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Fixed Point Iterative Schemes for Variational Inequality Problems

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Dedicated to Antonino Maugeri on the occasion of his 70th birthday.

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We apply fixed point iterative schemes to variational inequality problems, via admissible perturbations of projection operators in real Hilbert spaces. Then, we prove some convergence theorems, extending and complementing the results in the existing literature. In particular, we deal with the class of α-co-coercive operators with application to general equilibrium problems.

Keywords: Hilbert space, Krasnoselskij-type iterative scheme, projected dynamical system, projection operator, variational inequality problem.

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1. Introduction

In a wide class of evolutionary processes, the problem of computing the solutions of an initial value problem is encountered. Here, we consider projected dynamical systems in the sense of [7] and references therein. Precisely, a projected dynamical system is an operator which solves the initial value problem:

$$\text{(P)} \qquad \frac{dx}{dt}(t) = \Pi_{\mathbb{K}}\left(x(t), -F(x(t))\right), \quad x(0) = x_0 \in \mathbb{K},\ t \in [0, +\infty[,$$

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where $\mathbb{K}$ is a convex polyhedral set in $\mathbb{R}^n$, $F\colon \mathbb{K} \to \mathbb{R}^n$ and $\Pi_{\mathbb{K}}\colon \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is given as follows

$$\Pi_{\mathbb{K}}(x, -F(x)) = \lim_{t\to 0^+} \frac{P_{\mathbb{K}}(x - tF(x)) - x}{t},$$

by using the directional derivative in the sense of Gâteaux of the projection operator $P_{\mathbb{K}}\colon \mathbb{R}^n \to \mathbb{K}$. The definition of $\Pi_{\mathbb{K}}$ here means that

$$\Pi_{\mathbb{K}}(x, -F(x)) = \mathrm{Proj}_{T_{\mathbb{K}}(x)}(-F(x)),$$

where $\mathrm{Proj}_{T_{\mathbb{K}}(x)}$ is the metric projection onto the tangent cone of $\mathbb{K}$ at x, according to a famous result by Zarantonello [23]. Moreover, the set of critical points of (P), i.e. the solutions such that $\frac{dx}{dt}(t) = 0$, coincides with the set of solutions of a variational inequality problem in the sense of Stampacchia [22].

In view of this equivalence, established by Dupuis and Nagurney [9], we investigate the extension and application of fixed point iterative schemes to variational inequality problems, via admissible perturbations of projection operators in real Hilbert spaces.

Let K be a nonempty closed and convex subset of a real Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and $S\colon H \to H$ be a nonlinear operator. Then, we consider the following variational inequality problem, say (VIP) for short:

Problem 1.1. ([22]) *Let $S\colon H \to H$. Find $u \in K$ such that $\langle Su, v - u\rangle \geq 0$ for all $v \in K$.*

An equivalent formulation is $Su + N_K(u) \ni 0$, where $N_K(u)$ denotes the normal cone of K at u. See the book by Dontchev and Rockafellar [8] for various features related to that formulation.

Notice that $u \in K$ is a solution of (VIP) if and only if $u \in VI(K, S)$, where $VI(K, S) = \{u \in K : \langle Su, v - u\rangle \geq 0, \text{ for all } v \in K\}$ is the solution set of Problem 1.1.

The interest for such a kind of mathematical tool is due to the fact that a wide class of problems, including obstacle, equilibrium and economics arising in various areas of pure and applied sciences, can be equivalently reformulated as (VIPs). On the other hand, various numerical approximation methods are applicable for solving variational inequalities. Here, we consider the well-known metric projection operator.

Thus, we establish some convergence theorems in order to solve variational inequality problems involving nonlinear operators in real Hilbert spaces. As a sample model, we present results for Krasnoselskij-type fixed point iterative schemes. In particular, we deal with the class of α-co-coercive operators with application to general equilibrium problems.

2. Preliminaries

The development of the theory of fixed point iterative schemes is an actual and exciting research topic, which is strongly devoted to the convergence and stability analysis of numerical methods of approximation theory; some references can be found in [1, 2, 4, 20]. It is well-known that different problems, arising in pure and applied sciences, can be restated as an equivalent fixed point problem. Let $T\colon K \to K$ be a given operator, solving a fixed point problem means to get a point $x \in K$ such that $x = Tx$.

A sequence $\{x_n\} \subset K$ converges (strongly) to x if and only if $\|x_n - x\| \to 0$ as $n \to +\infty$. Also, a sequence $\{x_n\} \subset K$ converges weakly to x if and only if $\langle x_n, u\rangle \to \langle x, u\rangle$, for each $u \in K$.

We recall that a cornerstone iterative scheme, the so-called Picard iteration at starting point x_0, generates the sequence $\{x_n\}$ of successive approximations of a point x, by putting $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$, with given x_0. Now, there is a rich literature related to convergence, stability and establishment of a priori and a posteriori estimation errors of this scheme, see for instance [4, 5, 18]. Without aim of completeness, we recall three iterative schemes to provide the reader some background, see again [4, 5, 18].

First, we consider the *Krasnoselskij iteration scheme*, which generates the sequence $\{x_n\} \subset K$ given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \tag{1}$$

where $\lambda \in [0, 1]$. Clearly, by putting $\lambda = 1$ in (1), we get the *Picard iteration scheme.*

Secondly, let $\{\lambda_n\} \subset [0, 1]$ be a sequence of real numbers. We cite the *Mann iterative scheme* [15], which generates the sequence $\{x_n\} \subset K$ given by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

Finally, the *Ishikawa iterative scheme* [13] generates the sequence $\{x_n\} \subset K$ given by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Obviously, for $\beta_n = 0$, (3) reduces to (2), while, for $\alpha_n = \lambda$, (2) reduces to (1).

Also, an operator $T\colon K \to K$ is called:

(i) *Lipschitz continuous* if there exists a constant $k_1 > 0$ such that, for $x, y \in K$,

$$\|Tx - Ty\| \leq k_1 \|x - y\|.$$

(ii) *Nonexpansive* (or 1-*Lipschitz*) if, for $x, y \in K$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

(iii) *Monotone* if, for $x, y \in K$,

$$\langle Tx - Ty, x - y\rangle \geq 0.$$

(iv) *Strongly monotone* if there exists a constant $k_2 > 0$ such that, for $x, y \in K$,

$$\langle Tx - Ty, x - y\rangle \geq k_2\|x - y\|^2.$$

(v) *Demi-compact* if whenever $\{x_n\}$ is a bounded sequence in H and $\{Tx_n - x_n\}$ is strongly convergent, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is strongly convergent, see [17];

(vi) *α-co-coercive* if there exists a constant $\alpha > 0$ such that, for all $x, y \in K$,

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2.$$

Remark 2.1. Let T be α-co-coercive. By using the Cauchy-Schwarz inequality, we get

$$\|Tx - Ty\|\|x - y\| \geq \langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2 \geq 0.$$

So an α-co-coercive operator is Lipschitz continuous and monotone. In particular, a 1-co-coercive operator is nonexpansive and monotone.

One of the most important fixed point theorems for nonexpansive mappings in Banach space setting is known as the Browder-Göhde-Kirk theorem, see [10]. Here, we recall a version of this theorem in a Hilbert space setting, for our further use.

Theorem 2.2. (see [4], Theorem 3.1) *Let K be a nonempty bounded closed and convex subset of a real Hilbert space H and $T\colon K \to K$ a nonexpansive operator. Then T has at least one fixed point in K.*

Recently, Rus [20] presented an approach to fixed point iterative schemes based on the notion of admissible perturbation of mappings, see also [5, 6]. Some related notions are as follows.

Definition 2.3. ([20]) A mapping $G\colon K \times K \to K$ is called *admissible* if it satisfies the following two conditions:

(i) $G(x, x) = x$, for all $x \in K$;
(ii) $G(x, y) = x$ implies $x = y$.

Definition 2.4. ([20]) If $T\colon K \to K$ is a given mapping and $G\colon K \times K \to K$ is an admissible mapping, then the mapping $T_G\colon K \to K$, defined by

$$T_G x = G(x, Tx), \qquad \text{for all } x \in K, \tag{4}$$

is called the *admissible perturbation* of T.

Remark 2.5. ([3]) Let $Fix(T) = \{x \in K : x = Tx\}$ denote the set of all fixed points of mapping $T\colon K \to K$. Notice that, if $T_G\colon K \to K$ is the admissible perturbation of T, then T_G and T have the same set of fixed points, that is, $Fix(T_G) = Fix(T)$.

We recall the following example from [20], for our further use.

Example 2.6. ([20]) Let K be a nonempty closed and convex subset of a real Hilbert space H, $\lambda \in]0,1[$, $T\colon K \to K$ and $G\colon K \times K \to K$ be defined by

$$G(x,y) = (1-\lambda)x + \lambda y, \quad x, y \in K.$$

Then G is an admissible mapping and $T_G x = G(x, Tx)$ is an admissible perturbation of T. Here, T_G is also known as *Krasnoselskij perturbation* of T.

Definition 2.7. ([20]) We consider a nonlinear mapping $T\colon K \to K$ and an admissible operator $G\colon K \times K \to K$. Then the iterative scheme $\{x_n\}$, given by $x_0 \in K$ and

$$x_{n+1} = G(x_n, Tx_n), \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \tag{5}$$

is called the *Krasnoselskij iterative scheme* corresponding to G; we say *GK-iterative scheme* for short.

Definition 2.8. Let $G\colon K \times K \to K$ be an admissible operator. We say that G is affine Lipschitzian if there exists a constant $\mu \in [0,1]$ such that

$$\|G(x_1,y_1) - G(x_2,y_2)\| \le \|\mu(x_1 - x_2) + (1-\mu)(y_1 - y_2)\|, \tag{6}$$

for all $x_1, x_2, y_1, y_2 \in K$.

3. Main results

In this section, by following classical arguments in Nonlinear Analysis, we discuss the solvability of Problem 1.1. First, we consider the well-known metric projection operator.

We recall that for each $u \in H$, there exists a unique nearest point in K, say $P_K u$, such that $\|u - P_K u\| \le \|u - v\|$ for all $v \in K$. The operator P_K is called *projection* from H to K and is useful to establishing the existence of solution of (VIP). In fact, we have the following fundamental lemmas.

Lemma 3.1. *Let $z \in H$. An element $u \in K$ satisfies for all $y \in K$ the inequality $\langle u - z, y - u\rangle \ge 0$, if and only if $u = P_K z$.*

Lemma 3.2. *Let $S\colon H \to H$. An element $u \in K$ is, for all $v \in K$, a solution of $\langle Su, v - u\rangle \ge 0$, if and only if $u = P_K(u - \rho Su)$, with $\rho > 0$.*

Also, the operator P_K is *nonexpansive*, that is $\|P_K u - P_K v\| \leq \|u - v\|$ for all $u, v \in K$.

Now, Problem 1.1 can be solved by a fixed point iterative scheme as that shown in the following algorithm.

Algorithm 3.3. *Let $S\colon H \to H$ be a nonlinear operator. Given $u_0 \in K$, compute iteratively $u_{n+1} = P_K(u_n - \rho S u_n)$, with $\rho > 0$ and $n \in \mathbb{N} \cup \{0\}$.*

This basic formulation received the interest of many researchers and consequently a lot of modified algorithms appeared in the literature, see for instance [12]. Next, we give our results.

Theorem 3.4. *Let K be a nonempty bounded closed and convex subset of a real Hilbert space H, $T\colon K \to K$ a nonexpansive and demi-compact operator, and $G\colon K \times K \to K$ an affine Lipschitzian admissible operator with constant $\lambda \in]0,1[$. Then T has a fixed point in K and the GK-iterative scheme $\{x_n\}$, given by $x_0 \in K$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

converges to a fixed point of T.

Proof. Let $T_G\colon K \to K$ be the admissible perturbation operator associated with operator T, that is, $T_G x = G(x, Tx)$, for all $x \in K$. First, we show that T_G is nonexpansive. Indeed, since G is affine Lipschitzian admissible, then there exists a constant $\lambda \in]0,1[$ such that

$$\|G(x, Tx) - G(y, Ty)\| \leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|,$$

for all $x, y \in K$. From the facts above, we can write

$$\begin{aligned}\|T_G x - T_G y\| &= \|G(x, Tx) - G(y, Ty)\| \\ &\leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\| \\ &\leq (1-\lambda)\|x-y\| + \lambda\|Tx - Ty\|.\end{aligned}$$

Since T is a nonexpansive operator, it follows immediately that

$$\|T_G x - T_G y\| \leq \|x - y\|,$$

for all $x, y \in K$, and hence the operator T_G is nonexpansive.

Now, as a direct consequence of Theorem 2.2, we observe that T has at least one fixed point. Recall that $Fix(T) = Fix(T_G)$.

Next, let $x^* \in Fix(T)$. We prove that $\lim_{n\to+\infty} \|x_n - Tx_n\| = 0$.

Then, since G is affine Lipschitzian admissible, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 = \\ &= \|G(x_n, Tx_n) - G(x^*, Tx^*)\|^2 \leq \|(1-\lambda)(x_n - x^*) + \lambda(Tx_n - x^*)\|^2 \\ &= (1-\lambda)^2\|x_n - x^*\|^2 + 2\lambda(1-\lambda)\langle Tx_n - x^*, x_n - x^*\rangle + \lambda^2\|Tx_n - x^*\|^2.\end{aligned}$$

Now, consider $\alpha \geq 0$ with $\alpha^2 = \lambda(1-\lambda)$ and write

$$\alpha(x_n - Tx_n) = \alpha(x_n - x^*) - \alpha(Tx_n - x^*)$$

along with

$$\alpha^2\|x_n - Tx_n\|^2 = \alpha^2\|x_n - x^*\|^2 - 2\alpha^2\langle Tx_n - x^*, x_n - x^*\rangle + \alpha^2\|Tx_n - x^*\|^2.$$

By routine calculations, since T is nonexpansive and $x^* = Tx^*$, we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 + \alpha^2\|x_n - Tx_n\|^2 \leq \\ &\leq (1-\lambda)^2\|x_n - x^*\|^2 + 2\lambda(1-\lambda)\langle Tx_n - x^*, x_n - x^*\rangle + \lambda^2\|Tx_n - x^*\|^2 + \\ &\quad + \alpha^2\|x_n - x^*\|^2 - 2\alpha^2\langle Tx_n - x^*, x_n - x^*\rangle + \alpha^2\|Tx_n - x^*\|^2 \\ &\leq [(1-\lambda)^2 + \lambda^2 + 2\alpha^2]\|x_n - x^*\|^2 + [2\lambda(1-\lambda) - 2\alpha^2]\langle Tx_n - x^*, x_n - x^*\rangle, \end{aligned}$$

which leads to

$$\|x_{n+1} - x^*\|^2 + \alpha^2\|x_n - Tx_n\|^2 \quad \leq \quad \|x_n - x^*\|^2,$$

or, equivalently,

$$\alpha^2\|x_n - Tx_n\|^2 \quad \leq \quad \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

By summing up the above inequality for $n = 0$ to $n = N$, we get

$$\begin{aligned} \lambda(1-\lambda)\sum_{n=0}^{N}\|x_n - Tx_n\|^2 &\leq \sum_{n=0}^{N}[\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2] = \\ &= \|x_0 - x^*\|^2 - \|x_{N+1} - x^*\|^2 \leq \|x_0 - x^*\|^2 < +\infty. \end{aligned}$$

It follows that

$$\lim_{n\to+\infty}\|x_n - Tx_n\| = 0. \tag{7}$$

Since T is demi-compact, we deduce that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to+\infty} x_{n_i} = z$. Again, since T is nonexpansive we have $\lim_{i\to+\infty} Tx_{n_i} = Tz$ and by (7) we get $Tz = z$. Finally, since T_G is nonexpansive, the convergence of $\{x_n\}$ is a consequence of the fact that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \in \mathbb{N} \cup \{0\}$. □

Remark 3.5. Considering the above proof, we note that the conclusions of Theorem 3.4 remain true by assuming only that T is demi-compact at zero. Precisely, an operator $T : K \to K$ is demi-compact at zero if, whenever $\{x_n\}$ is a bounded sequence in K and $\lim_{n\to+\infty}(Tx_n - x_n) = 0$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is strongly convergent to some $x \in K$ such that $x = Tx$.

Example 3.6. Let $H = \mathbb{R}$, $K = [-\pi/2, \pi/2]$, $G : K \times K \to K$ as in Example 2.6, $T : K \to K$ be given by $Tx = \cos x$ and consider the starting point $x_0 = -0.6$. Clearly, $G\colon K \times K \to K$ is affine Lipschitzian admissible, $T\colon K \to K$ is nonexpansive and demi-compact. In Figure 2 we give eight iterations of the GK-iterative scheme $\{x_n\}$ with starting point $x_0 = -0.6$ and different values of λ.

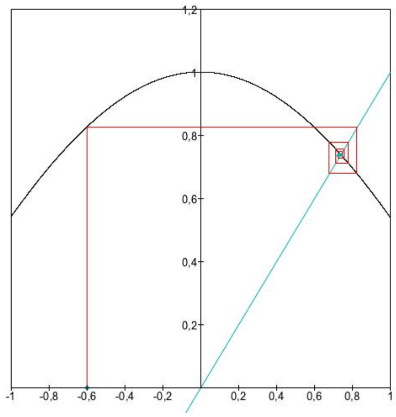

Figure 1: $x_0 = -0.60000$, $x^* = 0.73908$, *toll* $= 0.00001$.

I	$\lambda = 1/2$	$\lambda = 3/4$	$\lambda = 1$
0	-0.60000	-0.60000	-0.60000
1	0.11267	0.46900	0.82534
2	0.55316	0.78627	0.67831
3	0.70202	0.72644	0.77863
4	0.73278	0.74227	0.71187
5	0.73805	0.73827	0.75714
6	0.73891	0.73929	0.72680
7	0.73906	0.73903	0.74730
8	0.73908	0.73910	0.73353

Figure 2: Comparison of different values of λ.

Remark 3.7. There are various classes of operators satisfying the hypothesis of demi-compactness in the literature; here we recall some of the most used ones (see also Opial [16] and Petryshyn [19]):

(i) T is compact;

(ii) T has closed range and its inverse T^{-1} exists and is continuous;

(iii) T can be represented as $T = \widehat{T} + \widetilde{T}$, where $\widehat{T}$ is compact and $\widetilde{T}$ is such that $(I - \widetilde{T})^{-1}$ exists and is continuous on the domain of values $range(I - \widetilde{T})$.

As an immediate consequence of Theorem 3.4, we have the following result of existence of solution for Problem 1.1.

Theorem 3.8. *Let K be a nonempty bounded closed and convex subset of a real Hilbert space H, $S\colon H \to H$ a nonlinear operator and $G\colon K \times K \to K$ an affine Lipschitzian admissible operator with constant $\lambda \in]0, 1[$. Assume that*

$$\|x - y - \rho(Sx - Sy)\| \leq \|x - y\|, \text{ for all } x, y \in K, \text{ with } \rho > 0. \tag{8}$$

Then Problem 1.1 has a solution in K and the GK-iterative scheme $\{u_n\}$, given by $u_0 \in K$ and

$$u_{n+1} = G(u_n, P_K(u_n - \rho S u_n)), \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \text{ with } \rho > 0,$$

converges to a point $u \in VI(K, S)$, provided that $P_K(I - \rho S)$ is demi-compact.

Proof. Define $T\colon K \to K$ by $Tx = P_K(x - \rho Sx)$ so that, by Lemma 3.2, $u \in K$ is a solution of Problem 1.1 if and only if $u = Tu$. We show that T is a nonexpansive operator.

Since P_K is a nonexpansive operator, it follows that

$$\|Tx - Ty\| = \|P_K(x - \rho Sx) - P_K(y - \rho Sy)\| \leq \|x - y - \rho(Sx - Sy)\|. \quad (9)$$

Thus, by (8) and (9), we obtain immediately that

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in K$, and hence the operator T is nonexpansive. Thus, the conclusions of Theorem 3.8 hold true as a direct consequence of Theorem 3.4. $\square$

Remark 3.9. Let $S = k\rho^{-1}(T + I)$, where $T\colon K \to K$ is a nonexpansive operator, $k \in]0, 1[$ with $\rho > 0$. Then, for all $x, y \in K$, we have

$$\begin{aligned} \|x - y - \rho(Sx - Sy)\| &= \|x - y - \rho k\rho^{-1}(Tx + x - Ty - y)\| \\ &= \|(1 - k)(x - y) - k(Tx - Ty)\| \\ &\leq (1 - k)\|x - y\| + k\|Tx - Ty\| \\ &\leq \|x - y\|. \end{aligned}$$

So, condition (8) holds true.

Remark 3.10. If $P_K(I - \rho S)$ is a linear bounded operator with $\|P_K(I - \rho S)\| < 1$, then $(I - P_K(I - \rho S))^{-1}$ exists and is bounded. So $P_K(I - \rho S)$ satisfies (iii) of Remark 3.7 and hence is demi-compact. $\square$

By removing the hypothesis of demi-compactness in Theorem 3.4, we can obtain a weak convergence result. Precisely, we have the following theorem (see also Theorem 3.4 of [4]).

Theorem 3.11. *Let K be a nonempty bounded closed and convex subset of a real Hilbert space H, $T\colon K \to K$ a nonexpansive operator and $G\colon K \times K \to K$ an affine Lipschitzian admissible operator with constant $\lambda \in]0, 1[$. Then T has a fixed point in K and the GK-iterative scheme $\{x_n\}$, given by $x_0 \in K$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

converges weakly to a fixed point of T.

As a consequence of Theorem 3.11, we state the following result of existence of solution for Problem 1.1.

Theorem 3.12. *Let K be a nonempty bounded closed and convex subset of a real Hilbert space H, $S\colon H \to H$ a nonlinear operator and $G\colon K \times K \to K$ an affine Lipschitzian admissible operator with constant $\lambda \in]0, 1[$. Assume that*

(8) *holds true. Then Problem* 1.1 *has a solution in* K *and the* GK*-iterative scheme* $\{u_n\}$*, given by* $u_0 \in K$ *and*

$$u_{n+1} = G(u_n, P_K(u_n - \rho S u_n)), \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \text{ with } \rho > 0,$$

converges weakly to a point $u \in VI(K, S)$.

4. α-co-coercive operator and application to equilibrium problem

The same conclusions of Theorem 3.8 hold true if we assume that S is α-co-coercive and $\rho \in [0, 2\alpha]$ instead of condition (8). Indeed, under the above conditions, Iiduka proved that the operator $P_K(I - \rho S)$ is nonexpansive, see [12]. We can state the following result.

Theorem 4.1. *Let* K *be a nonempty bounded closed and convex subset of a real Hilbert space* H*,* $S\colon H \to H$ *an* α*-co-coercive operator and* $G\colon K \times K \to K$ *an affine Lipschitzian admissible operator with constant* $\lambda \in]0, 1[$*. Then Problem* 1.1 *has a solution in* K *and the* GK*-iterative scheme* $\{u_n\}$*, given by* $u_0 \in K$ *and*

$$u_{n+1} = G(u_n, P_K(u_n - \rho S u_n)), \quad \forall\, n \in \mathbb{N} \cup \{0\}, \text{ with } \rho > 0,$$

converges to a point $u \in VI(K, S)$*, provided that* $P_K(I - \rho S)$*, with* $\rho \in [0, 2\alpha]$*, is demi-compact.*

An interesting question, related to data dependence and stability analysis of the solution set of a (VIP), is to establish what is the effect of a perturbation of the operator $S\colon H \to H$ on the solution $u \in VI(K, S)$. We have the following theorem.

Theorem 4.2. *Let* $u \in VI(K, S)$ *and* $u^* \in VI(K, S^*)$ *with* $S^* u^* \neq S u^*$*, where* $S^*\colon H \to H$ *denotes another operator which can be seen as a perturbation of* S*. If* S *is an* α*-co-coercive operator, then*

$$\|u - u^*\| \geq \alpha \frac{\|Su - Su^*\|^2}{\|S^* u^* - Su^*\|}.$$

Proof. Let $u \in VI(K, S)$ and $u^* \in VI(K, S^*)$ so that the following inequalities hold true for all $v \in K$:

$$\langle Su, v - u \rangle \geq 0 \quad \text{and} \quad \langle S^* u^*, v - u^* \rangle \geq 0.$$

In particular, we have

$$\langle Su, u^* - u \rangle \geq 0 \quad \text{and} \quad \langle S^* u^*, u - u^* \rangle \geq 0.$$

It follows easily that

$$\langle S^* u^* - Su, u^* - u \rangle \leq 0,$$

or equivalently,

$$\langle S^* u^* - Su + Su^* - Su^*, u^* - u \rangle \leq 0,$$

which leads to

$$\langle S^*u^* - Su^*, u - u^* \rangle \geq \langle Su - Su^*, u - u^* \rangle.$$

Now, by using the Cauchy-Schwarz inequality and the α-co-coercivity of S, we obtain

$$\|S^*u^* - Su^*\| \|u - u^*\| \geq \alpha \|Su - Su^*\|^2$$

and hence we get

$$\|u - u^*\| \geq \alpha \frac{\|Su - Su^*\|^2}{\|S^*u^* - Su^*\|}. \qquad \square$$

We conclude with a typical application of the above results, as it appears in the existing literature, see [11, 14]. More precisely, we formalize a practical situation of support and market regulamentation.

We consider a commodity exploited from natural resources, a number of agents and the government. Clearly, the goals of agents and government are different and hence we deal with a bi-level problem. The agents have to maximize a profit function, by purchasing the commodity from the resource markets S_i, with $i = 1, \dots, m$, and selling it at the demand markets D_j, with $j = 1, \dots, n$. This problem is identified as the lower-level problem. On the other hand, the government has the primary purpose to ensure the stability of economic and social development, by preventing the natural resource from over-exploitation and supplying to the demand markets. The adopted policy of the government is a result of price-based instruments: tax exemptions and subsidies. This problem is identified as the upper-level problem. For the sake of completeness, we give the formulations of both the problems, but demonstrate the α-co-coercivity of the upper-level problem.

Therefore, we have:

- $x \in \mathbb{R}^{mn}_+$, where x_{ij} is the non-negative commodity shipment from S_i to D_j,
- $s \in \mathbb{R}^m_+$, where s_i is the amount of exploitation at the resource market S_i, $s_i = \sum_{j=1}^n x_{ij}$,
- $d \in \mathbb{R}^n_+$, where d_j is the amount of supply at the demand market D_j, $d_j = \sum_{i=1}^m x_{ij}$,
- $g \in \mathbb{R}^m_+$, where g_i is the purchase-price of the commodity at the resource market S_i, say $g_i = g_i(s_i)$,
- $h \in \mathbb{R}^n_+$, where h_j is the sale-price of the commodity at the demand market D_j, say $h_j = h_j(d_j)$,
- $t \in \mathbb{R}^{mn}_+$, where t_{ij} is the transaction cost (transportation is also included) between S_i and D_j, say $t_{ij} = t_{ij}(x_{ij})$,
- $y \in \mathbb{R}^m_+$, where y_i is the resource tax levied at the resource market S_i for preventing over-exploitation,

– $z \in \mathbb{R}^n_+$, where z_j is the subsidy granted at the demand market D_j for guaranteeing necessary supply.

Since most of the transaction cost is due to transportation, it is not restrictive to assume that the cost cannot be decreased by increasing the quantity of shipments. For this assertion, the road congestion plays a crucial role. On the other hand, it is well-known that the purchase-price moves up by increasing demand and the sale-price goes down by increasing supply. Consequently, in the sequel we assume that:

(i) The transaction cost t_{ij} is a non-decreasing function with respect to the shipment x_{ij}, $i = 1, \dots, m$ and $j = 1, \dots, n$,

(ii) g_i is strongly increasing with respect to s_i, $i = 1, \dots, m$,

(iii) h_j is strongly decreasing with respect to d_j, $j = 1, \dots, n$.

Next, for a given policy of the government, say $u = (y, z)$, the agents have to maximize their profit by satisfying the equilibrium:

$$g_i + y_i + t_{ij} \begin{cases} \geq h_j + z_j, & \text{if } x_{ij} = 0, \\ = h_j + z_j, & \text{if } x_{ij} > 0, \end{cases}$$

where $i = 1, \dots, m$ and $j = 1, \dots, n$.

Then, by using classical arguments in the theory of spatial price equilibrium problem, see also [21], the lower-level problem can be formalized in the form of a (VIP) as follows:

Problem 4.3. *Find $x \in \mathbb{R}^{mn}_+$ such that*

$$\langle t(x) + M^T[g(Mx) + y] - N^T[h(Nx) + z], v - x\rangle \geq 0$$

for all $v \in \mathbb{R}^{mn}_+$, where $M \in \mathbb{R}^{m \times nm}_+$ is the diagonal matrix with all entries equal to $e_n \in \mathbb{R}^n_+$, $N \in \mathbb{R}^{n \times nm}_+$ is the vector with all entries equal to the identity matrix $I_n \in \mathbb{R}^{n \times n}_+$ and $e_n \in \mathbb{R}^n_+$ is the vector with all entries equal to one.

Finally, we consider the upper-level problem to ensure the stability of economic and social development.

Let $s^{\max} \in \mathbb{R}^m_+$ be the maximal exploitation of the resource markets and $d^{\min} \in \mathbb{R}^n_+$ be the minimal supply on the demand markets, with $\sum_{i=1}^m s_i^{\max} \geq \sum_{j=1}^n d_j^{\min}$. The goal of the government is to find an optimal tax vector $y^* \in \mathbb{R}^m_+$ and subsidy vector $z^* \in \mathbb{R}^n_+$ such that $u^* = (y^*, z^*)$ is a solution of

$$\begin{cases} y^* \geq 0, \quad s^{\max} - s^* \geq 0, \quad \langle s^{\max} - s^*, y^*\rangle = 0, \\ z^* \geq 0, \quad d^* - d^{\min} \geq 0, \quad \langle d^* - d^{\min}, z^*\rangle = 0, \end{cases}$$

or equivalently, $$u \geq 0, \quad F(u) \geq 0, \quad \langle F(u), u\rangle = 0, \tag{10}$$

where $u = (y, z)$ and $F(u) = (s^{\max} - s(u), d(u) - d^{\min})$.

The operator $F\colon \mathbb{R}_+^{m+n} \to \mathbb{R}_+^{m+n}$ is not explicitly known, but its values $F(u)$ are available by observation for given u.

Remark 4.4. The system of inequalities and equalities (10) is properly called a *complementarity problem.* The variational inequality problem contains the complementarity problem as a special case. □

We show that Theorems 4.1 and 4.2 are applicable to the restriction of the operator F to a nonempty compact convex subset $\mathbb{K}$ of $\mathbb{R}_+^{m+n}$. Indeed, we have the following result.

Theorem 4.5. *Let $F\colon \mathbb{R}_+^{m+n} \to \mathbb{R}_+^{m+n}$ be the operator defined in* (10)*. If the assumptions* (i)-(iii) *hold true, then F is α-co-coercive, that is*

$$\langle F(u) - F(\tilde{u}), u - \tilde{u}\rangle \geq \alpha \|F(u) - F(\tilde{u})\|^2 \quad \textit{for all } u, \tilde{u} \in \mathbb{R}_+^{m+n}. \tag{11}$$

Proof. By routine considerations and using (i)-(iii), there exists a constant $\alpha > 0$ such that

$$\langle t(x) - t(\tilde{x}), x - \tilde{x}\rangle \geq 0 \quad \text{for all } x, \tilde{x} \in \mathbb{R}_+^{mn}, \tag{12}$$

$$\langle g(s) - g(\tilde{s}), s - \tilde{s}\rangle \geq \alpha \|s - \tilde{s}\|^2 \quad \text{for all } s, \tilde{s} \in \mathbb{R}_+^{m}, \tag{13}$$

$$\langle h(\tilde{d}) - h(d), d - \tilde{d}\rangle \geq \alpha \|d - \tilde{d}\|^2 \quad \text{for all } d, \tilde{d} \in \mathbb{R}_+^{n}. \tag{14}$$

Let $\overline{u} = (\overline{y}, \overline{z}) \in VI(\mathbb{R}_+^{m+n}, F)$ and $\tilde{u} = (\tilde{y}, \tilde{z}) \in VI(\mathbb{R}_+^{m+n}, F)$. Denote by $\overline{x}$ and $\tilde{x}$ the solutions of Problem 4.3 corresponding to $\overline{u}$ and $\tilde{u}$, respectively. By putting $y = \overline{y}$, $z = \overline{z}$ and $v = \tilde{x}$ in the variational inequality of Problem 4.3, we have

$$\langle t(\overline{x}) + M^T[g(M\overline{x}) + \overline{y}] - N^T[h(N\overline{x}) + \overline{z}], \tilde{x} - \overline{x}\rangle \geq 0. \tag{15}$$

Analogously, by putting $y = \tilde{y}$, $z = \tilde{z}$ and $v = \overline{x}$, we have

$$\langle t(\tilde{x}) + M^T[g(M\tilde{x}) + \tilde{y}] - N^T[h(N\tilde{x}) + \tilde{z}], \overline{x} - \tilde{x}\rangle \geq 0. \tag{16}$$

By combining (15) and (16), since $\overline{s} = M\overline{x}$, $\overline{d} = N\overline{x}$, $\tilde{s} = M\tilde{x}$ and $\tilde{d} = N\tilde{x}$, we get

$$(\overline{y} - \tilde{y})^T(\tilde{s} - \overline{s}) + (\overline{z} - \tilde{z})^T(\overline{d} - \tilde{d}) \geq$$
$$\geq (\overline{x} - \tilde{x})^T(t(\overline{x}) - t(\tilde{x})) + (\overline{s} - \tilde{s})^T(g(\overline{s}) - g(\tilde{s})) - (\overline{d} - \tilde{d})^T(h(\overline{d}) - h(\tilde{d})).$$

Therefore, by using (12)–(14), we obtain

$$(\overline{y} - \tilde{y})^T(\tilde{s} - \overline{s}) + (\overline{z} - \tilde{z})^T(\overline{d} - \tilde{d}) \geq \alpha[\|s - \tilde{s}\|^2 + \|d - \tilde{d}\|^2],$$

which leads to (11), since $u = (y, z)$ and $F(\overline{u}) - F(\tilde{u}) = (\tilde{s} - \overline{s}, \overline{d} - \tilde{d})$. □

In view of Theorem 4.5, we state the following result without proof.

Theorem 4.6. *Let* $\mathbb{K}$ *be a nonempty compact convex subset of* $\mathbb{R}_+^{m+n}$*, the operator* $F\colon \mathbb{R}_+^{m+n} \to \mathbb{R}_+^{m+n}$ *as defined in* (10) *and* $G\colon \mathbb{K}\times\mathbb{K}\to\mathbb{K}$ *an affine Lipschitzian admissible operator with constant* $\lambda \in]0,1[$*. Then the problem of finding* $u\in\mathbb{K}$ *such that* $\langle Fu, v-u\rangle \geq 0$ *for all* $v\in\mathbb{K}$*, has a solution in* $\mathbb{K}$ *and the* $G\mathbb{K}$*-iterative scheme* $\{u_n\}$*, given by* $u_0\in\mathbb{K}$ *and*

$$u_{n+1} = G(u_n, P_K(u_n - \rho F u_n)), \quad \forall\, n \in \mathbb{N}\cup\{0\}, \ \textit{with } \rho > 0,$$

converges to a point $u \in VI(\mathbb{K}, F)$*, provided that* $P_K(I-\rho F)$*, with* $\rho\in[0,2\alpha]$*, is demi-compact.*

5. Conclusion

We proposed an implicit approach, based on admissible perturbations of operators, for establishing the convergence of fixed point iterative schemes in real Hilbert spaces. By using the projection operator, we apply our technique to the solution of a general variational inequality problem. Based on the equivalence in [9], one can easily reformulate our results for an initial value problem of projected differential equation (P). Clearly, analogous results may be obtained for many other practical problems associated to variational inequalities; this is a direction for further research.

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