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Diametral Diameter Two Properties in Banach Spaces

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The aim of this note is to provide several new variants of the diameter two properties for Banach spaces. We study such properties looking for the abundance of diametral points, which holds in the setting of Banach spaces with the Daugavet property, for example. For this, we introduce the diametral diameter two properties in Banach spaces, showing for these new properties stability results, inheritance to subspaces and characterizations in terms of finite rank projections.

Keywords: Diameter two properties, diametral points, finite rank projections.

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1. Introduction

We recall that a slice in the unit ball of a Banach space X is a nonempty intersection of the unit ball with an open half-space. The Banach space X satisfies the *strong diameter two property* (SD2P), respectively *diameter two property*

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(D2P), *slice-diameter two property* (LD2P), if every finite convex combination of slices, respectively every nonempty relatively weakly open subset, every slice, in the unit ball of X has diameter 2. The weak-star slice diameter two property (w^*-LD2P), weak-star diameter two property (w^*-D2P) and weak-star strong diameter two property (w^*-SD2P) for a dual Banach space are defined as usual, changing slices by w^*-slices and weakly open subsets by w^*- open subsets in the unit ball. It is known that the above six properties are extremely different as it is proved in [6].

Even though the theory for Banach spaces with diameter two properties is quite young, a lot of nice results have appeared in the last few years (e.g. [1, 5, 6, 7, 11]). Moreover, it turns out that there are quite a lot of examples of Banach spaces with such properties as infinite-dimensional C^*-algebras [5], non-reflexive M-embedded spaces [14] and Daugavet spaces [15]. The latest example is quite important because Banach spaces with the Daugavet property actually satisfy the diameter two properties in a stronger way. Indeed, as it was pointed out in [13], Banach spaces with Daugavet property verify that each slice of the unit ball S has diameter two and each norm-one element of S is *diametral*, i.e. given $x \in S \cap S_X$ it follows that

$$diam(S) = \sup_{y \in S} \|y - x\|. \tag{1}$$

We will say that a Banach space X has the *diametral local diameter two property* (DLD2P) whenever X verifies the above condition. In dual Banach spaces, we will define the *weak-star diametral local diameter two property* (w^*-DLD2P) in the natural way replacing the concept of slice with the one of weak-star slice. Notice that this property was originally introduced in [13], under the name *space with bad projections* and also studied in [2, 4], under the name *local diameter 2 property +*. It is known that this property is different from the Daugavet property [13], that it is stable by taking ℓ_p sums [13] and that it is inherited by almost isometric ideals [4]. Let X be a Banach space and $Y \subseteq X$ subspace. According to [3], we will say that Y is an *almost isometric ideal in* X if for each $\varepsilon > 0$ and $E \subseteq X$ a finite-dimensional subspace there exists a linear and bounded operator $T\colon E \longrightarrow Y$ satisfying the following conditions:

1. $T(e) = e$ for each $e \in E \cap Y$.
2. For each $e \in E$ one has $\dfrac{1}{1+\varepsilon}\|e\| \leq \|T(e)\| \leq (1+\varepsilon)\|e\|$.

Note that, by a perturbation argument, a subspace is an almost isometric ideal if, and only if, its closure is an almost isometric ideal, so there is no loss of generality by assuming that such subspaces are closed. Moreover, in [3] it is proved that each diameter two property as well as Daugavet property are inherited to almost isometric ideals from the whole space.

The paper is organized as follows. In Section 2 we will define the *diametral diameter two property* in Definition 2.1 and, after studying connections with classical diameter two properties and Daugavet one, we will exhibit characterizations of this new property in terms of finite-rank projections and weakly convergent nets. We will also study how this property is preserved by taking ℓ_p sums and by taking closed subspaces. In Section 3 we will define the *diametral strong diameter two property* in a natural way (see Definition 3.1). Again we will get stability results by taking ℓ_p sums and by considering closed subspaces which remind us a behavior very close to the one of the Daugavet property. Finally, Section 4 is devoted to exhibit some open problems and remarks.

We shall introduce some notation. We consider real Banach spaces. B_X (resp. S_X) denotes the *closed unit ball* (resp. sphere) of the Banach space X. If Y is a subspace of a Banach space X,X^* stands for the *dual space* of X and Y° denotes the *annihilator* of Y. A *slice* of a bounded subset C of X is a set of the form

$$S(C, f, \alpha) := \{x \in C : \ f(x) > M - \alpha\},$$

where $f \in X^*$, $f \neq 0$, $M = \sup_{x \in C} f(x)$ and $\alpha > 0$. If $X = Y^*$ is a dual space for some Banach space Y and C is a bounded subset of X, a w^*-*slice* of C is a set of the form

$$S(C, y, \alpha) := \{f \in C : f(y) > M - \alpha\},$$

where $y \in Y$, $y \neq 0$, $M = \sup_{f \in C} f(y)$ and $\alpha > 0$. w (resp. w^*) denotes the *weak* (resp. *weak-star*) *topology* of a Banach space.

Finally, we recall that a Banach space X is called *strongly regular* if every nonempty closed, convex and bounded subset of X has finite convex combinations of slices (equivalently relatively weakly open subsets) with arbitrarily small diameter. Also, a point a in a closed, convex and bounded subset C of X is called a *point of strong regularity* for C if there are finite convex combinations of slices of C containing a with arbitrarily small diameter. The concept of strong regularity has been well studied in [10] and it is a weaker property than the well known Radon-Nikodym property.

2. Diametral diameter two property and stability results

We shall start by giving the following definition.

Definition 2.1. Let X be a Banach space. We will say that X has the *diametral diameter two property* (DD2P) if given a non-empty relatively weakly open subset W of B_X, $x \in W \cap S_X$ and $\varepsilon \in \mathbb{R}^+$ there exists $y \in W$ such that

$$\|x - y\| > 2 - \varepsilon. \tag{2}$$

If X is a dual Banach space we will say that X has the *weak-star diametral diameter two property* (w^*-DD2P) if given a non-empty relatively weakly-star open subset W of B_X, $x \in W \cap S_X$ and $\varepsilon \in \mathbb{R}^+$ there exists $y \in W$ satisfying (2).

From [15, Lemma 2.3] we get that each Banach space enjoying the Daugavet property satisfies the DD2P. However, there are Banach spaces with the DD2P which do not enjoy the Daugavet property.

Example 2.2. Let X be the renorming of $C([0,1])$ given in [2] satisfying that X is midpoint locally uniformly rotund, has the DLD2P and X fails the SD2P. Then X fails the Daugavet property. However, X has the DD2P because X has the DLD2P and an easy application of the well known Choquet lemma [9, Lemma 3.40].

It is known that a Banach space X has the D2P if, and only if, X^{**} has the w^*-D2P. However, this fact is far from being true for the DD2P. Indeed, applying the weak-star lower semicontinuity of a bidual norm is easy to get the following result.

Proposition 2.3. *Let X be a Banach space. If X^{**} has the w^*-*DD2P*, then X has the* DD2P.

Remark 2.4. The converse of Proposition 2.3 is not true. Indeed consider $X := \mathcal{C}([0,1])$. Now X has the DD2P as being a Daugavet space. However, B_{X^*} has denting points (this follows from the well known fact that $\mathcal{C}([0,1])^* = \ell_1([0,1])\oplus_1\mathcal{C}([0,1])^*$ and that $\ell_1([0,1])$ enjoys the RNP), so X^* fails the DLD2P and, consequently, X^{**} fails the w^*-DLD2P [4, Theorem 3.5].

Now we shall provide several characterizations of the DD2P. First of all, we shall show a useful characterization of the DD2P in terms of weakly convergent nets which will be used in order to prove the stability of the DD2P by ℓ_p sums.

Proposition 2.5. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the* DD2P.

(2) *For each $x \in S_X$ there exists a net $\{x_s\} \subset B_X$ which converges weakly to x and such that $\|x - x_s\| \to 2$.*

Proof. (1) $\Rightarrow$ (2). Consider $\mathcal{U}$ to be a neighborhood system of x in the weak topology relative to B_X. Now, for each $U \in \mathcal{U}$ and every $\varepsilon \in \mathbb{R}^+$, choose $x_{(U,\varepsilon)} \in U$ such that $\|x - x_{(U,\varepsilon)}\| \geq 2 - \varepsilon$. Such $x_{(U,\varepsilon)}$ exists because X has the DD2P. Now, considering in $\mathcal{U} \times \mathbb{R}^+$ the partial order given by the reverse inclusion in $\mathcal{U}$ and the inverse natural order in $\mathbb{R}$, we conclude that $\{x_{(U,\varepsilon)}\}$ is a net in B_X which is weakly convergent to x. It is also clear that $\|x - x_{(U,\varepsilon)}\| \to 2$.

(2) $\Rightarrow$ (1). Pick W a non-empty relatively weakly open subset of B_X, $x \in W \cap S_X$ and $\varepsilon \in \mathbb{R}^+$ and let us prove that there exists $y \in W$ such that $\|x - y\| > 2 - \varepsilon$. By assumption there exists a net $\{x_s\}$ in B_X such that

$$\|x - x_s\| \to 2, \quad \text{and} \quad x_s \xrightarrow{w} x.$$

From both convergences then there exists s such that $x_s \in W$ and $\|x - x_s\| > 2 - \varepsilon$. Now (1) follows choosing $y := x_s$. □

For dual Banach spaces we have the following characterization of the w^*-DD2P similar to the above one.

Corollary 2.6. *Let X be a dual Banach space. The following assertions are equivalent:*

(1) *X has the w^*-DD2P.*

(2) *For each $x \in S_X$ there exists a net $\{x_s\}$ in B_X which converges to x in the weak-star topology such that $\|x - x_s\| \to 2$.*

Remark 2.7. In view of Proposition 2.5, Daugavet property can also be easily characterized in terms of weakly convergent nets. Indeed, it is straighforward to prove from [15, Lemma 2.3] that a Banach space X has the Daugavet property if, and only if, given $x, y \in S_X$ there exists a net $\{y_s\}$ in B_X weakly convergent to y such that $\|x - y_s\| \to 2$.

In [13] a characterization of DLD2P in terms of the behavior of rank-one projections in a Banach space is proved. It turns out to be also true that DD2P can be characterized regarding the behaviour of the rank-one projections. This characterization strongly relies on the following known result which will be used several times in the following.

Lemma 2.8. [13, Lemma 2.1]. *Let X be a Banach space. Consider $x^* \in S_{X^*}, \varepsilon \in \mathbb{R}^+$ and $x \in S(B_X, x^*, \varepsilon) \cap S_X$. Then, given $0 < \delta < \varepsilon$, there exists $y^* \in S_{X^*}$ such that $x \in S(B_X, y^*, \delta) \subseteq S(B_X, x^*, \varepsilon)$.*

Proposition 2.9. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the* DD2P.

(2) *For each $x_1^*, \ldots, x_n^* \in S_{X^*}$ and $x \in X$ such that $x_i^*(x) \neq 0$, if we define*

$$p_i := x_i^* \otimes \frac{x}{x_i^*(x)} \quad \forall i \in \{1, \ldots, n\}$$

one has that, for each $\varepsilon \in \mathbb{R}^+$, there exists $y \in B_X$ such that

$$\|y - p_i(y)\| > 2 - \varepsilon \quad \forall i \in \{1, \ldots, n\}$$

and

$$\frac{x_i^*(y)}{x_i^*(x)} \geq 0 \quad \forall i \in \{1, \ldots, n\}$$

(3) *Given a weak-star slice $S := S(B_{X^*}, x, \delta)$ of B_{X^*} and $x_1^*, \ldots, x_n^* \in S \cap S_{X^*}$ there exist $y^* \in S$ and $y \in S_X$ such that*

$$(x_i^* - y^*)(y) > 2 - \delta \quad \forall i \in \{1, \ldots, n\}.$$

Proof. (1) $\Rightarrow$ (2). Consider $x_1^*, \ldots, x_n^* \in S_{X^*}$, $x \in X$ such that $x_i^*(x) \neq 0$ for each $i \in \{1, \ldots, n\}$ and define $p_i := x_i^* \otimes \frac{x}{x_i^*(x)}$ for each $i \in \{1, \ldots, n\}$.

Consider $0 < \varepsilon < 2$. Note that

$$\frac{x}{\|x\|} \in W := \left\{ y \in B_X \ : \left| \frac{x_i^*(y)}{x_i^*(x)} - \frac{1}{\|x\|} \right| \|x\| < \frac{\varepsilon}{2} \right\},$$

where W is a relatively weakly open subset of B_X. Moreover $\frac{x}{\|x\|} \in S_X$. As X has the DD2P we can ensure the existence of an element $y \in W$ such that $\left\| y - \frac{x}{\|x\|} \right\| > 2 - \frac{\varepsilon}{2}$.

Now, on the one hand, as $y \in W$ one has, for every $i \in \{1, \ldots, n\}$,

$$\left| \frac{x_i^*(y)}{x_i^*(x)} - \frac{1}{\|x\|} \right| \|x\| < \frac{\varepsilon}{2} \Rightarrow \frac{x_i^*(y)}{x_i^*(x)} > \frac{1}{\|x\|} - \frac{\varepsilon}{2\|x\|} = \frac{1 - \frac{\varepsilon}{2}}{\|x\|} \geq 0.$$

On the other hand, given $i \in \{1, \ldots, n\}$, it follows

$$\|y - p_i(y)\| \geq \left\| y - \frac{x}{\|x\|} \right\| - \left\| \frac{x}{\|x\|} - p_i(y) \right\| > 2 - \frac{\varepsilon}{2} - \left\| \frac{x}{\|x\|} - \frac{x_i^*(y)}{x_i^*(x)} x \right\| > 2 - \varepsilon$$

since $y \in W$. So (2) follows.

(2) $\Rightarrow$ (1). Consider a non-empty relatively weakly open subset

$$W := \bigcap_{i=1}^{n} S(B_X, y_i^*, \varepsilon_i)$$

of B_X and pick $x \in W \cap S_X$. In order to prove that X has the DD2P choose $0 < \varepsilon < \min_{1 \leq i \leq n} \varepsilon_i$. By Lemma 2.8 we can find, for each $i \in \{1, \ldots, n\}$, a functional $x_i^* \in S_{X^*}$ such that

$$x \in S(B_X, x_i^*, \varepsilon) \subseteq S(B_X, y_i^*, \varepsilon_i) \quad \forall i \in \{1, \ldots, n\},$$

and so $x \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \varepsilon) \subseteq W$. Consider $\eta \in \mathbb{R}^+$ small enough to satisfy $x_i^*(x)(1 - \eta) > 1 - \varepsilon$ for each $i \in \{1, \ldots, n\}$. For each $i \in \{1, \ldots, n\}$ define

$$p_i := x_i^* \otimes \frac{x}{x_i^*(x)}.$$

From the hypothesis we can find $y \in B_X$ such that

$$\|y - p_i(y)\| > 2 - \eta \ \ \forall i \in \{1, \ldots, n\} \quad \text{and} \quad \frac{x_i^*(y)}{x_i^*(x)} \geq 0.$$

Now, on the one hand, one has

$$1 - \eta < \|p_i(y)\| = \left\| \frac{x_i^*(y)}{x_i^*(x)} x \right\| = \frac{x_i^*(y)}{x_i^*(x)} \quad \forall i \in \{1, \ldots, n\}.$$

So $x_i^*(y) > (1-\eta)x_i^*(x) > 1-\varepsilon$ for each $i \in \{1,\ldots,n\}$ and, consequently, $y \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \varepsilon) \subseteq W$. Moreover, chosen $i \in \{1,\ldots,n\}$, it follows

$$\begin{aligned}\|y-x\| \geq \|y-p_i(y)\| - \|p_i(y)-x\| &> 2-\eta-\left|\frac{x_i^*(y)}{x_i^*(x)}-1\right| \\ &= 2-\eta-\frac{|x_i^*(y)-x_i^*(x)|}{x_i^*(x)} > 2-\eta-\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

As $0<\varepsilon<\min_{1\leq i\leq n}\varepsilon_i$ was arbitrary, we reached the desired result.

(1) $\Rightarrow$ (3). Let S and $x_1^*,\ldots,x_n^*$ be as in the hypothesis and pick $0<\eta<\delta$. Now, given $i \in \{1,\ldots,n\}$, one has

$$x_i^* \in S \Leftrightarrow x_i^*(x) > 1-\delta \Leftrightarrow x \in S(B_X, x_i^*, \delta).$$

So $x \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \delta) \cap S_X$. As X has the DD2P then there exists $y \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \delta) \cap S_X$ such that

$$\|x-y\| > 2-\eta \Rightarrow \ \exists\, y^* \in S_{X^*} \ / \ y^*(x)-y^*(y) > 2-\eta \Rightarrow \begin{cases} y^*(x) > 1-\eta, \\ y^*(y) < -1+\eta. \end{cases}$$

So $y^*(x) > 1-\delta$ and thus $y^* \in S$. In addition, given $i \in \{1,\ldots,n\}$, it follows

$$(x_i^*-y^*)(y) = x_i^*(y)-y^*(y) > 1-\delta+1-\eta = 2-\delta-\eta.$$

From the arbitrariness of $0<\eta<\delta$ we get the desired result by a perturbation argument, if necessary.

(3) $\Rightarrow$ (1). Let $W := \bigcap_{i=1}^{n} S(B_X, y_i^*, \varepsilon_i)$ be a non-empty relatively weakly open subset of B_X and consider $x \in W \cap S_X$. Pick $0<\delta<\min_{1\leq i\leq n}\varepsilon_i$. From Lemma 2.8 we can find, for each $i \in \{1,\ldots,n\}$, an element $x_i^* \in S_X$ such that

$$x \in S(B_X, x_i^*, \delta) \subseteq S(B_X, y_i^*, \varepsilon_i)$$

holds for each $i \in \{1,\ldots,n\}$. Now $x_1^*,\ldots,x_n^* \in S(B_{X^*}, x, \delta)$. From assumptions we can find $y^* \in S(B_{X^*}, x, \delta)$ and $y \in S_X$ such that $(x_i^*-y^*)(y) > 2-\delta$ holds for each $i \in \{1,\ldots,n\}$. Now, on the one hand, for every $i \in \{1,\ldots,n\}$, the following holds:

$$x_i^*(y) > 1-\delta \Rightarrow y \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \delta) \subseteq W.$$

On the other hand, as $y^* \in S(B_{X^*}, x, \delta)$, it follows that

$$\|x-y\| \geq y^*(x)-y^*(y) > 1-\delta+1-\delta = 2(1-\delta).$$

From the arbitrariness of $0<\delta<\min_{1\leq i\leq n}\varepsilon_i$ we conclude that X has the DD2P, as desired. □

Remark 2.10. Note that, given rank one projections $p_1, \ldots, p_n$ as in the above proposition, one has $\|I - p_i\| \geq 2$ whenever X enjoys the DLD2P [13]. However, if X also satisfies the DD2P these projections can be "normed" by a common point of the space.

A dual version of the above proposition is the following

Proposition 2.11. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X^* has the w^*-*DD2P.

(2) *For each $x_1, \ldots, x_n \in S_X$ and $x^* \in X^*$ such that $x^*(x_i) \neq 0$, if we define*

$$p_i := \frac{x^*}{x^*(x_i)} \otimes x_i \quad \forall i \in \{1, \ldots, n\}$$

one has that, for each $\varepsilon \in \mathbb{R}^+$, there exists $y^* \in B_{X^*}$ such that

$$\|y^* - p_i(y^*)\| > 2 - \varepsilon \quad \forall i \in \{1, \ldots, n\}$$

and

$$\frac{y^*(x_i)}{x^*(x_i)} \geq 0 \quad \forall i \in \{1, \ldots, n\}.$$

(3) *Given $S := S(B_X, x^*, \delta)$ a slice of B_X and $x_1, \ldots, x_n \in S \cap S_X$ there exist $y \in S$ and $y^* \in S_{X^*}$ such that*

$$y^*(x_i - y) > 2 - \delta \quad \forall i \in \{1, \ldots, n\}.$$

It is known that DLD2P is stable under taking ℓ_p-sums. Indeed, given two Banach spaces X and Y and $1 \leq p \leq \infty$, the Banach space $X \oplus_p Y$ has the DLD2P if, and only if, X and Y enjoy the DLD2P [13, Theorem 3.2].

Our aim is to establish the same result for the DD2P. We shall begin with the stability result:

Theorem 2.12. *Let X, Y be Banach spaces which satisfy the* DD2P *and let $1 \leq p \leq \infty$. Then $X \oplus_p Y$ enjoys the* DD2P.

Proof. Define $Z := X \oplus_p Y$, pick $(x_0, y_0) \in S_Z$. We apply Proposition 2.5.

On the one hand, if $p = \infty$, then either $\|x_0\| = 1$ or $\|y_0\| = 1$. Assume, with no loss of generality, that $\|x_0\| = 1$. As X has the DD2P then there exists $\{x_s\}$ a net in B_X such that $x_s \to x_0$ in the weak topology of X and $\|x_0 - x_s\| \to 2$.

Then we have that $(x_s, y_0) \to (x_0, y_0)$ in the weak topology of B_Z. Note that, from the definition of the norm on Z, we have that each term of the above net belongs to B_Z. In addition, given s one has

$$2 \geq \|(x_0, y_0) - (x_s, y_0)\|_\infty = \max\{\|x_0 - x_s\|, \|y_0\|\} \geq \|x_0 - x_s\|.$$

As $\|x_0 - x_s\| \to 2$ we conclude that $\|(x_0, y_0) - (x_s, y_0)\| \to 2$.

On the other hand, assume $p < \infty$. As $(x_0, y_0) \in S_Z$ we have that

$$(\|x_0\|^p + \|y_0\|^p)^{\frac{1}{p}} = 1.$$

Now x_0 is an element of $\|x_0\|S_X$. As X has the DD2P then by Proposition 2.5 there exists a net $\{x_s\}_{s\in S}$ in $\|x_0\|B_X$ such that $x_s \to x_0$ in the weak topology of X and $\|x_0 - x_s\| \to 2\|x_0\|$.

In addition, as Y also has the DD2P, then there exists a net $\{y_t\}_{t\in T}$ in $\|y_0\|B_Y$ such that $y_t \to y_0$ in the weak topology of Y and such that $\|y_0 - y_t\| \to 2\|y_0\|$.

Now we have $\{(x_s, y_t)\}_{(s,t)\in S\times T} \to (x_0, y_0)$ in the weak topology of Z. Moreover, given $s \in S$ and $t \in T$, one has

$$\|(x_s, y_t)\|_p = (\|x_s\|^p + \|y_t\|^p)^{\frac{1}{p}} \leq (\|x_0\|^p + \|y_0\|^p)^{\frac{1}{p}} = 1,$$

so $(x_s, y_t) \in B_Z$ for each $s \in S, t \in T$. Finally, given $s \in S$ and $t \in T$, it follows

$$\|(x_0, y_0) - (x_s, y_t)\|_p = (\|x_0 - x_s\|^p + \|y_0 - y_t\|^p)^{\frac{1}{p}} \to ((2\|x_0\|)^p + (2\|y_0\|)^p)^{\frac{1}{p}} = 2,$$

and the proof is complete. $\square$

Now let us prove the converse of the above result.

Proposition 2.13. *Let X, Y be Banach space and define $Z := X \oplus_p Y$ for $1 \leq p \leq \infty$. If X fails to have* DD2P *so does Z.*

Proof. As X fails the DD2P then there exists a non-empty relatively weak open subset U of B_X, $x_0 \in U \cap S_X$ and $\varepsilon_0 \in \mathbb{R}^+$ such that

$$\|x_0 - y\| \leq 2 - \varepsilon_0 \ \forall y \in U.$$

Now we shall argue by cases:

Case 1. If $p = \infty$ define the weak open subset of B_Z given by

$$W := \{(x, y) \in B_Z \ : \ x \in U\},$$

and pick $(x_0, 0) \in W$. Then, as $x \in U \cap B_X$, one has for each $(x, y) \in W$

$$\|(x_0, 0) - (x, y)\| = \max\{\|x_0 - x\|, \|y\|\} \leq \max\{2 - \varepsilon_0, 1\} < 2\,.$$

Case 2. If $p < \infty$ then, given $\varepsilon \in \mathbb{R}^+$, there exists $\delta > 0$ such that

$$\left.\begin{array}{c} 1-\delta < |r| \leq 1, |s| \leq 1, \\ (|r|^p + |s|^p)^{\frac{1}{p}} \leq 1 \end{array}\right\} \Rightarrow |s|^p < \varepsilon. \tag{3}$$

Define

$$W := \{(x, y) \in B_Z \ : \ x \in U \text{ and } \|x\| > 1 - \delta\},$$

which is a weakly open subset of B_Z from the lower weakly semicontinuity of the norm on X. Consider $(x_0, 0) \in W$. Now, given $(x, y) \in W$, we have from (3) that $\|y\|^p \leq \varepsilon$. In addition, as $x \in U$, we conclude $\|x - x_0\| \leq 2 - \varepsilon_0$. Hence

$$\|(x_0, 0) - (x, y)\| = (\|x - x_0\|^p + \|y\|^p)^{\frac{1}{p}} \leq ((2 - \varepsilon_0)^p + \varepsilon)^{\frac{1}{p}}.$$

So, taking ε small enough, we conclude that $\sup_{(x,y) \in W} \|(x_0, 0) - (x, y)\| < 2$. This dichotomy finishes the proof. □

Even though Example 2.2 shows that Daugavet property and DD2P are different, above results provide us more examples of such Banach spaces. Indeed, given $1 < p < \infty$, $Z := X \oplus_p Y$ has the DD2P and fails to have Daugavet property (actually, Z fails the SD2P [1, Theorem 3.2]) whenever X, Y are Banach spaces with the DD2P).

Finally we shall study the following problem: when does a subspace of a Banach space having the DD2P inherit the DD2P ? In order to give a partial answer, it has been recently proved in [8] that the D2P is hereditary to finite-codimensional subspaces. Bearing in mind the ideas of the proof of that result, we can prove the following

Theorem 2.14. *Let X be a Banach space which satisfies the* DD2P. *If Y is a closed subspace of X such that X/Y is finite-dimensional then Y has the* DD2P.

Proof. Consider

$$W := \{y \in Y \ : \ |y_i^*(y - y_0)| < \varepsilon_i \ \forall i \in \{1, \ldots, n\}\},$$

for $n \in \mathbb{N}, \varepsilon_i \in \mathbb{R}^+, y_i^* \in Y^*$ for each $i \in \{1, \ldots, n\}$ and $y_0 \in Y$ such that

$$W \cap B_Y \neq \emptyset.$$

Pick $y \in W \cap S_Y$ and let us find, for each $\delta \in \mathbb{R}^+$, a point $z \in W \cap B_Y$ such that $\|y - z\| > 2 - \delta$. To this aim pick an arbitrary $\delta \in \mathbb{R}^+$. Assume that $y_i^* \in X^*$ for each $i \in \{1, \ldots, n\}$. Observe that there is no loss of generality from the Hahn-Banach theorem. Define

$$U := \{x \in X \ : \ |y_i^*(x - y_0)| < \varepsilon_i \ \forall i \in \{1, \ldots, n\}\},$$

which is a weakly open set in X such that $U \cap B_X \neq \emptyset$.

Let $p\colon X \longrightarrow X/Y$ be the quotient map, which is a $w - w$ open map. Then $p(U)$ is a weakly open set in X/Y. In addition

$$\emptyset \neq p(U \cap B_X) \subseteq p(U) \cap p(B_X) \subseteq p(U) \cap B_{X/Y}.$$

Defining $A := p(U) \cap B_{X/Y}$, then A is a non-empty relatively weakly open and convex subset of $B_{X/Y}$ which contains zero. Hence, as X/Y is finite-dimensional, we can find a weakly open set V of X/Y, in fact a ball centered at 0, such that $V \subset A$ and that

$$diam(V \cap p(U) \cap B_{X/Y}) = diam(V) < \frac{\delta}{16}. \tag{4}$$

As $V \subset A$ then $B := p^{-1}(V) \cap U \cap B_X \neq \emptyset$. Hence B is a non-empty relatively weakly open subset of B_X. Moreover $y \in p^{-1}(V)$ because $p(y) = 0 \in V$, so $y \in B \cap S_X$. Using that X satisfies the DD2P we can ensure the existence of $v \in B$ such that

$$\|v - y\| > 2 - \frac{\delta}{16}. \tag{5}$$

Note that $v \in B$ implies $p(v) \in V = V \cap P(U) \cap B_{X/Y}$. In view of (4) it follows

$$\|p(v)\| \leq diam(V \cap p(U) \cap B_{X/Y}) < \frac{\delta}{16}.$$

Hence there exists $u \in Y$ such that $\|u - v\| < \frac{\delta}{16}$ and so $\|u\| < 1 + \frac{\delta}{16}$. Letting $z = \frac{u}{\|u\|}$, we have that

$$\begin{aligned} \|v - z\| &\leq \|u - v\| + \left\| u - \frac{u}{\|u\|} \right\| < \frac{\delta}{16} + \|u\| |\|u\| - 1| \\ &< \frac{\delta}{16} + \left(1 + \frac{\delta}{16}\right) \frac{\delta}{16} = \frac{\delta}{16}\left(2 + \frac{\delta}{16}\right). \end{aligned}$$

So

$$\|v - z\| < \frac{\delta}{4}. \tag{6}$$

Note that, given $i \in \{1, \ldots, n\}$, bearing in mind (6) one has

$$|y_i^*(z - y_0)| \leq |y_i^*(z - v)| + |y_i^*(v - y_0)| \leq \|y_i^*\| \frac{\delta}{4} + \varepsilon_i,$$

using that $v \in U$. Thus, if we define

$$W_\delta := \left\{ y \in Y \ : \ |y_i^*(y - y_0)| < \varepsilon_i + \|y_i^*\| \frac{\delta}{4} \ \forall i \in \{1, \ldots, n\} \right\}$$

it follows that $v \in W_\delta \cap B_Y$. On the other hand, in view of (5) and (6), we can estimate

$$\|y - z\| \geq \|y - v\| - \|v - z\| > 2 - \frac{\delta}{16} - \frac{\delta}{4} > 2 - \delta.$$

From here we can conclude the desired result. Indeed, for each $i \in \{1, \ldots, n\}$, we can find $\widehat{\varepsilon_i} \in \mathbb{R}^+$ and $\delta_0 \in \mathbb{R}^+$ such that

$$\widehat{\varepsilon_i} + \delta_0 \|y_i^*\| < \varepsilon_i \quad \forall i \in \{1, \ldots, n\},$$

and that

$$y \in \widehat{W} := \{z \in Y \ : \ |y_i^*(z - y_0)| < \widehat{\varepsilon_i} \ \forall i \in \{1, \ldots, n\}\}.$$

For $0 < \delta < \delta_0$ one has

$$\widehat{W_\delta} := \left\{ y \in Y \ : \ |y_i^*(y - y_0)| < \widehat{\varepsilon_i} + \|y_i^*\| \frac{\delta}{4} \ \ \forall i \in \{1, \ldots, n\} \right\} \subseteq W.$$

The arbitrariness of δ in the above argument allows us to conclude the desired result. □

As in [8] for the w^*-D2P we can prove a stability result for the w^*-DD2P.

Corollary 2.15. *Let X be a Banach space and let $Y \subseteq X$ be a closed subspace. If X^* has the w^*-*DD2P *and Y is finite-dimensional then $(X/Y)^*$ has the w^*-*DD2P.

Proof. Consider a weakly-star open subset W of $Y^\circ = (X/Y)^*$ such that

$$W \cap B_{Y^\circ} \neq \emptyset,$$

and pick $z^* \in W \cap S_{Y^\circ}$. Now we can extend W to a weak-star open subset of X^*, say U, as it is done in Theorem 2.14 satisfying $z^* \in U \cap S_{X^*}$.

Let $p\colon X^* \longrightarrow X^*/Y^\circ$ be the quotient map, which is a $w^* - w^*$ open map. Then $p(U)$ is a weakly-star open set of X^*/Y° which meets with B_{X^*/Y°.

If we define $A := p(U) \cap B_{X^*/Y^\circ}$, then we have that A is a relatively weak-star open and convex subset of B_{X^*/Y° which contains to zero.

As $X^*/Y^\circ = Y^*$ is finite-dimensional, we can find V a weak-star open set of X^*/Y°, in fact a ball centered at zero, such that $V \subset A$ and whose diameter is as closed to zero as desired.

From here, it is straightforward to check that computations of Theorem 2.14 work and allow us to conclude that

$$\sup_{x^* \in W \cap B_{Y^\circ}} \|z^* - x^*\| = 2,$$

so $Y^\circ = (X/Y)^*$ has the w^*-DD2P as desired. □

As we have pointed out in the Introduction, the D2P is inherited to almost isometric ideals from the whole space [3, Proposition 3.2]. Now, following similar ideas, we get the following

Proposition 2.16. *Let X be a Banach space and let $Y \subseteq X$ a closed almost isometric ideal. If X has the* DD2P*, so does Y.*

Proof. Take $n = 1$ in the proof of Proposition 3.12 □

3. Diametral strong diameter two property and stability results

Now we shall introduce the natural strengthening of the SD2P in the same way the DD2P is defined.

Definition 3.1. Let X be a Banach space. We will say that X has the *diametral strong diameter two property* (DSD2P) if given C a finite convex combination of non-empty relatively weakly open subsets of B_X, $x \in C$ and $\varepsilon \in \mathbb{R}^+$ then there exists $y \in C$ such that

$$\|x - y\| > 1 + \|x\| - \varepsilon. \tag{7}$$

If X is a dual space, we will say that X has the *weak-star diametral strong diameter two property* (w^*-DSD2P) if given C a finite convex combination of non-empty relatively weakly-star open subsets of B_X, $x \in C$ and $\varepsilon \in \mathbb{R}^+$ then there exists $y \in C$ satisfying (7).

Remark 3.2. On the one hand, note that the above definition extends the strong diameter two property from the Bourgain lemma [10].

On the other hand, the condition (2) is replaced with (7) to get the implication DSD2P ⇒ SD2P. Indeed, consider X to be the Banach space of Example 2.2 and $C := \sum_{i=1}^n \lambda_i W_i$ a finite convex combination of non-empty relatively weakly open subsets of B_X. If $C \cap S_X \neq \emptyset$ then there exists $x := \sum_{i=1}^n \lambda_i x_i \in C \cap S_X$. As X is a strictly convex space we conclude that $x_1 = x_2 = \ldots = x_n$. Consequently $x \in \bigcap_{i=1}^n W_i \subseteq C$ and, as X has the DD2P, we can find, for each $\varepsilon > 0$, an element $y \in \bigcap_{i=1}^n W_i \subseteq X$ such that $\|y - x\| > 2 - \varepsilon$. However, X fails the SD2P.

As in the DD2P, the first example of Banach space with the DSD2P comes from Daugavet spaces.

Example 3.3. Daugavet Banach spaces enjoy the DSD2P.

Proof. Consider X to be a Banach space enjoying the Daugavet property. From the proof of [15, Lemma 2.3] it follows that given C a finite convex combination of non-empty relatively weakly open subsets of B_X, $x \in S_X$ and

$\varepsilon \in \mathbb{R}^+$ we can find $y \in C$ such that

$$\|x + y\| > 2 - \varepsilon.$$

From here let us prove that X enjoys the DSD2P. To this aim pick a finite convex combination of non-empty relatively weakly open subsets $C := \sum_{i=1}^{n} \lambda_i W_i$ of B_X. Let $x \in C$ such that $x \neq 0$. From Daugavet property we can find $y \in \sum_{i=1}^{n} \lambda_i(-W_i)$ such that

$$\left\| \frac{x}{\|x\|} + y \right\| > 2 - \varepsilon.$$

Now $-y \in C$. Moreover

$$\begin{aligned} \|x - (-y)\| \geq \left\| \frac{x}{\|x\|} + y \right\| - \left\| \frac{x}{\|x\|} - x \right\| &> 2 - \varepsilon - \|x\| \left| \frac{1}{\|x\|} - 1 \right| \\ &= 2 - \varepsilon - |1 - \|x\|| = 2 - \varepsilon - 1 + \|x\| = 1 + \|x\| - \varepsilon. \end{aligned}$$

In order to conclude the proof assume that $0 \in C$. As $diam(C) = 2$ (see the proof of [7, Lemma 2.3]) we can find $x, y \in C$ such that

$$\left.\begin{array}{c} \|x - y\| > 2 - \varepsilon \\ \|x\| \leq 1 \end{array}\right\} \Rightarrow \|y - 0\| = \|y\| > 1 - \varepsilon = 1 + \|0\| - \varepsilon.$$

From the arbitrariness of C we conclude that X has the DSD2P. □

Given a Banach space X, it is true that X has the DSD2P whenever X^{**} has the w^*-DSD2P by a similar argument to the one given in Proposition 2.3. Again, the converse is not true, because the example exhibited in Remark 2.4 also works for the DSD2P.

Moreover, DSD2P admits a characterization in terms of weakly convergent nets as DD2P does. Indeed, we have the following

Proposition 3.4. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the* DSD2P*.*

(2) *For each $x_1, \ldots, x_n \in B_X$ and each $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^{n} \lambda_i = 1$ it follows that, for each $i \in \{1, \ldots, n\}$, there exists $\{x_s^i\}$ a net in B_X weakly convergent to x_i such that*

$$\left\{ \left\| \sum_{i=1}^{n} \lambda_i (x_i - x_s^i) \right\| \right\} \to 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|.$$

Proof. (1) $\Rightarrow$ (2). Pick $\mathcal{U}$ a system of neighborhoods of 0. Now, for each $U \in \mathcal{U}$ and $\varepsilon \in \mathbb{R}^+$, pick $x_{(U,\varepsilon)}^i$ for each $i \in \{1, \ldots, n\}$ such that

$$x_{(U,\varepsilon)}^i \in (x_i + U) \cap B_X$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i (x_i - x^i_{(U,\varepsilon)}) \right\| > 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon,$$

which can be done because X has the DSD2P.

Now it is quite clear that, given $i \in \{1, \ldots, n\}$, $\{x_{(U,\varepsilon)}\}_{(U,\varepsilon)\in\mathcal{U}\times\mathbb{R}^+}$ is a net in B_X weakly convergent to x_i. Moreover, it is clear that

$$\left\{ \left\| \sum_{i=1}^{n} \lambda_i (x_i - x^i_{(U,\varepsilon)}) \right\| \right\} \to 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|$$

from triangle inequality.

(2) $\Rightarrow$ (1). Is similar to the proof of Proposition 2.5. □

Now we can establish a dual version for the result above.

Proposition 3.5. *Let X be a dual Banach space. The following assertions are equivalent:*

(1) *X has the w^*-DSD2P.*
(2) *For each $x_1, \ldots, x_n \in B_X$ and each $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n \lambda_i = 1$ it follows that, for each $i \in \{1, \ldots, n\}$, there exists $\{x^i_s\}$ a net in B_X convergent to x_i in the weak-star topology of X such that*

$$\left\{ \left\| \sum_{i=1}^{n} \lambda_i (x_i - x^i_s) \right\| \right\} \to 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|.$$

Now, as we have done in Theorem 2.12 for the DD2P, we will focus on analyzing the DSD2P in the ℓ_p sum of two Banach spaces. As every Banach space enjoying the DSD2P has the strong diameter two property, we conclude that the ℓ_p sum of two Banach spaces does not have the DSD2P whenever $1 < p < \infty$ [1, Theorem 3.2]. Nevertheless, we will prove that, as well as happens with Daugavet spaces, DSD2P has a nice behavior in the case $p = \infty$. We shall begin proving the following necessary condition.

Proposition 3.6. *Let X, Y be Banach spaces and assume that $X \oplus_p Y$ has the* DSD2P *for $p \in \{1, \infty\}$. Then X and Y enjoy the* DSD2P*.*

Proof. In order to prove the proposition, assume that X does not satisfy the DSD2P. Then there exists a convex combination of non-empty relatively weakly open subsets $C := \sum_{i=1}^n \lambda_i \bigcap_{j=1}^{n_i} S(B_X, x^*_{ij}, \eta_{ij})$ of B_X, an element $\sum_{i=1}^n \lambda_i x_i \in C$ and $\varepsilon \in \mathbb{R}^+$ satisfying that

$$\left\| \sum_{i=1}^{n} \lambda_i (x_i - y_i) \right\| \leq 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon \quad \forall \sum_{i=1}^{n} \lambda_i y_i \in C. \tag{8}$$

Obviously we will assume the non-trivial case (i.e. $\sum_{i=1}^{n}\lambda_i x_i \neq 0$), so we can assume, taking $\delta<\varepsilon$ if necessary, that $\|\sum_{i=1}^{n}\lambda_i x_i\|-\varepsilon\geq 0$.

Using Lemma 2.8 as many times as necessary we can assume that all the η_{ij} are equal (say η) and that $\eta<\frac{\varepsilon}{2}$ $\forall i\in\{1,\ldots,n\}$. Define

$$\mathcal{C}:=\sum_{i=1}^{n}\lambda_i\bigcap_{j=1}^{n_i}S(B_{X\oplus_p Y},(x_{ij}^*,0),\eta).$$

If $p=1$ we have from [1, Theorem 3.1, equation (3.1)] that

$$\mathcal{C}\subseteq C\times\eta B_Y. \tag{9}$$

So consider $\sum_{i=1}^{n}\lambda_i(x_i,0)\in\mathcal{C}$ and pick $\sum_{i=1}^{n}\lambda_i(x_i',y_i')\in\mathcal{C}$. Then

$$\left\|\sum_{i=1}^{n}\lambda_i((x_i,0)-(x_i',y_i'))\right\|=\left\|\sum_{i=1}^{n}\lambda_i(x_i-x_i')\right\|+\left\|\sum_{i=1}^{n}\lambda_i y_i'\right\|.$$

Now on the one hand, from (8), we have the inequality

$$\left\|\sum_{i=1}^{n}\lambda_i(x_i-x_i')\right\|\leq 1+\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|-\varepsilon.$$

On the other hand we have from (9) the following

$$\left\|\sum_{i=1}^{n}\lambda_i y_i'\right\|<\eta.$$

So combining both previous inequalities and keeping in mind that $\eta<\frac{\varepsilon}{2}$ we conclude

$$\left\|\sum_{i=1}^{n}\lambda_i((x_i,0)-(x_i',y_i'))\right\|\leq 1+\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|-\frac{\varepsilon}{2}.$$

From the arbitrariness of $\sum_{i=1}^{n}\lambda_i(x_i',y_i')\in\mathcal{C}$ we conclude that $X\oplus_1 Y$ fails the DSD2P, so we are done in the case $p=1$.

The case $p=\infty$ is easier. Indeed, pick $\sum_{i=1}^{n}\lambda_i(x_i',y_i')\in\mathcal{C}$. Then

$$\begin{aligned}\left\|\sum_{i=1}^{n}\lambda_i((x_i,0)-(x_i',y_i'))\right\|&=\max\left\{\left\|\sum_{i=1}^{n}\lambda_i(x_i-x_i')\right\|,\left\|\sum_{i=1}^{n}\lambda_i y_i\right\|\right\}\\&\leq\max\left\{1+\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|-\varepsilon,1\right\}=1+\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|-\varepsilon,\end{aligned}$$

where the last inequality holds from the assumption $\|\sum_{i=1}^{n}\lambda_i x_i\|-\varepsilon\geq 0$.

Hence, $X\oplus_\infty Y$ does not have the DSD2P, so we are done. $\square$

Now we shall establish the converse of the result above for $p = \infty$.

Theorem 3.7. *Let X, Y be a Banach spaces. If X and Y have the* DSD2P *so does $Z := X \oplus_\infty Y$.*

Proof. Pick $n \in \mathbb{N}$, $(x_1, y_1), \ldots, (x_n, y_n) \in B_Z$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n \lambda_i = 1$. In order to prove that Z has the DSD2P we shall use Proposition 3.4. As

$$\left\| \sum_{i=1}^n \lambda_i (x_i, y_i) \right\|_\infty = \max \left\{ \left\| \sum_{i=1}^n \lambda_i x_i \right\|, \left\| \sum_{i=1}^n \lambda_i y_i \right\| \right\},$$

then either

$$\left\| \sum_{i=1}^n \lambda_i (x_i, y_i) \right\|_\infty = \left\| \sum_{i=1}^n \lambda_i x_i \right\| \quad \text{or} \quad \left\| \sum_{i=1}^n \lambda_i (x_i, y_i) \right\|_\infty = \left\| \sum_{i=1}^n \lambda_i y_i \right\|.$$

We shall assume, without loss of generality, the validity of the latter equality. Now, as X has the DSD2P, we have from Proposition 3.4 that, for each $i \in \{1, \ldots, n\}$, there exists a net $\{x_s^i\}$ weakly convergent to x_i such that

$$\left\| \sum_{i=1}^n \lambda_i (x_i - x_s^i) \right\| \to 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\|.$$

Now we have that $(x_s^i, y_i) \to (x_i, y_i)$ in the weak topology of Z for each $i \in \{1, \ldots, n\}$. Moreover, from the definition of the norm on Z, we have that $(x_s^i, y_i) \in B_Z$ for each $i \in \{1, \ldots, n\}$ and for each s. Finally, given s one has

$$\begin{aligned} 1 + \left\| \sum_{i=1}^n \lambda_i (x_i, y_i) \right\|_\infty &\geq \left\| \sum_{i=1}^n \lambda_i ((x_i, y_i) - (x_s^i, y_i)) \right\|_\infty \\ \geq \left\| \sum_{i=1}^n \lambda_i (x_i - x_s^i) \right\| \to 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\| &= 1 + \left\| \sum_{i=1}^n \lambda_i (x_i, y_i) \right\|_\infty. \end{aligned}$$

So $\|\sum_{i=1}^n \lambda_i ((x_i, y_i) - (x_s^i, y_i))\|_\infty \to 1 + \|\sum_{i=1}^n \lambda_i (x_i, y_i)\|_\infty$. Consequently, Z has the DSD2P applying Proposition 3.4, so we are done. □

Finally we will analyze the inheritance of DSD2P to subspaces. Again in [8] it is proved that given X a Banach space with the SD2P and $Y \subseteq X$ a closed subspace such that X/Y is strongly regular, then Y has the SD2P. Following similar ideas we have the following

Theorem 3.8. *Let X be a Banach space and $Y \subseteq X$ be a closed subspace. If X has the* DSD2P *and X/Y is strongly regular then Y also has the* DSD2P.

Proof. Let $C := \sum_{i=1}^n \lambda_i W_i = \sum_{i=1}^n \lambda_i \left\{ y \in B_Y : |y_{ij}^*(y - y_0^i)| < \eta_{ij} \quad 1 \leq j \leq n_i \right\}$

be a finite convex combination of non-empty relatively weakly open subsets of B_Y, where $y^*_{ij} \in B_{Y^*}$ for each $i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\}$ and $y^i_0 \in Y$ for each $i \in \{1, \ldots, n\}$. Pick $\sum_{i=1}^n \lambda_i x_i \in C$, $\varepsilon \in \mathbb{R}^+$ and let us prove that there exists $\sum_{i=1}^n \lambda_i y_i \in C$ such that

$$\left\| \sum_{i=1}^n \lambda_i (x_i - y_i) \right\| > 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\| - \varepsilon.$$

To this aim pick $0 < \delta$ such that

$$|y^*_{ij}(x_i - y^i_0)| + \frac{\delta}{32} + \left(1 + \frac{\delta}{32}\right)\left(1 - \frac{1}{1 + \frac{\delta}{32}}\right) < \eta_{ij} \quad \forall i \in \{1, \ldots, n\} \tag{10}$$

and

$$\frac{\delta}{16} + 2\left(1 + \frac{\delta}{32}\right)\left(1 - \frac{1}{1 + \frac{\delta}{32}}\right) + \frac{\delta}{4} < \varepsilon. \tag{11}$$

Let $\pi\colon X \longrightarrow X/Y$ be the quotient map. We have no loss of generality, by Hahn-Banach theorem, if we assume that $y^*_{ij} \in B_{X^*}$ for each $i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\}$. Consider $\mathcal{W}_i$ the non-empty relatively weakly open subset of B_X defined by W_i for each $i \in \{1, \ldots, n\}$.

For each $i \in \{1, \ldots, n\}$ consider $A_i := \pi(\mathcal{W}_i)$, which is a convex subset of $B_{X/Y}$ containing to zero. By [10, Proposition III.6] then $\overline{A_i}$ is equal to the closure of the set of its strongly regular points. As a consequence, for each $i \in \{1, \ldots, n\}$, there exists a_i a strongly regular point of $\overline{A_i}$ such that

$$\|a_i\| < \frac{\delta}{32}. \tag{12}$$

For every $i \in \{1, \ldots, n\}$ we can find $m_i \in \mathbb{N}, \mu^i_1, \ldots, \mu^i_{m_i} \in]0, 1]$ such that $\sum_{j=1}^{m_i} \mu^i_j = 1$ and $(a^i_1)^*, \ldots, (a^i_{m_i})^* \in S_{(X/Y)^*}, \alpha^i_j \in \mathbb{R}^+$ satisfying that

$$a_i \in \sum_{j=1}^{m_i} \mu^i_j (S(B_{X/Y}, (a^i_j)^*, \alpha^i_j) \cap \overline{A_i})$$

and

$$diam\left(\sum_{j=1}^{m_i} \mu^i_j (S(B_{X/Y}, (a^i_j)^*, \alpha^i_j) \cap \overline{A_i})\right) < \frac{\delta}{32}. \tag{13}$$

It is clear that, for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$, one has

$$S(B_{X/Y}, (a^i_j)^*, \alpha^i_j) \cap A_i \neq \emptyset \Rightarrow S(B_X, \pi^*((a^i_j)^*), \alpha^i_j) \cap \mathcal{W}_i \neq \emptyset.$$

Notice that we can not still apply the hypothesis because we do not know whether $\sum_{i=1}^n \lambda_i x_i \in \mathcal{C} := \sum_{i=1}^n \lambda_i \sum_{j=1}^{m_i} \mu^i_j (\mathcal{W}_i \cap S(B_X, \pi^*((a^i_j)^*), \alpha^i_j))$. Now, in order to finish the proof, we need to find points in $\mathcal{C}$ close enough to $\sum_{i=1}^n \lambda_i x_i$. This will be done in the following claim.

Claim 3.9. *We can find, for each* $i \in \{1, \ldots, n\}$*, an element* $z_i \in B_X$ *such that* $\sum_{i=1}^{n} \lambda_i z_i \in \mathcal{C}$ *and that*

$$\left\| \sum_{i=1}^{n} \lambda_i (x_i - z_i) \right\| < \frac{\delta}{32} + \left(1 + \frac{\delta}{32}\right) \left(1 - \frac{1}{1 + \frac{\delta}{32}}\right). \tag{14}$$

Proof. Pick $i \in \{1, \ldots, n\}$. As $\|\pi(x_i) - a_i\| = \|a_i\| < \frac{\delta}{32}$ we can find $z_i \in X$ such that $\pi(z_i) = a_i$ and such that

$$\|x_i - z_i\| < \frac{\delta}{32}. \tag{15}$$

Now $a_i \in \sum_{j=1}^{m_i} \mu_j^i (S(B_{X/Y}, (a_j^i)^*, \alpha_j^i) \cap \overline{A_i})$ so, for each $j \in \{1, \ldots, m_i\}$, we can find $b_{ij} \in S(B_{X/Y}, (a_j^i)^*, \alpha_j^i) \cap \overline{A_i}$ such that $a_i = \sum_{j=1}^{m_i} \mu_j^i b_{ij}$. For each $j \in \{1, \ldots, m_i\}$ we can find, considering a perturbation argument if necessary, an element $z_{ij} \in B_X$ such that $\pi(z_{ij}) = b_{ij}$. Finally, as $\pi(z_i) - \sum_{j=1}^{m_i} \mu_j^i \pi(z_{ij}) = 0$ we can find, by definition of the norm on X/Y, an element $y_i \in Y$ such that

$$z_i = \sum_{j=1}^{m_i} \mu_j^i z_{ij} + y_i, \tag{16}$$

and

$$\|y_i\| < \frac{\delta}{32}. \tag{17}$$

We shall prove that $\sum_{i=1}^{n} \lambda_i \frac{z_i}{1+\frac{\delta}{32}}$ does the work. First of all we have

$$\left\| \sum_{i=1}^{n} \lambda_i z_i \right\| \leq \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| + \sum_{i=1}^{n} \lambda_i \|x_i - z_i\| \overset{(15)}{<} 1 + \frac{\delta}{32}.$$

So $\frac{z_i}{1+\frac{\delta}{32}} \in B_X$ for each $i \in \{1, \ldots, n\}$.

Moreover, given $i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\}$, one has

$$\begin{aligned} \left| y_{ij}^* \left(\frac{z_i}{1 + \frac{\delta}{32}} - y_0^i \right) \right| &\leq |y_{ij}^*(x_i - y_0^i)| + \|x_i - z_i\| + \left\| z_i - \frac{z_i}{1 + \frac{\delta}{32}} \right\| \\ &\overset{(15)}{<} |y_{ij}^*(x_i - y_0^i)| + \frac{\delta}{32} + \left(1 + \frac{\delta}{32}\right) \left(1 - \frac{1}{1 + \frac{\delta}{32}}\right) \overset{(10)}{<} \eta_{ij}. \end{aligned}$$

Finally, pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$. Then, by (16), one has

$$z_i = \sum_{j=1}^{m_i} \mu_j^i (z_{ij} + y_i).$$

In addition: $\pi^*(a_{ij}^*)(z_{ij}+y_i)=a_{ij}^*(b_{ij})+a_{ij}^*(\pi(y_i))$.

On the one hand, as $y_i\in Y$ then $\pi(y_i)=0$. On the other hand $a_{ij}^*(b_{ij})>1-\alpha_j^i$. Now, up to consider a smaller positive number in (15) (check that the choice of b_{ij} does not depend on the one of $z_1,\ldots,z_n$), we can assume that $a_{ij}^*(b_{ij})>(1-\alpha_j^i)\left(1+\frac{\delta}{32}\right)$, so $\sum_{i=1}^n\lambda_i\frac{z_i}{1+\frac{\delta}{32}}\in\mathcal{C}$. Now the claim follows just considering $\frac{z_i}{1+\frac{\delta}{32}}$ instead of z_i. □

Now, as X has the DSD2P, we can find $\sum_{i=1}^n\lambda_i z_i'\in\mathcal{C}$ such that

$$\left\|\sum_{i=1}^n\lambda_i(z_i-z_i')\right\|>1+\left\|\sum_{i=1}^n\lambda_i z_i\right\|-\frac{\delta}{32}. \tag{18}$$

Given $i\in\{1,\ldots,n\}$ we have that

$$\pi(z_i')\in\sum_{j=1}^{m_i}\mu_j^i(S(B_{X/Y},(a_j^i)^*,\alpha_j^i)\cap\overline{A_i}),$$

so

$$\|\pi(z_i')\|\leq\|a_i\|+diam\left(\sum_{j=1}^{m_i}\mu_j^i(S(B_{X/Y},(a_j^i)^*,\alpha_j^i)\cap\overline{A_i})\right)\overset{(12)(13)}{<}\frac{\delta}{16}.$$

Now, as it is done in Proposition 2.14, we can find $y_i\in B_Y$ such that

$$\|y_i-z_i'\|<\frac{\delta}{4}. \tag{19}$$

On one hand, given $i\in\{1,\ldots,n\}$ and $j\in\{1,\ldots,n_i\}$, one has

$$|y_{ij}^*(y_i-y_0)|\leq|y_{ij}^*(z_i'-y_0)|+|y_{ij}^*(y_i-z_i)|<\eta_{ij}+\frac{\delta}{4}.$$

$$\begin{aligned}\left\|\sum_{i=1}^n\lambda_i(x_i-y_i)\right\| &\geq \left\|\sum_{i=1}^n\lambda_i(z_i-z_i')\right\|-\left\|\sum_{i=1}^n\lambda_i(x_i-z_i)\right\|-\left\|\sum_{i=1}^n\lambda_i(y_i-z_i')\right\|\\
&\overset{(18)}{>}1+\left\|\sum_{i=1}^n\lambda_i z_i\right\|-\frac{\delta}{32}-\sum_{i=1}^n\lambda_i\|y_i-z_i'\|-\left\|\sum_{i=1}^n\lambda_i(z_i-x_i)\right\|\\
&\overset{(14)(19)}{>}1+\left\|\sum_{i=1}^n\lambda_i z_i\right\|-\frac{\delta}{32}-\frac{\delta}{4}-\frac{\delta}{32}-\left(1+\frac{\delta}{32}\right)\left(1-\frac{1}{1+\frac{\delta}{32}}\right)\\
&\overset{(14)}{>}1+\left\|\sum_{i=1}^n\lambda_i x_i\right\|-\frac{\delta}{16}-\frac{\delta}{4}-2\left(1+\frac{\delta}{32}\right)\left(1-\frac{1}{1+\frac{\delta}{32}}\right)\\
&\overset{(11)}{>}1+\left\|\sum_{i=1}^n\lambda_i x_i\right\|-\varepsilon.\end{aligned}$$

From the arbitrariness of ε we conclude that Y has the DSD2P by a perturbation argument similar to the one done in Proposition 2.14. $\square$

Now we have a weak-star version of Theorem 3.8.

Corollary 3.10. *Let X be a Banach space and let $Y \subseteq X$ be a closed subspace. If X^* has the w^*-*DSD2P *and Y is reflexive then $(X/Y)^*$ has the w^*-*DSD2P.

Proof. Consider $C := \sum_{i=1}^n \lambda_i W_i$ a finite convex combination of non-empty relatively weakly-star open subsets of B_{Y° and pick $\sum_{i=1}^n \lambda_i z_i^* \in C$.

Define $\mathcal{W}_i$ to be the weak-star open subset of B_{X^*} defined by W_i for each $i \in \{1, \ldots, n\}$.

Let $\pi\colon X^* \longrightarrow X^*/Y^\circ$ the quotient map and define $A_i := \pi(\mathcal{W}_i)$.

As $X^*/Y^\circ = Y^*$ is reflexive, then X^*/Y° is strongly regular, so we can find, for each $i \in \{1, \ldots, n\}$, a_i a point of strong regularity of $\overline{A_i}$ whose norm is as close to zero as desired. Given $i \in \{1, \ldots, n\}$, as a_i is a point of strong regularity, we can find a finite convex combination of slices containing a_i and whose diameter is as small as wanted. In addition, because of reflexivity of X^*/Y°, convex combination of slices are indeed finite convex combination of weak-star slices, so we can actually find finite convex combination of weak-star slices containig to a_i and whose diameter is a closed to zero as desired for each $i \in \{1, \ldots, n\}$.

Using the previous ideas, the result can be concluded following word by word the proof of Theorem 3.8. $\square$

Now we shall prove the inheritance of the DSD2P to almost isometric ideals. This result relies on the following known theorem.

Theorem 3.11. [3, Theorem 1.4] *Let X be a Banach space and let $Y \subseteq X$ be an almost ideal in X. Then there exists a Hahn-Banach operator $\varphi\colon Y^* \longrightarrow X^*$ such that, for each $\varepsilon > 0$, for each finite-dimensional subspace $E \subseteq X$ and each finite-dimensional subspace $F \subseteq Y^*$, there exists $T\colon E \longrightarrow Y$ verifying the following conditions:*

(1) *$T(e) = e$ for each $e \in E \cap Y$.*

(2) *For each $e \in E$ one has*

$$\frac{1}{1+\varepsilon}\|e\| \leq \|T(e)\| \leq (1+\varepsilon)\|e\|.$$

(3) *For each $e \in E$ and $f \in F$ it follows: $\varphi(f)(e) = f(T(e))$.*

Now we are ready to prove the following proposition.

Proposition 3.12. *Let X be a Banach space and let $Y \subseteq X$ be a closed almost isometric ideal. If X has the* DSD2P*, so does Y.*

Proof. Pick $C := \sum_{i=1}^{n} \lambda_i \bigcap_{j=1}^{n_i} S(B_Y, y_{ij}^*, \alpha_{ij})$ to be a finite convex combination of non-empty relatively weakly open subsets of B_Y, choose $\sum_{i=1}^{n} \lambda_i y_i \in C$ and pick $\varepsilon > 0$. Our aim is to find $\sum_{i=1}^{n} \lambda_i z_i \in C$ such that $\|\sum_{i=1}^{n} \lambda_i (y_i - z_i)\| > 1 + \|\sum_{i=1}^{n} \lambda_i y_i\| - \varepsilon$. Assume, with no loss of generality, that $\max\limits_{1\leq i\leq n} \max\limits_{1\leq j\leq n_i} y_{ij}^*(y_i) < 1$.

Choose $\mu_0 > 0$ such that

$$0 < \mu < \mu_0 \Rightarrow \frac{y_{ij}^*(y_i)}{1+\mu} > 1 - \alpha_{ij} \quad \forall i \in \{1, \ldots, n\},\ j \in \{1, \ldots, n_i\}. \tag{20}$$

and

$$0 < \mu < \mu_0 \Rightarrow \frac{\frac{1}{1+\mu}\left(1 + \|\sum_{i=1}^{n} \lambda_i y_i\| - \mu\right) - \mu}{1+\mu} > 1 + \left\|\sum_{i=1}^{n} \lambda_i y_i\right\| - \varepsilon. \tag{21}$$

Now consider $0 < \mu < \mu_0$ and a Hahn-Banach operator $\varphi\colon Y^* \longrightarrow X^*$ satisfying the properties described in Theorem 3.11. Define

$$\widehat{C} := \sum_{i=1}^{n} \lambda_i \bigcap_{j=1}^{n_i} S(B_X, \varphi(y_{ij}^*), 1 - y_{ij}^*(y_i)).$$

As X has the DSD2P and clearly $\sum_{i=1}^{n} \lambda_i y_i \in \widehat{C}$ we can conclude the existence of an element $\sum_{i=1}^{n} \lambda_i x_i \in \widehat{C}$ such that

$$\left\|\sum_{i=1}^{n} \lambda_i (y_i - x_i)\right\| > 1 + \left\|\sum_{i=1}^{n} \lambda_i y_i\right\| - \mu. \tag{22}$$

Now for μ, $E := span\{x_1, \ldots, x_n,\ y_1, \ldots, y_n\} \subseteq X$ and $F := span\{y_{ij}^* \ /\ i \in \{1, \ldots, n\},\ j \in \{1, \ldots, n_i\}\} \subseteq Y^*$ consider T the operator satisfying the properties described in Theorem 3.11. Given $i \in \{1, \ldots, n\}$ one has

$$\|T(x_i)\| \leq (1+\mu)\|x_i\| \leq 1+\mu.$$

So, if we define $z := \sum_{i=1}^{n} \lambda_i \frac{T(x_i)}{1+\mu}$, it is clear that $z \in B_Y$. We will prove that indeed $z \in C$. To this aim, pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n_i\}$. Now

$$\begin{aligned} y_{ij}^*\left(\frac{T(x_i)}{1+\mu}\right) &= \frac{y_{ij}^*(T(x_i))}{1+\mu} = \frac{\varphi(y_{ij}^*)(x_i)}{1+\mu} > \frac{1 - (1 - y_{ij}^*(y_i))}{1+\mu} \\ &= \frac{y_{ij}^*(y_i)}{1+\mu} \overset{(20)}{>} 1 - \alpha_{ij}. \end{aligned}$$

Thus $z \in C$. Finally, we have that

$$\left\|\sum_{i=1}^{n} \lambda_i y_i - z\right\| = \left\|\sum_{i=1}^{n} \lambda_i \left(y_i - \frac{T(x_i)}{1+\mu}\right)\right\| = \frac{\|\sum_{i=1}^{n} \lambda_i (y_i - T(x_i) + \mu y_i)\|}{1+\mu}$$

$$\geq \frac{\|T\left(\sum_{i=1}^{n} \lambda_i (y_i - x_i)\right)\| - \mu \sum_{i=1}^{n} \lambda_i \|y_i\|}{1+\mu} > \frac{\frac{1}{1+\mu}\|\sum_{i=1}^{n} \lambda_i (y_i - x_i)\| - \mu}{1+\mu}$$

$$\overset{(22)}{>} \frac{\frac{1}{1+\mu}\left(1 + \|\sum_{i=1}^{n} \lambda_i y_i\| - \mu\right) - \mu}{1+\mu} \overset{(21)}{>} 1 + \left\|\sum_{i=1}^{n} \lambda_i y_i\right\| - \varepsilon.$$

As ε was arbitrary we conclude that Y has the DSD2P. $\square$

4. Some remarks and open questions.

Let us consider the following diagram

$$\begin{array}{ccccccc} \text{DP} & \overset{(1)}{\Longrightarrow} & \text{DSD2P} & \overset{(2)}{\Longrightarrow} & \text{DD2P} & \overset{(3)}{\Longrightarrow} & \text{DLD2P} \\ & & \Downarrow (4) & & \Downarrow (5) & & \Downarrow (6) \\ & & w^* - \text{DSD2P} & \overset{(7)}{\Longrightarrow} & w^* - \text{DD2P} & \overset{(8)}{\Longrightarrow} & w^* - \text{DLD2P} \end{array}$$

where the last row only makes sense in dual Banach spaces. By Example 2.2 or Theorem 2.12, neither the converse implication of (2) nor the one of (7) holds. In addition, there are Banach spaces with the Daugavet property whose dual unit ball have denting points (e.g. $\mathcal{C}([0,1])$). Consequently, the converses of (4),(5) and (6) are not true.

However, it remains open the following

Question 4.1. Does the converse of (1),(3) or (8) hold?

It is known that a Banach space X has the DLD2P if, and only if, X^* has the w^*-DLD2P. This fact arise two questions.

Question 4.2. Let X be a Banach space. Is it true that X has the DD2P if, and only if, X^* has the w^*-DD2P?

A similar question remains open for the DSD2P.

Question 4.3. Let X be a Banach space. Is it true that X has the DSD2P if, and only if, X^* has the w^*-DSD2P?

Remark 4.4. In a preprint version of this paper it was posed as an open question whether the DSD2P is stable by taking ℓ_1-sums. Note that a positive answer was obtained in [12].

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