

Alexandrov Curvature of Convex Hypersurfaces in Hilbert Space Forms

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Received: April 3, 2016

Accepted: May 6, 2017

It is shown that convex hypersurfaces in Hilbert space forms have corresponding lower bounds on Alexandrov curvature. This extends earlier work of Buyalo, Alexander, Kapovitch, and Petrunin for convex hypersurfaces in Riemannian manifolds of finite dimension.

Keywords: Convex hypersurfaces, Alexandrov curvature, Hilbert spaces.

2010 Mathematics Subject Classification: 53C45, 53C23.

1. Introduction

One of the primary applications of metric geometry is the extension of results from Riemannian geometry to more general settings, such as infinite-dimensional manifolds. We will consider here generalization of classical results about convex hypersurfaces in a Riemannian manifold. For the infinite-dimensional setting, we will restrict our attention to generalized space forms: Hilbert manifolds of constant curvature. For $\kappa = 0$, a Hilbert space form is simply a smooth Hilbert manifold based on a model Hilbert space, such that each chart is a local isometry. For $\kappa > 0$, the isometries come instead from the model space of a sphere of radius $1/\kappa^2$ in a Hilbert space. Finally, if $\kappa < 0$, the corresponding model space is hyperbolic, obtained via direct generalization of any of the standard models of hyperbolic n -space. We do not require Hilbert spaces to be separable.

Theorem 1.1. *If C is an open set in a Hilbert space form H of curvature κ and $\overline{C}$ is locally convex, then ∂C is a Alexandrov space of curvature $\geq \kappa$ under the induced length metric.*

Questions of this sort go back to [2], where Alexandrov defined Alexandrov curvature and showed that it characterizes boundaries of locally convex bodies in $\mathbb{R}^3$. This was generalized by Buyalo to the case of locally convex sets of full dimension in a Riemannian manifold in [5], where existence of a lower bound

on curvature is shown. Alexander, Kapovitch, and Petrunin verified the result for the same lower bound on curvature of the ambient manifold in [1].

The proof of Theorem 1.1 relies on approximating ∂C by smooth manifolds, where the connection between curvature and convexity is well understood. Due to the infinite dimension of H , we cannot smooth ∂C by integrating over H against a mollifier. As currently known C^∞ smoothing operators for infinite dimensional spaces do not preserve convexity, we proceed by integrating over a suitably chosen finite dimensional subspace. Lemma 3.2 shows this can be done in such a way that the curvature of ∂C is controlled by the curvature of smooth, finite-dimensional approximating manifolds. A similar approximation of infinite-dimensional curvature by finite dimensional curvature is outlined in [6]. We specify convexity as a local notion in order to allow, for example, large metric balls in flat tori. The argument is local in nature, so this generality comes at no cost.

Definitions are set up in Section 2. The flat case of Theorem 1.1 is covered in Sections 3 and 4, with the spherical case dealt with in Section 5. The proof of Theorem 1.1 for $\kappa \geq 0$ is intended to be self-contained. The hyperbolic case is handled in Section 6 where dimension reduction is combined with the Riemannian result.

2. Basic definitions

We begin by defining curvature in the sense of Alexandrov. There are several equivalent definitions. We will find it most convenient to work with comparison angles.

Definition 2.1. For three points x, y, z in a metric space (X, d) and $\kappa \in \mathbb{R}$, the *comparison angle* $\tilde{\angle}_\kappa xyz$ is defined as the corresponding angle in a triangle of side lengths $d(x, y), d(y, z), d(z, x)$ in the constant curvature κ plane, hyperbolic plane, or sphere. (In the case $\kappa > 0$, the comparison angle is only defined for triangles of perimeter $< 2\pi/\sqrt{\kappa}$.)

(X, d) is called a *length space* if the distance between any two points equals the infimum of the lengths of paths between them.

Definition 2.2. A length space (X, d) is said to have *Alexandrov curvature* $\geq \kappa$ if X is locally complete and every $x \in X$ has a neighborhood U_x which satisfies the quadruple condition:

$$\tilde{\angle}_\kappa bac + \tilde{\angle}_\kappa cap + \tilde{\angle}_\kappa pab \leq 2\pi$$

for any quadruple $(a; b, c, p)$ of distinct points in U_x .

If X is a Riemannian manifold, then Alexandrov curvature $\geq \kappa$ is equivalent to sectional curvature $\geq \kappa$.

It will also be helpful to fix notation for polygonal paths.

Definition 2.3. For two points p, q in a vector space V , $\sigma_{pq}: [0, 1] \rightarrow V$ denotes the constant speed linear path $\sigma_{pq}(t) = (1 - t)p + tq$.

Definition 2.4. A path $\tau: [0, 1] \rightarrow V$ is called a *polygonal path* if it can be written in the form

$$\tau(t) = \sum_{i=1}^{k-1} \sigma_{p_i p_{i+1}}(kt - i) 1_{[i/k, (i+1)/k]}(t)$$

for some set of points $p_1, \dots, p_k \in V$. Here 1_A denotes the characteristic function of the set A .

3. Approximation by smooth manifolds

In this section, we prove two technical lemmas which allow us to approximate C^1 convex functions f on a Hilbert space by convex functions that are smooth on a finite-dimensional linear subspace. This enables us to control the Alexandrov curvature of graph_f , the graph of f in $H \times \mathbb{R}$, via the sectional curvature of the approximating smooth graphs.

Lemma 3.1. *Let $f: V \rightarrow (X, d)$ be a λ -bi-Lipschitz map from a Banach space V onto a metric space (X, d) . For any rectifiable curve $\sigma: [0, 1] \rightarrow X$ and any $\varepsilon > 0$, there exists a polygonal path $\tau: [0, 1] \rightarrow V$ such that $f \circ \tau(0) = \sigma(0)$, $f \circ \tau(1) = \sigma(1)$, $\forall t \in [0, 1]$, $|\sigma(t) - f \circ \tau(t)| < \varepsilon$ and $|l(\sigma) - l(f \circ \tau)| < \varepsilon$.*

Proof. For each rectifiable curve $\sigma_0: [0, 1] \rightarrow X$ and $\varepsilon > 0$, define

$$B_\varepsilon(\sigma_0) = \{\sigma: [0, 1] \rightarrow X; \forall t \in [0, 1], d(\sigma_0(t), \sigma(t)) < \varepsilon, |l(\sigma_0) - l(\sigma)| < \varepsilon\}.$$

For each rectifiable curve $\sigma_0: [0, 1] \rightarrow V$ and $\varepsilon > 0$, define

$$\beta_\varepsilon(\sigma_0) = \{\sigma: [0, 1] \rightarrow V; \forall t \in [0, 1], |\sigma_0(t) - \sigma(t)| < \varepsilon, |l(\sigma_0) - l(\sigma)| < \varepsilon\}.$$

Fix a rectifiable curve $\sigma_0: [0, 1] \rightarrow V$ and $\varepsilon > 0$. For any $\sigma \in \beta_\varepsilon(\sigma_0)$, for all $t \in [0, 1]$,

$$|\sigma_0(t) - \sigma(t)| < \varepsilon \implies |f \circ \sigma_0(t) - f \circ \sigma(t)| < \lambda\varepsilon.$$

Furthermore,

$$\begin{aligned} |l(f \circ \sigma_0) - l(f \circ \sigma)| &\leq |l(\sigma_0) - l(\sigma)| + |l(f \circ \sigma_0) - l(\sigma_0)| + |l(f \circ \sigma) - l(\sigma)| \\ &< \varepsilon + l(f \circ \sigma_0) + l(\sigma_0) + l(f \circ \sigma) + l(\sigma) \\ &\leq \varepsilon + \lambda l(\sigma_0) + l(\sigma_0) + \lambda l(\sigma) + l(\sigma) \\ &< \varepsilon + (\lambda + 1)l(\sigma_0) + (\lambda + 1)(l(\sigma_0) + \varepsilon) \\ &< 2(\lambda + 1)(\varepsilon + l(\sigma_0)). \end{aligned}$$

So for $\varepsilon' = 2(\lambda + 1)(\varepsilon + l(\sigma_0))$ we get $\beta_\varepsilon(\sigma_0) \subset f^{-1}(B_{\varepsilon'}(f \circ \sigma_0))$. By a similar argument, for any rectifiable curve $\sigma_0: [0, 1] \rightarrow X$ and $\varepsilon > 0$,

$$f^{-1}(B_\varepsilon(\sigma_0)) \subset \beta_{\varepsilon'}(f^{-1} \circ \sigma_0),$$

for $\varepsilon' = 2(\lambda + 1)(\varepsilon + l(\sigma_0))$. Thus the β 's and $f^{-1}(B)$'s determine equivalent topologies on the space of rectifiable curves $\sigma: [0, 1] \rightarrow V$. Polygonal paths are dense under the β -topology, so they are dense under the $f^{-1}(B)$ -topology. $\square$

We restrict to Hilbert spaces in the next lemma because we are interested in spaces with curvature bounds. For a Banach space with bounded sectional curvature, the homothety isomorphism shows all positive multiples of the curvature bound to hold as well. This provides a curvature bound of 0, leading to the parallelogram law and proving that the Banach space is in fact a Hilbert space.

Lemma 3.2. *Let $f: \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz convex function, where Ω is a domain in a Hilbert space H . For any $x_0 \in \Omega$, there exists $R > 0$ such that Y , the graph of f over $B_R(x_0)$, satisfies the quadruple condition*

$$\tilde{Z}_0 bac + \tilde{Z}_0 cap + \tilde{Z}_0 pab \leq 2\pi$$

for any quadruple $(a; b, c, p)$ of distinct points, under the induced length metric d from $H \times \mathbb{R}$.

Proof. The function f is locally Lipschitz, hence Lipschitz for some Lipschitz constant $L \geq 1$. Local curvature bounds of the same type imply global curvature bounds, so we may assume without loss of generality that f is L -Lipschitz in all of Ω . Let $\hat{f}: \Omega \rightarrow \hat{f}(\Omega) \subset \text{graph}_f$ be defined by $\hat{f}(x) = (x, f(x))$, and note that $\hat{f}$ is $\sqrt{1 + L^2}$ -bi-Lipschitz. Choose $R > 0$ such that $B_{3R}(x_0) \subset \Omega$. Suppose that $(a; b, c, p)$ is a quadruple of distinct points such that

$$\tilde{Z}_0 bac + \tilde{Z}_0 cap + \tilde{Z}_0 pab = 2\pi + \varepsilon_0 > 2\pi,$$

where $(a; b, c, p) = (\hat{f}(a'); \hat{f}(b'), \hat{f}(c'), \hat{f}(p'))$ and $a', b', c', p' \in B_R(x_0)$. The comparison angles vary continuously in the intrinsic distances, so there exists $\varepsilon > 0$ such that if $(A; B, C, D)$ is a quadruple of points in some other metric space (X_1, d_1) with

$$\begin{aligned} |d(a, b) - d_1(A, B)| &< \varepsilon, & |d(a, c) - d_1(A, C)| &< \varepsilon, & |d(a, p) - d_1(A, P)| &< \varepsilon, \\ |d(b, c) - d_1(B, C)| &< \varepsilon, & |d(b, p) - d_1(B, P)| &< \varepsilon, & |d(c, p) - d_1(C, P)| &< \varepsilon, \end{aligned}$$

then

$$\tilde{Z}_0 BAC + \tilde{Z}_0 CAP + \tilde{Z}_0 PAB = 2\pi + (\varepsilon_0/2) > 2\pi.$$

By Lemma 3.1, we may approximate $d(a, b)$ by the length of the image under $\hat{f}$ of a polygonal path τ_1 determined by some points $a' = q_1, q_2, \dots, q_{k_1-1}$ and

$b' = q_{k_1} \in B_{2R}(x_0)$ such that

$$d(a, b) + (\varepsilon/3) \geq \sum_{i=1}^{k_1-1} l(\widehat{f} \circ \sigma_{q_i q_{i+1}}) = l(\widehat{f} \circ \tau_1) \geq d(a, b).$$

Similarly, we may approximate $d(a, c)$ by the image under $\widehat{f}$ of a polygonal path determined by points $a' = q_{k_1+1}, q_{k_1+2}, \dots, c' = q_{k_2} \in B_{2R}(x_0)$ such that

$$d(a, c) + (\varepsilon/3) \geq \sum_{i=k_1+1}^{k_2-1} l(\widehat{f} \circ \sigma_{q_i q_{i+1}}) \geq d(a, c).$$

Continue in this manner choosing $q_{k_2+1}, q_{k_2+2}, \dots, q_{k_3}, \dots, q_{k_6}$ to approximate the remaining four intrinsic distances.

The $k_6 + 1$ points $q_1, \dots, q_{k_6}, x_0$ lie in a k_6 -dimensional subspace of H , which we will identify as $\mathbb{R}^n$, $n = k_6$. Let $\varphi_\delta: \mathbb{R}^n \rightarrow \mathbb{R}$ be the standard C^∞ mollifier supported on the δ -ball, and define $f_\delta: B_{5R/2}(x_0) \rightarrow \mathbb{R}$ by $f_\delta = f * \varphi_\delta$, where the convolution occurs in the $\mathbb{R}^n$ -variables and $\delta < R/2$. Let $\widehat{f}_\delta(x) = (x, f_\delta(x))$. As f is assumed to be convex and L -Lipschitz, it is easy to check the following properties:

- (1) $f_\delta|_{B_{2LR}(x_0) \cap \mathbb{R}^n}$ is C^∞ ;
- (2) $f_\delta|_{B_{2LR}(x_0) \cap \mathbb{R}^n}$ is L -Lipschitz;
- (3) $f_\delta \rightarrow f$ pointwise as $\delta \rightarrow 0$;
- (4) For every rectifiable curve $\sigma: [0, 1] \rightarrow B_{2R}(x_0)$, $l(\widehat{f}_\delta \circ \sigma) \rightarrow l(\widehat{f} \circ \sigma)$. This convergence is uniform on sets $\{\sigma: [0, 1] \rightarrow B_{2R}(x_0); l(\sigma) < C\}$ with $C \in \mathbb{R}$;
- (5) f_δ is convex.

Let Y_δ denote the graph of f_δ over $B_{2R}(x_0)$ with metric d_δ induced by $H \times \mathbb{R}$, and let $Y_{\delta,n}$ denote the graph of f_δ over $B_{2R}(x_0) \cap \mathbb{R}^n$ with metric $d_{\delta,n}$ induced by $\mathbb{R}^n \times \mathbb{R}$. Note that $f_\delta|_{B_{2R}(x_0) \cap \mathbb{R}^n}$ is a C^∞ convex function over a domain in $\mathbb{R}^n$, so $Y_{\delta,n}$ is a Riemannian manifold of nonnegative sectional curvature. In particular, it satisfies the quadruple condition.

We supposed that $(a; b, c, p)$ in Y does not satisfy the quadruple condition. We will obtain a contradiction by showing

$$\begin{aligned} |d(a, b) - d_{\delta,n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(b'))| &< \varepsilon, & |d(a, c) - d_{\delta,n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(c'))| &< \varepsilon, \\ |d(a, p) - d_{\delta,n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(p'))| &< \varepsilon, & |d(b, c) - d_{\delta,n}(\widehat{f}_\delta(b'), \widehat{f}_\delta(c'))| &< \varepsilon, \\ |d(b, p) - d_{\delta,n}(\widehat{f}_\delta(b'), \widehat{f}_\delta(p'))| &< \varepsilon, & |d(c, p) - d_{\delta,n}(\widehat{f}_\delta(c'), \widehat{f}_\delta(p'))| &< \varepsilon. \end{aligned}$$

Let $C = d(a, b) + d(a, c) + \dots + d(c, p) + \varepsilon$. Choosing δ_0 small with respect to C , we have for all $\delta < \delta_0$,

$$\tau \in \{\sigma: [0, 1] \rightarrow B_{2R}(x_0); l(\sigma) < C\} \implies |l(\widehat{f}_\delta \circ \tau) - l(\widehat{f} \circ \tau)| \leq \varepsilon/3.$$

Recall that τ_1 is the polygonal path determined by $q_1, \dots, q_{k_1}$.

$$l(\tau_1) \leq l(\widehat{f} \circ \tau_1) = \sum_{i=1}^{k_1-1} l(\widehat{f} \circ \sigma_{q_i q_{i+1}}) \leq d(a, b) + (\varepsilon/3) < C,$$

so $l(\widehat{f}_\delta \circ \tau_1) \leq l(\widehat{f} \circ \tau_1) + (\varepsilon/3)$ for $\delta < \delta_0$. $\widehat{f}_\delta \circ \tau_1: [0, 1] \rightarrow Y_{\delta, n}$ is a path from $\widehat{f}_\delta(a')$ to $\widehat{f}_\delta(b')$, so

$$d_{\delta, n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(b')) \leq l(\widehat{f}_\delta \circ \tau_1) \leq l(\widehat{f} \circ \tau_1) + (\varepsilon/3) \leq d(a, b) + (2\varepsilon/3).$$

Applying Lemma 3.1 again, choose $\tau_2: [0, 1] \rightarrow B_{2R}(x_0)\tilde{Z}_0 p\mathbb{R}^n$ such that

$$d_{\delta, n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(b')) \geq l(\widehat{f}_\delta \circ \tau_2) - (\varepsilon/6).$$

Note that

$$l(\tau_2) \leq l(\widehat{f}_\delta \circ \tau_2) \leq d_{\delta, n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(b')) + (\varepsilon/6) \leq d(a, b) + (5\varepsilon/6) < C.$$

For $\delta < \delta_0$,

$$l(\widehat{f}_\delta \circ \tau_2) \geq l(\widehat{f} \circ \tau_2) - (\varepsilon/3),$$

so

$$d_{\delta, n}(\widehat{f}_\delta(a'), \widehat{f}_\delta(b')) > l(\widehat{f} \circ \tau_2) - \varepsilon \geq d(a, b) - \varepsilon.$$

The remaining inequalities follow in a similar manner, for the same choice of C and δ_0 . So for $\delta < \delta_0$, the quadruple $(\widehat{f}_\delta(a'); \widehat{f}_\delta(b'), \widehat{f}_\delta(c'), \widehat{f}_\delta(p'))$ violates the quadruple condition in the Riemannian manifold of nonnegative sectional curvature $Y_{\delta, n}$. Therefore our original assumption is false and Y satisfies the quadruple condition. $\square$

4. The flat case

Proposition 4.1. *If C is an open set in a Hilbert space H and $\overline{C}$ is locally convex, then ∂C is a Alexandrov space of curvature ≥ 0 under the induced length metric.*

Proof. We must prove the quadruple condition holds in a neighborhood of every $x_0 \in \partial C$. Let $C' = B_{2\rho}(x_0) \cap C$, where ρ is chosen small enough to make C' convex. Note that the intrinsic balls of radius ρ about x_0 are the same for C and C' . Choose a point $y \in C'$, and $r \in (0, \rho/2)$ such that $B_{2r}(y) \subset C'$. Let H' be the hyperplane through x_0 with normal vector $y - x_0$. For any $x \in H' \cap B_{2r}(x_0)$, let L_x be the line through x spanned by $y - x_0$. $L_x \cap C'$ is convex and C' is open and bounded, so $L_x \cap C'$ is a bounded interval. $x + (y - x_0) \in L_x \cap C'$, so $L_x \cap C' \neq \emptyset$. Considering $y - x_0$ as the upward direction, let $f(x)$ denote the $\mathbb{R}$ -coordinate of the bottom endpoint of $L_x \cap C'$ in $H' \times \mathbb{R}$. $f: H' \cap B_{2r}(x_0) \rightarrow \mathbb{R}$

is then a convex function, as the epigraph is convex. Furthermore, the graph of f is a neighborhood of x_0 in $\partial C'$, and thus also in ∂C since $2r < \rho$.

Using convexity and C^1 smoothness of $d(x, C')$ away from C' , we may approximate f by C^1 convex functions g_ε . By Lemma 3.2, the graph of g_ε over $H' \cap B_R(x_0)$ satisfies the quadruple condition for $R = r/3$. The graph of f over $H' \cap B_R(x_0)$ then satisfies the quadruple condition by continuity. $\square$

Curvature bounds are local in nature, allowing an immediate generalization to manifolds.

Corollary 4.2. *If C is an open set in a flat Hilbert space form H of curvature and $\overline{C}$ is locally convex, then ∂C is a Alexandrov space of curvature ≥ 0 under the induced length metric.*

Assuming only local convexity for $\overline{C}$ allows sets like a closed ball of radius $3/8$ in a flat torus $\mathbb{R}^\infty/\mathbb{Z}^\infty$.

5. The spherical case

For the unit sphere S^∞ centered at the origin in a Hilbert space H , if $C \subset S^\infty$ is a connected open set, strictly contained in the upper hemisphere, with $\overline{C}$ convex, then the cone K generated by the set $\overline{C}$,

$$K = \{\lambda x; \lambda \geq 0, x \in \overline{C}\},$$

is also convex. By Proposition 4.1, ∂K is an Alexandrov space of curvature ≥ 0 . ∂K is the cone over ∂C , so ∂C is an Alexandrov space of curvature ≥ 1 by a well-known result in Alexandrov geometry. (See Theorem 4.7.1 in [4], for example.)

Theorem 1.1 for $\kappa > 0$ now follows by appropriate localization and rescaling.

6. The hyperbolic case

As above, it suffices to consider a small ball B in our hyperbolic model space $\mathbb{H}^\infty$. Taking the Poincaré half-space model, for example, we see the Lipschitz constant of a chart for B is bounded. This allows us to continue to use Lemma 3.1.

By Azagra and Ferrera's generalization [3] of Lasry and Lion's inf-sup convolution [7], we may approximate ∂C by C^1 convex hypersurfaces. It therefore suffices to prove Theorem 1.1 for $C^1 \partial C$.

Using our coordinate chart, we may locally view ∂C as the graph of a function in Hilbert space. Mimicking the argument for Lemma 3.2, we then reduce to a k_6 -dimensional subspace $\mathbb{H}^{k_6}$ of $\mathbb{H}^\infty$ corresponding to a hyperbolic polygonal approximation. The key here is that $k_6 + 1$ points in a hyperbolic space determine a k_6 -dimensional hyperbolic space. Having reduced to finite dimension,

we obtain Alexandrov curvature $\geq \kappa$ by [1]. (We cannot simplify to the smooth case directly due to the lack of a powerful enough convolution operator.)

In order to approximate the original comparison angle sum, note that we no longer have an $\widehat{f}_\delta$ due to the application of [1]. We are left only to estimate terms of the form $|d(a, b) - d_n(a, b)|$. By adding more dimensions back into $\mathbb{R}^n$, we obtain $d(a, b)$ as the sup of all such $d_n(a, b)$ as $n \rightarrow \infty$. Increasing the size of $\mathbb{R}^n$ does not alter a, b, c, p, ε , so for large enough $\mathbb{R}^n$ we obtain the same contradiction, proving the last case of Theorem 1.1.

The hyperbolic argument works in more general settings. The Fubini-Study metric on projective Hilbert space, for example, has enough symmetry locally for smooth finite-dimensional convex hulls of any finite collection of close enough points. In general, one has:

Theorem 6.1. *If C is an open set in a Riemannian Hilbert manifold H of sectional curvature $\geq \kappa$, $\overline{C}$ is locally convex, and every finite collection of points in H is contained in a convex finite-dimensional submanifold, then ∂C is a Alexandrov space of curvature $\geq \kappa$ under the induced length metric.*

Unfortunately, understanding with any generality the dimension of convex hulls even in finite-dimensional spaces remains an open problem.

References

- [1] S. Alexander, V. Kapovitch, A. Petrunin: An optimal lower curvature bound for convex hypersurfaces in Riemannian manifolds, *Illinois J. Math.* 52(3) (2008) 1031–1033.
- [2] A.D. Alexandrov: *Vnutrennyaya Geometriya Vypuklykh Poverhnostei*, OGIZ, Moscow-Leningrad (1948).
- [3] D. Azagra, J. Ferrera: Inf-convolution and regularization of convex functions on Riemannian manifolds of nonpositive curvature, *Rev. Mat. Complut.* 19(2) (2006) 323–345.
- [4] D. Burago, Y. Burago, S. Ivanov: *A Course in Metric Geometry*, *Grad. Studies in Mathematics* 33, American Math. Society, Providence (2001).
- [5] S. V. Buyalo: Shortest paths on convex hypersurfaces of a Riemannian space, *Studies in topology II*, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)* 66 (1976) 114–132, 207.
- [6] S. Halbeisen: On tangent cones of Alexandrov spaces with curvature bounded below, *Manuscripta Math.* 103(2) (2000) 169–182.
- [7] J.-M. Lasry, P.-L. Lions: A remark on regularization in Hilbert spaces, *Israel J. Math.* 55(3) (1986) 257–266.