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An Evolutionary Structure of Convex Quadrilaterals. Part III

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We introduce an evolutionary structure of Euclidean networks for boundary convex quadrilaterals in the two dimensional Euclidean Space (Botanological network) which has two roots, one main branch and two branches. A botanological network is a weighted full Steiner tree which is enriched by a collection of instantaneous images of the process of photosynthesis, by assuming mass flow continuity.

Keywords: Weighted Fermat-Torricelli problem, weighted Fermat-Torricelli point, Botanological network, weighted Steiner minimal tree, inverse weighted Fermat-Torricelli problem, convex quadrilateral.

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1. Introduction

Let $A_1A_2A_3A_4$ be a convex quadrilateral with $A_i(x_i, y_i)$ and a positive real number (weight) B_i in A_i, $A_0(x_0, y_0)$ and $A_{0'}(x_{0'}, y_{0'})$ be two points in $\mathbb{R}^2$ with given weights B_0 in A_0 and $B_{0'}$ in $A_{0'}$, a_i be the Euclidean distance A_0A_i for $i = 1, 2$, a_i be the Euclidean distance $A_{0'}A_i$ for $i = 3, 4$, and l be the Euclidean distance $A_0A_{0'}$.

The generalized Gauss problem for convex quadrilaterals in $\mathbb{R}^2$ reads:

Problem 1.1. *Find $A_0(x_0, y_0)$ and $A_{0'}(x_{0'}, y_{0'})$, such that*

$$f(a_1, a_2, a_3, a_4, l) = \sum_{i=1}^{4} B_i a_i + \frac{B_0 + B_{0'}}{2} l \ \to \ min. \tag{1}$$

In 2013, we solve explicitly the generalized Gauss problem for convex quadrilaterals in $\mathbb{R}^2$ ([4, Theorem 2.2, p. 486–488]).

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The solution of the generalized Gauss problem is a weighted full Steiner tree for boundary quadrilaterals which has been first studied for equal weights in [1].

By letting $l = 0$ in (1), we obtain the weighted Fermat-Torricelli problem for convex quadrilaterals in $\mathbb{R}^2$. We call the solution of the weighted Fermat-Torricelli problem for convex quadrilaterals a weighted Fermat-Torricelli tree structure of degree four for the case where A_0 (weighted Fermat-Torricelli point) is located at the interior of $A_1A_2A_3A_4$.

By letting $l = 0$ and $B_i = 0$ for $i \in \{1, 2, 3, 4\}$ in (1), we obtain the weighted Fermat-Torricelli problem for a triangle in $\mathbb{R}^2$. We call the solution of the weighted Fermat-Torricelli problem for a triangle a weighted Fermat-Torricelli tree structure of degree three for the case where A_0 (weighted Fermat-Torricelli point) is located at the interior of $\triangle A_iA_jA_k$, for $i, j, k = 1, 2, 3, 4$ and $i \neq j \neq k$. The position of the weighted Fermat-Torricelli tree of degree four for boundary convex quadrilaterals is given in [3, Problem 2.1,pp. 412-413]. It is worth mentioning that a mechanical solution of the weighted Fermat-Torricelli problem for a triangle has been obtained by S.Gueron and R. Tessler who also establish the solution of the inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$ ([2, p. 449]).

In this paper, we introduce a generalized weighted full Steiner tree of degree three for boundary convex quadrilaterals (Botanological network), by modelling the process of photosynthesis in a plant which has two roots one main branch and two branches.

2. A botanological network for boundary convex quadrilaterals in the two-dimensional Euclidean Space.

We start by stating the inverse weighted full Steiner problem for boundary convex quadrilaterals in $\mathbb{R}^2$ which is an important generalization of the inverse weighted Fermat-Torricelli problem for convex quadrilaterals studied in [3, Section 4, pp. 416-425].

Problem 2.1. *Given two points A_0, $A_{0'}$, which belong to the interior of the quadrilateral $A_1A_2A_3A_4$ in $\mathbb{R}^2$, does there exist a unique set of positive weights $\bar{B}_i$, $(i = 0, 0', 1, 2, 3, 4)$ such that*

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = c = const, \tag{2}$$

for which A_0 and $A_{0'}$ minimize

$$f(A_0, A_{0'}) = \sum_{i=1}^{4} \bar{B}_i a_i + \frac{\bar{B}_0 + \bar{B}_{0'}}{2} l \tag{3}$$

under the following condition for the weights:

$$\bar{B}_i + \bar{B}_j = \bar{B}_0 + \bar{B}_{0'} + \bar{B}_k + \bar{B}_m \tag{4}$$

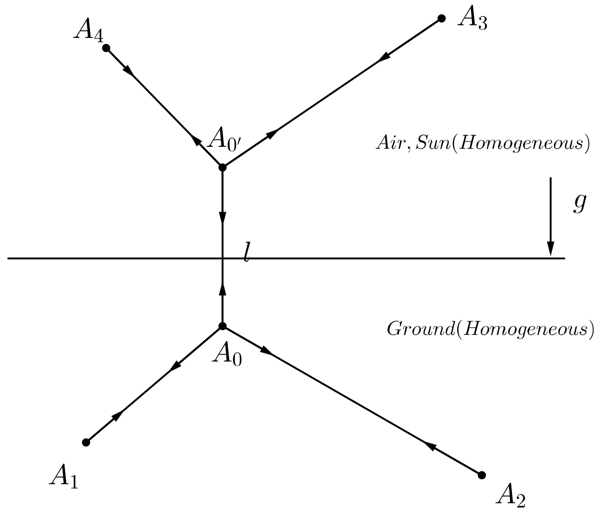

Figure 2.1: A botanological network of a boundary convex quadrilateral in $\mathbb{R}^2$.

for $i, j, k, m = 1, 2, 3, 4$ and $i \neq j \neq k \neq m$ (Inverse weighted full Steiner problem for quadrilaterals).

A negative answer to the inverse weighted full Steiner problem for convex quadrilaterals in $\mathbb{R}^2$ is given by the following theorem which gives the dynamic plasticity equations of a two-way full variable weighted Steiner tree for a boundary convex quadrilateral. The dynamic plasticity of convex quadrilaterals yields the same position of a full variable weighted Steiner tree with respect to a fixed boundary convex quadrilateral $A_1A_2A_3A_4$.

We denote by α_{ijk} the angle $\angle A_iA_jA_k$, for $i, j, k = 0, 0', 1, 2, 3, 4$.

Theorem 2.2. (Dynamic plasticity of a full variable weighted Steiner tree) *The dynamic plasticity of a two-way variable weighted Steiner tree with respect to $A_1A_2A_3A_4$ is given by the following system of equations:*

$$\frac{\bar{B}_1}{\sin\alpha_{200'}} = \frac{\bar{B}_2}{\sin\alpha_{100'}} = \frac{\bar{B}_0 + \bar{B}_{0'}}{-2\sin(\alpha_{200'} + \alpha_{100'})} \tag{5}$$

$$\frac{\bar{B}_3}{\sin\alpha_{00'4}} = \frac{\bar{B}_4}{\sin\alpha_{00'3}} = \frac{\bar{B}_0 + \bar{B}_{0'}}{-2\sin(\alpha_{00'4} + \alpha_{00'3})} \tag{6}$$

$$\bar{B}_1 + \bar{B}_2 = \bar{B}_0 + \bar{B}_{0'} + \bar{B}_3 + \bar{B}_4, \ \ and \tag{7}$$

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = 1\,. \tag{8}$$

Proof. Let $A_0 = (x_0, y_0)$ and $A_{0'} = (x_{0'}, y_{0'})$. By differentiating the objective function (3) w.r. to x_0 and y_0 we obtain that A_0 is the weighted Fermat-Torricelli point for $\triangle A_1A_2A_{0'}$, with weights $\bar{B}_1$, $\bar{B}_2$, $\frac{\bar{B}_0+\bar{B}_{0'}}{2}$, which corresponds to the vertices A_1, A_2 and $A_{0'}$, respectively. By applying the inverse weighted

Fermat-Torricelli problem for $\triangle A_1A_2A_{0'}$ we obtain (5) ([2, p. 449], [3, Corollary 4.3,p. 417]).

By differentiating the objective function (3) w.r.t. $x_{0'}$ and $y_{0'}$ we obtain that $A_{0'}$ is the weighted Fermat-Torricelli point for $\triangle A_3A_4A_0$, with weights $\bar{B}_3$, $\bar{B}_4$, $\frac{\bar{B}_0+\bar{B}_{0'}}{2}$, which corresponds to the vertices A_3, A_4 and A_0, respectively. By applying the inverse weighted Fermat-Torricelli problem for $\triangle A_3A_4A_0$ we obtain (6) ([2, p. 449], [3, Corollary 4.3,p. 417]). The equations (7) and (8) are taken from the inverse full weighted Steiner problem for $A_1A_2A_3A_4$ by considering the full variable weighted Steiner tree as a two-way communication network in $\mathbb{R}^2$. $\square$

We proceed by describing an instantaneous image of the process of photosynthesis referring to a plant which starts to grow and produces two roots on the ground, one main branch from the ground to the air and two branches in the air (Fig. 2.1).

First, we assume that the air and the ground are homogeneous and that the evolution of the main branch occurs in the vertical direction w.r.t. the ground.

We consider the following two main steps in order to give the two fundamental conditions of mass flow continuity:

Step 1: (First way of communication)
Water is transferred from the roots A_1A_0 and A_2A_0 with weights $(B_1)_w$ and $(B_2)_w$, leaving a storage $(B_0)_w$ at A_0 and $(B_{0'})_w$ at $A_{0'}$ and continues to be transferred to the branches $A_{0'}A_4$ and $A_{0'}A_3$ with weights $(B_4)_w$ and $(B_3)_w$, respectively.

Step 2: (Second way of communication)
Air and Sun create the quantities of photosynthesis $(B_4)_{PH}$ and $(B_3)_{PH}$ which start to move from the branches $A_4A_{0'}$ and $A_3A_{0'}$ leave a storage $(B_{0'})_{PH}$ at $A_{0'}$ and $(B_0)_{PH}$ at A_0 and continue to be transferred to the roots A_0A_1 and A_0A_2 with weights $(B_1)_{PH}$ and $(B_2)_{PH}$, respectively.

From the Steps 1 and 2 we derive two conditions of mass flow continuity:

$$(B_1)_w + (B_2)_w = (B_0)_w + (B_{0'})_w + (B_3)_w + (B_4)_w, \text{ and} \tag{9}$$

$$(B_1)_{PH} + (B_2)_{PH} + (B_0)_{PH} + (B_{0'})_{PH} = (B_3)_{PH} + (B_4)_{PH}. \tag{10}$$

Therefore, the objective function (1) takes the form:

$$\sum_{i=1}^{4}(B_i)_w a_i + [(B_1)_w + (B_2)_w - (B_0)_w]l + \sum_{i=1}^{4}(B_i)_{PH}a_i + \\ +[(B_3)_{PH} + (B_4)_{PH} - (B_{0'})_{PH}]\, l \;\to\; min\,. \tag{11}$$

Definition 2.3. A *botanological network* for a boundary convex quadrilateral $A_1A_2A_3A_4$ is a full weighted Steiner tree in $\mathbb{R}^2$, such that the weights $(B_i)_w$ and $(B_j)_{PH}$ for $i,j = 0, 0', 1, 2, 3, 4$ minimize the objective function (2) and satisfy the two conditions of mass flow continuity (9) and (10).

By applying the inverse weighted Steiner problem for $A_1A_2A_3A_4$, and the two steps of the evolution of a plant with two roots, one main branch vertical to the ground and two branches in the air, we obtain the dynamic plasticity of a botanological network with respect to the boundary quadrilateral $A_1A_2A_3A_4$. We let $\tilde{B_i} = (B_i)_w + (B_i)_{PH}$, $\tilde{B_0} = (B_0)_w - (B_0)_{PH}$, $\tilde{B_{0'}} = (B_{0'})_w - (B_{0'})_{PH}$ and $\bar{B_j} = \frac{\tilde{B_j}}{\sum_{j=1}^4 \tilde{B_j}}$, for $i = 1, 2, 3, 4$, $j = 0, 0', 1, 2, 3, 4$.

Theorem 2.4. *The dynamic plasticity of a botanological network with respect to $A_1A_2A_3A_4$ is given by the following system of equations:*

$$\frac{\bar{B}_1}{\sin\alpha_{200'}} = \frac{\bar{B}_2}{\sin\alpha_{100'}} = \frac{\mathcal{C}}{-2\sin(\alpha_{200'} + \alpha_{100'})}\,, \tag{12}$$

$$\frac{\bar{B}_3}{\sin\alpha_{00'4}} = \frac{\bar{B}_4}{\sin\alpha_{00'3}} = \frac{\mathcal{C}}{-2\sin(\alpha_{00'4} + \alpha_{00'3})}\,, \tag{13}$$

$$\bar{B}_1 + \bar{B}_2 = \bar{B}_0 + \bar{B}_{0'} + \bar{B}_3 + \bar{B}_4\,, \tag{14}$$

$$\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = 1\,, \tag{15}$$

where

$$\mathcal{C} = 1 - \frac{(B_0)_w - (B_{0'})_w}{\sum_{i=1}^4 \tilde{B}_i} - \frac{(B_{0'})_{PH} - (B_0)_{PH}}{\sum_{i=1}^4 \tilde{B}_i}\,. \tag{16}$$

Proof. Taking into account (9) and (10) the objective function (2) takes the form:

$$f(A_0, A_{0'}) = \tag{17}$$

$$= \sum_{i=1}^4((\tilde{B_i})a_i + l((B_1)_w + (B_2)_w - (B_0)_w + (B_3)_{PH} + (B_4)_{PH} - (B_{0'})_{PH})$$

or

$$f(A_0, A_{0'}) = \tag{18}$$

$$= \sum_{i=1}^4((\tilde{B_i})a_i + \frac{l}{2}(2((B_1)_w + (B_2)_w - (B_0)_w + (B_3)_{PH} + (B_4)_{PH} - (B_{0'})_{PH}))$$

or

$$\frac{1}{\sum_{i=1}^4 \tilde{B}_i} f(A_0, A_{0'}) = \sum_{i=1}^4 \bar{B}_i a_i + \tag{19}$$

$$+ \frac{l}{2\sum_{i=1}^4 \tilde{B}_i}\left(2\left[(B_1)_w + (B_2)_w - (B_0)_w + (B_3)_{PH} + (B_4)_{PH} - (B_{0'})_{PH}\right]\right)$$

or
$$\frac{1}{\sum_{i=1}^{4}\tilde{B}_i}f(A_0,A_{0'})=\sum_{i=1}^{4}\bar{B}_i a_i+\frac{l}{2}\mathcal{C}\,, \tag{20}$$

where $\mathcal{C}$ is given by (16).

Then, by applying Theorem 2.2 for $g(A_0,A_{0'})=\frac{1}{\sum_{i=1}^{4}\tilde{B}_i}f(A_0,A_{0'})$ and using $\bar{B}_0+\bar{B}_{0'}=\mathcal{C}$, we derive the dynamic plasticity equations (12) and (13). □

Let $A_1A_2A_3A_4$ be either a rectangle or an isosceles trapezoid where A_1A_2 is parallel to A_3A_4 and $a_{14}=a_{23}$ in $\mathbb{R}^2$.

Proposition 2.5. *If $(B_1)_w=(B_2)_w$, $(B_3)_w=(B_4)_w$ and $(B_1)_{PH}=(B_2)_{PH}$, $(B_3)_{PH}=(B_4)_{PH}$, then*

$$\alpha_{200'}=\alpha_{100'} \quad \textit{and} \quad \alpha_{30'0}=\alpha_{40'0}. \tag{21}$$

Proof. The equalities $(B_1)_w=(B_2)_w$, $(B_3)_w=(B_4)_w$ and $(B_1)_{PH}=(B_2)_{PH}$, $(B_3)_{PH}=(B_4)_{PH}$ yield $\tilde{B}_1=\tilde{B}_2$ and $\tilde{B}_3=\tilde{B}_4$, or

$$\bar{B}_1=\bar{B}_2 \quad \text{and} \quad \bar{B}_3=\bar{B}_4. \tag{22}$$

By replacing (22) in (12) and (13), we obtain (21). □

Corollary 2.6. *Given $(B_i)_w$, $(B_i)_{PH}$ for $i=0,0',1,2,3,4$ where $(B_1)_w=(B_2)_w$, $(B_3)_w=(B_4)_w$ and $(B_1)_{PH}=(B_2)_{PH}$, $(B_3)_{PH}=(B_4)_{PH}$, the angles $\alpha_{100'}$ and $\alpha_{40'0}$ are given by the following two relations:*

$$\alpha_{200'}=\arccos\left(\frac{\mathcal{C}}{-4\bar{B}_1}\right) \tag{23}$$

and
$$\alpha_{30'0}=\arccos\left(\frac{\mathcal{C}}{-4\bar{B}_4}\right). \tag{24}$$

Proof. From Proposition 2.5 we get:

$$\alpha_{200'}=\alpha_{100'} \quad \text{and} \quad \alpha_{30'0}=\alpha_{40'0}. \tag{25}$$

By replacing (25) in (12) and (13) we get:

$$\frac{\bar{B}_1}{\sin\alpha_{200'}}=\frac{\mathcal{C}}{-2\sin(2\alpha_{200'})} \tag{26}$$

and
$$\frac{\bar{B}_4}{\sin\alpha_{30'0}}=\frac{\mathcal{C}}{-2\sin(2\alpha_{30'0})} \tag{27}$$

which yield (23) and (24). □

References

[1] K. Bopp: Über das kürzeste Verbindungssystem zwischen vier Punkten, Dissertation, Göttingen (1879).

[2] S. Gueron, R. Tessler: The Fermat-Steiner problem, Amer. Math. Monthly 109 (2002) 443–451.

[3] A. N. Zachos, G. Zouzoulas: An evolutionary structure of convex quadrilaterals, J. Convex Analysis 15(2) (2008) 411–426.

[4] A. N. Zachos, G. Zouzoulas: An evolutionary structure of convex quadrilaterals. Part II, J. Convex Analysis 20(2) (2013) 483–493.