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Schur-Convexity of a General Closed Four-Point Quadrature Formula

Iva Franjić
Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
ifranjic@pbf.hr

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Schur-convexity property of a general closed four-point quadrature formula is studied and necessary and sufficient conditions for this type of quadrature formula to have this property are given. If a quadrature formula is Schur-convex, this implies the following: if one considers two intervals which share the same midpoint, the error of a Schur-convex quadrature formula is always smaller on a smaller interval.

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1. Introduction

Throughout this paper, I will denote a non-empty open interval in $\mathbb{R}$. Our focus point is the function $K\colon I^2 \to \mathbb{R}$ defined as

$$\begin{aligned} K(x,y) = \frac{1}{y-x}\int_x^y f(t)dt - \left(\frac{1}{2} - Q\right)\Big[f(x)+f(y)\Big] - \\ - \ Q\Big[f\Big(\lambda x + (1-\lambda)y\Big) + f\Big((1-\lambda)x + \lambda y\Big)\Big] \end{aligned} \tag{1}$$

for $(x,y) \in I^2,\ x \neq y,\ \lambda \in (0,1)$. For $x = y \in I$, we set $K(x,x) = 0$. The values of this function can obviously be interpreted as the error of a general closed four-point quadrature formula. This type of quadrature formulas has already been studied in [1], where sharp estimates of the error were given. In this paper, necessary and sufficient conditions for the function K to be Schur-convex will be established. The property of Schur-convexity was first introduced in 1923. by Issai Schur. Let us recall its definition.

Definition 1.1. A function $\Phi\colon A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur-convex* on A if $\Phi(x_1, x_2, \ldots, x_n) \leq \Phi(y_1, y_2, \ldots, y_n)$ for every $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and

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$\mathbf{y} = (y_1, y_2, \ldots, y_n) \in A$ with $\mathbf{x} \prec \mathbf{y}$, that is,

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \quad \text{and} \quad \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \ \text{ for } \ k = 1, 2, \ldots, n-1,$$

where $x_{[i]}$ denotes the ith largest component in $\mathbf{x}$. A function Φ is said to be *Schur-concave* on A if $(-\Phi)$ is Schur-convex.

Our main result states that the function K defined in (1) is Schur-convex on I^2 if and only if it is non-negative on I^2 and analogously, it is Schur-concave on I^2 if and only if it is negative on I^2 (this is only a part of the main result; the properties of K are also proven to be influenced by the function f being 4-convex or 4-concave). So, let us now assume that K is Schur-convex (and therefore positive). Let us consider two intervals $[x_1, x_2]$ and $[y_1, y_2]$ such that $x_1, x_2, y_1, y_2 \in I$, $x_1 < x_2$ and $y_1 < y_2$. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ and assume $\mathbf{x} \prec \mathbf{y}$, which implies that $x_1 + x_2 = y_1 + y_2$ and $x_2 \leq y_2$. From there it follows that these two intervals share the same midpoint and that $[x_1, x_2] \subseteq [y_1, y_2]$. Since K is Schur-convex, we have $K(x_1, x_2) \leq K(y_1, y_2)$. This in turn implies that the error of this quadrature formula is smaller on a shorter interval than on the larger one. If K was Schur-concave, then the situation would be reversed: the error would be smaller on a larger interval due to $K(x_1, x_2) \geq K(y_1, y_2)$. But, since K is Schur-concave on I^2 if and only if it is negative on I^2, we conclude that the absolute value of the error of this quadrature formula is always smaller on a smaller interval (of the two which share the same midpoint).

2. Auxiliary results

In this section, a few crucial lemmas are presented. Without loss of generality, throughout the rest of the paper, we assume $x < y$ and $0 < \lambda \leq 1/2$. In the opposite case, all results follow immediately after making obvious modifications. We introduce the notation

$$z_{1-\lambda} = (1-\lambda)x + \lambda y \quad \text{and} \quad z_\lambda = \lambda x + (1-\lambda)y.$$

Lemma 2.1. *Let $f \colon I \to \mathbb{R}$ be such that $f \in C^4(I)$. Then for $x, y \in I$, $x < y$ and $\lambda \in (0, 1/2]$, we have*

$$\int_x^y f(t)dt - (y-x)\Big(\frac{1}{2} - Q\Big)\Big(f(x) + f(y)\Big) - (y-x)Q\big(f(z_{1-\lambda}) + f(z_\lambda)\big) = \\ = \int_x^y F_4(\lambda, t) f^{(4)}(t)dt, \tag{2}$$

where

$$Q = \frac{1}{12\lambda(1-\lambda)}, \tag{3}$$

$$F_4(\lambda, t) = \begin{cases} p(t), & x \le t \le z_{1-\lambda}, \\ q(t), & z_{1-\lambda} \le t \le z_\lambda, \\ r(t), & z_\lambda \le t \le y \end{cases} \tag{4}$$

and

$$p(t) = \frac{(t-x)^3}{24}\left(t - 2y + x + \frac{y-x}{3\lambda(1-\lambda)}\right), \tag{5}$$

$$\begin{aligned} q(t) &= \frac{1}{24}t^4 - \frac{x+y}{12}t^3 + \frac{x^2+y^2+(4-6\lambda)xy}{24(1-\lambda)}t^2 + \\ &\quad + \frac{(3\lambda-2)(x^2y+xy^2) - \lambda(x^3+y^3)}{24(1-\lambda)}t + \\ &\quad + \frac{\lambda^2(y-x)^4 + 3\lambda xy(y-x)^2 + 3x^2y^2(1-\lambda)}{72(1-\lambda)}, \end{aligned} \tag{6}$$

$$r(t) = -\frac{(y-t)^3}{24}\left(t + y - 2x - \frac{y-x}{3\lambda(1-\lambda)}\right). \tag{7}$$

Proof. Set

$$J(\lambda, t) = \begin{cases} p(t) = \frac{1}{4!}t^4 + a_3t^3 + a_2t^2 + a_1t + a_0, & x \le t \le z_{1-\lambda}, \\ q(t) = \frac{1}{4!}t^4 + b_3t^3 + b_2t^2 + b_1t + b_0, & z_{1-\lambda} < t \le z_\lambda, \\ r(t) = \frac{1}{4!}t^4 + c_3t^3 + c_2t^2 + c_1t + c_0, & z_\lambda < t \le y, \end{cases}$$

where $a_i,\ b_i,\ c_i \in \mathbb{R},\ i = 0, 1, 2, 3$ have to be determined so that (2) holds, that is, so that $J(\lambda, t) = F_4(\lambda, t)$. Integration by parts readily gives

$$\begin{aligned} \int_x^y J(\lambda, t) f^{(4)}(t)dt &= \int_x^y f(t)dt + \\ &+ \sum_{k=0}^{3}(-1)^{k+1}\left[r^{(3-k)}(y)f^{(k)}(y) - p^{(3-k)}(x)f^{(k)}(x)\right] + \\ &+ \sum_{k=0}^{3}(-1)^{k+1}\left[q^{(3-k)}(z_\lambda) - r^{(3-k)}(z_\lambda)\right] f^{(k)}(z_\lambda) + \\ &+ \sum_{k=0}^{3}(-1)^{k+1}\left[p^{(3-k)}(z_{1-\lambda}) - q^{(3-k)}(z_{1-\lambda})\right] f^{(k)}(z_{1-\lambda}). \end{aligned} \tag{8}$$

In order for (2) to be valid, the following conditions need to be satisfied:

$$r(y) = r'(y) = r''(y) = p(x) = p'(x) = p''(x) = 0,$$

$$r'''(y) = -p'''(x) = (y-x)(1/2 - Q),$$

$$q^{(3-k)}(z_\lambda) = r^{(3-k)}(z_\lambda), \quad k = 1, 2, 3,$$

$$q'''(z_\lambda) - r'''(z_\lambda) = (y-x)Q,$$

$$p^{(3-k)}(z_{1-\lambda}) = q^{(3-k)}(z_{1-\lambda}), \quad k = 1, 2, 3,$$
$$p'''(z_{1-\lambda}) - q'''(z_{1-\lambda}) = (y - x)\, Q\,.$$

The unique solution to this system of linear equations is:

$$a_3 = c_3 = \frac{(6\lambda^2 - 6\lambda + 1)(y + x)}{72\lambda(1-\lambda)}, \qquad b_3 = -\frac{x + y}{12},$$
$$a_2 = \frac{x^2 - (6\lambda^2 - 6\lambda + 1)xy}{24\lambda(1-\lambda)}, \qquad b_2 = \frac{x^2 + y^2 + (4 - 6\lambda)xy}{24(1-\lambda)},$$
$$c_2 = \frac{y^2 - (6\lambda^2 - 6\lambda + 1)xy}{24\lambda(1-\lambda)}, \qquad Q = \frac{1}{12\lambda(1-\lambda)}$$

$$a_1 = \frac{(6\lambda^2 - 6\lambda + 1)x^2 y - (2\lambda^2 - 2\lambda + 1)x^3}{24\lambda(1-\lambda)},$$
$$b_1 = \frac{(3\lambda - 2)(x^2 y + xy^2) - \lambda(x^3 + y^3)}{24(1-\lambda)},$$
$$c_1 = \frac{(6\lambda^2 - 6\lambda + 1)xy^2 - (2\lambda^2 - 2\lambda + 1)y^3}{24\lambda(1-\lambda)},$$
$$a_0 = \frac{(3\lambda^2 - 3\lambda + 1)x^4 - (6\lambda^2 - 6\lambda + 1)x^3 y}{72\lambda(1-\lambda)},$$
$$b_0 = \frac{\lambda^2(y - x)^4 + 3\lambda xy(y - x)^2 + 3x^2y^2(1-\lambda)}{72(1-\lambda)},$$
$$c_0 = \frac{(3\lambda^2 - 3\lambda + 1)y^4 - (6\lambda^2 - 6\lambda + 1)xy^3}{72\lambda(1-\lambda)}. \qquad \square$$

An interesting fact is that the weight Q from (3) is the same as the weight obtained in [1], where general closed four-point quadrature formulas were also considered (on the interval $[0, 1]$) and the weight was chosen in such a way to produce the formulas with the maximum degree of exactness. In the aforementioned paper, the main tool used in the proofs was the (extended) Euler formula - the well known formula for the expansion of an arbitrary function in Bernoulli polynomials; details on Bernoulli polynomials can be found, for example, in [3]. It was shown that (2) can be derived via the Euler formula and that the function F_4 defined in (4) expressed through Bernoulli polyonomials takes the form:

$$F_4(\lambda, t) = \frac{(y - x)^4}{4!}\big(G_4(\lambda, t) - G_4(\lambda, x)\big), \quad t \in [x, y], \quad \text{where}$$

$$G_4(\lambda,t) = \frac{1}{12\lambda(1-\lambda)}\left[B_4^*\left(\frac{z_{1-\lambda}-t}{y-x}\right)+B_4^*\left(\frac{z_\lambda-t}{y-x}\right)\right]- \\ -\frac{6\lambda^2-6\lambda+1}{6\lambda(1-\lambda)}B_4\left(\frac{y-t}{y-x}\right),$$

B_4 is the 4th Bernoulli polynomial: $B_4(t) = t^4 - 2t^3 + t^2 - 1/30$ and B_4^* is the continuous periodic function of period 1 such that $B_4^*(t) = B_4(t),\ 0 \le t < 1$.

Furthermore, it was shown that for $t \in [x, y]$, we have

$$\begin{aligned} &F_4(\lambda,t) \ge 0, \quad \text{if } \lambda \in (0,\ 1/2-\sqrt{3}/6], \quad \text{and} \\ &F_4(\lambda,t) \le 0, \quad \text{if } \lambda \in [1/3,\ 1/2]\,. \end{aligned} \tag{9}$$

However, it turns out that (9) holds for a larger interval of values of λ, but since in [1] a more general case was considered, this was overlooked. The extension of this result will be the content of our next lemma. For the sake of completeness, we give the whole proof.

Lemma 2.2. *Let $x, y \in \mathbb{R}$, $x < y$. The function $F_4\colon (0, 1/2] \times [x, y] \to \mathbb{R}$ defined in* (4)*, with p, q, r as in* (5), (6), (7)*, has constant sign for $\lambda \in \big(0,\ 1/2-\sqrt{3}/6\big] \cup \big[15/32-\sqrt{33}/32,\ 1/2\big]$ and $t \in [x, y]$. More precisely, we have*

$$F_4(\lambda,t) \ge 0, \qquad \textit{for } \lambda \in \left(0,\ \frac{3-\sqrt{3}}{6}\right] \ \textit{and } t \in [x,y], \tag{10}$$

while

$$F_4(\lambda,t) \le 0, \qquad \textit{for } \lambda \in \left[\frac{15-\sqrt{33}}{32},\ \frac{1}{2}\right] \ \textit{and } t \in [x,y]. \tag{11}$$

Proof. Since F_4 is symmetrical about the midpoint in t, that is, we have $F_4(\lambda,\ y+x-t) = F_4(\lambda,\ t)$ for $t \in [x, (x+y)/2]$, only the sign for $t \in [x,\ (x+y)/2]$ needs to be considered.

Part 1. Let $t \in [x,\ z_{1-\lambda}]$. Elementary calculation shows that

$$t_0 = 2y - x - \frac{y-x}{3\lambda(1-\lambda)} \ \le\ x \ \Leftrightarrow\ \lambda \in (0,\ 1/2-\sqrt{3}/6],$$

where $(1/2-\sqrt{3}/6) \approx 0.211325$, and on the other hand

$$t_0 \ \ge\ z_{1-\lambda} \ \Leftrightarrow\ s(\lambda) = 3\lambda^3 - 9\lambda^2 + 6\lambda - 1 \ge 0. \tag{12}$$

The function s is increasing for $\lambda \in (0,\ 1-\sqrt{3}/3)$ and decreasing for $\lambda \in (1-\sqrt{3}/3,\ 1/2]$ and further, we have $s(1/2-\sqrt{3}/6) < 0,\ \ s(15/32-\sqrt{33}/32) > 0,$

$s(1-\sqrt{3}/3) > 0$ and $s(1/2) > 0$. Note that $15/32 - \sqrt{33}/32 \approx 0.289232 < 1 - \sqrt{3}/3 \approx 0.42265$. From there we conclude that there exists a unique $\lambda_0 \in (1/2 - \sqrt{3}/6,\ 15/32 - \sqrt{33}/32)$ such that $s(\lambda_0) = 0$ and such that $s(\lambda) \geq 0$ for $\lambda \in [\lambda_0,\ 1/2]$.

In conclusion of Part 1, for $t \in [x,\ z_{1-\lambda}]$ we have

$$F_4(\lambda, t) \geq 0 \quad \Leftrightarrow \quad \lambda \in \left(0,\ \frac{3-\sqrt{3}}{6}\right],$$

$$F_4(\lambda, t) \leq 0 \quad \Leftrightarrow \quad \lambda \in [\lambda_0,\ 1/2]\,, \quad \text{where} \quad \frac{3-\sqrt{3}}{6} < \lambda_0 < \frac{15-\sqrt{33}}{32}\,.$$

Part 2. Let $t \in [z_{1-\lambda},\ (x+y)/2]$. Here we have

$$\frac{\partial F_4(\lambda, t)}{\partial t} = \frac{1}{6}\left(t - \frac{x+y}{2}\right)\left(t^2 - (x+y)t + xy + \frac{\lambda(y-x)^2}{2(1-\lambda)}\right).$$

The discriminant of the quadratic polynomial is $D = \dfrac{1-3\lambda}{1-\lambda}(y-x)^2$.

(1) For $\lambda \geq 1/3$, we have $D \leq 0$ and so $\partial F_4(\lambda, t)/\partial t \leq 0$, which implies that $F_4(\lambda, t)$ is decreasing in t. Now,

$$F_4(\lambda,\ z_{1-\lambda}) \leq 0 \quad \text{will imply} \quad F_4(\lambda, t) \leq 0, \quad \text{and}$$
$$F_4\left(\lambda,\ \frac{x+y}{2}\right) \geq 0 \quad \text{will imply} \quad F_4(\lambda, t) \geq 0.$$

We have

$$F_4(\lambda,\ z_{1-\lambda}) = -\frac{\lambda^2(y-x)^4}{72(1-\lambda)} s(\lambda)\,, \tag{13}$$

where $s(\lambda)$ is as in (12). So,

$$F_4(\lambda,\ z_{1-\lambda}) \leq 0 \quad \Leftrightarrow \quad s(\lambda) \geq 0 \quad \Leftrightarrow \quad \lambda \in [\lambda_0,\ 1/2], \tag{14}$$

where λ_0 is the unique zero of the function s, same as in Part 1 of the proof. Since $s(1/3) > 0$, it follows that $1/2 - \sqrt{3}/6 < \lambda_0 < 1/3$ and we can now conclude that for $\lambda \in [1/3,\ 1/2]$, we have $F_4(\lambda, t) \leq 0$.

(2) For $\lambda < 1/3$, we have $D > 0$ and the real zero smaller than $(x+y)/2$ is

$$t_1 = \frac{x+y}{2} - \frac{y-x}{2}\sqrt{\frac{1-3\lambda}{1-\lambda}}$$

and we have

$$t_1 \leq z_{1-\lambda} \quad \Leftrightarrow \quad 2\lambda^2 - 4\lambda + 1 \geq 0 \quad \Leftrightarrow \quad \lambda \in (0,\ 1 - \sqrt{2}/2],$$

where $(1 - \sqrt{2}/2) \approx 0.292893$.

(i) For $\lambda \in (0,\ 1-\sqrt{2}/2]$, we have $\partial F_4(\lambda,t)/\partial t \geq 0$ and so $F_4(\lambda,t)$ is here increasing in t. Now,

$$F_4(\lambda,\ z_{1-\lambda}) \geq 0 \quad \text{will imply} \quad F_4(\lambda,t) \geq 0, \quad \text{and}$$
$$F_4\left(\lambda,\ \frac{x+y}{2}\right) \leq 0 \quad \text{will imply} \quad F_4(\lambda,t) \leq 0.$$

We have

$$\begin{aligned} F_4\left(\lambda, \frac{x+y}{2}\right) &= \frac{(16\lambda^2-15\lambda+3)(y-x)^4}{1152(1-\lambda)} \leq 0 \\ &\Leftrightarrow \quad \lambda \in \left[\frac{15-\sqrt{33}}{32},\ \frac{1}{2}\right], \end{aligned} \tag{15}$$

where $(15/32-\sqrt{33}/32) \approx 0.289232$, and so we conclude that for $\lambda \in \left[15/32-\sqrt{33}/32,\ 1-\sqrt{2}/2\right]$, we have $F_4(\lambda,t) \leq 0$. On the other hand, recall (13). We have

$$F_4(\lambda,\ z_{1-\lambda}) \geq 0 \quad \Leftrightarrow \quad s(\lambda) \leq 0 \quad \Leftrightarrow \quad \lambda \in (0,\ \lambda_0]$$

and so we conclude that for $\lambda \in (0,\ \lambda_0]$, we have $F_4(\lambda,t) \geq 0$. Note that $s(15/32-\sqrt{33}/32) > 0$, so $\lambda_0 < 15/32-\sqrt{33}/32$.

(ii) For $\lambda \in (1-\sqrt{2}/2,\ 1/3)$, we have $z_{1-\lambda} < t_1 < (x+y)/2$. Now we have $\partial F_4(\lambda,t)/\partial t < 0$ for $t \in [z_{1-\lambda},\ t_1)$ and $\partial F_4(\lambda,t)/\partial t > 0$ for $t \in (t_1,\ (x+y)/2)$. Let us consider

$$F_4(\lambda,t_1) = \frac{\lambda^2(1-4\lambda)(y-x)^4}{288(1-\lambda)^2}.$$

Obviously, $F_4(\lambda,t_1) < 0$ for $\lambda \in (1-\sqrt{2}/2,\ 1/3)$. So, in order for F_4 to have constant sign, we need to have $F_4(\lambda,\ z_{1-\lambda}) \leq 0$ and $F_4\left(\lambda,\ (x+y)/2\right) \leq 0$. Recalling (2), we see that $F_4\left(\lambda, (x+y)/2\right) \leq 0$ for $\lambda \in (1-\sqrt{2}/2,\ 1/3)$. Furthermore, from (14) and the fact that $\lambda_0 < 15/32-\sqrt{33}/32$ (see (i)), it follows that $F_4(\lambda,\ z_{1-\lambda}) \leq 0$ for $\lambda \in (1-\sqrt{2}/2,\ 1/3)$. Thus, for $\lambda \in (1-\sqrt{2}/2,\ 1/3)$, we have $F_4(\lambda,t) \leq 0$.

In conclusion of Part 2, for $t \in [z_{1-\lambda},\ (x+y)/2]$, we have

$$F_4(\lambda,t) \geq 0 \quad \Leftrightarrow \quad \lambda \in (0,\ \lambda_0], \ \text{ where } \frac{3-\sqrt{3}}{6} < \lambda_0 < \frac{15-\sqrt{33}}{32};$$
$$F_4(\lambda,t) \leq 0 \quad \Leftrightarrow \quad \lambda \in \left[\frac{15-\sqrt{33}}{32},\ \frac{1}{2}\right].$$

Finally, merging Part 1 and Part 2, produces (10) and (11). □

Lemma 2.3. *Let $f\colon I \to \mathbb{R}$ be such that $f \in C^4(I)$. Then for $x, y \in I$, $x < y$ and $\lambda \in (0, 1/2]$, we have*

$$\int_x^y f(t)dt - \frac{y-x}{2}\big(f(x)+f(y)\big) + \frac{(y-x)^2}{2}\left(\frac{1}{2} - Q\right)\big(f'(y) - f'(x)\big) + \\ + \frac{(y-x)^2}{2} Q(1-2\lambda)\big(f'(z_\lambda) - f'(z_{1-\lambda})\big) = \int_x^y H_4(\lambda, t) f^{(4)}(t)dt, \tag{16}$$

where

$$Q = \frac{1}{12\lambda(1-\lambda)}, \tag{17}$$

$$H_4(\lambda, t) = \begin{cases} p(t), & x \le t \le z_{1-\lambda}, \\ q(t), & z_{1-\lambda} \le t \le z_\lambda, \\ r(t), & z_\lambda \le t \le y \end{cases} \tag{18}$$

and

$$p(t) = \frac{(t-x)^2}{24}\left((y-t)^2 - \frac{(y-x)^2(1-2\lambda)^2}{2\lambda(1-\lambda)}\right), \tag{19}$$

$$q(t) = \frac{1}{24}t^4 - \frac{x+y}{12}t^3 + \frac{2xy - (3\lambda-2)(x^2+y^2)}{24(1-\lambda)}t^2 + \\ + \frac{(2\lambda-1)(x^3+y^3) - xy(x+y)}{24(1-\lambda)}t + \\ + \frac{\lambda(1-2\lambda)(y-x)^4 + 2(1-2\lambda)xy(y-x)^2 + 2(1-\lambda)x^2y^2}{48(1-\lambda)}, \tag{20}$$

$$r(t) = \frac{(y-t)^2}{24}\left((t-x)^2 - \frac{(y-x)^2(1-2\lambda)^2}{2\lambda(1-\lambda)}\right). \tag{21}$$

Proof. Recall (8). In order for (16) to be valid, the following set of conditions have to be satisfied:

$$\begin{aligned} & r(y) = r'(y) = p(x) = p'(x) = 0, \\ & r''(y) = p''(x) = \frac{(y-x)^2}{2}\left(\frac{1}{2} - Q\right), \\ & r'''(y) = -p'''(x) = \frac{y-x}{2}, \\ & q^{(3-k)}(z_\lambda) = r^{(3-k)}(z_\lambda), \qquad k = 0, 2, 3, \\ & q''(z_\lambda) - r''(z_\lambda) = \frac{(y-x)^2}{2} Q(1-2\lambda), \\ & p^{(3-k)}(z_{1-\lambda}) = q^{(3-k)}(z_{1-\lambda}), \qquad k = 0, 2, 3, \\ & p''(z_{1-\lambda}) - q''(z_{1-\lambda}) = -\frac{(y-x)^2}{2} Q(1-2\lambda). \end{aligned}$$

The unique solution to this system of linear equations is:

$$a_3 = b_3 = c_3 = -\frac{x+y}{12}, \qquad Q = \frac{1}{12\lambda(1-\lambda)},$$

$$a_2 = c_2 = \frac{6\lambda(1-\lambda)(x^2+y^2)-(y-x)^2}{48\lambda(1-\lambda)}, \quad b_2 = \frac{2xy-(3\lambda-2)(x^2+y^2)}{24(1-\lambda)},$$

$$a_1 = \frac{(1-2\lambda)^2 x(y-x)^2 - 2\lambda(1-\lambda)xy(x+y)}{24\lambda(1-\lambda)},$$

$$b_1 = \frac{(2\lambda-1)(x^3+y^3)-xy(x+y)}{24(1-\lambda)},$$

$$c_1 = \frac{(1-2\lambda)^2 y(y-x)^2 - 2\lambda(1-\lambda)xy(x+y)}{24\lambda(1-\lambda)},$$

$$a_0 = \frac{2\lambda(1-\lambda)x^2y^2-(1-2\lambda)^2x^2(y-x)^2}{48\lambda(1-\lambda)},$$

$$b_0 = \frac{\lambda(1-2\lambda)(y-x)^4+2(1-2\lambda)xy(y-x)^2+2(1-\lambda)x^2y^2}{48(1-\lambda)},$$

$$c_0 = \frac{2\lambda(1-\lambda)x^2y^2-(1-2\lambda)^2y^2(y-x)^2}{48\lambda(1-\lambda)}. \qquad \square$$

Remark 2.4. Note that the weight Q in (3) obtained in Lemma 2.1 is the same as the one in (17) obtained in Lemma 2.3.

Remark 2.5. Using the Euler identity (see [3]) and a technique similar to the one applied in [1], it can be shown that

$$\begin{aligned} H_4(\lambda, t) \;=\; & \frac{(y-x)^4}{24}\Bigg\{B_4\left(\frac{y-t}{y-x}\right) - B_4 \\ & - \frac{1-2\lambda}{6\lambda(1-\lambda)}\left[B_3^*\left(\frac{z_\lambda - t}{y-x}\right) - B_3^*\left(\frac{z_{1-\lambda}-t}{y-x}\right) - 2B_3(1-\lambda)\right]\Bigg\} \end{aligned}$$

Lemma 2.6. *Let $x, y \in \mathbb{R}$, $x < y$. The function $H_4\colon (0, 1/2] \times [x,y] \to \mathbb{R}$ defined in* (18)*, with p, q, r as in* (19), (20), (21), *has constant sign for $\lambda \in \big(0,\ 1/2-\sqrt{3}/6\big] \cup \big[15/32-\sqrt{33}/32,\ 1/2\big]$ and $t \in [x,y]$. More precisely, we have*

$$H_4(\lambda, t) \le 0, \qquad \textit{for } \lambda \in \left(0,\ \frac{3-\sqrt{3}}{6}\right] \textit{ and } t \in [x,y], \tag{22}$$

while

$$H_4(\lambda, t) \ge 0, \qquad \textit{for } \lambda \in \left[\frac{15-\sqrt{33}}{32},\ \frac{1}{2}\right] \textit{ and } t \in [x,y]. \tag{23}$$

Proof. Again we deal with a function which is symmetrical about the midpoint in t, that is, $H_4(\lambda,\ y+x-t) = H_4(\lambda,\ t)$ for $t \in [x, (x+y)/2]$, so it is enough to consider its sign for $t \in [x, (x+y)/2]$.

Part 1. Let $t \in [x,\ z_{1-\lambda}]$. Set

$$g(t) = (y-t)^2 - \frac{(y-x)^2(1-2\lambda)^2}{2\lambda(1-\lambda)}.$$

The function g is decreasing. So,

$$g(x) \le 0 \;\;\Rightarrow\;\; g(t) \le 0 \;\;\Rightarrow\;\; H_4(\lambda,\ t) \le 0,$$
$$g(z_{1-\lambda}) \ge 0 \;\;\Rightarrow\;\; g(t) \ge 0 \;\;\Rightarrow\;\; H_4(\lambda,\ t) \ge 0.$$

We have

$$g(x) = -\frac{(6\lambda^2-6\lambda+1)(y-x)^2}{2\lambda(1-\lambda)} \;\le\; 0 \;\;\Leftrightarrow\;\; \lambda \in \left(0,\ \frac{3-\sqrt{3}}{6}\right],$$

and on the other hand

$$g(z_{1-\lambda}) = -\frac{(y-x)^2}{2\lambda(1-\lambda)}\, w(\lambda) \;\ge\; 0 \;\;\Leftrightarrow\;\; w(\lambda) \le 0, \tag{24}$$

where
$$w(\lambda) = 2\lambda^4 - 6\lambda^3 + 10\lambda^2 - 6\lambda + 1. \tag{25}$$

The function w is convex for all λ, which implies that w' is increasing. Further, we have $w'(1/2-\sqrt{3}/6) < 0$ and $w'(1/2) > 0$, so from there we conclude that there exists a unique $\lambda_1 \in (1/2-\sqrt{3}/6,\ 1/2)$ such that $w'(\lambda_1) = 0$ and such that w is decreasing for $\lambda \in (0, \lambda_1)$ and increasing for $\lambda \in (\lambda_1,\ 1/2]$. Furthermore, we have $w(1/2-\sqrt{3}/6) > 0$ and $w(1/2) < 0$. This implies that there exists a unique $\lambda^* \in (1/2-\sqrt{3}/6,\ \lambda_1)$ such that $w(\lambda^*) = 0$. Thus, we conclude that (24) holds for $\lambda \in [\lambda^*,\ 1/2]$.

In conclusion of Part 1, for $t \in [x,\ z_{1-\lambda}]$ we have

$$H_4(\lambda, t) \le 0 \;\;\Leftrightarrow\;\; \lambda \in (0,\ 1/2-\sqrt{3}/6],$$
$$H_4(\lambda, t) \ge 0 \;\;\Leftrightarrow\;\; \lambda \in [\lambda^*,\ 1/2]\,, \;\; \text{where } \; \lambda^* > 1/2-\sqrt{3}/6.$$

Part 2. Let $t \in [z_{1-\lambda},\ (x+y)/2]$. Here we have

$$\frac{\partial H_4(\lambda,t)}{\partial t} = \frac{1}{6}\left(t - \frac{x+y}{2}\right)\left(t^2 - (x+y)t + xy + \frac{(1-2\lambda)(y-x)^2}{2(1-\lambda)}\right).$$

The discriminant of the quadratic polynomial is

$$D = \frac{3\lambda-1}{1-\lambda}(y-x)^2\,.$$

(1) For $\lambda \leq 1/3$, we have $D \leq 0$ and so $\partial H_4(\lambda,t)/\partial t \leq 0$, which implies that $H_4(\lambda,t)$ is decreasing in t. We have

$$H_4\left(\lambda,\ \frac{x+y}{2}\right) = -\frac{(16\lambda^2-15\lambda+3)(y-x)^4}{384(1-\lambda)} \geq 0$$
$$\Leftrightarrow\quad \lambda \in \left[\frac{15-\sqrt{33}}{32},\ \frac{1}{2}\right], \tag{26}$$

so we conclude that $H_4(\lambda,t) \geq 0$ for $\lambda \in \left[15/32-\sqrt{33}/32,\ 1/3\right]$. On the other hand, H_4 is continuous, so we recall from Part 1 that

$$H_4\left(\lambda,\ z_{1-\lambda}\right) \leq 0 \quad\Leftrightarrow\quad g(z_{1-\lambda}) \leq 0 \quad\Leftrightarrow\quad w(\lambda) \geq 0,$$

with $w(\lambda)$ as in (25). Since $w(15/32-\sqrt{33}/32) < 0$, it follows that we have $H_4(\lambda,t) \leq 0$ for $\lambda \in (0,\ \lambda^*]$, where $\lambda^* \in (1/2-\sqrt{3}/6,\ 15/32-\sqrt{33}/32)$.

(2) For $\lambda > 1/3$, we have $D > 0$ and the real zero smaller than $(x+y)/2$ is

$$t_1 = \frac{x+y}{2} - \frac{y-x}{2}\sqrt{\frac{3\lambda-1}{1-\lambda}}$$

and we have

$$t_1 \leq z_{1-\lambda} \quad\Leftrightarrow\quad v(\lambda) = 2\lambda^3-4\lambda^2+4\lambda-1 \geq 0\,.$$

The function v is increasing and $v(1/3) < 0$ and $v(1/2) > 0$. Therefore, there exists a unique λ_2 such that $v(\lambda_2) = 0$ and $\lambda_2 \in (1/3,\ 1/2)$.

(i) For $\lambda \in (1/3,\ \lambda_2)$, we have $z_{1-\lambda} < t_1 < (x+y)/2$. It follows that $\partial H_4(\lambda,t)/\partial t < 0$ for $t \in [z_{1-\lambda},\ t_1)$ and $\partial H_4(\lambda,t)/\partial t > 0$ for $t \in (t_1,\ (x+y)/2)$. Let us consider

$$H_4(\lambda,\ t_1) = -\frac{(1-2\lambda)(2\lambda^2-4\lambda+1)(y-x)^4}{96(1-\lambda)^2}.$$

It is elementary to check that $H_4(\lambda,\ t_1) \geq 0$ for $\lambda \in (1/3,\ 1/2]$ and so we conclude that $H_4(\lambda,\ t) \geq 0$ for $\lambda \in (1/3,\ \lambda_2)$.

(ii) For $\lambda \in [\lambda_2, 1/2]$, we have $\partial H_4(\lambda,t)/\partial t \geq 0$ and so $H_4(\lambda,t)$ is increasing in t. Note that from (26) it follows that $H_4\left(\lambda,\ (x+y)/2\right) \geq 0$ for these λ, so the only possibility for H_4 to have a constant sign here is to ensure that $H_4(\lambda,\ z_{1-\lambda}) \geq 0$. Since $H_4(\lambda,\ z_{1-\lambda}) \geq 0 \Leftrightarrow w(\lambda) \leq 0$ and $w(1/3) < 0$, we conclude that $H_4(\lambda,t) \geq 0$ for $\lambda \in [\lambda_2,\ 1/2]$.

In conclusion of Part 2, for $t \in [z_{1-\lambda},\ (x+y)/2]$ we have

$$H_4(\lambda,t) \le 0 \quad \Leftrightarrow \quad \lambda \in (0,\ \lambda^*], \ \text{ where } \frac{3-\sqrt{3}}{6} < \lambda^* < \frac{15-\sqrt{33}}{32},$$
$$H_4(\lambda,t) \ge 0 \quad \Leftrightarrow \quad \lambda \in \left[\frac{15-\sqrt{33}}{32},\ \frac{1}{2}\right].$$

Finally, merging Part 1 and Part 2 produces (22) and (23). □

The last lemma provides a useful characterization of Schur-convexity (see [4]).

Lemma 2.7. *Let $\Phi\colon I^n \to \mathbb{R}$ be a continuous symmetric function. If Φ is differentiable on I^n, then Φ is Schur-convex on I^n if and only if*

$$(x_i - x_j)\left(\frac{\partial \Phi}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}\right) \ge 0$$

for all $x_i, x_j \in I,\ i \neq j,\ i,j = 1,2,\ldots,n$. The function Φ is Schur-concave if and only if the reversed inequality sign holds.

3. Main result

Theorem 3.1. *Let $f\colon I \to \mathbb{R}$ be such that $f \in C^4(I)$ and let $Q = \dfrac{1}{12\lambda(1-\lambda)}$. If $\lambda \in \big(0,\ 1/2 - \sqrt{3}/6\big]$, then the following statements are equivalent:*

(i) *The function K defined in* (1) *is Schur-convex on I^2.*

(ii) *For all $x, y \in I$, $x < y$, we have*

$$\frac{1}{y-x}\int_x^y f(t)dt \ \ge\ \left(\frac{1}{2} - Q\right)\big(f(x) + f(y)\big) + Q\big(f(z_{1-\lambda}) + f(z_\lambda)\big).$$

(iii) *The function f is 4-convex on I.*

Furthermore, if $\lambda \in \big[15/32 - \sqrt{33}/32,\ 1/2\big]$, then the following statements are equivalent: (i), (ii) *and*

(iv) *The function f is 4-concave on I.*

Proof. We prove (i)⇔(iii) and (ii)⇔(iii); the proofs of (i)⇔(iv) and (ii)⇔(iv) are analogous. So, let $\lambda \in \big(0,\ 1/2 - \sqrt{3}/6\big]$.

(i)⇔(iii): Assume f is 4-convex on I, that is, let (iii) hold. In order to prove (i), that is, that K is Schur-convex on I^2, it only needs to be verified that the condition from Lemma 2.7 is valid. Simple calculation gives

$$\frac{(y-x)^2}{2}\left(\frac{\partial K}{\partial y} - \frac{\partial K}{\partial x}\right) = -\frac{(y-x)^2}{2}Q(1-2\lambda)\big(f'(z_\lambda) - f'(z_{1-\lambda})\big) +$$
$$+\frac{y-x}{2}\big(f(x) + f(y)\big) - \int_x^y f(t)dt - \frac{(y-x)^2}{2}\left(\frac{1}{2} - Q\right)\big(f'(y) - f'(x)\big).$$

From Lemma 2.3 it now follows

$$\frac{(y-x)^2}{2}\left(\frac{\partial K}{\partial y}-\frac{\partial K}{\partial x}\right)=\int_x^y -H_4(\lambda,t)f^{(4)}(t)dt. \tag{27}$$

Applying Lemma 2.6 yields (i). Conversely, assume that K is Schur-convex on I^2. Applying Lemma 2.7 yields that both expressions in (27) are positive. Since we know from Lemma 2.6 that $H_4(\lambda,t)$ has constant sign for $t\in[x,y]$, we can apply the Mean Value Theorem to deduce that there exists a point $\xi\in[x,y]$ such that

$$f^{(4)}(\xi)\int_x^y -H_4(\lambda,t)dt \;\geq\; 0,$$

for all $x,y\in I,\ x<y$. Since $f\in C^4(I)$ and x and y are arbitrary, we now conclude that f is 4-convex, which proves (iii).

(ii)⇔(iii): Assume f is 4-convex on I, that is, let (iii) hold. From Lemma 2.2 it follows that

$$(y-x)\,K(x,y)=\int_x^y F_4(\lambda,t)f^{(4)}(t)dt. \tag{28}$$

Applying Lemma 2.2 implies (ii). Conversely, let (ii) hold, that is, assume $K(x,y)$ is positive for all $x,y\in I,\ x<y$. This implies that expressions on both sides of (28) are positive and since from Lemma 2.2 we know that $F_4(\lambda,t)$ has constant (positive) sign for $t\in[x,y]$, we can again apply the Mean Value Theorem to deduce that there exists a point $\eta\in[x,y]$ such that

$$f^{(4)}(\eta)\int_x^y F_4(\lambda,t)dt \;\geq\; 0,$$

for all $x,y\in I,\ x<y$. Since $f\in C^4(I)$ and x and y are arbitrary, we now conclude that f is 4-convex, which proves (iii). □

Remark 3.2. Note that the implication (i)⇒(ii) can also be proven directly. Assume K is Schur-convex on I^2. Since $\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(x,y)$, we have in consequence $K\left(\frac{x+y}{2},\frac{x+y}{2}\right)\leq K(x,y)$ for all x, $y\in I$, $x<y$, that is,

$$0\;\leq\;\frac{1}{y-x}\int_x^y f(t)dt\;-\;\left(\frac{1}{2}-Q\right)\big(f(x)+f(y)\big)-Q\big(f(z_{1-\lambda})+f(z_\lambda)\big).$$

Remark 3.3. Analogous results still hold if the function K is assumed to be Schur-concave and the function f 4-concave (that is, 4-convex in the second part) - simply apply Theorem 3.1 for $-K$ and $-f$.

Remark 3.4. Since n-convex functions are continuous, they can be represented as a uniform limit of a sequence of the corresponding Bernstein polynomials (see for example [5]). The Bernstein polynomials of n-convex functions are also n-convex. Also, if the corresponding Bernstein polynomials are n-convex, so is the function f. Having this in mind, the implications (iii)⇒(i)⇒(ii) and (iv)⇒(i)⇒(ii) in Theorem 3.1 can be proven without the assumption that $f \in C^4(I)$. The conjecture is that the implications (i)⇒(iii), (i)⇒(iv), (ii)⇒(iii), and (ii)⇒(iv) remain valid even without the regularity condition $f \in C^4(I)$.

Remark 3.5. Theorem 3.1 contains results for several well known quadrature formulas.

- for $\lambda = 1/2$ one obtains the error of the Simpson formula:

$$\frac{1}{y-x}\int_x^y f(t)dt - \frac{1}{6}f(x) - \frac{2}{3}f\left(\frac{x+y}{2}\right) - \frac{1}{6}f(y)$$

- for $\lambda = 1/3$ one obtains the error of the Simpson 3/8 formula:

$$\frac{1}{y-x}\int_x^y f(t)dt - \frac{1}{8}f(x) - \frac{3}{8}f\left(\frac{2x+y}{3}\right) - \frac{3}{8}f\left(\frac{x+2y}{3}\right) - \frac{1}{8}f(y).$$

- for $\lambda = 1/2 - \sqrt{3}/6 \quad \Leftrightarrow \quad Q = 1/2$ one obtains the error of the Gauss two-point formula, using $a = 3 + \sqrt{3}$ and $b = 3 - \sqrt{3}$:

$$\frac{1}{y-x}\int_x^y f(t)dt - \frac{1}{2}f\left(\frac{ax+by}{6}\right) - \frac{1}{2}f\left(\frac{bx+ay}{6}\right).$$

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