^+(\rho)} (1 + |Dv|^2)^{p_2(\rho)/2} dx \le c_3 \Big[\Big(\frac{\rho}{2R}\Big)^{m-\varepsilon} \int_{B^+(R)} (1 + |Dv|^2)^{p_2(2R)/2} dx + \\ + \rho^{m - mp_2(2R)/s} \Big(\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \Big)^{p_2(2R)/s} \Big]. \tag{23}$$

Proof. We consider the shorter p_2 instead of $p_2(2R)$ and we define the following frozen functional

$$\mathcal{S}_0(w) := \int_{B^+(R)} |Dw|^{p_2} dx. \tag{24}$$

where we let $w \in W^{1,p_2}(B^+(R))$ be a minimizer of $\mathcal{S}_0$ with $w = v$ on $\partial B^+(R)$. Bearing in mind inequality (22) and Proposition 2.3, we have

$$v \in W^{1,(1+\delta)p(x)}(B^+(R)) \subset W^{1,(1+\delta/2)p_2}(B^+(R)).$$

Let us now consider Proposition 2.4 for $D = B^+(R)$ and $S = R/k$ for a suitable $k > 0$. Then, we have

$$\int_{B^+(R)} |Dw|^{(1+\delta/2)p_2} dx \le c \int_{B^+(R)} (1 + |Du|^2)^{(1+\delta/2)p_2/2} dx. \tag{25}$$

Moreover, from the boundary regularity results for minimizers of functionals of standard growth (see e.g. [10]), we have, for any $k \in (0,1)$

$$\int_{B_\rho^+} |Dw|^{p_2} dx \ \leq c\Big[\Big(\big(\frac{\rho}{R}\big)^m + k \Big) \int_{B^+(R)} |Dw|^{p_2} dx + \\ + k^{1-p_2} R^{m(1-p_2/s)} \Big(\int_{B^+(2R)} |Dh|^s dx \Big)^{p_2/s} \Big], \tag{26}$$

where c is a constant depending only on the boundary condition.

The minimality of v implies, reasoning as in [7], that

$$\mathcal{S}_0(v) - \mathcal{S}_0(w) \geq c \int_{B^+(R)} (|Dv| + |Dw|)^{p_2-2} |Dv - Dw|^2 dx,$$

and then

$$\int_{B^+(R)} |Dv - Dw|^{p_2} dx \leq \mathcal{S}_0(v) - \mathcal{S}_0(w). \tag{27}$$

Taking into account that v is a minimizer of $\mathcal{S}_{p(x)}$, we have

$$\mathcal{S}_0(v) - \mathcal{S}_0(w) \leq \mathcal{S}_0(v) - \mathcal{S}_{p(x)}(v) + \mathcal{S}_{p(x)}(w) - \mathcal{S}_0(w). \tag{28}$$

Following the method used in [7], $\forall \varepsilon > 0 \ \exists\, C(\varepsilon) > 0$:

$$|t^r - t^s| \leq C(\varepsilon)(s-r)\big(1 + t^{(1+\varepsilon)s}\big), \qquad \forall t \geq 0 \ \text{ and } \ s \geq r \geq 1, \tag{29}$$

and then, from (22), Hölder continuity of $p(x)$ and Proposition 2.3, it follows that

$$\begin{aligned} |\mathcal{S}_0(v) - \mathcal{S}_{p(x)}(v)| \leq & \ c_3 R^\sigma \int_{B^+(R)} (1 + |Dv|^2)^{(1+\delta)p(x)/2} dx \\ \leq & \ cR^{\sigma - m\delta} \Big(\int_{B^+(2R)} (1 + |Dv|^2)^{p(x)/2} dx \Big)^{1+\delta} \\ & + cR^\sigma \int_{B^+(2R)} (1 + |Dh|^2)^{(1+\delta)p(x)/2} dx \\ \leq & \ cR^{\sigma - m\delta} \int_{B^+(2R)} (1 + |Dv|^2)^{p(x)/2} dx \\ & + cR^{\sigma + m(1-(1+\delta)p_2/s)} \Big(\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \Big)^{(1+\delta)p_2/s}. \end{aligned} \tag{30}$$

Analogously, from Proposition 2.4 with $p(x) = p_2$, $v = w$, $h = v$ and the latter half of the above estimates, we get

$$\begin{aligned}
|\mathcal{S}_{p(x)}(w) - \mathcal{S}_0(w)| \le\ & cR^{\sigma} \int_{B^+(R)} (1 + |Dw|^2)^{(1+\delta/2)p_2/2} dx \\
\le\ & cR^{\sigma} \int_{B^+(R)} (1 + |Dv|^2)^{(1+\delta/2)p_2/2} dx \\
\le\ & cR^{\sigma - m\delta} \int_{B^+(2R)} (1 + |Dv|^2)^{p(x)/2} dx + \\
& + cR^{\sigma + m(1-(1+\delta)p_2/s)} \Big(\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \Big)^{(1+\delta)p_2/s}. \qquad (31)
\end{aligned}$$

Combining (27), (28), (30) and (31), we get

$$\begin{aligned}
\int_{B^+(R)} |Dv - Dw|^{p_2} dx \le\ & cR^{\sigma - m\delta} \int_{B^+(2R)} (1 + |Dv|^2)^{p(x)/2} dx + \\
& + cR^{\sigma + m(\sigma-(1+\delta)p_2/s)} \Big(\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \Big)^{(1+\delta)p_2/s}. \qquad (32)
\end{aligned}$$

Then we have

$$\begin{aligned}
& \int_{B^+(\rho)} |Dv|^{p_2} dx \le \\
& \le\ c\Big[\Big(\frac{\rho}{R}\Big)^m + k \Big] \int_{B^+(2R)} |Dw|^{p_2} dx + ck^{1-p_2} R^{m(1-p_2/s)} \Big(\int_{B^+(R)} |Dh|^s dx \Big)^{p_2/s} + \\
& \qquad + cR^{\sigma - m\delta} \int_{B^+(2R)} (1 + |Dv|^2)^{p(x)/2} dx + \\
& \qquad + cR^{\sigma + m(1-(1+\delta)p_2/s)} \Big(\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \Big)^{(1+\delta)p_2/s}.
\end{aligned}$$

Choosing $R \le 1$ sufficiently small, recalling that $\sigma - m\delta p_2/s \ge \sigma - m\delta$, and using the minimality of w, we have

$$\begin{aligned}
& \int_{B^+(\rho)} (1 + |Dv|^2)^{p_2(\rho)/2} dx \le \int_{B^+(\rho)} (1 + |Dv|^2)^{p_2(2R)} dx \le \\
& \le\ c\Big[\Big(\frac{\rho}{R}\Big)^m + k + R^{\sigma - m\delta} \Big] \int_{B^+(2R)} (1 + |Dv|^2)^{p_2(2R)/2} dx + \\
& \qquad + c(k^{1-p_2} + 1) R^{m - p_2(2R)m/s} \Big(\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \Big)^{p_2/s}.
\end{aligned}$$

Taking into account [15, Lemma 5.13] and choosing k and R sufficiently small, we obtain (23). $\square$

References

[1] E. Acerbi, G. Mingione: Regularity results for a class of functionals with nonstandard growth, Arch. Ration. Mech. Anal. 156(2) (2001) 121–140.

[2] E. Acerbi, G. Mingione: Regularity results for a class of quasiconvex functionals with nonstandard growth, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 30(2) (2001) 311–339.

[3] E. Acerbi, G. Mingione: Regularity results for stationary electro-rheological fluids, Arch. Ration. Mech. Anal. 164 (2002) 213–259.

[4] P. Acquistapace: On BMO-regularity for linear elliptic systems, Annali Mat. Pura Appl. 161(4) (1992) 231–269.

[5] S. Campanato: Sistemi Ellittici in Forma di Divergenza. Regolarità all'Interno, Quaderni Scuola Normale, Pisa (1980).

[6] F. Chiarenza, M. Frasca, P. Longo: $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc. 336 (1993) 841–853.

[7] A. Coscia, G. Mingione: Hölder continuity of the gradient of $p(x)$-harmonic mappings, C. R. Acad. Sci. Paris, Sér. I Math. 328(4) (1999) 363–368.

[8] A. Coscia, M. Mucci: Integral representation and Γ-convergence of variational integrals with $p(x)$-growth, ESAIM Control Optim. Calc. Var. 7 (2002) 495–519.

[9] J. Daněček, E. Viszus: $L^{2,\lambda}$-Regularity for minima of variational integrals, Boll. Unione Mat. Ital. 8, 6-B (2003), 39–48.

[10] F. Duzaar, J. F. Grotowski, M. Kronz: Partial and full boundary regularity for minimizers of functionals with nonquadratic growth, J. Convex Analysis 11(2) (2004) 437–476.

[11] M. Eleuteri: Hölder continuity results for a class of functionals with nonstandard growth, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), 7(1) (2004) 129–157.

[12] M. Eleuteri, J. Habermann: Calderón-Zygmund type estimates for a class of obstacle problems with $p(x)$ growth, J. Math. Anal. Appl. 372(1) (2010) 140–161.

[13] M. Eleuteri, J. Habermann: A Hölder continuity result for a class of obstacle problems under non standard growth conditions, Math. Nachr. 284(11-12) (2011) 1404–1434.

[14] M. Giaquinta: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of Math. Studies, Princeton Univ. Press, Princeton (1983).

[15] M. Giaquinta, L. Martinazzi: An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs, Lecture Notes 11, Scuola Normale Superiore, Pisa (2012).

[16] Q. Huang: Estimates on the generalized Morrey Spaces $L_{\phi}^{p,\lambda}$ and BMO for linear elliptic systems, Indiana Univ. Math. J. 45(2) (1996) 397–439.

[17] F. John, L. Nirenberg: On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961) 415–476.

[18] J. Jost, M. Meier: Boundary regularity for minima of certain quadratic functionals, Math. Ann. 262(4) (1983) 549–561.

[19] N. V. Krylov: Second-order elliptic equations with variably partially VMO coefficients, J. Functional Analysis 257(6) (2009) 1695–1712.

[20] G. Lieberman: A mostly elementary proof of Morrey space estimates for elliptic and parabolic equations with VMO coefficients, J. Functional Analysis 201(2) (2003) 457–479.

[21] P. Marcellini: Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, Arch. Rat. Mech. Analysis 105(3) (1989) 267–284.

[22] L.-E. Persson, M. A. Ragusa, N. Samko, P. Wall: Commutators of Hardy operators in vanishing Morrey spaces, AIP Conference Proceedings 1493 (2012) 859–866.

[23] M. A. Ragusa: Local Hölder regularity for solutions of elliptic systems, Duke Math. J. 113(2) (2002) 385–397.

[24] M. A. Ragusa: Cauchy-Dirichlet problem associated to divergence form parabolic equations, Comm. Contemp. Math. 6(3) (2004) 377–393.

[25] M. A. Ragusa: Homogeneous Herz spaces and regularity results, Nonlinear Analysis: Theory, Methods and Applications 71(12) (2009) e1909–e1914.

[26] M. A. Ragusa: Embeddings for Morrey-Lorentz spaces, J. Optim. Theory Appl. 154(2) (2012) 491–499.

[27] M. A. Ragusa: Necessary and sufficient condition for a VMO function, Appl. Math. Comput. 218(24) (2012) 11952–11958.

[28] M. A. Ragusa, A. Tachikawa: Partial regularity of the minimizers of quadratic functionals with VMO coefficients, J. London Math. Soc. 72 (2005) 609–620.

[29] M. A. Ragusa, A. Tachikawa: On continuity of minimizers for certain quadratic growth functionals, J. Math. Soc. Japan 57(3) (2005) 691–700.

[30] M. A. Ragusa, A. Tachikawa, H. Takabayashi: Partial regularity for $p(x)$-harmonic maps, Trans. Amer. Math. Soc. 365 (2013) 3329–3353.

[31] M. A. Ragusa, A. Tachikawa: On interior regularity of minimizers of p(x)-energy functionals, Nonlinear Analysis 93 (2013) 162–167.

[32] M. A. Ragusa, A. Tachikawa: Boundary regularity of minimizers of $p(x)$-energy functionals, Annales de l'Institut Henri Poincaré, Analyse non Linéaire 33(2) (2016) 451–476.

[33] K. Rajagopal, M. Ružička: Mathematical modeling of electrorheological materials, Contin. Mech. Thermodyn. 13 (2001) 59–78.

[34] M. Ružička: Electrorheological Fluids: Modeling and Mathematical Theory Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin (2000).

[35] D. Sarason: Functions of vanishing mean oscillation, Trans. Amer. Math. Soc. 207 (1975) 391–405.

[36] A. Tachikawa: On the singular set of minimizers of $p(x)$-energies, Calc. Var. Partial Differential Equations 50 (2014) 145–169.

[37] V. V. Zhikov: On the averaging of nonlinear variational problems in punctured domains, Dokl. Akad. Nauk 345(2) (1995) 156–160.

[38] V. V. Zhikov: On some variational problems, Russian J. Math. Phys. 5 (1997) 105–116.