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Inverse Problem in Convex Optimization with Linear Homogeneous Constraints

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We characterize solutions of maximization problems under linear constraints that are positively homogeneous of degree one. Necessary and sufficient conditions are given for functions to arise as solutions of such kind of problems. At the same time, we consider the integration problem of homogeneous differential forms in order to solve the inverse problem.

Keywords: Inverse problem, multi-constraint maximization, value function, homogeneous differential forms, integrability.

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1. Introduction

Inverse problems have very important applications in different fields such as economics, physics, medical sciences,...etc. In this article, we treat an inverse problem arising in convex optimization. Such problems originate, in particular, in microeconomic theory of consumer behavior. This kind of problems are known as characterization problems of consumer's demand functions.

In [1], the author considered the multi-constraint maximization problem

$$\max \quad f(x) \quad \text{subject to} \quad Ax \leq C(A)$$

where $x \in \mathbb{R}^n$, A is an $m \times n$ matrix of rank m and $C(A) = (C^1(a^1), ..., C^m(a^m))$ is a given mapping, $C^i(a^i) : \mathbb{R}^n \to \mathbb{R}$ and a^i is the ith row of the matrix A. Hence, the ith constraint takes the form $a^i x = C^i(a^i)$. Necessary and sufficient conditions were given in [1] for a given function to be a solution of this kind of problems. This means that, the inverse problem is solved and solutions of this problem were characterized.

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We consider the same problem in this paper, but we assume that each component of the mapping $C(A)$ is homogeneous of degree one (henceforth, 1-homogeneous). The homogeneity assumption is important as it implies that the solution of this optimization problem $x(A)$ is 0-homogeneous in a^i for all $i = 1, ..., m$, whereas the Lagrange multiplier of the ith constraint $\lambda_i(A)$ is -1-homogeneous in a^i and 0-homogeneous in a^k, for all $k \neq i$.

The homogeneous case requires a different approach because the method used in [1] does not apply directly to this case. Therefore, we characterize solutions of the above problem when the "linear" constraints are 1-homogeneous. As a by product, we treat the integrability problem of homogeneous differential forms. Moreover, we use Lie derivative to characterize homogeneous differential forms. This result will be used to solve the integrability problem.

This family of inverse problems in convex optimization originates from consumer theory in microeconomics. The first result goes back to Slutsky [11] who gave conditions that characterize solutions of the individual's problem $\max u(x)$ subject to $p^t x \leq I$, where u is the utility function of the individual, $p \in \mathbb{R}^n$ is the price vector and $I > 0$ is the income. The results of Slutsky were then generalized to the multi-constraint case, $\max u(x)$ subject to $Ax \leq Y$, see [3]. Another line of research in this domain started by considering, again, a single constraint problem whose constraint takes the form $p^t x \leq w(p)$, see [2], and then generalized to the multi-constraint case $Ax \leq C(A)$, see [1], where the function $C(A)$ is not 1-homogeneous.

Our approach for the solution depends on the value function, $V(A)$, of the above problem which is defined as $V(A) = \max_x\{f(x)|Ax \leq C(A)\}$. This functions has important homogeneity and convexity properties that will be given below. The inverse problem is mainly an integration one; that is, we need to find the necessary and sufficient conditions for the existence of $m + 1$ functions $V, \lambda_1, ..., \lambda_m$ such that

$$\frac{\partial V}{\partial a_j^i} = \lambda_i \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) \tag{1}$$

This important expression, obtained using the envelope theorem, relates the value function to the solution and the Lagrange multipliers of the optimization problem. As we shall see, we need to consider a family of functions $\lambda_{ik} = \lambda_i/\lambda_k$ that will constitute an important part of the necessary and sufficient conditions for integration.

In addition to (1), the functions V, $\lambda_1, ..., \lambda_m$ should satisfy additional convexity, homogeneity and positivity conditions. Consequently, the integration problem will be split into mathematical integration problem and economic integration problem.

- **Mathematical integration.** Given a function $x(A)$ and a family of functions λ_{ik}, $1 \leq i, k \leq m$, what are the necessary and sufficient conditions for the existence of $m+1$ functions $\lambda_1, ..., \lambda_m$ and V that satisfy (1). Moreover, the function $V(A)$ should be 0-homogeneous in a^i, λ_i should be (-1)-homogeneous in a^i and 0-homogeneous in a^k for all $k \neq i$, $i = 1, ..., m$.

- **Economic integration.** In addition to the mathematical integration, we impose the following additional conditions on the functions that satisfy (1); the functions λ_i are strictly positive and the function V is quasi-convex with respect to each a^i for all $i = 1, ..., m$.

Throughout this paper, we employ exterior differential techniques to solve the inverse problem. A good reference to these techniques is the book by Bryant et al. [5], see also the book of Chiappori and Ekeland [7] which includes several applications of exterior differential calculus techniques and a user's guide for these techniques. We get local results; that is, the functions involved in the integration problem are defined in a neighbourhood of some given point. We define a family of differential forms and set up an integration problem using these forms. The solution of this integration problem, then, requires solving a nonlinear system of partial differential equations.

The rest of the article will be organized in the following way. In the next section we review the definition and basic properties of Lie derivative and we consider the integrability of homogeneous differential forms. Then, the model is introduced with some basic results. After that, necessary and sufficient conditions for mathematical integration are given. After mathematical integration, the economic integration problem is solved. Then, the results are applied to the single constraint case and duality is discussed. Finally, as an application to our results, an example from the economic theory of individual excess demands is given.

2. Lie derivative and integration of homogeneous forms

In this section we consider an n-dimensional manifold $\mathcal{M}$ with coordinate system $(x^1, ..., x^n)$ and $X \in T\mathcal{M}$ is a vector field belonging to the tangent bundle, $T\mathcal{M}$, of $\mathcal{M}$.

Definition 2.1. Let D be a cone in $\mathbb{R}^n$, $f\colon D \to \mathbb{R}$ be a function defined on D. Then, f is said to be *homogenous of degree* $k \in \mathbb{R}$, if for any real number $t > 0$, $f(tx^1, ..., tx^n) = t^k f(x^1, ..., x^n)$, $\forall x \in D$. A differential p-form

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p}(x) dx^{i_1} \wedge ... \wedge dx^{i_p}$$

is called k-homogeneous if the functions $\omega_{i_1 \cdots i_p}(x)$ are k-homogeneous.

Homogeneous functions are characterized by Euler's formula which states that a C^1 function f is k-homogeneous on a cone $D \subset \mathbb{R}^n$ if and only if $x^t \nabla f(x) = kf(x)$ for all $x \in D$.

Definition 2.2. Let $\mathcal{M}$ be a manifold, $X \in T\mathcal{M}$ be a vector field on $\mathcal{M}$ and $\Lambda^k(\mathcal{M})$ be the set of k-forms on $\mathcal{M}$. The *interior product* $\iota_X \colon \Lambda^k(\mathcal{M}) \to \Lambda^{k-1}(\mathcal{M})$ is defined by

$$(\iota_X\omega)(Y_1, ..., Y_{k-1}) = \omega(X, Y_1, ..., Y_{k-1})$$

The differential form $\iota_X\omega$ is also called the *contraction* of $\omega \in \Lambda^k(\mathcal{M})$ with X.

The interior product satisfies $\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^k\omega \wedge \iota_X\eta$ for any $\omega \in \Lambda^k(\mathcal{M})$ and $\eta \in \Lambda^l(\mathcal{M})$.

Now, we recall the definition of Lie derivative and some properties of it. We adopt the notations used by Olver, [10].

Definition 2.3. Let X be a vector field on a manifold $\mathcal{M}$ and ω be a vector field or a differential form defined on $\mathcal{M}$. The *Lie derivative* of ω with respect to X is the object whose value at $x \in \mathcal{M}$ is

$$\mathcal{L}_X\omega = \lim_{t\to 0} \frac{\phi_t^*(\omega\mid_{\phi_t(x)}) - \omega\mid_x}{t} = \left.\frac{d}{dt}\right|_{t=0} \phi_t^*(\omega\mid_{\phi_t(x)}),$$

where $\phi_t(x)$ is the flow of the vector field X and ϕ^* refers to the pull-back of ϕ.

The Lie derivative of an ordinary function f is just the contraction of the exterior derivative of f with the vector field X; that is, $\mathcal{L}_X f = \iota_X df$.

Lemma 2.4. *Let ω be a differential form of degree k and X be any vector field, then the Lie derivative has the following properties:*

(1) *$\mathcal{L}_X\omega$ is of the same degree as ω,*
(2) *$d(\mathcal{L}_X\omega) = \mathcal{L}_X d\omega$,*
(3) *$\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$,*
(4) *$\mathcal{L}_X(\omega \wedge \theta) = \mathcal{L}_X\omega \wedge \theta + \omega \wedge \mathcal{L}_X\theta$,*
(5) *$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega$,*

where ι is the interior product between ω and X and d is the exterior derivative.

For the rest of this section, we denote by $X \in T\mathcal{M}$ the vector field defined by

$$X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} \tag{2}$$

where $\mathcal{M}$ is a manifold and $(x^1, ..., x^n)$ is a coordinate system of $\mathcal{M}$. The next theorem gives a characterization of homogeneous differential forms using Lie derivative.

Theorem 2.5. *A differential p-form $\omega(x)$ is k-homogeneous if and only if $\mathcal{L}_X\omega = (k + p)\omega$.*

The proof follows using identity (3) of Lemma 2.4 and simple calculation. Before proceeding, we give the following theorem, known as Darboux theorem, that will be used repeatedly

Theorem 2.6. *Let $\omega(x)$ be a given differential 1-form. If $\omega \wedge d\omega = 0$ in the neighbourhood of some point $\bar{x}$ then there exists two functions $f(x)$ and $g(x)$, defined locally in a neighbourhood of $\bar{x}$ such that $\omega = fdg$. Conversely, if $\omega = fdg$ then $\omega \wedge d\omega = 0$.*

2.1. Integrability of homogeneous differential forms

We discuss now the integration of homogeneous differential forms. Given a differential form ω, when can it be written as $\omega = fdg$ in the neighbourhood of some point $\bar{x} \in \mathcal{M}$ where f and g are homogeneous functions of certain degrees. Let us start by the following lemma

Lemma 2.7. *If ω is a C^1, k-homogeneous 1-form such that $\iota_X\omega = 0$ then*

$$\iota_X d\omega = (k+1)\omega$$

Proof. k-homogeneity implies that $\mathcal{L}_X d\omega = (k+1)\omega$. On the other hand $(k+1)\,\omega = \mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$. But $\iota_X\omega = 0$, so $\iota_X d\omega = (k+1)\,\omega$. □

Building on this lemma we prove the next result:

Theorem 2.8. *Let ω be a C^1 differential 1-form such that $\iota_X\omega = 0$. Then $\omega \wedge d\omega = 0$ with ω k-homogeneous if and only if there is a 1-form β such that $d\omega = \beta \wedge \omega$ and $\iota_X\beta = k+1$*

Proof. If $d\omega = \beta \wedge \omega$, then $\omega \wedge d\omega = 0$. We have, $\iota_X d\omega = (\iota_X\beta)\omega - \beta(\iota_X\omega) = (k+1)\,\omega$. But $\mathcal{L}_X d\omega = \iota_X d\omega + d(\iota_X\omega) = (k+1)\,\omega$. This proves the k-homogeneity of ω. Conversely, if $\omega \wedge d\omega = 0$, then there exists a 1-form β such that $d\omega = \beta \wedge \omega$. Hence, $\iota_X d\omega = (\iota_X\beta)\omega - \beta(\iota_X\omega)$ by using Lemma 2.7, $\iota_X d\omega = (\iota_X\beta)\omega = (k+1)\,\omega$. Then $\iota_X\beta = k+1$. □

The next theorem answers the question of homogeneous decomposition of homogeneous differential form $\omega = fdg$

Theorem 2.9. *Let ω be a C^1, k-homogeneous differential 1-form such that $\omega \wedge d\omega = 0$ in a neighbourhood $\mathcal{U}$ of some point $\bar{x}$. Then, there exists a $(k+1)$-homogeneous function f and a 0-homogeneous function g, defined in a possibly smaller neighbourhood $\mathcal{V} \subset \mathcal{U}$ such that $\omega(x) = f(x)dg(x)$.*

Proof. Suppose that $\omega \wedge d\omega = 0$. Then, there exist two functions f and g such that $\omega = fdg$. Since $\iota_X\omega = 0$, then $\iota_X dg = 0$; that is, g is 0-homogeneous. We have also, $d\omega = df \wedge dg$, and $dg = \omega/f$. It follows that $d\omega = \frac{df}{f} \wedge \omega$. Apply the vector field X to both sides of the previous equation and use Lemma 2.7

to get $(k+1)\,\omega = \iota_X \frac{df}{f}\omega$. Thus, $\iota_X df = (k+1)f$, which proves that $f(x)$ is $(k+1)$-homogeneous. $\square$

3. Setting up The Model

We consider a multi-constraint maximization problem of the form

$$(\mathcal{P}) \qquad \max_x f(x) \;\text{ subject to }\; Ax = C(A),$$

where f is a function that satisfies certain regularity and convexity conditions given below, A is an $m \times n$ matrix of rank m and $C\colon \mathbb{R}^{mn} \to \mathbb{R}^m$ is a given mapping. The i^{th} constraint takes the form $a^i x = C^i(A)$ where a^i is the i^{th} row of the matrix A. Define the Lagrangian function

$$L(x,\lambda) = f(x) + \sum_{k=1}^{m} \lambda_k \Big(C^k(A) - \sum_{l=1}^{n} a^k_l x^l \Big)$$

with $x \in \mathbb{R}^n$, and $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_{++}$ is the vector of Lagrange multipliers. Define the value function of this problem by

$$V(A) = \max_x \left\{ f(x) + \sum_{k=1}^{m} \lambda_k \Big(C^k(A) - \sum_{l=1}^{n} a^k_l x^l \Big) \right\} \tag{3}$$

Differentiating the function $V(A)$ with respect to a^i_j and using the envelope theorem we get

$$\frac{\partial V}{\partial a^i_j} = \sum_{k=1}^{m} \lambda_k \frac{\partial C^k}{\partial a^i_j} - \lambda_i x^j. \tag{4}$$

We suppose that C^k is a function of the vector a^k only, where a^k is the k^{th} row of the matrix A. Moreover, we assume that each component of the mapping $C(A)$ is homogeneous of degree one. This implies, in particular, that the functions $x(A)$ and $V(A)$ are homogeneous of degree zero with respect to each a^i, $i = 1, ..., m$ and the Lagrange multiplier corresponding to the i^{th} constraint, $\lambda_i(A)$ is homogeneous of degree -1 with respect to a^i, and is homogeneous of degree 0 with respect to a^k for $k \neq i$. Throughout this paper, we adopt the following assumptions

Assumption 3.1. For each $i \in \{1, ..., m\}$, we assume that the function C^i has the following properties:

(a) $C^i\colon \mathbb{R}^n \to \mathbb{R}$ is a function of a^i only;
(b) C^i is a convex function of a^i;
(c) C^i is of class C^2;
(d) C^i is homogeneous of degree one in a^i; that is, $a^i D_{a^i} C^i - C^i(a^i) = 0$.

We consider the following assumptions on the objective function f.

Assumption 3.2. Assume the function f satisfies the following conditions:
(a) f is strictly increasing with respect to each coordinate of the vector x;
(b) the Hessian matrix $D_x^2 f$ is negative definite on the subspace $\{D_x f\}^\perp$;
(c) f is of class C^2.

Assumption 3.1(a) implies that $D_{a^i}C^k = 0$ if $i \neq k$, equation (4) becomes

$$\frac{\partial V}{\partial a_j^i} = \lambda_i\left(\frac{\partial C^i}{\partial a_j^i} - x^j\right). \tag{5}$$

Define a family of differential 1-forms $\omega^1, ..., \omega^m$ by

$$\omega^i = \sum_{j=1}^{n}\left(\frac{\partial C^i}{\partial a_j^i} - x^j\right) da_j^i. \tag{6}$$

It follows that the differential of V, can be written as:

$$dV = \sum_{i=1}^{m} \lambda_i \omega^i.$$

Notice that

$$d\omega^i = \sum_{j,l} \frac{\partial^2 C^i}{\partial a_l^i \partial a_j^i} da_l^i \wedge da_j^i - \sum_{j,k,l} \frac{\partial x^j}{\partial a_l^k} da_l^k \wedge da_j^i.$$

The coefficients in the first summation are symmetric, so we end up with

$$d\omega^i = -\sum_{j,k,l} \frac{\partial x^j}{\partial a_l^k} da_l^k \wedge da_j^i.$$

Differentiating both sides of the ith constraint with respect to a_j^i and rearranging, we get:

$$\frac{\partial C^i}{\partial a_j^i} - x^j = \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i. \tag{7}$$

Using this result, the 1-form ω^i can be written as

$$\omega^i = \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i da_j^i. \tag{8}$$

Now, our inverse problem can be stated as follows:

- We observe the function $x^j(A)$, $j = 1, ..., n$ from $\mathbb{R}^{mn}$ to $\mathbb{R}^n$.
- Then we define the functions $C^i(a^i) = a^i x(A)$.
- We observe also a family of positive functions λ_{ik} using symmetry conditions that will be given below.
- Our goal is to find a function $f(x)$, by first finding the value function $V(A)$, such that $x(A) \in \operatorname{argmax} \{f(x)|Ax = C(A)\}$ and $V(A) = f(x(A))$.

We restrict the domains of the indices as, $1 \leq i, k, k', s, t \leq m$ and $1 \leq j, j', l, l', r \leq n$. In what follows, δ^i_k denotes the Kronecker symbol which equals one if $i = k$ and zero otherwise.

3.1. Some preliminary results

In this section, we give some important preliminary results that will be used to solve the inverse problem. Define the family of function λ_{ik} by $\lambda_{ik} := \lambda_i/\lambda_k$, $1 \leq i < k \leq m$. These functions appear in the conditions for mathematical and economic integration. The following lemma, for which we omit the proof, is easily obtained

Lemma 3.3. *Let $V(A)$ be the value function defined in* (3) *and λ_i be the Lagrange multiplier corresponding to constraint i of problem* ($\mathcal{P}$)*. Then,*

(a) *$V(A)$ is 0-homogenous in a^i for all $i = 1, ..., m$;*
(b) *$V(A)$ is quasi-convex with respect to each a^i for all $i = 1, ..., m$;*
(c) *λ_i is -1-homogeneous in a^i and 0-homogeneous in a^k for all $k \neq i$.*

The next lemma follows from homogeneity properties of λ_i, $i = 1, ..., m$.

Lemma 3.4. *For any $i, k = 1, ..., m$, $1 \leq i \neq k \leq m$, the function λ_{ik} has the following homogeneity properties:*

(a) *homogeneous of degree -1 in a^i;*
(b) *homogeneous of degree 1 in a^k;*
(c) *homogeneous of degree 0 in $a^{k'}$, $k' \neq i, k$.*

That is,

$$\sum_l \frac{\partial \lambda_{ik}}{\partial a^{k'}_l} a^{k'}_l = \lambda_{ik}(\delta^{k'}_k - \delta^{k'}_i). \tag{9}$$

The implicit function theorem gives the following result:

Lemma 3.5. *If the function $f(x)$ satisfies Assumption* (3.2) *and $C(A)$ is of class C^2, then the maps $A \to x(A)$ and $A \to \lambda(A)$ are of class C^2.*

By differentiating the ith constraint and using 0-homogeneity of x with respect to a^i, we get the following

Lemma 3.6. *Let $x(A)$ be a solution of a multi-constraint maximization problem of the above type, then*

(1) $\sum\limits_{l=1}^{n} \frac{\partial x^l}{\partial a^i_j} a^k_l = 0 \quad \textit{if} \quad i \neq k.$

(2) $\sum\limits_{j=1}^{n} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) a^i_j = (a^i)^T (D_{a^i} x) a^i = 0.$

In the rest of this paper we state our main results. We solve the inverse problem and start by solving the mathematical integration problem in the next section.

4. Mathematical integration: necessary and sufficient conditions

In this section we solve the mathematical problem. We express necessary and sufficient conditions for mathematical integration as a set of nonlinear partial differential equations for the observable functions λ_{ik} and x. Define a family of 1-forms Ω_k, $k = 1, ..., m$, by

$$\Omega_k = \sum_{i=1}^{m} \lambda_{ik} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) da^i_j.$$

Then, Ω_k can be expressed as

$$\Omega_k = \sum_{s=1}^{m} \lambda_{sk} \omega^s, \tag{10}$$

where ω^s is the 1-form defined by (6) or the equivalent form (8). Equation (5) implies that

$$\frac{1}{\lambda_k} dV = \Omega_k. \tag{11}$$

The 1-forms $\Omega_1, ..., \Omega_m$ are defined using only observable functions and are collinear to the same gradient dV. One can easily show that $\Omega_i \wedge \Omega_k = 0$ for all $i, k = 1, ...m$ which proves that the 1-forms $\Omega_1, ..., \Omega_m$ are proportional. This proportionality implies that if $\Omega_i \wedge d\Omega_i = 0$ for some i then $\Omega_k \wedge d\Omega_k = 0$ for any $k \neq i$. Moreover, by virtue of Darboux theorem 2.6, $\Omega_k \wedge d\Omega_k = 0$ if and only if $\Omega_k = \mu_k dV$, see [1] for details. Notice that the equation $\Omega_k \wedge d\Omega_k = 0$ gives the necessary and sufficient conditions for mathematical integration. Our objective now is to write these conditions explicitly over the functions $\lambda_{ik}(A)$, $1 \leq i < k \leq m$, and $x(A)$. The following theorem gives the required explicit integrability conditions

Theorem 4.1. *Given the family of 1-forms $\Omega_1, ..., \Omega_m$. Then $\Omega_k \wedge d\Omega_k = 0$ if and only if*

(a) *there exists a set of $m \times n$ matrices $R_1, R_2, ..., R_m$ that satisfy the conditions $\sum_{l=1}^{n} R^l_{kk'}(A) a^{k'}_l = \delta^{k'}_k$ for all $k, k' = 1, ..., m$;*

(b) *for all* $1 \leq i, s \leq m$, $1 \leq j, l \leq n$ *the following conditions are satisfied:*

$$\frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} - R_{ks}^l(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i = \\ = \frac{\partial \lambda_{sk}}{\partial a_j^i} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s - \lambda_{sk} \frac{\partial x^l}{\partial a_j^i} - R_{ki}^j(A) \lambda_{sk} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s. \tag{12}$$

Proof. Recall that $\Omega_k \wedge d\Omega_k = 0$ if and only if there exists a 1-form α_k such that

$$d\Omega_k = \alpha_k \wedge \Omega_k. \tag{13}$$

The 1-form α_k can be identified (mod Ω_k). Notice that

$$\Omega_k = \sum_{i=1}^m \lambda_{ik} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^i da_j^i.$$

Define a family of vector fields by $\xi^i = \sum_{j=1}^n a_j^i \frac{\partial}{\partial a_j^i}$. Then,

$$\iota_{\xi^{k'}} \Omega_k = \lambda_{k'k} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a_j^{k'}} a_r^{k'} a_j^{k'} = 0.$$

To find a 1-form α_k that satisfies the equation $d\Omega_k = \alpha_k \wedge \Omega_k$, we apply both sides of this equation to the vector field $\xi^{k'}$. So we have

$$\iota_{\xi^{k'}} d\Omega_k = (\iota_{\xi^{k'}} \alpha_k) \Omega_k - \alpha_k (\iota_{\xi^{k'}} \Omega_k). \tag{14}$$

But $\iota_{\xi^{k'}} \Omega_k = 0$, $\forall k, k'$. Therefore, equality (14) becomes

$$\iota_{\xi^{k'}} d\Omega_k = (\iota_{\xi^{k'}} \alpha_k) \Omega_k. \tag{15}$$

The exterior derivative of Ω_k is

$$d\Omega_k = \sum_{i,j,s,l} \lambda_{ik} \left(\frac{\partial^2 C^i}{\partial a_l^s \partial a_j^i} - \frac{\partial x^j}{\partial a_l^s} \right) da_l^s \wedge da_j^i + \sum_{i,j,s,l} \frac{\partial \lambda_{ik}}{\partial a_l^s} \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) da_l^s \wedge da_j^i.$$

Using (7), we can write $d\Omega_k$ as

$$d\Omega_k = \sum_{i,j,s,l} \left(\frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} \right) da_l^s \wedge da_j^i \, .$$

Now, we apply the 2-form $d\Omega_k$ to the vector field $\xi^{k'}$, we find that

$$\begin{aligned}\iota_{\xi^{k'}} d\Omega_k = &\sum_{i,j,l}\left(\frac{\partial\lambda_{ik}}{\partial a_l^{k'}}\sum_r \frac{\partial x^r}{\partial a_j^i}a_r^i - \lambda_{ik}\frac{\partial x^j}{\partial a_l^{k'}}\right)a_l^{k'}\,da_j^i - \\ &- \sum_{j,s,l}\left(\frac{\partial\lambda_{k'k}}{\partial a_l^s}\sum_r \frac{\partial x^r}{\partial a_j^{k'}}a_r^{k'} - \lambda_{k'k}\frac{\partial x^j}{\partial a_l^s}\right)a_j^{k'}\,da_l^s \\ = &\sum_{i,j,l}\frac{\partial\lambda_{ik}}{\partial a_l^{k'}}a_l^{k'}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i da_j^i + \lambda_{k'k}\sum_{j,s,l}\frac{\partial x^j}{\partial a_l^s}a_j^{k'}\,da_l^s.\end{aligned}$$

Then, using equation (9), we have

$$\iota_{\xi^{k'}} d\Omega_k = \delta_k^{k'}\sum_i\lambda_{ik}\sum_{r,j}\frac{\partial x^r}{\partial a_j^i}a_r^i da_j^i + \lambda_{k'k}\left(\sum_{j,l}\frac{\partial x^j}{\partial a_l^{k'}}a_j^{k'}\,da_l^{k'} - \sum_{j,r}\frac{\partial x^r}{\partial a_j^{k'}}a_r^{k'}\,da_j^{k'}\right).$$

We end up with

$$\iota_{\xi^{k'}} d\Omega_k = \delta_k^{k'}\sum_i\lambda_{ik}\sum_{r,j}\frac{\partial x^r}{\partial a_j^i}a_r^i da_j^i = \delta_k^{k'}\Omega_k\,. \tag{16}$$

Comparing (15) and (16), we conclude that

$$\iota_{\xi^{k'}} d\Omega_k = \iota_{\xi^{k'}}\alpha_k\sum_i\lambda_{ik}\sum_{r,j}\frac{\partial x^r}{\partial a_j^i}a_r^i da_j^i = \delta_k^{k'}\sum_i\lambda_{ik}\sum_{r,j}\frac{\partial x^r}{\partial a_j^i}a_r^i da_j^i\,.$$

Consequently, the differential 1-form α_k must satisfy $\iota_{\xi^{k'}}\alpha_k = \delta_k^{k'}$. The 1-form α_k can be expressed in terms of the basis of 1-forms, da_j^i, $1 \le i \le m$, $1 \le j \le n$, on $\mathbb{R}^{mn}$ as

$$\alpha_k = \sum_{s,l} R_{ks}^l(A)da_l^s\,.$$

Using this expression of α_k, condition $\iota_{\xi^{k'}}\alpha_k = \delta_k^{k'}$ means that

$$\sum_{l=1}^n R_{kk'}^l(A)a_l^{k'} = \delta_k^{k'}\,.$$

Now the equation $d\Omega_k = \alpha_k \wedge \Omega_k$ can be expressed as

$$\sum_{i,j,s,l}\left(\frac{\partial\lambda_{ik}}{\partial a_l^s}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i - \lambda_{ik}\frac{\partial x^j}{\partial a_l^s}\right)da_l^s\wedge da_j^i = \sum_{i,j,s,l}\left(R_{ks}^l(A)\lambda_{ik}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i\right)da_l^s\wedge da_j^i$$

or equivalently as

$$\sum_{i,j,s,l}\left(\frac{\partial\lambda_{ik}}{\partial a_l^s}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i - \lambda_{ik}\frac{\partial x^j}{\partial a_l^s} - R_{ks}^l(A)\lambda_{ik}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i\right)da_l^s\wedge da_j^i = 0\,. \tag{17}$$

Write the previous equation as $\sum\limits_{i,j,s,l}(\Gamma_k)^{sl}_{ij}da^s_l\wedge da^i_j=0$, where

$$(\Gamma_k)^{sl}_{ij}=\frac{\partial\lambda_{ik}}{\partial a^s_l}\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r-\lambda_{ik}\frac{\partial x^j}{\partial a^s_l}-R^l_{ks}(A)\lambda_{ik}\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r.$$

Then equation (17) is satisfied if and only if $(\Gamma_k)^{sl}_{ij}=(\Gamma_k)^{ij}_{sl}$ for any given $k\in\{1,...,m\}$ and all $1\le i,s\le m$ and $1\le j,l\le n$. So, we get the required symmetry conditions. □

Notice that the 1-forms Ω_k, $k=1,...,m$, are not homogeneous with respect to any of the vectors $a^1,...,a^m$. One can readily verify that

$$\mathcal{L}_{\xi^{k'}}\Omega_k=(1+\delta^{k'}_k)\Omega_k-\lambda_{k'k}\omega^{k'}.$$

In particular, $\mathcal{L}_{\xi^k}\Omega_k=2\Omega_k-\omega^k$.

Corollary 4.2. *Suppose that the conditions* (12) *are satisfied. Then*

(a) $S_i=S_i^T$, *for all* $i=1,...,m$, *where* S_i *is the* $n\times n$ *matrix whose* jl*-entry is given by*

$$S^{jl}_i=\frac{\partial x^j}{\partial a^i_l}-R^j_{ii}(A)\sum_r\frac{\partial x^r}{\partial a^i_l}a^i_r\,,$$

(b) $$\frac{\partial x^l}{\partial a^i_j}+R^j_{ki}(A)\sum_r\frac{\partial x^r}{\partial a^k_l}a^k_r-\tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a^i_{j'}}a^k_{j'}\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r=$$
$$=\lambda_{ik}\left(\frac{\partial x^j}{\partial a^k_l}-\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a^k_l}a^k_{j'}\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r\right),$$

where $\tau_{ik}=(\sum\limits_{r,j'}\frac{\partial x^r}{\partial a^i_{j'}}a^i_ra^k_{j'})^{-1}=(a^i(D_{a^i}x^r)(a^k)^t)^{-1}$.

Proof. If $s=i=k$ then, using the fact that $\lambda_{ii}=1$, the relations (12) boil down to the following symmetry conditions

$$\frac{\partial x^j}{\partial a^i_l}-R^j_{ii}(A)\sum_r\frac{\partial x^r}{\partial a^i_l}a^i_r=\frac{\partial x^l}{\partial a^i_j}-R^l_{ii}(A)\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r\,.$$

So we get (a). To prove (b), it suffices to take $s=k$ and $i\neq k$ in (12) which results in this case in

$$\frac{\partial\lambda_{ik}}{\partial a^k_l}\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r-\lambda_{ik}\frac{\partial x^j}{\partial a^k_l}-R^l_{kk}(A)\lambda_{ik}\sum_r\frac{\partial x^r}{\partial a^i_j}a^i_r= \tag{18}$$
$$=-\frac{\partial x^l}{\partial a^i_j}-R^j_{ki}(A)\sum_r\frac{\partial x^r}{\partial a^k_l}a^k_r\,.$$

Now, multiply both sides of the last equality by a_j^k, summing over j,

$$\frac{\partial \lambda_{ik}}{\partial a_l^k}\sum_{r,j'}\frac{\partial x^r}{\partial a_{j'}^i}a_r^i a_{j'}^k - \lambda_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k - R_{kk}^l(A)\lambda_{ik}\sum_{r,j'}\frac{\partial x^r}{\partial a_{j'}^i}a_r^i a_{j'}^k =$$

$$= -\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k - \sum_{j'}R_{ki}^{j'}(A)a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k\,.$$

But, $\sum_{j'} R_{ki}^{j'}(A)a_{j'}^k = 0$. Therefore, we have

$$\frac{\partial \lambda_{ik}}{\partial a_l^k}\sum_{r,j'}\frac{\partial x^r}{\partial a_{j'}^i}a_r^i a_{j'}^k - \lambda_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k - R_{kk}^l(A)\lambda_{ik}\sum_{r,j'}\frac{\partial x^r}{\partial a_{j'}^i}a_r^i a_{j'}^k = -\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\,.$$

Multiplication with τ_{ik} leads to

$$\frac{\partial \lambda_{ik}}{\partial a_l^k} = \lambda_{ik}\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k + \lambda_{ik}R_{kk}^l(A) - \tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\,. \tag{19}$$

Substituting this expressiono back into (18), we get

$$\lambda_{ik}\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i + \lambda_{ik}R_{kk}^l(A)\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i - \tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i -$$

$$-\lambda_{ik}\frac{\partial x^j}{\partial a_l^k} - R_{kk}^l(A)\lambda_{ik}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i = -\frac{\partial x^l}{\partial a_j^i} - R_{ki}^j(A)\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k\,.$$

Then,

$$\lambda_{ik}\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i - \tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i - \lambda_{ik}\frac{\partial x^j}{\partial a_l^k} =$$

$$= -\frac{\partial x^l}{\partial a_j^i} - R_{ki}^j(A)\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k\,.$$

Rearranging, this condition can be written as

$$\frac{\partial x^l}{\partial a_j^i} + R_{ki}^j(A)\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k - \tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i =$$

$$= \lambda_{ik}\Bigg(\frac{\partial x^j}{\partial a_l^k} - \tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i\Bigg).$$

This completes the proof. □

We can use condition (b) to determine the functions λ_{ik}.

5. Economic integration

In this section, we solve the economic integration problem. We start by giving the following theorem.

Theorem 5.1. *Given a function $x(A) \in \mathbb{R}^n$ that is 0-homogeneous in a^i, $i = 1, ..., m$ and a family of strictly positive functions λ_{ik}, $1 \le i, k \le m$, all of class C^2 that satisfies the homogeneity conditions of Lemma 3.4 defined in a neighbourhood $\mathcal{V}$ of some point $\bar{A}$ such that $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ for all λ_{ik}, $1 \le i, k, s, t \le m$. Then, there exist $m+1$ functions $\mu_1, ..., \mu_k$ and V, defined in a possibly smaller neighbourhood $\mathcal{U} \subset \mathcal{V}$, such that $\mu_k dV = \Omega_k$ if and only if the conditions (12) are satisfied in $\mathcal{V}$.*

Proof. Given the functions $x(A)$ and λ_{ik}, $1 \le i, k \le m$ as in the statement of the theorem. Define a family of 1-forms Ω_k, $k = 1, ..., m$ as above. Now, $\Omega_k \wedge d\Omega_k = 0$ if and only if there exist two functions μ_k and V_k such that $\Omega_k = \mu_k dV_k$. Then we have $\Omega_i \wedge \Omega_k = \mu_i \mu_k dV_i \wedge dV_k = 0$. Therefore, $dV_k = \phi_{ik}(A) dV_i$, $\forall i, k = 1, ..., m$ for some function ϕ_{ik}. So we can set $dV_1 = ... = dV_m = dV$. □

The conditions $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ are satisfied by the functions $\lambda_{ik} = \lambda_i/\lambda_k$. These conditions are required to prove that $\Omega_i \wedge \Omega_k = 0$ for any $i, k = 1, ..., m$, for the differential forms defined by (10), see [1]. We recall the following three results from [1] for which we give brief proofs for the convenience of the reader.

Lemma 5.2. *Suppose that $V(A)$ is the value function, $x(A)$ is a solution and $\lambda(A)$ is the associated vector of Lagrange multipliers for problem $(\mathcal{P})$. Then we have*

$$D^2_{a^i} V(A) = \lambda_i(A)(D^2_{a^i} C^i(a^i) - D_{a^i} x(A)) + D_{a^i}\lambda_i(A)(D_{a^i} C^i(a^i) - x)^T. \quad (20)$$

Moreover, the $n \times n$ matrix $D^2_{a^i} C^i(a^i) - D_{a^i} x(A)$ is symmetric and positive semi-definite on $\{D_{a^i} V\}^\perp$.

Proof. By differentiating equation (5) and using the fact the value function V is quasi-convex with respect to a^i, the result follows. □

Lemma 5.3. *Let $x(A)$ be a solution of problem $(\mathcal{P})$ and $C(A) = Ax(A)$. Then*

$$\sum_r \frac{\partial^2 x^r}{\partial a^k_l \partial a^i_j} a^s_r + \frac{\partial x^l}{\partial a^i_j} \delta^s_k + \frac{\partial x^j}{\partial a^k_l} \delta^i_s = \frac{\partial^2 C^s}{\partial a^k_l \partial a^i_j} \delta^i_s \delta^k_s. \quad (21)$$

Moreover, if $C^i(a^i)$ is a convex function then the $n \times n$ matrix $\mathbf{M}^i$, where

$$\mathbf{M}^i_{jl} = \sum_r \frac{\partial^2 x^r}{\partial a^i_l \partial a^i_j} a^i_r + \frac{\partial x^l}{\partial a^i_j}_k + \frac{\partial x^j}{\partial a^i_l},$$

is symmetric and positive semi-definite.

Proof. The result follows by differentiating the s^{th} constraint firstly with respect to a_j^i and then with respect to a_l^k. Positivity follows from the convexity of $C^i(a^i)$. $\square$

Lemma 5.4. *Let $x(A) \in \mathbb{R}^n$ be a given function, define $C(A)$ by $C(A) = Ax(A)$. Then the matrix $T^i = a^i \otimes D^2_{a^i} x(A) + D_{a^i} x(A)$ whose ij-entry is*

$$T^i_{jl} = \sum_r \frac{\partial^2 x^r}{\partial a_l^i \partial a_j^i} a_r^i + \frac{\partial x^l}{\partial a_j^i},$$

is symmetric and positive semi-definite on the subspace $\{(D_{a^i} x)^t (a^i)^t\}^\perp$.

Proof. From the above equation (21) we have $T^i + D_{a^i} x = D^2_{a^i} C^i$. Using this equation and (5), we get $D^2_{a^i} V = \lambda_i T^i + \frac{1}{\lambda_i}(D_{a^i}\lambda_i)(D_{a^i} V)^T$. The result follows from the last equality, the quasi-convexity of V with respect to a^i and the result that $D_{a^i} V = \lambda_i (a^i D_{a^i} x)^t$. $\square$

The following theorem solves the economic integration problem.

Theorem 5.5. *Let $x(A) \in \mathbb{R}^n$, $\lambda_{ik}(A) > 0$ be given functions defined on a neighbourhood $\mathcal{U}$ of some point $\bar{A} \in \mathbb{R}^{mn}$ where x is zero-homogeneous in a^i, and λ_{ik} satisfy the homogeneity conditions of Lemma (3.4). Define $C(A) = Ax(A)$. Suppose that the following conditions are satisfied in $\mathcal{U}$ for all $i, k = 1, ..., m$.*

(a) $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ *for all* $1 \le i, k, s, t \le m$.

(b)

$$\frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} - R^l_{ks}(A)\lambda_{ik} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i =$$

$$= \frac{\partial \lambda_{sk}}{\partial a_j^i} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s - \lambda_{sk} \frac{\partial x^l}{\partial a_j^i} - R^j_{ki}(A)\lambda_{sk} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s$$

(c) *The matrix $\mathbf{M}^i$, $i = 1, ..., m$, is positive semi-definite.*

(d) *The restriction of the matrix T^i to $\{(D_{a^i} x)^t (a^i)^T\}^\perp$, $i = 1, ..., m$, is positive semi-definite.*

Then, there exist positive functions $\lambda_1, ..., \lambda_m$ and a function V which is quasi-convex with respect to a^i for each i, defined in a neighbourhood $\mathcal{V} \subset \mathcal{U}$ such that

$$D_{a^i} V = \lambda_i (D_{a^i} C^i - x).$$

Moreover, $V(A)$ is 0-homogeneous in a^i, $\lambda(A)$ is (-1)-homogeneous in a^i and 0-homogeneous in a^k for any $k \neq i$, $i = 1, ..., m$.

Proof. Condition (3) implies that the function $C^i(a^i)$ is convex. Consider the family of 1-forms $\Omega_1, ..., \Omega_m$ defined by (10). Conditions (2) are equivalent to

$\Omega_k \wedge d\Omega_k = 0$. Using Darboux theorem 2.6, the last equation is satisfied if and only if there exist two functions μ_k and V such that $\mu_k dV = \Omega_k$. Note that V is independent of k. Therefore, we have

$$\mu_k \, \frac{\partial V}{\partial a^i_j} = \lambda_{ik} \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a^i_j} \, a^i_r \, . \tag{22}$$

The function V is 0-homogeneous in a^i for all i. This can be verified by multiplying the last equation by a^i and summing over j. Now, multiply both sides of equation (22) by a^k_j, and adding up

$$\mu_k \sum_j \frac{\partial V}{\partial a^i_j} a^k_j = \lambda_{ik} \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r \; a^k_j \, .$$

Using $\tau_{ik} = (\sum\limits_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'})^{-1}$, where τ_{ik} is homogeneous of degree -1 in a^i, homogeneous of degree 0 in a^k, and homogeneous of degree -1 in $a^{k'}$, $\forall\, k' \neq i, k$, we can write

$$\mu_k = \frac{\lambda_{ik}}{\tau_{ik}} \, \frac{1}{a^k D_{a^i} V} \; , \quad i \neq k.$$

It follows that

$$\tau_{ik} \; \mu_k \; a^k D_{a^i} V = \lambda_{ik}(A) \; > \; 0 \tag{23}$$

for all A in sufficiently small neighbourhood of some point $\bar{A}$. Substituting for λ_{ik} in (22), we get

$$\mu_k dV = \sum_i \mu_k \tau_{ik} (a^k D_{a^i} V) \Big(\frac{\partial C^i}{\partial a^i_j} - x^j \Big) da^i_j \, .$$

Canceling μ_k of the last equation, and setting

$$\tau_{ik}(a^k D_{a^i} V) = \frac{\sum\limits_j \frac{\partial V}{\partial a^i_j} a^k_j}{\sum\limits_{r,j} \frac{\partial x^r}{\partial a^i_j} a^k_r a^i_r} \, .$$

Define a family of functions λ_i, $1 \leq i \leq m$ by $\lambda_i = \tau_{ik}(a^k D_{a^i} V)$, for some $k \neq i$. It follows from (23) that $\lambda_i(A) > 0$ in a neighbourhood of $\bar{A}$. Moreover, λ_i is (-1)-homogeneous in a^i and 0-homogeneous in a^k for all $k \neq i$. Then

$$dV = \sum_{i=1}^{m} \lambda_i \Big(\frac{\partial C^i}{\partial a^i_j} - x^j \Big) da^i_j \, .$$

It remains to prove that the function V has the required positivity conditions. Note that

$$\frac{\partial^2 V}{\partial a^s_l \partial a^i_j} = \sum_{i=1}^{m} \lambda_i \Big(\sum_{r=1}^{n} \frac{\partial^2 x^r}{\partial a^s_l \partial a^i_j} a^i_r + \frac{\partial x^l}{\partial a^i_j} \delta^i_s \Big) + \frac{\partial \lambda_i}{\partial a^s_l} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r \, .$$

Using (21), we can write $D_A^2 V$ as

$$\frac{\partial^2 V}{\partial a_l^s \partial a_j^s} = \lambda_s T_{jl}^s + \frac{\partial \lambda_s}{\partial a_l^s} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^s} a_r^s \,. \tag{24}$$

Take a vector $\varrho \in \{D_{a^s} V\}^{\perp}$, that is, ϱ satisfies the condition

$$\sum_{j=1}^{n} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^s} a_r^s \varrho_j = 0 \,.$$

It follows that

$$\sum_{j,l=1}^{n} \frac{\partial^2 V}{\partial a_l^s \partial a_j^s} \varrho_j \varrho_l = \lambda_s \sum_{j,l=1}^{n} T_{jl}^s \varrho_j \varrho_l \geq 0 \,.$$

We conclude that the matrix $D_{a^s}^2 V$ is positive semi-definite on $\{D_{a^s} V\}^{\perp}$, that is, V is quasi-convex with respect to a^s. □

6. Single constraint case and integration of homogeneous forms

The optimization problem in the single constraint case takes the form max $f(x)$ subject to $a^t x = C(a)$. The solution $x(a)$ as well as the value function are 0-homogeneous with respect to $a \in \mathbb{R}^n$ while the Lagrange multiplier is (-1)-homogeneous. Define the 1-form ω as

$$\omega = \sum_{i=1}^{n} \left(\frac{\partial C}{\partial a_j} - x^j \right) da_j \tag{25}$$

and the vector field ξ as

$$\xi = \sum_{j=1}^{n} a_j \frac{\partial}{\partial a_j} \,.$$

Then, as above, we have

$$\omega(a) = \frac{1}{\lambda(a)} dV(a) \,.$$

Notice that ω is collinear to the gradient of a 0-homogeneous function, $V(a)$, with 1-homogeneous coefficient function $1/\lambda(a)$. Moreover, $\iota_\xi \omega = 0$. Using Theorem 2.8 and Theorem 4.1 we get the following result

Theorem 6.1. *Given a C^2 function $x(a) \in \mathbb{R}^n$, define $C(a) = a^t x(a)$ and ω as in* (25)*. There exist a 1-homogeneous function $\mu(a)$ and a 0-homogeneous function $V(a)$ defined in the neighbourhood, $\mathcal{U}$, of some point $\bar{a} \in \mathbb{R}^n$ such that $\omega(a) = \mu(a) dV(a)$ if and only of there exists a vector R such that $a^t R = 1$ and*

$$\frac{\partial x^l}{\partial a_j} - R^l \sum_{k=1}^{n} \frac{\partial x^k}{\partial a_j} a_k = \frac{\partial x^j}{\partial a_l} - R^l \sum_{k=1}^{n} \frac{\partial x^k}{\partial a_l} a_k, \quad \forall\ 1 \leq j, l \leq n \,. \tag{26}$$

Moreover, if the matrix $T = a \otimes D_a^2 x(x) + D_a x(a)$ *is positive semi-definite on* $\{(D_a x(a))^t a\}^\perp$ *then the function* $\mu(a) > 0$ *and* $V(a)$ *is quasiconvex.*

Proof. Conditions (26) are equivalent to $\omega \wedge d\omega = 0$, which is, in turn, equivalent to the existence of two functions $\mu(a)$ and $V(a)$ such that $\omega(a) = \mu(a)dV(a)$. Since $\iota_\xi \omega = 0$, Theorem 2.8 implies that $\mu(a)$ is 1-homogeneous and $V(a)$ is 0-homogeneous. The decomposition $\omega = \mu dV$ gives

$$\sum_{k=1}^{n} \frac{\partial x^k}{\partial a_j} a_k = \mu(a) \frac{\partial V}{\partial a_j} \,.$$

Differentiating, we get

$$T = \mu(a) D_a^2 V(a) + (D_a \mu(a))(D_a V(a))^t$$

in the space $\{(D_a x)^t a\}^\perp = \{D_a V\}^\perp$, $T = \mu(a) D_a^2 V(a)$. We conclude that the function $V(a)$ is quasiconvex. The integration is local, which means that the functions $\mu(a)$ and $V(a)$ are defined in a neighbourhood of some point $\bar{a} \in \mathbb{R}^n$. We can assume, without loss of generality, that $\mu(\bar{a}) > 0$, for if $\mu(\bar{a}) < 0$ then we can write $\omega = \nu(a) dv(a)$ in a neighbourhood of $\bar{a}$ with $\nu(a) = -\mu(a)$ and $v(a) = -V(a)$. □

7. Duality

After solving the mathematical economic integration problems, we get functions $\lambda_1, ...\lambda_m$ and V that have the required homogeneity and positivity properties. The question now is how to get a concave (or quasi-concave) objective function.

In the single constraint case, given a quasi-convex function $V(a)$, let $f(x)$ be defined by

$$f(x) = \min_a \{V(a) | a^t x \leq c(a)\} \,.$$

The first order conditions for a minimum are

$$\frac{\partial V}{\partial a_j} = \lambda \left(\frac{\partial C}{\partial a_j} - x^j \right) \,,$$

and since V is quasi-convex, they are also sufficient, assuming V has a non-vanishing gradient. Therefore,

$$x(a) \in \arg\max \{f(x) | a^t x \leq c(a)\} \,.$$

Moreover, the function $f(x)$ is quasiconcave in the neighbourhood of some point $\bar{x} = x(\bar{a})$. Let $\alpha \in \mathbb{R}$ such that $f(x_1) \geq \alpha$ and $f(x_2) \geq \alpha$. Then

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) &= \min_a \{V(a) | a^t \left(\frac{x_1 + x_2}{2}\right) \leq C(a)\} \\ &\geq \min_{i=1,2} \{V(a) | a^t x_i \leq C(a)\} = \min_{i=1,2} f(x_i) = a \,, \end{aligned}$$

see [6], Proposition 11. In the general multi-constraint case , a function $f(x)$ can de defined similarly as

$$f(x) = \min_{A}\{V(A)|Ax \leq C(A)\}\,.$$

The function $f(x)$ is not necessarily quasiconcave. One can define a class of functions that is stable under duality. Details can be found in [1] with minor modifications.

8. Example: Characterization of excess demand functions

Our results generalize some well-known results concerning the characterization of individual excess demand functions, see Chiappori and Ekeland [8] and Geanakoplos and Polemarchakis [9]. Other characterization results are given in Aloqeili [4], in which the author gives conditions that characterize excess demands when initial endowments are not fixed[1]. Throughout this sections, we denote the independent variables by $p \in \mathbb{R}^n_{++}$ as they represent prices. Using standard notations, the individual (Marshallian) demand function solves the utility maximization problem

$$\max_{x}\; u(x) \;\text{ subject to }\; p^t x = p^t w\,,$$

where u is the individual's demand function and $w \in \mathbb{R}^n_{++}$ is the initial endowment of the consumer. The function $C(p) = p^t w$ is, therefore, the individual's income function. Note that, $C(p)$ is 1-homogeneous, so the above optimization problem belongs to the kind of problems we are considering here. The individual's excess demand function is defined as $z(p) = x(p, p^t w) - w$. The previous maximization problem can be written as

$$\max_{z}\; U(z) \;\text{ subject to }\; p^t z = 0\,.$$

Clearly, the solution of this problem $z(p)$, which is called the individual excess demand functions, is 0-homogeneous and satisfies Walras' law $p^t z(p) = 0$. Other conditions are given in the proposition below. The characterization problem of individual excess demands was considered, among others, by Geanakoplos and Polemarchakis in [9] and is completely solved by Chiappori and Ekeland in [8]. We refer to this article for a detailed exposition. For the sake of comparison, we give the following result from the previous references, Proposition 2 in [8] and Proposition A in [9]:

Proposition 8.1. *Let $z(p)$ be an excess demand function, then $z(p)$ is continuously differentiable on $\mathbb{R}^n_{++}$, 0-homogeneous and and satisfies $p.z(p) = 0$. Moreover, $D_p z(p) = K(p) - v(p)z(p)^t$ such that (i) $K(p)$ is symmetric and negative semi-definite, (ii) rank$(K(p)) = n - 1$, (iii) $p^t K(p) = K(p)p = 0$, (iv) $p^t v(p) = 1$.*

[1]The authors thank an anonymous referee for pointing out this example as an application of their results.

The interested reader is referred to [8] and [9] and the references therein for details.

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