

Bootstrapping the Mazur–Orlicz–König Theorem and the Hahn–Banach Lagrange Theorem

Stephen Simons

Department of Mathematics, University of California,
Santa Barbara, CA 93106-3080, U.S.A.
stesim38@gmail.com

Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

Received: December 25, 2015

Accepted: August 4, 2016

We give some extensions of König’s extension of the Mazur–Orlicz theorem. These extensions include generalizations of a surprising recent result of Sun Chuanfeng, and generalizations to the product of more than two spaces of the “Hahn–Banach–Lagrange” theorem.

Keywords: Sublinear functional, convex function, affine function, Hahn–Banach theorem, Mazur–Orlicz–König theorem.

2010 Mathematics Subject Classification: 46A22, 46N10.

1. Introduction

In this paper, all vector spaces are *real*. We shall use the terms *sublinear*, *linear*, *convex*, *concave* and *affine* in their usual senses.

This paper is about extensions of the Mazur–Orlicz theorem, which first appeared in [5]: *Let E be a vector space, $S: E \rightarrow \mathbb{R}$ be sublinear and C be a nonempty convex subset of E . Then there exists a linear map $L: E \rightarrow \mathbb{R}$ such that $L \leq S$ on E and $\inf_C L = \inf_C S$.* Early improvements and applications of this result were given, in chronological order, by Sikorski [6], Pták [4], König [1], Landsberg–Schiretzek [3] and König [2].

By a *convex–affine version* of a known result we mean that it corresponds to the known result with the word *sublinear* in the hypothesis replaced by *convex* and the word *linear* in the conclusion replaced by *affine*. It is important to note that a convex–affine version of a known result is not necessarily a *generalization* of it because an affine function dominated by a sublinear functional is not necessarily linear. Recently, Sun Chuanfeng established the following convex–affine version of the Mazur–Orlicz theorem: *Let E be a vector space, $f: E \rightarrow \mathbb{R}$ be convex*

and C be a nonempty convex subset of E . Then there exists an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and $\inf_C A = \inf_C f$. This seems to be a more difficult result than the original Mazur–Orlicz theorem.

The “Hahn–Banach–Lagrange” theorem, an existence theorem for linear functionals on a vector space that generalizes the Mazur–Orlicz theorem, first appeared in [7], and the analysis was refined in [8] and [9]. The idea behind this result is to provide a unified and relatively nontechnical framework for treating the main existence theorems for continuous linear functionals in linear and nonlinear functional analysis, convex analysis, Lagrange multiplier theory and minimax theory. Applications were also given to the theory of monotone multifunctions. In many cases, the Hahn–Banach–Lagrange Theorem leads to necessary and sufficient conditions instead of the more usual known sufficient conditions for the existence of these functionals, and also leads to sharp numerical bounds for the norm of the functional obtained. We will give an analysis of the Hahn–Banach–Lagrange theorem in Section 5.

We now give a short outline of the analysis in this paper. Section 2 contains the generalization due to König of the Mazur–Orlicz theorem, which has the advantage that effort does not have to be expended to prove that certain sets are convex. The proof is similar to that of the original Mazur–Orlicz theorem, but somewhat more technical.

If E is a vector space and $f: E \rightarrow \mathbb{R}$ is convex, we use f to define implicitly a sublinear function $S_f: E \times \mathbb{R} \rightarrow [0, \infty[$, to which we will apply the results of Section 2. S_f is defined in Lemma 3.2, and its properties are explored in Lemma 3.4. The main result of this section, Theorem 3.5, gives a method for the construction of affine functions.

In Section 4, we discuss the result of Sun Chuanfeng’s mentioned above.

Theorem 4.3 contains a generalization of this result in which the convex subset is replaced by any subset Z satisfying equation (17). Sun Chuanfeng’s original result is established in Corollary 4.4.

The original Hahn–Banach–Lagrange theorem used a convex set, C , a sublinear functional S , and two functions, j and k . See Corollary 5.4 for the simplest formulation of this kind of result. Corollary 5.3 contains a version in which the convex set C is replaced by a set Z with no algebraic structure. These results are discussed here as consequences of Corollary 5.2 and Theorem 5.1, which are results on n (≥ 2) vector spaces instead of just 2. Theorem 5.1 is obtained by a very simple bootstrapping procedure from Lemma 2.1.

The question now arises whether there are convex-affine results in the spirit of Sun Chuanfeng’s theorem that are similar to Corollary 5.3 and Corollary 5.4. We give two such results, in Theorem 6.1 and Corollary 6.2. It would be nice if there were a result analogous to Theorem 6.1 or Corollary 6.2 for n (≥ 3) vector spaces. We explain in Remark 6.3 why we think that this is unlikely.

The author would like to express his sincere thanks to Sun Chuanfeng for sending him a preprint of [10].

2. On the existence of linear functionals

Lemma 2.1 (Mazur–Orlicz–König theorem). *Let E be a nonzero vector space, $S: E \rightarrow \mathbb{R}$ be sublinear and D be a nonempty subset of E such that*

$$\text{for all } d_1, d_2 \in D, \text{ there exists } d \in D \text{ such that } S(d - \tfrac{1}{2}d_1 - \tfrac{1}{2}d_2) \leq 0.$$

Then there exists a linear map $L: E \rightarrow \mathbb{R}$ such that $L \leq S$ on E and

$$\inf_D L = \inf_D S.$$

Proof. See König, [1, Basic Theorem, p. 583]. □

3. An implicitly defined sublinear functional

Notation 3.1. We introduce some notation to simplify the expressions in what follows. We suppose that E is a vector space and $f: E \rightarrow \mathbb{R}$ is convex. Let $\underline{f} := f - f(0) - 1$, so that

$$\underline{f} \text{ is convex and } \underline{f}(0) = -1. \tag{1}$$

If $x \in E$, let $f_x:]0, \infty[\rightarrow \mathbb{R}$ be defined by $f_x(\mu) := \mu \underline{f}(x/\mu)$. We know from [11, Theorem 2.1.5(v), pp. 46–49] that $\underline{f}$ is continuous on any one-dimensional subspace of E , and so f_x is continuous. If $0 < \mu < \nu$ then $0 < \mu/\nu < 1$, and so

$$\begin{aligned} \underline{f}(x/\nu) &= \underline{f}((1 - \mu/\nu)0 + (\mu/\nu)(x/\mu)) \\ &\leq (1 - \mu/\nu)(-1) + (\mu/\nu)\underline{f}(x/\mu) = \mu/\nu - 1 + (\mu/\nu)\underline{f}(x/\mu). \end{aligned}$$

Multiplying by ν , we see that

$$0 < \mu < \nu \implies f_x(\nu) + \nu \leq f_x(\mu) + \mu. \tag{2}$$

Consequently,

$$f_x \text{ is continuous and strictly decreasing, and } \lim_{\nu \rightarrow \infty} f_x(\nu) = -\infty. \tag{3}$$

Lemma 3.2. *Let $(x, \alpha) \in E \times \mathbb{R}$ and $I(x, \alpha) := \{\mu > 0: f_x(\mu) < \alpha\}$. Then:*

- (a) $I(x, \alpha)$ is a semi-infinite open subinterval of $]0, \infty[$.
- (b) If $\inf I(x, \alpha) > 0$ then $\inf I(x, \alpha)$ is the unique value of σ with $f_x(\sigma) = \alpha$.
- (c) We define the function $S_f: E \times \mathbb{R} \rightarrow [0, \infty[$ by $S_f(x, \alpha) := \inf I(x, \alpha)$. Then

$$S_f(x, \alpha) \leq 0 \iff \text{for all } \rho \geq 0, f(\rho x) - \alpha \rho \leq f(0). \tag{4}$$

If $S_f(x, \alpha) > 0$ then $S_f(x, \alpha)$ is uniquely determined by the implicit equality

$$S_f(x, \alpha) \underline{f}(x/S_f(x, \alpha)) = \alpha \text{ (or equivalently, } (x, \alpha)/S_f(x, \alpha) \in \text{graph } \underline{f}). \tag{5}$$

Proof. (a) This follows from (3).

(b) By hypothesis, there exists ν such that $0 < \nu < \inf I(x, \alpha)$, from which $f_x(\nu) \geq \alpha$. From the intermediate value theorem and (3), there exists a unique $\sigma > 0$ such that $f_x(\sigma) = \alpha$. Now if $\mu \in I(x, \alpha)$ then $f_x(\mu) < \alpha = f_x(\sigma)$, and so $\mu > \sigma$. Consequently, $\inf I(x, \alpha) \geq \sigma$. On the other hand, for all $n \geq 1$, $\sigma + 1/n > \sigma$, hence $f_x(\sigma + 1/n) < f_x(\sigma) = \alpha$, and so $\sigma + 1/n \in I(x, \alpha)$. Consequently, $\inf I(x, \alpha) \leq \sigma$. Thus $\inf I(x, \alpha) = \sigma$, as required.

(c) We now establish (4). Since $f(0x) - \alpha 0 = f(0)$, putting $\nu = 1/\rho$,

$$\begin{aligned} \text{for all } \rho \geq 0, f(\rho x) - \alpha \rho \leq f(0) &\iff \text{for all } \rho > 0, f(\rho x) - \alpha \rho \leq f(0) \\ \iff \text{for all } \nu > 0, f(x/\nu) - \alpha/\nu \leq f(0) &\iff \text{for all } \nu > 0, f_x(\nu) + \nu \leq \alpha, \\ &\text{and, further,} \\ S_f(x, \alpha) \leq 0 &\iff \text{for all } \mu > 0, \mu \in I(x, \alpha) \iff \text{for all } \mu > 0, f_x(\mu) < \alpha. \end{aligned}$$

It is clear by comparing the two sets of implications above that “ $\Leftarrow$ ” in (4) is satisfied. If, on the other hand, for all $\mu > 0$, $f_x(\mu) < \alpha$ and $\nu > 0$, we take $0 < \mu < \nu$. From (2), $f_x(\nu) + \nu \leq f_x(\mu) + \mu < \alpha + \mu$. Letting $\mu \rightarrow 0$, we see that $f_x(\nu) + \nu \leq \alpha$, and so “ $\Rightarrow$ ” in (4) is also satisfied. This gives (4), and (5) is immediate from (b). $\square$

Definition 3.3. If $\alpha \in \mathbb{R}$ then α^- is the “negative part” of α , that is to say, $\alpha^- = \max(-\alpha, 0)$. We write $\text{hyp } \underline{f}$ for the *hypograph* of $\underline{f}$, that is to say the set $\{(x, \alpha) \in E \times \mathbb{R} : \underline{f}(x) \geq \alpha\}$.

Lemma 3.4 (Some properties of S_f). *We first give the values of S_f on the “vertical axis”, the hypograph of $\underline{f}$ and the graph of $\underline{f}$:*

$$\text{For all } \alpha \in \mathbb{R}, S_f(0, \alpha) = \alpha^-. \quad (6)$$

$$\text{For all } (x, \alpha) \in \text{hyp } \underline{f}, S_f(x, \alpha) \geq 1. \quad (7)$$

$$\text{For all } x \in E, S_f(x, \underline{f}(x)) = 1. \quad (8)$$

$$S_f \text{ is sublinear, that is to say } S_f(0, 0) = 0. \quad (9)$$

$$(x, \alpha) \in E \times \mathbb{R} \text{ and } \lambda > 0 \implies S_f(\lambda x, \lambda \alpha) = \lambda S_f(x, \alpha); \quad (10)$$

and

$$(x, \alpha) \text{ and } (y, \gamma) \in E \times \mathbb{R} \implies S_f(x, \alpha) + S_f(y, \gamma) \geq S_f(x + y, \alpha + \gamma). \quad (11)$$

Proof. (6) follows from the observation that, for all $\mu > 0$, $f_0(\mu) := -\mu$, and so $I(0, \alpha) = \{\mu > 0 : -\mu < \alpha\} =]\alpha^-, \infty[$. If $S_f(x, \alpha) < 1$ then $1 \in I(x, \alpha)$ and so $\underline{f}(x) = f_x(1) < \alpha$. (7) follows from this. (8) is immediate from (7) and the “uniqueness” in (5). (9) is immediate from (6).

From Lemma 3.2(b), $I(\lambda x, \lambda \alpha) := \{\mu > 0: \underline{f}(\lambda x/\mu) < \lambda \alpha/\mu\}$. Setting $\nu = \mu/\lambda$, so that $\mu = \lambda \nu$, $I(\lambda x, \lambda \alpha) := \{\lambda \nu: \nu > 0, \underline{f}(x/\nu) < \alpha/\nu\} = \lambda I(x, \alpha)$. (10) follows by taking the infima of both sides.

Let $\mu \in I(x, \alpha)$ and $\nu \in I(y, \gamma)$. Then $\mu \underline{f}(x/\mu) < \alpha$ and $\nu \underline{f}(y/\nu) < \gamma$. Thus

$$\begin{aligned} \underline{f}\left(\frac{x+y}{\mu+\nu}\right) &= \underline{f}\left(\frac{\mu(x/\mu) + \nu(y/\nu)}{\mu+\nu}\right) \leq \frac{\mu}{\mu+\nu} \underline{f}\left(\frac{x}{\mu}\right) + \frac{\nu}{\mu+\nu} \underline{f}\left(\frac{y}{\nu}\right) \\ &< \frac{\mu}{\mu+\nu} \frac{\alpha}{\mu} + \frac{\nu}{\mu+\nu} \frac{\gamma}{\nu} = \frac{\alpha+\gamma}{\mu+\nu}. \end{aligned}$$

Consequently, $\mu + \nu \in I(x+y, \alpha+\gamma)$, and so $\mu + \nu \geq S_f(x+y, \alpha+\gamma)$.

(11) now follows by taking the infimum over μ and ν . $\square$

We now come to the main result of this section.

Theorem 3.5 (The existence of affine maps). *Let $\emptyset \neq B \subset E \times \mathbb{R}$,*

$$\inf_{(b,\beta) \in B} [f(b) + \beta] \in \mathbb{R}, \quad (12)$$

and, for all $(b_1, \beta_1), (b_2, \beta_2) \in B$, there exists $(b, \beta) \in B$ such that

$$\text{for all } \rho \geq 0, \quad f\left(\rho\left[b - \frac{1}{2}b_1 - \frac{1}{2}b_2\right]\right) + \rho\left[\beta - \frac{1}{2}\beta_1 - \frac{1}{2}\beta_2\right] \leq f(0). \quad (13)$$

Then there exists an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and

$$\inf_{(b,\beta) \in B} [A(b) + \beta] = \inf_{(b,\beta) \in B} [f(b) + \beta].$$

Proof. Let $\delta := \inf_{(b,\beta) \in B} [f(b) + \beta] \in \mathbb{R}$. For all $(b, \beta) \in B$, $f(b) + \beta \geq \delta$, and so $(b, \delta - \beta - f(0) - 1) \in \text{hyp} \underline{f}$. Let $D := \{(b, \delta - \beta - f(0) - 1)\}_{(b,\beta) \in B}$. Thus $D \subset \text{hyp} \underline{f}$, and so (7) implies that $\inf_D S_f \geq 1$. From (13) and (4), for all $(b_1, \beta_1), (b_2, \beta_2) \in B$, there exists $(b, \beta) \in B$ such that

$$S_f\left(b - \frac{1}{2}b_1 - \frac{1}{2}b_2, \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2 - \beta\right) \leq 0,$$

which implies that

$$\text{for all } d_1, d_2 \in D, \quad \text{there exists } d \in D \text{ such that } S_f\left(d - \frac{1}{2}d_1 - \frac{1}{2}d_2\right) \leq 0.$$

From Lemma 2.1, with $S = S_f$ and E replaced by $E \times \mathbb{R}$, there exists a linear map $L: E \times \mathbb{R} \rightarrow \mathbb{R}$ such that $L \leq S_f$ on $E \times \mathbb{R}$ and $\inf_D L = \inf_D S_f \geq 1$. From algebraic considerations, there exist a linear map $\Lambda: E \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that, for all $(x, \alpha) \in E \times \mathbb{R}$, $L(x, \alpha) = \Lambda(x) - \lambda \alpha$. Thus, from (8),

$$\text{for all } x \in E, \quad \Lambda(x) - \lambda \underline{f}(x) = L(x, \underline{f}(x)) \leq S_f(x, \underline{f}(x)) = 1, \quad (14)$$

and, for all $(b, \eta) \in D$, $\Lambda(b) - \lambda \eta = L(b, \eta) \geq \inf_D L \geq 1$, from which

$$\text{for all } (b, \beta) \in B, \quad \Lambda(b) - 1 + \lambda(f(0) + 1 + \beta) \geq \lambda \delta. \quad (15)$$

From (6), $-\lambda = \Lambda(0) - \lambda 1 = L(0, 1) \leq S_f(0, 1) = 0$, so $\lambda \geq 0$. Now, if we had $\lambda = 0$ then, from (14), for all $x \in E$, $\Lambda(x) \leq 1$. Consequently, $\Lambda = 0$, which would contradict (15). Thus $\lambda > 0$. We write A for the affine function $\Lambda/\lambda - 1/\lambda + f(0) + 1$. If we divide (14) by λ and rearrange the terms we see that, for all $x \in E$, $(\Lambda/\lambda)(x) - 1/\lambda \leq \underline{f}(x)$, from which $A(x) \leq \underline{f}(x) + f(0) + 1 = f(x)$. Thus $A \leq f$ on E . Consequently,

$$\inf_{(b,\beta) \in B} [A(b) + \beta] \leq \inf_{(b,\beta) \in B} [f(b) + \beta]. \quad (16)$$

Dividing (15) by λ , for all $(b, \beta) \in B$, $(\Lambda/\lambda)(b) - 1/\lambda + f(0) + 1 + \beta \geq \delta$. Thus $A(b) + \beta \geq \delta$. Consequently,

$$\inf_{(b,\beta) \in B} [A(b) + \beta] \geq \delta = \inf_{(b,\beta) \in B} [f(b) + \beta].$$

The required result follows by combining this with (16). $\square$

Remark 3.6. Another way of seeing the sublinearity of S_f is to note that S_f is the *Minkowski functional* of the strict epigraph of $\underline{f}$.

4. Sun Chuanfeng's theorem

Lemma 4.1. *Let E be a vector space, $f: E \rightarrow \mathbb{R}$ be convex and $x \in E$. Then there exists an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and $A(x) = f(x)$.*

Proof. Let $B = \{(x, 0)\}$. Then $\inf_{(b,\beta) \in B} [f(b) + \beta] = f(x)$ and, for all $\rho \geq 0$, $f(\rho[0]) + \rho[0] = f(0)$. The result follows from Theorem 3.5. $\square$

Remark 4.2. Lemma 4.1 says that the algebraic subdifferential of f at x is nonempty. This can also be deduced by applying the Hahn–Banach theorem to the sublinear functional introduced in [11, Theorem 2.1.13, pp. 55–56].

Theorem 4.3 (A generalization of Sun Chuanfeng's theorem). *Let E be a vector space, $f: E \rightarrow \mathbb{R}$ be convex, $\emptyset \neq Z \subset E$ and, for all $z_1, z_2 \in Z$, there exists $z \in Z$ such that*

$$\text{for all } \rho \geq 0, \quad f(\rho[z - \tfrac{1}{2}z_1 - \tfrac{1}{2}z_2]) \leq f(0). \quad (17)$$

Then there exists an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and $\inf_Z A = \inf_Z f$.

Proof. If $\inf_Z f = -\infty$ then the result is immediate from Lemma 4.1, so we can and will suppose that $\inf_Z f \in \mathbb{R}$. Let $B := Z \times \{0\}$. (12) follows since $\inf_{(b,\beta) \in B} [f(b) + \beta] = \inf_Z f$, and (13) is immediate from (17). Theorem 3.5 gives an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and

$$\inf_{z \in Z} [A(z) + 0] = \inf_{z \in Z} [f(z) + 0].$$

The desired result follows since $\inf_{z \in Z} [A(z) + 0] = \inf_Z A$ and, as we have already observed, $\inf_{(b,\beta) \in B} [f(b) + \beta] = \inf_Z f$. $\square$

Corollary 4.4 (Sun Chuanfeng’s theorem). *Let E be a vector space, the function $f: E \rightarrow \mathbb{R}$ be convex, and C be a nonempty convex (or even midpoint convex) subset of E . Then there exists an affine function $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and $\inf_C A = \inf_C f$.*

Proof. This is immediate from Theorem 4.3. $\square$

Remark 4.5. In a certain sense, the analysis presented in this section is actually only “half the story” because $B \subset E \times \{0\}$. We will discuss the “whole story” in Section 6. Specifically, Theorem 4.3 will be extended by Theorem 6.1, and Corollary 4.4 will be extended by Corollary 6.2.

5. Results of Hahn–Banach–Lagrange type

The main result in this section is Theorem 5.1, which represents a considerable generalization of the “Hahn–Banach–Lagrange” theorem, [9, Theorem 1.11, p. 21]. Theorem 5.1 is an easy consequence of Lemma 2.1. Corollary 5.2–5.4 are simple special cases of Theorem 5.1. See [9] for a discussion of the many consequences of Corollaries 5.3 and 5.4.

Theorem 5.1. *For all $m = 1, \dots, n$, let E_m be a vector space and $S_m: E_m \rightarrow \mathbb{R}$ be sublinear. Let Z be a nonempty set, for all $m = 1, \dots, n$, $j_m: Z \rightarrow E_m$ and, for all $z_1, z_2 \in Z$, there exists $z \in Z$ such that*

$$\sum_{m=1}^n S_m(j_m(z) - \tfrac{1}{2}j_m(z_1) - \tfrac{1}{2}j_m(z_2)) \leq 0.$$

Then, for all $m = 1, \dots, n$, there exists a linear map $L_m: E_m \rightarrow \mathbb{R}$ such that $L_m \leq S_m$ on E_m and

$$\inf_Z \sum_{m=1}^n [L_m \circ j_m] = \inf_Z \sum_{m=1}^n [S_m \circ j_m].$$

Proof. Define $j: Z \rightarrow E_1 \times \dots \times E_n$ by $j(z) := (j_1(z), \dots, j_n(z))$, and define the sublinear map $S: E_1 \times \dots \times E_n \rightarrow \mathbb{R}$ by $S(x_1, \dots, x_n) := \sum_{m=1}^n S_m(x_m)$. Then the conditions of Lemma 2.1 are satisfied with $E := E_1 \times \dots \times E_n$ and $D := j(Z)$. From Lemma 2.1, there exists a linear map $L: E \rightarrow \mathbb{R}$ so that $L \leq S$ on E and $\inf_D L = \inf_D S$. For all $m = 1, \dots, n$, there exists a linear map $L_m: E_m \rightarrow \mathbb{R}$ such that, for all $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$, $L(x_1, \dots, x_n) = \sum_{m=1}^n L_m(x_m)$. Now let $1 \leq m \leq n$, $y \in E_m$ and $w = (0, \dots, y, \dots, 0)$, where the “ y ” is in the m^{th} place. Then $L_m(y) = L(w) \leq S(w) = S_m(y)$, from which $L_m \leq S_m$ on E_m . The result follows since $\inf_Z \sum_{m=1}^n [L_m \circ j_m] = \inf_Z L \circ j = \inf_D L$ and $\inf_Z \sum_{m=1}^n [S_m \circ j_m] = \inf_Z S \circ j = \inf_D S$. $\square$

Corollary 5.2. *For all $m = 1, \dots, n$, let E_m be a vector space and $S_m: E_m \rightarrow \mathbb{R}$ be sublinear. Let C be a nonempty convex subset of a vector space and, for all $m = 1, \dots, n$, $j_m: C \rightarrow E_m$ be affine. Then, for all $m = 1, \dots, n$, there exists a linear map $L_m: E_m \rightarrow \mathbb{R}$ such that $L_m \leq S_m$ on E_m and*

$$\inf_C \sum_{m=1}^n [L_m \circ j_m] = \inf_C \sum_{m=1}^n [S_m \circ j_m].$$

Proof. This is immediate from Theorem 5.1. $\square$

The original “Hahn–Banach–Lagrange” theorem appeared in (among other places) [9, Theorem 1.11, p. 21]. Corollary 5.3 below is a generalization of the generalization of this that appeared in [9, Theorem 1.13, p. 22].

Corollary 5.3. *Let E be a vector space and $S: E \rightarrow \mathbb{R}$ be sublinear. Let $Z \neq \emptyset$, $j: Z \rightarrow E$, $k: Z \rightarrow \mathbb{R}$ and, whenever $z_1, z_2 \in Z$, there exists $z \in Z$ such that*

$$S(j(z) - \tfrac{1}{2}j(z_1) - \tfrac{1}{2}j(z_2)) + k(z) - \tfrac{1}{2}k(z_1) - \tfrac{1}{2}k(z_2) \leq 0.$$

Then there exists a linear map $L: E \rightarrow \mathbb{R}$ such that $L \leq S$ on E and

$$\inf_Z [L \circ j + k] = \inf_Z [S \circ j + k].$$

Proof. We apply Theorem 5.1 with $n = 2$, $E_1 = E$, $E_2 = \mathbb{R}$, $S_1 = S$, $S_2 = I_{\mathbb{R}}$, the identity map on $\mathbb{R}$, $j_1 = j$ and $j_2 = k$. Thus there exist a linear map $L: E \rightarrow \mathbb{R}$ and a linear map $M: \mathbb{R} \rightarrow \mathbb{R}$ such that $L \leq S$ on E , $M \leq I_{\mathbb{R}}$ on $\mathbb{R}$ and $\inf_Z [L \circ j + M \circ k] = \inf_Z [S \circ j + M \circ k]$. The result follows since, as is easily seen, $M = I_{\mathbb{R}}$. $\square$

We have included Corollary 5.4 below so that it can be compared with the convex–affine result contained in Corollary 6.2.

Corollary 5.4. *Let E be a vector space and $S: E \rightarrow \mathbb{R}$ be sublinear. Let C be a nonempty convex subset of a vector space, $j: C \rightarrow E$ be affine and $k: C \rightarrow \mathbb{R}$ be convex. Then there exists a linear map $L: E \rightarrow \mathbb{R}$ such that $L \leq S$ on E and*

$$\inf_C [L \circ j + k] = \inf_C [S \circ j + k].$$

Proof. This is immediate from Corollary 5.3. $\square$

6. A convex–affine result of Hahn–Banach–Lagrange type

Our next result is a convex–affine version of Corollary 5.3, and its consequence, Corollary 6.2, is a convex–affine version of Corollary 5.4.

Theorem 6.1. (A convex–affine result of Hahn–Banach–Lagrange type)
Let E be a vector space, $f: E \rightarrow \mathbb{R}$ be convex, $Z \neq \emptyset$, $j: Z \rightarrow E$, $k: Z \rightarrow \mathbb{R}$ and, whenever $z_1, z_2 \in Z$, there exists $z \in Z$ such that, for all $\rho \geq 0$,

$$f(\rho[j(z) - \tfrac{1}{2}j(z_1) - \tfrac{1}{2}j(z_2)]) + \rho[k(z) - \tfrac{1}{2}k(z_1) - \tfrac{1}{2}k(z_2)] \leq f(0). \quad (18)$$

Then there exists an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and

$$\inf_Z [A \circ j + k] = \inf_Z [f \circ j + k].$$

Proof. If $\inf_Z [f \circ j + k] = -\infty$ then the result is immediate from Lemma 4.1, so we can and will suppose that $\inf_Z [f \circ j + k] \in \mathbb{R}$. Let $B = \{(j(z), k(z))\}_{z \in Z}$.

(12) now follows since $\inf_{(b,\beta) \in B} [f(b) + \beta] = \inf_Z [f \circ j + k]$, and (13) is immediate from (18). Theorem 3.5 now provides an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and $\inf_{(b,\beta) \in B} [A(b) + \beta] = \inf_{(b,\beta) \in B} [f(b) + \beta]$. The desired result follows since $\inf_{(b,\beta) \in B} [A(b) + \beta] = \inf_Z [A \circ j + k]$ and, as was noted above, $\inf_{(b,\beta) \in B} [f(b) + \beta] = \inf_Z [f \circ j + k]$. $\square$

Corollary 6.2. *Let E be a vector space, $f: E \rightarrow \mathbb{R}$ be convex, C be a nonempty convex subset of a vector space, $j: C \rightarrow E$ be affine, and $k: C \rightarrow \mathbb{R}$ be convex. Then there exists an affine map $A: E \rightarrow \mathbb{R}$ such that $A \leq f$ on E and*

$$\inf_C [A \circ j + k] = \inf_C [f \circ j + k].$$

Proof. This is immediate from Theorem 6.1. $\square$

Remark 6.3. It is tempting to try to find an analog of Theorem 6.1 for $n \geq 3$ convex functions instead of 2 in the general spirit of Theorem 5.1. The problem is that the technique used in the proof of Theorem 5.1 of progressively setting all the values of x_m other than one particular one to be 0 and using the fact that linear and sublinear maps vanish at 0 does not seem available in the convex–affine case.

References

- [1] H. König: On certain applications of the Hahn–Banach and minimax theorems, Arch. Math. 21 (1970) 583–591.
- [2] H. König: Some basic theorems in convex analysis, in: Modern Applied Mathematics. Optimization and Operations Research, edited by B. Korte, North-Holland, Amsterdam (1982) 107–144.
- [3] M. Landsberg, W. Schirotzek: Mazur–Orlicz type theorems with applications, Math. Nachr. 79 (1977) 331–341.
- [4] V. Pták: On a theorem of Mazur and Orlicz, Studia Math. 15 (1956) 365–366.
- [5] S. Mazur, W. Orlicz: Sur les espaces métriques linéaires II, Studia Math. 13 (1953) 137–179.
- [6] R. Sikorski: On a theorem of Mazur and Orlicz, Studia Math. 13 (1953) 180–182.
- [7] S. Simons: A new version of the Hahn–Banach theorem, Arch. Math. 80 (2003) 630–646.
- [8] S. Simons: The Hahn–Banach–Lagrange theorem, Optimization 56 (2007) 149–169.
- [9] S. Simons: From Hahn–Banach to Monotonicity, Lecture Notes in Mathematics 1693, second edition, Springer-Verlag, Berlin (2008).
- [10] Sun Chuanfeng: The Mazur–Orlicz theorem for convex functionals, J. Convex Analysis 24 (2017) 763–768.
- [11] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).