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# Normal Regularity of Weakly Convex Sets in Asymmetric Normed Spaces

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We consider duality mappings in asymmetric normed spaces and study their properties. We also prove the Fréchet normal regularity and the proximally normal regularity of weakly convex sets in asymmetric normed spaces.

*Keywords:* Duality mapping, weakly convex set, Fréchet normal regular, proximally normal regular.

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## 1. Introduction

Weakly convex sets are often considered in literature under different names. In 1959 Federer, in order to extend the Steiner polynomial formula on larger classes, considered "sets with positive reach" in $\mathbb{R}^n$ [19]. Further the notion of weakly convex sets in finite dimensional Euclidean spaces was considered by Vial [34]. Vial considered weakly convex sets from a more "geometrical" perspective, as sets with a supporting ball. He proved the sufficient condition for a set to be a manifold. Now the class of sets with positive reach and the class of weakly convex sets in the sense of Vial are known to be equivalent in $\mathbb{R}^n$.

The properties of functions can be considered in terms of their epigraphs. This approach was used in [12] by Clark, Stern and Wolenski to study proximally smooth functions and sets in a Hilbert space.

Further Poliquin and Rockafellar [30] introduced prox-regular sets that can be viewed as the "directionally local" version of proximally smooth sets. Later Poliquin, Rockafellar and Thibault [31] considerd prox-regular sets in Hilbert spaces and proved that in a Hilbert space the class of uniformly prox-regular sets coincides with the class of proximally smooth sets. In Banach spaces the

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prox-regular sets were considered in [5] and the equivalence of the uniform prox-regularity and the weak convexity was shown for spaces with convexity and smoothness moduli of power type. In [3] this result was generalized for uniformly smooth and uniformly convex Banach spaces. A fairly complete survey of the properties of weakly convex sets in a Hilbert space is given in [20].

The motivation for studying weakly convex sets is that the class of weakly convex sets is much wider than the class of convex sets, but shares many useful properties with the latter. The weakly convex sets may be used, for example, in differential inclusions (see, e.g. [33]), in the gradient projection method ([1]), in differential games ([23]), set valued mappings theory ([2]).

Mordukhovich introduced the concept of normal regularity [26]. The applications of regularity of sets in Hilbert spaces may be found in [8, 33]. In Banach spaces normal regularity may be used, for example, to prove necessary conditions for the existence of a linearly suboptimal solution of multiobjective problems [28]. The notion of normal regularity for sets is closely related to that of lower regularity for functions: normal regularity of a set is equivalent to lower regularity of its indicator function. Lower regularity of some functions arising in applications was studied in [25].

Note that nonconvex sets may fail to be regular. Therefore, we want to describe the largest class of sets, which possesses this property. In [31] the proximal normal regularity for prox-regular sets in Hilbert spaces was shown. Further, in [6] it was shown that in uniformly smooth and uniformly convex Banach spaces the prox-regular sets are proximally normal regular.

We consider normal regularity for weakly convex sets. Weak convexity of sets is considered with respect to (w.r.t.) a quasiball in a Banach space. A quasiball is a convex closed (may be unbounded) set that contains a neighbourhood of zero. According to the Baire category theorem a set in a Banach space is a quasiball iff it is a closed convex absorbing set. The Miskowski function of a quasiball is an asymmetric seminorm. Asymmetric normed spaces were studied in [13], [14], [29]. Approximation problems and weakly convex sets in such spaces were studied in [21, 22, 24]. The properties of weakly convex sets with respect to an unbounded quasiball may be used for studying optimization problems with weakly convex functions on a global level.

In [6] the normal regularity of locally prox-regular sets in uniformly smooth and uniformly convex Banach spaces is shown. In this paper we prove that the weakly convex sets with respect to a quasiball are normally regular if the quasiball is smooth in some sense (some special convexity assumptions are also made). We will need to generalize the notions of proximal normals and Mordukhovich normal cone.

The rest of the paper is organized as follows. In the second section some definitions and notations used throughout the paper are given. In the third section duality mappings with respect to a quasiball are introduced and some auxiliary results are presented. In the fourth part the characterization of a boundedly uniformly smooth quasiball is given. In the fifth part the main theorem on the normal and Fréchet regularity of weakly convex sets with respect to a quasiball is proven. We also give examples that prove that the assertions in our theorems are essential.

## 2. Notations and Definitions

Let $E$ be a Banach space. Through $\operatorname{int} X$, $\partial X$ and $\overline{X}$ we denote the *interior, boundary* and *closure* of the set $X \subset E$. The value of the functional $p \in E^*$ on the vector $x \in E$ is denoted by $\langle p, x \rangle$.

In this paper we consider the notion of asymmetric seminorm, generated by a quasiball (the Minkowski function of this quasiball), where a *quasiball* is a closed convex set $M \subset E$ such that $0 \in \operatorname{int} M$.

Note that the quasiball $M$ is a ball with respect to some norm, equivalent to the norm of the space $E$, iff it is bounded w.r.t. the norm of $E$ and symmetric, i.e. $-M = M$.

The *Minkowski function* of a quasiball $M \subset E$ is the function $\mu_M \colon E \to [0; +\infty)$ defined by

$$\mu_M(x) = \inf \{t > 0 \ : \ x \in tM\}, \quad \forall x \in E.$$

A function $\mu \colon E \to [0; +\infty)$ is an *asymmetric seminorm* [14] if it is *positive homogeneous*

$$\mu(\lambda x) = \lambda \mu(x), \quad \forall x \in E, \quad \forall \lambda \geq 0$$

and *subadditive*

$$\mu(x + y) \leq \mu(x) + \mu(y), \quad \forall x, y \in E.$$

**Remark 2.1.** A continuous function $\mu \colon E \to [0; +\infty)$ is an asymmetric seminorm iff it is the Minkowski function of some quasiball.

Let $M \subset E$ be a quasiball. The *M-distance* from a point $x \in E$ to a set $A \subset E$ is defined as

$$\varrho_M(x, A) = \inf_{a \in A} \mu_M(x - a).$$

Let us recall that the *Minskowski sum* of sets $A \subset E$ and $B \subset E$ is

$$A + B = \left\{ a + b \;\middle|\; a \in A, \ b \in B \right\}.$$

**Remark 2.2.** By definition, we have that

$$\varrho_M(x, A) = \inf \left\{ t > 0 \middle| \ x \in A + tM \right\} = \inf \left\{ t > 0 \middle| \ A \bigcap (x - tM) \neq \emptyset \right\}.$$

A *ball* of radius $d \geq 0$ with centre at $a \in E$ is $\mathfrak{B}_d(a) = \{x \in E : \|x - a\| \leq d\}$. Further, we denote $\mathfrak{B}_d$ the ball of radius $d \geq 0$ with centre at $a = 0$.

**Remark 2.3.** In the case when $M$ is a unit ball of the space the $M$-distance is the usual distance between a point and a set: $\varrho_M(x,A) = \inf_{a\in A}\|x-a\|$.

For any quasiball $M \subset E$, we denote

$$\sigma_M = \inf_{x\in\partial M}\|x\|. \tag{1}$$

**Remark 2.4.** For any quasiball $M$ the Minkowski function $\mu_M(\cdot)$ is Lipschitz continuous with constant $\frac{1}{\sigma_M}$. This implies that for any set $A \subset E$ the function $\varrho_M(\cdot, A)$ satisfies the Lipschitz condition with the same constant.

We generalize the notion of metric projection for the case of an asymmetric seminorm. The *$M$-projection* of a point $x \in E$ on a set $A \subset E$ is, by definition,

$$P_M(x,A) = A\bigcap(x - \varrho_M(x,A)M).$$

## 3. Duality Mappings

In this section we study the subdifferential of the Minkowski function and duality mappings w.r.t. a quasiball.

Recall that the *Fenchel subdifferential* of a convex function $f\colon E \to \mathbb{R}\cup\{+\infty\}$ at $x_0 \in \operatorname{dom} f := \{x \in E : f(x) < +\infty\}$ is

$$\partial f(x_0) = \{p \in E^* :\ f(x) - f(x_0) \geq \langle p, x - x_0\rangle,\ \forall x \in E\}.$$

The *support function* of the set $M \subset E$ is

$$s(p,M) = \sup_{x\in M}\langle p,x\rangle, \quad p \in E^*.$$

The *normal cone* to a convex set $M \subset E$ at a point $x \in M$ is

$$N(x,M) = \{p \in E^* :\ \langle p,x\rangle = s(p,M)\}.$$

Let us define the *set of unit normals* to a quasiball $M \subset E$ at the point $x \in M$:

$$N^1(x,M) = \{p \in N(x,M) :\ s(p,M) = 1\}.$$

**Remark 3.1.** Let $M$ be a quasiball and $x_0 \in \partial M$. Then, from the Separation Theorem it follows that $N^1(x_0,M) \neq \emptyset$.

Recall that the *polar set* of $M \subset E$ is

$$M^\circ = \{p \in E^* :\ \langle p,x\rangle \leq 1,\ \forall x \in M\}.$$

**Lemma 3.2.** *Let $M$ be a quasiball in a Banach space $E$. Then*

(i) $\partial\mu_M(x_0) = N^1(x_0,M) = \{p \in E^* : \langle p,x_0\rangle = s(p,M) = 1\},\ \forall x_0 \in \partial M.$

*If $E$ is reflexive, then*

(ii) $\partial s(p_0,M) = \{x \in E : \langle p_0,x\rangle = \mu_M(x) = 1\},\ \forall p_0 \in E^* : s(p_0,M) = 1.$

**Proof.** The first relation of this lemma was proven in [18]. The second relation can be proved similarly. □

In this paragraph we generalize the notion of duality mapping. Recall [7] that the duality mapping $J\colon E \to 2^{E^*}$ is defined as follows $J(x) = \{p \in E^* :\ \langle p, x\rangle = \|p\|\|x\|\}$. To generalize this notion, we substitute the norm of the space by the Minskowski function, and the norm of the dual space by the support function. Further, we generalize some results of Chapter 2 from [16] for a quasiball.

Let $M \subset E$ be a quasiball. Define the *duality mappings* $J_M\colon E \to 2^{E^*}$ and $J^*_M\colon E^* \to 2^E$ as follows. Given a point $x \in E$, if $\mu_M(x) \neq 0$, then we define

$$J_M(x) = \{p \in E^* : \langle p, x\rangle = s(p, M)\mu_M(x),\ s(p, M) = \mu_M(x)\},$$

otherwise we put $J_M(x) = \{0\}$.

Recall that the *barrier cone* to the set $M \subset E$ is the set

$$b(M) = \{p \in E^* :\ s(p, M) < +\infty\}.$$

Given a functional $p \in E^*$, if $p \in b(M) \setminus \{0\}$ we define

$$J^*_M(p) = \{x \in E : \langle p, x\rangle = s(p, M)\mu_M(x), \quad s(p, M) = \mu_M(x)\},$$

otherwise we put $J^*_M(p) = \{0\}$.

**Remark 3.3.** It follows from the definitions and Lemma 3.2 that

$$J_M(x) = N^1(x, M) = \partial\mu_M(x), \quad \forall x \in \partial M, \tag{2}$$

and, if $E^*$ is reflexive, that

$$J^*_M(p) = \partial s(p, M), \quad \forall p \in E^* :\ s(p, M) = 1. \tag{3}$$

In view of the positive homogeneity of the functions $\mu_M(\cdot)$ and $s(\cdot, M)$ we obtain the positive homogeneity of the duality mappings.

**Lemma 3.4.** *Let $M \subset E$ be a quasiball. Then for any $t > 0$, $x \in E$ and $p \in E^*$: $J_M(tx) = tJ_M(x)$, $J^*_M(tp) = tJ^*_M(p)$.*

**Lemma 3.5.** *Let $M \subset E$ be a quasiball. Then for any $x \in E$ the set $J_M(x)$ is non-empty.*

The proof of this lemma follows directly from Lemma 3.4 and Remarks 3.1, 3.3. □

A convex set $M \subset E$ is said to be *smooth* if $N^1(x, M)$ is a singleton for any $x \in \partial M$. Using Lemma 3.4, we get the following remark.

**Remark 3.6.** If the quasiball $M \subset E$ is smooth, then $J_M(x)$ is a singleton for any $x \in E$.

Recall that a set $M \subset E$ is *strictly convex* if for any different points $x, y \in M$ the inclusion $\frac{x+y}{2} \in \text{int } M$ holds.

**Lemma 3.7.** *Let $M$ be a quasiball in a reflexive Banach space and let $b(M)\backslash\{0\}$ be an open set. Then for any $p \in E^*$ the set $J^*_M(p)$ is nonempty. If, additionally, the quasiball $M$ is strictly convex, then the set $J^*_M(p)$ is a singleton.*

**Proof.** If $p \notin b(M)$ or $p = 0$ the definition implies that $J^*_M(p) = \{0\}$. Therefore we suppose that $p \in b(M) \setminus \{0\}$. Considering $\frac{p}{s(p,M)}$ instead of $p$ and using Lemma 3.4 we suppose that $s(p, M) = 1$. In view of Lemma 5.3 [21] there exists $x \in M$ such that $s(p, M) = \langle p, x\rangle$. Then, taking into account that $\langle p, x\rangle \leq s(p, M)\mu_M(x)$ we obtain that $\mu_M(x) = 1$, i.e. $x \in \partial M$. Thus, $\langle p, x\rangle = s(p, M) = 1 = \mu_M(x)$, and, therefore $x \in J^*_M(p)$. So, $J^*_M(p)$ is non-empty.

Suppose now that $M$ is strictly convex and it is not a singleton. Then there exist $x_1, x_2 \in \partial M, \quad x_1 \neq x_2$ such that $\langle p, x_i\rangle = \mu_M(x_i) = s(p, M) = 1$ for $i = 1, 2$. Note that for any $\lambda \in [0, 1]$ the inclusion $\lambda x_1 + (1 - \lambda)x_2 \in M$ holds and $\langle p, \lambda x_1 + (1 - \lambda)x_2\rangle = 1$. Taking into consideration that $\langle p, \lambda x_1 + (1 - \lambda)x_2\rangle \leq s(p, M)\mu_M(\lambda x_1 + (1 - \lambda)x_2)$ and $\mu_M(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda\mu_M(x_1) + (1 - \lambda)\mu_M(x_2) = 1$ we obtain that $\mu_M(\lambda x_1 + (1 - \lambda)x_2) = 1$ and, therefore, $\lambda x_1 + (1 - \lambda)x_2 \in \partial M$. Thus, $M$ is not strictly convex. Contradiction. $\square$

Let us prove that the assertion that $b(M)\backslash\{0\}$ is open is essential for $J^*_M(\cdot)$ to be non-empty for any $p \in E^*$. Put $E = \mathbb{R}^2$ and consider $M = \{(x, y) \in E : (x + 2)(y + 2) \geq 1, \quad x + 2 > 0\}$ and functional $\langle p, (x, y)\rangle = -\frac{x}{2}$. Let us prove that $s(p, M) = 1$. As $x > -2$, we have that $s(p, M) = \sup\{\langle p, (x, y)\rangle : (x, y) \in M\} \leq 1$. Consider now $(x_n, y_n) = (-2 + \frac{1}{n}, n - 2)$, $n \in \mathbb{N}$. Observe that $(x_n, y_n) \in M$ for all $n \in \mathbb{N}$ and for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\langle p, (x_n, y_n)\rangle > 1 - \varepsilon$. Therefore, $s(p, M) = 1$. However, for any $(x, y) \in M$ we have that $\langle p, (x, y)\rangle < 1$. Thus $\partial s(p, M) = \emptyset$ and, therefore, $J^*_M(p) = \emptyset$.

In case $J_M(x)$ or $J^*_M(p)$ is a singleton, we will denote the element of this set $j_M(x)$ or $j^*_M(p)$ accordingly.

**Lemma 3.8.** *Let $M$ be a smooth strictly convex quasiball and let $b(M) \setminus \{0\}$ be an open set. Then for any $p \in b(M)$ and for any $x \in \partial M$*

$$j_M(j^*_M(p)) = p, \quad j^*_M(j_M(x)) = x.$$

**Proof.** Let us prove the first equality. Fix a functional $p \in b(M)$. Lemma 3.7 implies that $J^*_M(p)$ is a singleton. If $p = 0$, then $j^*_M(p) = 0$ and the definition of $J_M(x)$ implies that $j_M(j^*_M(p)) = 0$. Therefore, we may consider that $p \neq 0$ and, thus, $s(p, M) > 0$ (as the quasiball $M$ contains a neighbourhood of zero). Assume without loss of generality that $s(p, M) = 1$.

Then the definition of $j^*_M(\cdot)$ implies that $\langle p, j^*_M(p)\rangle = 1$ and $\mu_M(j^*_M(p)) = s(p, M) = 1$. Thus, $j^*_M(p) \in \partial M$ and, taking into account the definition of

$J_M(\cdot)$, we have that $p \in J_M(j^*_M(p))$. Remark 3.6 implies that the set $J_M(j^*_M(p))$ is a singleton. Therefore, $p = j_M(j^*_M(p))$.

Now let us prove the second equality. Fix an arbitrary $x \in \partial M$. By Remark 3.6 the set $J_M(x)$ is a singleton. Then $\langle j_M(x), x\rangle = 1$ and $s(j_M(x), M) = \mu_M(x) = 1$. This and the definition of $J^*_M(\cdot)$ imply that $x \in J^*_M(j_M(x))$. Taking into account Lemma 3.7, we obtain that $x = j^*_M(j_M(x))$. □

## 4. Characterizations of a boundedly uniformly smooth quasiball

A quasiball $M \subset E$ is called *boundedly uniformly smooth* if

$$\lim_{t\to+0} \frac{\beta_M(t,R)}{t} = 0, \qquad \forall R > \sigma_M, \tag{4}$$

where $\sigma_M$ is defined by (1), and, for any $t \geq 0$ and $R > \sigma_M$,

$$\beta_M(t,R) = \sup\left\{ \tfrac{\mu_M(x+ty)+\mu_M(x-ty)}{2} - 1 \ :\ x \in \partial M \cap \mathfrak{B}_R,\ y \in \mathfrak{B}_1 \right\}. \tag{5}$$

**Remark 4.1.** If the quasiball $M \subset E$ is boundedly uniformly smooth, then it is smooth.

**Remark 4.2.** In the case $M = \mathfrak{B}_1$ the modulus $\beta_M(t,R)$ for $R > \sigma_M = 1$ does not depend on $R$ and coincides with the modulus of smoothness of the space: $\rho(t) = \sup\left\{ \frac{\|x+ty\|+\|x-ty\|}{2} - 1 :\ \|x\| = \|y\| = 1 \right\}$ (see e.g. [16]). So, the ball $\mathfrak{B}_1$ is boundedly uniformly smooth if and only if the space $E$ is uniformly smooth.

**Remark 4.3.** Let there be given a quasiball $M \subset E$. Suppose that $p \in J_M(x)$. Since $\mu_M(x) = s(p,M) \geq \sigma_M\|p\|$, then $\|p\| \leq \frac{\mu_M(x)}{\sigma_M}$.

**Lemma 4.4.** *Let $M \subset E$ be a quasiball. If the duality mapping $J_M(\cdot)$ is single-valued and uniformly continuous on $\partial M \cap \mathfrak{B}_R$ for any $R > \sigma_M$, where $\sigma_M$ is defined by (1), then $J_M(\cdot)$ is single-valued and uniformly continuous on $\mathfrak{B}_R$ for any $R > \sigma_M$.*

**Proof.** If $x \in E$ is such that $\mu_M(x) > 0$, then according to Lemma 3.4 and the assumption of the lemma the set $J_M(x) = \mu_M(x) J_M\big(\frac{x}{\mu_M(x)}\big)$ is a singleton. Otherwise $J_M(x) = \{0\}$ is a singleton also. For any $t > 0$, $R > \sigma_M$ let us define

$$\Omega(t,R) = \sup_{\substack{a,b\in\partial M\cap\mathfrak{B}_R,\\ \|a-b\|\leq t}} \|j_M(a) - j_M(b)\|.$$

By the assumption of the lemma $\Omega(t,R) \to 0$ as $t \to +0$ for any $R > \sigma_M$.

Fix any $\varepsilon > 0$. Using Remarks 4.3 and 2.4, for any $x, y \in E$ we get

$$\|j_M(x) - j_M(y)\| \le \|j_M(x)\| + \|j_M(y)\| \le \frac{\mu_M(x) + \mu_M(y)}{\sigma_M}$$
$$\le \frac{2\min\{\mu_M(x), \mu_M(y)\}}{\sigma_M} + \frac{\|x - y\|}{\sigma_M^2}.$$

Hence, $\|j_M(x) - j_M(y)\| < \varepsilon$ provided that $\min\{\mu_M(x), \mu_M(y)\} < \frac{\varepsilon\sigma_M}{4}$ and $\|x - y\| < \frac{\varepsilon\sigma_M^2}{2}$.

Now assume that $x, y \in \mathfrak{B}_R$ are such that $\min\{\mu_M(x), \mu_M(y)\} \ge \frac{\varepsilon\sigma_M}{4}$. We have

$$\|j_M(x) - j_M(y)\| = \left\|\mu_M(x) j_M\left(\frac{x}{\mu_M(x)}\right) - \mu_M(y) j_M\left(\frac{y}{\mu_M(y)}\right)\right\|$$
$$\le \mu_M(x)\left\|j_M\left(\frac{x}{\mu_M(x)}\right) - j_M\left(\frac{y}{\mu_M(y)}\right)\right\| + \left\|j_M\left(\frac{y}{\mu_M(y)}\right)\right\| \cdot \|\mu_M(x) - \mu_M(y)\|$$
$$\le \frac{\|x\|}{\sigma_M}\Omega\left(\left\|\frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)}\right\|, \max\left\{\frac{\|x\|}{\mu_M(x)}, \frac{\|y\|}{\mu_M(y)}\right\}\right) + \frac{\|x - y\|}{\sigma_M^2}.$$

Observing that for any $x, y \in \mathfrak{B}_R$ with $\min\{\mu_M(x), \mu_M(y)\} \ge \frac{\varepsilon\sigma_M}{4}$

$$\left\|\frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)}\right\| \le \frac{\|x - y\|}{\mu_M(x)} + \frac{\|y\|}{\mu_M(x)\mu_M(y)}|\mu_M(x) - \mu_M(y)|$$
$$\le \frac{4}{\varepsilon\sigma_M}\left(1 + \frac{4R}{\varepsilon\sigma_M^2}\right)\|x - y\|,$$

we obtain

$$\|j_M(x) - j_M(y)\| \le \frac{R}{\sigma_M}\Omega\left(\frac{4}{\varepsilon\sigma_M}\left(1 + \frac{4R}{\varepsilon\sigma_M^2}\right)\|x - y\|, \frac{4R}{\varepsilon\sigma_M}\right) + \frac{\|x - y\|}{\sigma_M^2}.$$

Choose $t \in \left(0, \frac{\varepsilon\sigma_M^2}{2}\right)$ sufficiently small so that $\frac{R}{\sigma_M}\Omega\left(\frac{4}{\varepsilon\sigma_M}\left(1 + \frac{4R}{\varepsilon\sigma_M^2}\right)t, \frac{4R}{\varepsilon\sigma_M}\right) < \frac{\varepsilon}{2}$, we get $\|j_M(x) - j_M(y)\| < \varepsilon$ for all $x, y \in \mathfrak{B}_R$ with $\|x - y\| < t$. □

A function $f\colon E \to \mathbb{R}$ is called *uniformly Fréchet differentiable* on the set $X \subset E$, if for any $x_0 \in X$ there exists the Fréchet derivative $f'(x_0) \in E^*$ and

$$\sup_{x_0 \in X, y \in \mathfrak{B}_1} \frac{|f(x_0 + \lambda y) - f(x_0) - \lambda\langle f'(x_0), y\rangle|}{\lambda} \to 0, \quad \lambda \to 0$$

**Remark 4.5.** Assume that the function $f\colon E \to \mathbb{R}$ is Gâteaux differentiable on the set $X_\varepsilon := X + \mathfrak{B}_\varepsilon$ with some $\varepsilon > 0$ and the Gâteaux derivative $f'(\cdot)$ is uniformly continuous on $X_\varepsilon$. Then $f$ is uniformly Fréchet differentiable on $X$.

**Lemma 4.6.** *Let $M$ be a quasiball. Assume that for any $R > \sigma_M$ the Minkowski function $\mu_M(\cdot)$ is uniformly Fréchet differentiable on $\partial M \bigcap \mathfrak{B}_R$. Then $M$ is a boundedly uniformly smooth set.*

**Proof.** Fix an arbitrary $R > \sigma_M$. As $\mu_M(\cdot)$ is uniformly Fréchet differentiable on $\partial M \bigcap \mathfrak{B}_R$, then

$$g(x, y, \lambda) = \frac{\mu_M(x + \lambda y) - \mu_M(x) - \langle \mu'_M(x), \lambda y\rangle}{\lambda} \to 0, \quad \lambda \to 0$$

uniformly for $x \in \partial M \bigcap \mathfrak{B}_R$, $y \in \partial \mathfrak{B}_1$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \partial M \bigcap \mathfrak{B}_R$, $y \in \mathfrak{B}_1$ and $0 < \lambda < \delta$, then

$$g(x, y, \lambda) + g(x, y, -\lambda) < \varepsilon.$$

Therefore, taking into account that $\mu_M(x) = 1$, we get that $\mu_M(x + \lambda y) + \mu_M(x - \lambda y) < 2\mu_M(x) + \varepsilon\lambda = 2 + \varepsilon\lambda$. Therefore, $\frac{\beta_M(\lambda, R)}{\lambda} < \varepsilon$ and $M$ is a boundedly uniformly smooth set. $\square$

**Lemma 4.7.** *Let $M$ be a boundedly uniformly smooth quasiball. Then $J_M(\cdot)$ is single-valued and uniformly continuous on $\partial M \bigcap \mathfrak{B}_R$ for any $R > \sigma_M$, where $\sigma_M$ is defined by* (1)*.*

**Proof.** Fix $\varepsilon > 0$ and $R > \sigma_M$. Remarks 3.6, 4.1, imply that $J_M(\cdot)$ is single-valued on $E$. Let us first prove that there exists $\delta > 0$ such that if $p, q \in M^\circ$ and $\|p - q\| > \varepsilon$, then $s(p + q, \partial M \bigcap \mathfrak{B}_R) < 2 - \frac{\varepsilon\delta}{2}$. We will refer to it as relation $(A)$.

As $M$ is a boundedly uniformly smooth set, then there exists $\delta > 0$ such that for any $x \in \partial M \bigcap \mathfrak{B}_R$ and $y \in \mathfrak{B}_\delta$ the inequality $\mu_M(x + y) + \mu_M(x - y) < 2 + \frac{\varepsilon\|y\|}{2}$ holds. Let $p, q \in M^\circ$ and $\|p - q\| > \varepsilon$. Consider $y \in \partial \mathfrak{B}_\delta$ such that $\langle p - q, y\rangle \geq \|y\|\varepsilon$. Then

$$\begin{aligned}
s(p + q, \partial M \textstyle\bigcap \mathfrak{B}_R) &= \sup_{x \in \partial M \bigcap \mathfrak{B}_R} \langle p + q, x\rangle \\
&= \sup_{x \in \partial M \bigcap \mathfrak{B}_R} \{\langle p, x + y\rangle + \langle q, x - y\rangle\} - \langle p - q, y\rangle \\
&\leq \sup_{x \in \partial M \bigcap \mathfrak{B}_R} \{\mu_M(x + y) + \mu_M(x - y)\} - \varepsilon\delta \\
&\leq 2 + \frac{\varepsilon\delta}{2} - \varepsilon\delta = 2 - \frac{\varepsilon\delta}{2}.
\end{aligned}$$

So, relation $(A)$ is proved.

Consider now some $u, v \in \partial M \bigcap \mathfrak{B}_R$ such that $\|u - v\| < \frac{\varepsilon\delta}{2}\sigma_M$, where $\delta$ is taken from relation $(A)$. Then $\mu_M(v - u) < \frac{\varepsilon\delta}{2}$.

Taking into account that $\langle f, m\rangle \leq s(f, M)\mu_M(m)$ for any $f \in E^*$, $m \in E$, we have

$$s(j_M(u) + j_M(v), \partial M \bigcap \mathfrak{B}_R) \geq \langle j_M(u), u\rangle + \langle j_M(v), u\rangle =$$
$$= 2 + \langle j_M(v), u - v\rangle \geq 2 - s(j_M(v), M)\mu_M(v - u) > 2 - \frac{\varepsilon\delta}{2}.$$

Therefore, taking into account that $j_M(u), j_M(v) \in M^\circ$ and using relation $(A)$, we obtain that $\|j_M(u) - j_M(v)\| \leq \varepsilon$. □

**Theorem 4.8.** *Let $M$ be a quasiball. The following assertions are equivalent:*

(1) *$M$ is a boundedly uniformly smooth set,*

(2) *$\mu_M(\cdot)$ is uniformly Fréchet differentiable on $\partial M \cap \mathfrak{B}_R$ for any $R > \sigma_M$*

(3) *the mapping $J_M(\cdot)$ is single-valued and uniformly continuous on $\partial M \cap \mathfrak{B}_R$ for any $R > \sigma_M$.*

(4) *the mapping $J_M(\cdot)$ is single-valued and uniformly continuous on $\mathfrak{B}_R$ for any $R > \sigma_M$*

**Proof.** Let us prove that (3)⇒(4). As $J_M(\cdot)$ is uniformly continuous on $\partial M \cap \mathfrak{B}_R$ for any $R > \sigma_M$, then by Lemma 4.4 it is uniformly continuous on $\mathfrak{B}_R$ for any $R > \sigma_M$. Thus, (3)⇒(4).

Now let us show that (4) ⇒ (2). According to equation (2) and bearing in mind that the mapping $J_M(\cdot)$ and the function $\mu_M(\cdot)$ are positively homogeneous, we obtain that $J_M(x) = \mu_M(x)\partial\mu_M(x)$ for any $x \in 2M \setminus \frac{1}{2}M$.

Fix any $R > \sigma_M$. Assertion (4) implies that the convex function $\mu_M(\cdot)$ is Gâteaux differentiable on the set $2M \setminus \frac{1}{2}M$ (see Proposition 5.3 [17]) and in view of Remark 2.4 the Gâteaux derivative $\mu'_M(\cdot)$ is uniformly continuous on $(2M \setminus \frac{1}{2}M) \cap \mathfrak{B}_{2R}$. Observing that $(\partial M \bigcap \mathfrak{B}_R) + \mathfrak{B}_\varepsilon \subset (2M \setminus \frac{1}{2}M) \cap \mathfrak{B}_{2R}$ for $\varepsilon = \frac{\sigma_M}{2}$ and applying Remark 4.5 we obtain that $\mu_M(\cdot)$ is uniformly Fréchet differentiable on $\partial M \cap \mathfrak{B}_R$. Thus, (4)⇒(2).

Implication (2)⇒(1) follows from Lemma 4.6. Lemma 4.7 proves implication (1)⇒(3). Thus the proof is complete. □

Theorem 4.8 generalizes the following Šmulian's and Diestel's results (see [32] and [16, Theorem 1, §IV, Chapter II]).

**Theorem 4.9.** *The following assertions are equivalent:*

(i) *the duality mapping $J(\cdot)$ is single-valued and uniformly continuous on $\partial\mathfrak{B}_1(0)$;*

(ii) *the norm of $E$ is uniformly Fréchet differentiable;*

(iii) *$E$ is uniformly smooth.*

## 5. Regularities for weakly convex sets

In the previous sections we have made few assumptions on the quasiball. However, further we will need some more properties of the quasiball, so that the existence and uniqueness of projection hold.

Though any uniformly convex set in a Banach space is bounded [4], we can adapt the notion of uniform convexity for unbounded sets by introducing the notion of bounded uniform convexity.

A set $M \subset E$ is called *boundedly uniformly convex* if $\delta_M(\varepsilon, R) > 0$ for any positive numbers $R$ and $\varepsilon$, and for all $x, y \in M \cap \mathfrak{B}_R$, where

$$\delta_M(\varepsilon, R) = \sup\left\{\delta \in \left[0, \frac{\varepsilon}{2}\right] : \|x - y\| \geq \varepsilon \ \Rightarrow\ \mathfrak{B}_\delta\left(\frac{x+y}{2}\right) \subset M\right\}. \qquad (6)$$

It is well known that the metric projection of a point on a convex closed set in a uniformly convex space exists. However, not for any boundedly uniformly convex quasiball $M$ the $M$-projection of a point on a closed convex set exists. Consider, for example, the epigraph of the hyperbola $M = \{(x, y) \in \mathbb{R}^2 : (x+2)(y+2) \geq 1,\ x+2 > 0\}$ in $\mathbb{R}^2$ and the half-plane $A = \{(x, y) \in \mathbb{R}^2 :\ y \leq 0\}$. Then the $M$-projection of the point $(0, 1)$ on the half-plane $A$ is empty. Thus we need some additional property of the quasiball to obtain the existence of $M$-projection.

A set $M \subset E$ is called *parabolic* if, for any vector $b \in E$, the set $\left(b + \frac{1}{2}M\right) \setminus M$ is bounded.

The term "parabolic" is due to the fact that the epigraph of a parabola possesses this property, and the epigraph of a hyperbola does not.

For any set $A \subset E$ let us define

$$S_M(A) = \{x \in E :\ 0 < \varrho_M(x, A) < \sup_{y \in E} \varrho_M(y, A)\},$$

$$T_M(A) = \{x \in E :\ \text{the set } P_M(x, A) \text{ is a singleton}\}.$$

Theorem 2.4 [21] implies the following proposition.

**Proposition 5.1.** *Let in a Banach space $E$ a quasiball $M$ be parabolic and boundedly uniformly convex and let $A \subset E$ be a closed set. Then the set $T_M(A)$ is everywhere dense in $S_M(A)$, i.e. $S_M(A) \subset \overline{T_M(A)}$.*

It is well known that if a point does not belong to a closed set, then the distance from this point to the set is greater than zero. However, for the $M$-distance it is not so. Consider, for example, the space $\mathbb{R}^2$, the set $A = \{(x, y) \in \mathbb{R}^2 :\ x = 0\}$ and the quasiball – the epigraph of parabola $M = \{(x, y) :\ y \geq x^2 - 1\}$. Then the $M$-distance from any point to the set $A$ is zero. This is the motivation for us to introduce the following definition.

A set $A \subset E$ is called *closed w.r.t. a quasiball* $M$ or $M$*-closed* if for any $x \in E \backslash A$ the inequality $\varrho_M(x, A) > 0$ holds.

If a set $A \subset E$ is closed w.r.t. a quasiball $M$, then $A$ is closed, while the converse statement may fail as in example above.

Consider the following cone. The *cone of* $M$*-∗-normals* to a set $A \subset E$ at a point $a \in \overline{A}$ is

$$N^*_M(a, A) = \{z \in E \ : \ \exists t > 0 : \ a \in P_M(a + tz, A)\}. \tag{7}$$

The *set of unit* $M$*-∗-normals* to a set $A \subset E$ at a point $a \in \overline{A}$ is

$$N^{1*}_M(a, A) = N^*_M(a, A) \cap \partial M. \tag{8}$$

**Remark 5.2.** In the case of a quasiball $M$, for a set $A$ at some point $a$ the norm of elements of $N^{1*}_M(a, A)$ may be unbounded. Consider, for example, the set $A = \{(x, y) : \quad x = 0, \quad y \geq 0\}$ in $\mathbb{R}^2$ and let $M = \{(x, y) : \ y \geq x^2 - 1\}$. Then the projection of any point $(x, y) \notin A$ is $a = (0, 0)$. Thus the set $N^{1*}_M(a, A)$ is unbounded.

So, let us consider the following definition.

A set $A \subset E$ is called $M$*-quasibounded* if it is $M$-closed and for any $R \geq 0$ the inequality $\kappa(R) < +\infty$ holds, where

$$\kappa(R) = \sup \left\{ \|z\| \ : \ z \in N^{1*}_M(a, A), \ a \in A \cap \mathfrak{B}_R \right\}. \tag{9}$$

In this section we will discuss diverse properties of various normal cones, but let us first recall the definition of Fréchet cone.

The *Fréchet normal cone* to the set $A$ at $x \in A$ is

$$N^F(x, A) = \{\xi \in E^* : \forall \gamma > 0 \ \exists \delta > 0 \ \forall a \in \mathfrak{B}_\delta(x) \bigcap A : \langle \xi, a - x\rangle \leq \gamma \|a - x\|\}.$$

The *proximal* $M$*-normal cone* to the set $A$ at a point $a \in \partial A$ is

$$N^P_M(a, A) = \{p \in b(M) : \ N^*_M(a, A) \bigcap J^*_M(p) \neq \emptyset\}. \tag{10}$$

The *Mordukhovich limiting cone* is

$$\begin{aligned} N^L_M(x, A) &= {}^{w^*-seq} \limsup_{y \to x} N^P_M(y, A) \\ &= \{{}^{w^*}\!-\lim_{n \to \infty} x^*_n : \ x^*_n \in N^P_M(x_n, A), \ x_n \in A, \ x_n \to x, \ n \to \infty\}, \end{aligned}$$

where ${}^{w^*}\!-\lim$ means the limit with respect to the weak* topology.

**Lemma 5.3.** *Let the set* $M$ *be a parabolic quasiball. Let the set* $A \neq E$ *be* $M$*-quasibounded,* $x \in A$ *and* $p \in N^F(x, A)$*. Then* $s(p, M) < +\infty$*.*

**Proof.** Suppose the contrary, that $s(p, M) = +\infty$. Without loss of generality, assume that $\|p\| = 1$.

Let $\sigma_M$ be defined by (1). As the set $A$ is $M$-quasibounded and $A \neq E$, then there exists $b \in E$ and $\alpha > 0$ such that $A \bigcap (b - \alpha M) = \emptyset$. The set $\left(x - \frac{\alpha}{2} M\right) \setminus (b - \alpha M)$ is bounded according to [21, Lemma 5.1] since the quasiball $M$ is parabolic. Thus, there exists $R > 0$ such that

$$\left(x - \frac{\alpha}{2} M\right) \setminus (b - \alpha M) \subset \mathfrak{B}_R. \tag{11}$$

Denote $C = \kappa(R)$, where $\kappa(R)$ is defined by (9). Since $A$ is $M$-quasibounded, we have $C < +\infty$.

As $s(p, M) = \sup\limits_{m \in M} \langle p, m \rangle = +\infty$, there exists $z \in \partial M$ such that

$$\langle p, z \rangle > 3C + \sigma_M. \tag{12}$$

Define

$$\gamma = \min \left\{ \frac{\langle p, z \rangle}{2\|z\|}, \frac{C}{\|z\| + \sigma_M + 2C} \right\}. \tag{13}$$

Since $p \in N^F(x, A)$, there exists $\delta > 0$ such that

$$\langle p, a - x \rangle \leq \gamma \|a - x\|, \quad \forall a \in \mathfrak{B}_\delta(x) \bigcap A. \tag{14}$$

Define

$$t = \min \left\{ \frac{\alpha \sigma_M}{2(3\sigma_M + \|z\|)}, \frac{\delta}{\|z\| + \sigma_M + 2C} \right\}. \tag{15}$$

Note that $x + tz \in \mathfrak{B}_\delta(x)$. Let us prove that $x + tz \notin A$. Suppose the contrary. Then, as by (15) $t \leq \frac{\delta}{\|z\| + \sigma_M + 2C}$ and, thus, $\|tz\| < \delta$, (14) and (13) imply that $t\langle p, z \rangle = \langle p, x + tz - x \rangle \leq t\gamma\|z\| < t\langle p, z \rangle$. The contradiction obtained proves that $x + tz \notin A$.

Since $A$ is $M$-closed, we have $\varrho_M(x + tz, A) > 0$. Using Remark 2.4 and equation (15) we see that $\varrho_M(x + tz, A) \leq \frac{t\|z\|}{\sigma_M} < \alpha \leq \varrho_M(b, A) \leq \sup_{y \in E} \varrho_M(y, A)$. Thus, $x + tz \in S_M(A)$. By Proposition 5.1 there exist $x_t \in E \setminus A$ such that

$$\|x + tz - x_t\| \leq \sigma_M t \tag{16}$$

and $u_t \in P_M(x_t, A)$.

The definition of the $M$-projection and Remark 2.4 imply that

$$\mu_M(x_t - u_t) = \varrho_M(x_t, A) \leq \mu_M(x_t - x) \leq \mu_M(tz) + t = 2t. \tag{17}$$

Consequently, in view of (15), (16), (17)

$$\mu_M(x-u_t) \le \mu_M(x_t-u_t) + \frac{\|x-x_t\|}{\sigma_M} \le 2t + \frac{1}{\sigma_M}\left(\|x+tz-x_t\| + \|tz\|\right)$$
$$\le \frac{(3\sigma_M + \|z\|)t}{\sigma_M} \le \frac{\alpha}{2}.$$

This means that $u_t \in x - \frac{\alpha}{2}M$. As $u_t \in A$ and $A\bigcap(b-\alpha M) = \emptyset$, we obtain that $u_t \in \left(x - \frac{\alpha}{2}M\right) \setminus (b-\alpha M)$ and, in view of inclusion (11), we have that $\|u_t\| \le R$.

Since $x_t - u_t \in N^*_M(u_t, M)$ by (9) we obtain that $\frac{\|x_t-u_t\|}{\mu_M(x_t-u_t)} \le \kappa(R) = C$. Using (17) we get $\|x_t - u_t\| \le 2Ct$. Consequently, by (15) we have $\|x-u_t\| \le \|x_t - x\| + 2Ct \le t(\|z\| + \sigma_M + 2C) \le \delta$.

In view of (14), we obtain that $\langle p, u_t - x\rangle \le \gamma\|u_t - x\| \le \gamma t(\|z\| + \sigma_M + 2C)$. Using (13) we have $\langle p, u_t - x\rangle \le Ct$. Taking into account that $\|p\| = 1$, we obtain

$$\langle p, tz\rangle \le \langle p, u_t - x\rangle + \|tz - u_t + x\| \le Ct + \|x + tz - x_t\| + \|x_t - u_t\|$$
$$\le Ct + \sigma_M t + 2Ct \le (3C + \sigma_M)t.$$

This contradicts (12). □

**Remark 5.4.** The assumption that $A$ is $M$-quasibounded is essential in Lemma 5.3. Indeed, let $E = \mathbb{R}^2$, $M = \{(x,y) \in \mathbb{R}^2 : y+1 \ge x^2\}$, $A = \{(x,y) \in \mathbb{R}^2\colon\ x \le 0,\ y \ge 0\}$, $z = (1,0)$ and the functional $p \in (\mathbb{R}^2)^*$ be defined as $\langle p, (x,y)\rangle = x$ for all $(x,y) \in \mathbb{R}^2$. Then $p \in N^F(z, A)$, yet $s(p, M) = +\infty$.

**Lemma 5.5.** *Let $M \subset E$ be a quasiball such that the Minkowski function $\mu_M(\cdot)$ is Fréchet differentiable at any $x \in \partial M$. Let there be given a set $A \subset E$ and a point $x_0 \in \partial A$. Then $N^P_M(x_0, A) \subset N^F(x_0, A)$.*

**Proof.** Fix any $p \in N^P_M(x_0, A) \setminus \{0\}$. By the definition of the proximal $M$-normal cone there exists $r > 0$ and $y \in J^*_M(p)$ such that $x_0 \in P_M(x_0 + ry, A)$. The latter inclusion means that

$$\mu_M(x_0 + ry - x) \ge \mu_M(ry), \quad \forall x \in A. \tag{18}$$

Since $y \in J^*_M(p)$, we have $\mu_M(y) = s(p, M) > 0$. Substituting $p$ with $\frac{p}{\mu_M(y)}$ and $y$ with $\frac{y}{\mu_M(y)}$ in view of conicity of $N^P_M(x_0, A)$ and $N^F(x_0, A)$ we can assume that $\mu_M(y) = s(p, M) = 1$. According to Lemma 3.2 (i), we obtain that $p \in \partial\mu_M(y)$. Using the Fréchet differentiability of $\mu_M(\cdot)$ at $y$ and the positive homogeneity of $\mu_M(\cdot)$, we obtain that $p$ is the Fréchet derivative of $\mu_M(\cdot)$ at point $ty$ for any $t > 0$. Fix any $\varepsilon > 0$. By the definition of the Fréchet derivative there exists $\delta > 0$ such that

$$\mu_M(x_0 + ry - x) - \mu_M(ry) - \langle p, x_0 - x\rangle \le \varepsilon\|x - x_0\|, \quad \forall x \in \mathfrak{B}_\delta(x_0).$$

Using (18), we have

$$\langle p, x - x_0 \rangle \le \varepsilon \|x - x_0\|, \quad \forall x \in A \bigcap \mathfrak{B}_\delta(x_0).$$

Thus, $p \in N^F(x_0, A)$. Taking into account that $0 \in N^F(x_0, A)$, we get $N^P_M(x_0, A) \subset N^F(x_0, A)$. □

**Remark 5.6.** The assumption on the smoothness of the quasiball in Lemma 5.5 is essential. Indeed, let $E = \mathbb{R}^2$, $M = \{(x, y) \subset \mathbb{R}^2 : (x-1)^2 + (y+1)^2 \le 4,\ (x+1)^2 + (y-1)^2 \le 4\}$ and $A = \{(x, y) \in \mathbb{R}^2 : x \le 0 \text{ or } y \le 0\}$. Let $z_0 = (0, 0)$ and $z_1 = (1, 1)$. Let the functional $p \in (\mathbb{R}^2)^*$ be defined by the equality $\langle p, (x, y)\rangle = x + y$ for all $(x, y) \in \mathbb{R}^2$. Then $z_0 \in P_M(z_0 + z_1, A)$, $z_1 \in J^*_M(p)$ and, thus, $p \in N^P_M(z_0, A)$. But $p \notin N^F(z_0, A)$, so the inclusion $N^P_M(x_0, A) \subset N^F(x_0, A)$ fails.

**Lemma 5.7.** *Let $E$ be a reflexive Banach space. Let the quasiball $M \subset E$ be boundedly uniformly convex and parabolic. Let the set $A \subset E$ be $M$-quasibounded, $a \in A$, $p \in N^F(a, A)$, $\|p\| = 1$. Let $b \in M$, $\langle p, b\rangle = s(p, M)$. Then there exist sequences $\{a_k\} \subset A$ and $\{b_k\} \subset M$ such that $\lim\limits_{k\to\infty} \|a_k - a\| = 0$, $\lim\limits_{k\to\infty} \|b_k - b\| = 0$ and $b_k \in N^{1*}_M(a_k, A)$ for any $k \in \mathbb{N}$.*

**Proof.** Fix an infinitesimal sequence $\{t_k\}$ of positive numbers. Let us show that there exists an index $k_0 \in \mathbb{N}$ such that $a + t_k b \notin A$ for any $k \ge k_0$. Suppose the contrary. Then we may select a subsequence from $\{t_k\}$ such that $a + t_k b \in A$ for all $k \in \mathbb{N}$. As $p \in N^F(a, A)$, there exists an infinitesimal sequence $\{\beta_k\}$ of positive numbers such that $\langle p, t_k b\rangle \le \beta_k \|t_k b\|$ for all $k \in \mathbb{N}$. Dividing this inequality by $t_k$ and passing to the limit as $k \to \infty$, we obtain the inequality $\langle p, b\rangle \le 0$, which contradicts the relations $\langle p, b\rangle = s(p, M) > 0$. Thus, there exists an index $k_0 \in \mathbb{N}$ such that $a + t_k b \notin A$ as $k \ge k_0$. In view of the fact that the set $A$ is $M$-closed, we have that $\varrho_M(a + t_k b, A) > 0$ for all $k \ge k_0$. According to Remark 2.4, we have that $\lim_{k\to\infty} \varrho_M(a + t_k b, A) = \varrho_M(a, A) = 0$. Therefore, there exists an index $k_1 \ge k_0$ such that $\varrho_M(a + t_k b, A) < \frac{1}{2} \sup_{y\in E} \varrho_M(y, A)$ for all $k \ge k_1$. Thus, $a + t_k b \in S_M(A)$ for all $k \ge k_1$. According to Lemma 3.2 (i), for any $k \ge k_1$ there exists $x_k \in T_M(A)$ such that

$$\|a + t_k b - x_k\| < t_k \varrho_M(x_k, A) \tag{19}$$

and $0 < \varrho_M(x_k, A) < \frac{1}{2} \sup_{y\in E} \varrho_M(y, A)$. Since $x_k \in T_M(A)$, then for any $k \ge k_1$ there exists $a_k \in P_M(x_k, A)$. For $k \ge k_1$ denote

$$\varrho_k = \varrho_M(x_k, A), \qquad b_k = \frac{x_k - a_k}{\varrho_k}.$$

Then $b_k \in N^{1*}_M(a_k, A)$ for $k \ge k_1$.

Since $\mu_M(x_k - a_k) = \varrho_M(x_k, A) < \frac{1}{2} \sup_{y\in E} \varrho_M(y, A)$ and $\lim_{k\to\infty} x_k = a$, we have that the parabolicity of the quasiball $M$ implies the boundedness of the sequence $\{a_k\}$. As the set $A$ is $M$-quasibounded, then $\sup_{k\geq k_1}\|b_k\| = C < +\infty$. Therefore,

$$\|a-a_k\| \leq \|a+t_kb-x_k\|+t_k\|b\|+\|x_k-a_k\| \leq t_k\varrho_k+t_k\|b\|+C\varrho_k,\ \forall k \geq k_1. \tag{20}$$

Remark 2.4 and inequality (19) imply that $\mu_M(x_k - a) \leq \mu_M(t_kb) + \frac{t_k\varrho_k}{\sigma_M}$. On the other hand, the inclusion $a \in A$ and the equality $\varrho_k = \varrho_M(x_k, A)$ imply that $\varrho_k \leq \mu_M(x_k - a)$. As $b \in \partial M$, then $\mu_M(b) = 1$. Thus,

$$\varrho_k \leq t_k + \frac{t_k\varrho_k}{\sigma_M}. \tag{21}$$

This and inequality (20) imply that

$$\lim_{k\to\infty}\|a_k - a\| = 0. \tag{22}$$

As $p \in N^F(a, A)$, there exists an infinitesimal sequence $\{\varepsilon_k\}$ of positive numbers such that for any $k \geq k_1$

$$\langle p, a_k - a\rangle \leq \varepsilon_k\|a - a_k\|.$$

Using the inequalities (19), (20), for $k \geq k_1$ we get that

$$\langle p, a_k - x_k + t_kb\rangle \leq \varepsilon_k\|a - a_k\| + t_k\varrho_k \leq \varepsilon_k(t_k\varrho_k + t_k\|b\| + C\varrho_k) + t_k\varrho_k.$$

Dividing this inequality by $\varrho_k$ and using the equality $b_k = \frac{x_k-a_k}{\varrho_k}$, for $k \geq k_1$ we obtain that

$$-\langle p, b_k\rangle + \frac{t_k}{\varrho_k}(\langle p, b\rangle - \varepsilon_k\|b\|) \leq t_k(1 + \varepsilon_k) + C\varepsilon_k.$$

As $\lim_{k\to\infty}\varepsilon_k = 0$, $\langle p, b\rangle > 0$, there exists an index $k_2 \geq k_1$ such that $\langle p, b\rangle - \varepsilon_k\|b\| > 0$ for all $k \geq k_2$. Using inequality (21), for $k \geq k_2$ we obtain that

$$-\langle p, b_k\rangle + \left(1 - \frac{t_k}{\sigma_M}\right)(\langle p, b\rangle - \varepsilon_k\|b\|) \leq t_k(1 + \varepsilon_k) + C\varepsilon_k.$$

Therefore,

$$\langle p, b - b_k\rangle \leq \Delta_k := t_k(1 + \varepsilon_k) + C\varepsilon_k + \frac{t_k}{\sigma_M}\langle p, b\rangle + \varepsilon_k\|b\|, \quad \forall k \geq k_2. \tag{23}$$

Since $\lim\limits_{k\to\infty}\Delta_k = 0$ and $b_k \in M$ it follows that $\lim\limits_{k\to\infty}\langle p, b_k\rangle = \langle p, b\rangle = s(p, M)$. As the set $M$ is boundedly uniformly convex, it implies that $\lim\limits_{k\to\infty}\|b_k - b\| = 0$. $\square$

Following [21], [22], [24] we define weakly convex sets.

We say that a set $A \subset E$ is *weakly convex* w.r.t. the quasiball $M \subset E$, if

$$A \bigcap \left( a + \frac{x-a}{\varrho_M(x,A)} - \text{int } M \right) = \emptyset, \quad \forall x \in E \setminus A, \forall a \in P_M(x,A).$$

In other words, the set $A \subset E$ is called *weakly convex w.r.t. the quasiball $M$*, iff

$$a \in P_M(a+z, A), \qquad \forall a \in A, \quad \forall z \in N_M^{1*}(a,A). \tag{24}$$

The following figure illustrates the definition of a weakly convex set.

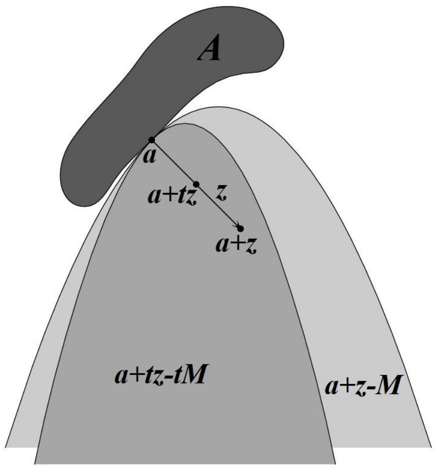


Figure 1.

**Theorem 5.8.** *Let $E$ be a reflexive Banach space. Let the quasiball $M \subset E$ be boundedly uniformly convex and parabolic. Let the set $A \subset E$ be $M$-quasi-bounded and weakly convex w.r.t. the quasiball $M$. Then for any $a \in \partial A$ we have that $N^F(a,A) \subset N_M^P(a,A)$.*

**Proof.** Fix $a \in \partial A$ and $p \in N^F(a,A)$, $\|p\| = 1$. According to Lemma 5.3 $s(p,M) < +\infty$. Since $M$ is parabolic, by Lemma 5.2 [21] the set $b(M) \setminus \{0\}$ is open. According to Lemma 3.7 there exists $b_0 \in J_M^*(p)$. Hence $b = \frac{b_0}{\mu_M(b_0)} \in M$ and $\langle p, b \rangle = s(p,M)$. In view of Lemma 5.7 there exist sequences $\{a_k\} \subset A$ and $\{b_k\} \subset M$ such that $\lim\limits_{k\to\infty} \|a_k - a\| = 0$, $\lim\limits_{k\to\infty} \|b_k - b\| = 0$ and $b_k \in N_M^{1*}(a_k, A)$ for any $k \in \mathbb{N}$. Using the weak convexity of the set $A$ w.r.t. $M$, we obtain that for all $k \in \mathbb{N}$

$$a_k + b_k \notin A + \text{int } M.$$

Therefore,

$$a + b \notin A + \text{int } M,$$

and, thus, $b \in N_M^{1*}(a,A)$. Hence $b_0 \in N_M^*(a,A) \bigcap J_M^*(p)$ and, consequently, $p \in N_M^P(a,A)$. Taking into account that $0 \in N_M^P(a,A)$, we obtain that $N^F(a,A) \subset N_M^P(a,A)$. $\square$

**Remark 5.9.** The assumption that $A$ is $M$-quasibounded is essential in Theorem 5.8. Indeed, let the quasiball $M$, the set $A$, the point $z$ be the same as in Remark 5.2 and let $p \in \partial\mathfrak{B}_1^*(0)$ be such that $\langle p, (x,y)\rangle = x$ for all $(x,y) \in \mathbb{R}^2$. The definition of set $A$ implies that $\langle p, (x,y) - (0,0)\rangle = 0$ for any $(x,y) \in A$. Then $p \in N^F(z,A)$, yet, as $s(p,M) = +\infty$, we obtain that $p \notin b(M)$ and, thus, $p \notin N_M^P(z,A)$.

By $\frac{d}{dx}f(x)$ we denote the Fréchet derivative of $f(\cdot)$ at $x$.

**Lemma 5.10.** *Let $M \subset E$ be a quasiball and the Minkowski functional $\mu_M(\cdot)$ be Fréchet differentiable at any $x_1 \in \partial M$. Then for any $x \in E$*

$$\frac{d}{dx}\mu_M^2(x) = 2j_M(x).$$

**Proof.** Fix any $x \in E$. First suppose that $\mu_M(x) \neq 0$. Since $x_1 = \frac{x}{\mu_M(x)}$ is a boundary point of $M$, it follows by Remark 3.3 that $J_M(x_1) = \partial\mu_M(x_1)$. As $\mu_M(\cdot)$ is Fréchet differentiable at $x_1$ we have $j_M(x_1) = \frac{d}{dx}\mu_M(x_1)$. Taking into account that functions $j_M(\cdot)$ and $\mu_M(\cdot)$ are positively homogeneous, we obtain that $j_M(x) = \mu_M(x)\frac{d}{dx}\mu_M(x) = \frac{1}{2}\frac{d}{dx}\mu_M^2(x)$.

Now let $\mu_M(x) = 0$. Using Remark 2.4 we get $\mu_M(x') = \mu_M(x') - \mu_M(x) \leq \frac{\|x'-x\|}{\sigma_M}$ for any $x' \in E$. Thus, $\mu_M^2(x') \leq \frac{\|x'-x\|^2}{\sigma_M^2} = o(\|x'-x\|)$ as $x' \to x$. Therefore, $\frac{d}{dx}\mu_M^2(x) = 0 = 2j_M(x)$. □

We say that a set $A$ at a point $x \in A$ is

- *Fréchet normal regular* w.r.t. a quasiball $M$ if $N^F(x,A) = N_M^L(x,A)$,
- *proximally normal regular* w.r.t. a quasiball $M$ if $N_M^P(x,A) = N_M^L(x,A)$.

**Theorem 5.11.** *Let $E$ be a reflexive Banach space. Let the quasiball $M$ be boundedly uniformly smooth, boundedly uniformly convex and parabolic. Let the set $A \subset E$ be $M$-quasibounded and $M$-weakly convex. Then $A$ is Fréchet normal regular and proximally normal regular w.r.t. $M$ at any $x \in A$, i.e.*

$$N^F(x,A) = N_M^P(x,A) = N_M^L(x,A).$$

**Proof.** By Remarks 3.6 and 4.1 for any $x \in E$ the set $J_M(x)$ is a singleton and we denote its element by $j_M(x)$ as usual. For any $R > \sigma_M$ define

$$\omega_R(\tau) = \sup\{\|j_M(u) - j_M(u')\| : u, u' \in \mathfrak{B}_R,\ \|u-u'\| \leq \tau\},\ \tau > 0. \quad (25)$$

Fix some arbitrary $R > \sigma_M$, $y \in \partial A$, $y' \in A$. Lemma 5.2 [21] states that the set $b(M)\setminus\{0\}$ is open. Thus, by Lemma 3.7 $J_M^*(\cdot)$ is single-valued, and for any $p \in E^*$ we will denote the element of $J_M^*(p)$ by $j_M^*(p)$. Consider $q \in N_M^P(y,A)$

such that $s(q, M) \leq 1$ and for any $\theta \in [0,1]$ the inclusion $j_M^*(q)+\theta(y-y') \in \mathfrak{B}_R$ holds. Let us prove that

$$\langle q, y'-y\rangle \leq \|y-y'\|\omega_R(\|y-y'\|). \tag{26}$$

By (10) we have that $j_M^*(q) \in N_M^*(y, A)$ and in view of (8) we get $\frac{j_M^*(q)}{s(q,M)} \in N_M^{1*}(y, A)$. Thus, as $A$ is weakly convex w.r.t. $M$, (24) implies that $y \in P_M(y + \frac{j_M^*(q)}{s(q,M)}, A)$. Since $s(q, M) \leq 1$, we obtain that $y \in P_M(y + j_M^*(q), A)$. Therefore, for any $y' \in A$

$$\mu_M(y + j_M^*(q) - y') \geq \mu_M(y + j_M^*(q) - y) = \mu_M(j_M^*(q)). \tag{27}$$

Recall that by Lemma 3.8 we have $j_M(j_M^*(q)) = q$. Lemma 5.10 implies that for any $y' \in A$

$$\mu_M^2(y + j_M^*(q) - y') = \mu_M^2(j_M^*(q)) + 2\int_0^1 \langle j_M(j_M^*(q) + \theta(y-y')), y-y'\rangle d\theta =$$

$$= \mu_M^2(j_M^*(q))+2\langle q, y-y'\rangle + 2\int_0^1\langle j_M(j_M^*(q) + \theta(y-y'))-j_M(j_M^*(q)), y-y'\rangle d\theta.$$

In view of (27), we obtain that

$$\langle q, y-y'\rangle + \|y-y'\| \int_0^1 \|j_M(j_M^*(q) + \theta(y-y')) - j_M(j_M^*(q))\| d\theta \geq 0.$$

Then, considering (25), we obtain (26).

Fix an arbitrary $x \in A$. Let us prove that $N_M^L(x, A) \subset N^F(x, A)$. Consider a functional $p \in N_M^L(x, A)$. If $p = 0$, then $p \in N^F(x, A)$.

Let $p \neq 0$. By the definition of the Mordukhovich limiting cone there exist sequences $\{x_n\}$ and $\{p_n\}$ such that $p_n \in N_M^P(x_n, A)$, $x_n \in A$, $x_n \to x$ and $p_n \overset{w}{\to} p$. As $p_n \overset{w}{\to} p$, there exists $\gamma > 0$ such that $\|p_n\| \leq \gamma$ for all $n \in \mathbb{N}$.

Let us prove that the sequence $\{s(p_n, M)\}$ is bounded. As $p_n \in N_M^P(x_n, A)$ we have $j_M^*(p_n) \in N_M^*(x_n, A)$ and hence $\frac{j_M^*(p_n)}{s(p_n,M)} \in N_M^{1*}(x_n, A)$ for any $n \in \mathbb{N}$. As $x_n \to x$, the sequence $\{x_n\}$ is bounded and the $M$-quasiboundedness of the set $A$ implies that there exists $C > 0$ such that $\left\|\frac{j_M^*(p_n)}{s(p_n,M)}\right\| \leq C$ for any $n \in \mathbb{N}$. Therefore,

$$s(p_n, M) = \left\langle p_n, \frac{j_M^*(p_n)}{s(p_n, M)}\right\rangle \leq \|p_n\|C \leq \gamma C. \tag{28}$$

Thus,

$$\|j_M^*(p_n)\| \leq Cs(p_n, M) \leq \gamma C^2. \tag{29}$$

Let there be given $\eta > 0$. Put $R = \gamma C^2 + \sigma_M$. By Theorem 4.8 $j_M(\cdot)$ is uniformly continuous on $\mathfrak{B}_R$ and, hence, $\omega_R(\tau) \to 0$ as $\tau \to +0$. Consequently, there exists $\varepsilon \in (0, \sigma_M)$ such that $\omega_R(\tau) \le \frac{\eta}{C\gamma}$ for any $\tau \in (0, \varepsilon)$. Consider an arbitrary $x' \in \mathfrak{B}_{\frac{\varepsilon}{2}}(x) \bigcap A$. Since $x_n \to x$ there exists $N > 0$ such that $\|x_n - x'\| < \varepsilon$ for any $n \ge N$.

Using (29), for any $n \ge N$ and $\theta \in [0,1]$ we obtain that $\|j_M^*(p_n) + \theta(x_n - x')\| \le \gamma C^2 + \varepsilon \le R$ and therefore $j_M^*(p_n) + \theta(x_n - x') \in \mathfrak{B}_R$.

By (28), we have that $s\left(\frac{p_n}{\gamma C}, M\right) \le 1$. Then, taking into account (26), for any $n \ge N$ we obtain that

$$\left\langle \frac{p_n}{\gamma C}, x' - x_n \right\rangle \le \|x' - x_n\| \omega_R(\|x' - x_n\|) \le \|x' - x_n\| \frac{\eta}{C\gamma}.$$

Therefore,

$$\langle p_n, x' - x_n \rangle \le \eta \|x' - x_n\|.$$

Passing to the limit as $n \to \infty$, we obtain the inequality

$$\langle p, x' - x \rangle \le \eta \|x' - x\|, \quad \forall x' \in \mathfrak{B}_{\frac{\varepsilon}{2}}(x) \bigcap A\,.$$

Then $p \in N^F(x, A)$ and, therefore, $N_M^L(x, A) \subset N^F(x, A)$.

By Theorem 5.8 and the definition of the limiting cone we have $N^F(x, A) \subset N_M^P(x, A) \subset N_M^L(x, A)$. Thus the proof is complete. $\square$

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