Journal of Convex Analysis
Volume 25 (2018), No. 3, 841–860

Complete Convergence and Strong Laws of Large Numbers for Double Arrays of Convex Compact Integrable Random Sets and Applications for Random Fuzzy Variables

Nguyen Van Quang
Institute for Computational Science and Technology (ICST), Ho Chi Minh City, Vietnam; and: Vinh University, Nghe An Province, Vietnam
nvquang@hotmail.com

Hoang Thi Duyen
Quang Binh University, Quang Binh Province, Vietnam
hoangduyen267@gmail.com

Received: October 6, 2016
Revised manuscript received: May 23, 2017
Accepted: May 31, 2017

We prove some fairly general results of the complete convergence for maximum partial double sums and strong law of large numbers for double arrays of real and Banach valued random variables. Then using the norm compactness of the expectation of convex compact integrable random sets and an embedding method, we improve several results for maximum partial double sums of convex compact integrable random sets under some new conditions on the support functions. We also provide a typical example illustrating this study. Further applications to the strong law of large numbers for random fuzzy convex upper semicontinuous variables are given.

Keywords: Strong law of large numbers, maximum partial sums, complete convergence, random set, embedding method.

2010 Mathematics Subject Classification: 28B20, 60F15, 54A20

1. Introduction

The concept of complete convergence for a sequence of random variables was introduced by Hsu and Robbins [10]. Moreover, they proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value noting that the variance of the summands is finite. This result has been generalized and extended in several directions by many authors. One can refer to Baum and Katz [2], Gut [8], Chen, Hu, Liu and Volodin [5], Bai, Chen and Sung [1], Quang and Duyen [14]. In particular, Bai, Chen and Sung [1] proved some results of the complete

ISSN 0944-6532 / $ 2.50

convergence and strong law of large numbers (SLLN) of pairwise independent and compactly uniformly integrable in the Cesàro sense random variables (see Theorems 1.1, 3.1). This results were generalized to the sequence of convex compact integrable random sets by relaxing the condition of compactly uniform integrability in the Cesàro sense (see [14], Theorem 4.3). Moreover, the various convergence results on the SLLN for random fuzzy convex upper semicontinuous variables was provided. However, the same problems for double arrays have not been studied yet.

In the present paper, at first we establish the complete convergence for maximum partial double sums of real and Banach valued random variables. Then using the norm compactness of the expectation of convex compact integrable random sets and an embedding method, we provide several results for maximum partial double sums of convex compact integrable random sets with respect to the Hausdorff distance. We apply our techniques to prove the SLLN for random fuzzy convex upper semicontinuous variables. Consequently, our results extend some theorems given by Bai, Chen and Sung [1], Puri and Ralescu [13], Quang and Duyen [14].

The paper is organized as follows: In Section 2 we set our notations and definitions and summarize needed results. Section 3 is devoted to the complete convergence for maximum partial double sums and the SLLN for double arrays of random variables. Section 4 generalized some results in Section 3 to random sets under some new conditions, in particular, an illustrative example is provided. Futher applications for random fuzzy variables are given in Section 5.

2. Preliminaries

Some basic definitions and properties will be briefly listed. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathbf{E}, \|.\|)$ be a real separable Banach space with the dual space $\mathbf{E}^*$, and $\overline{B}_{\mathbf{E}}$ (resp. $\overline{B}_{\mathbf{E}^*}$) is the closed unit ball of $\mathbf{E}$ (resp. $\mathbf{E}^*$). We denote by $c(\mathbf{E})$ the family of all nonempty closed subsets of $\mathbf{E}$, $wk(\mathbf{E})$ (resp. $k(\mathbf{E})$) the family of all nonempty weakly compact (resp. compact) subsets of $\mathbf{E}$, and $cwk(\mathbf{E})$ (resp. $ck(\mathbf{E})$) is the family of all nonempty convex weakly compact (resp. convex compact) subsets of $\mathbf{E}$. For any subsets $A, B \in k(\mathbf{E})$ (or $ck(\mathbf{E})$) and $\lambda \in \mathbb{R}$, the *Minkowski's addition* and *scalar multiplication* respectively in $k(\mathbf{E})$ (or $ck(\mathbf{E})$) are defined by

$$A + B := \{a + b : a \in A, b \in B\},$$

$$\lambda A := \{\lambda a : a \in A\}.$$

Neither $k(\mathbf{E})$ nor $ck(\mathbf{E})$ is a linear space, but they become separable complete metric spaces when the Hausdorff distance between two sets A, B of $k(\mathbf{E})$ (or $ck(\mathbf{E})$) is defined as

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Also, the following notation is used, $|A| = d_H(A, \{0\}) = \sup_{a \in A} \|a\|$.

A $c(\mathbf{E})$-valued mapping $X\colon \Omega \rightrightarrows \mathbf{E}$ is a closed valued random set, if X is measurable, i.e., for every $B \in c(\mathbf{E})$, the set $X^-(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\} \in \mathcal{F}$ (see [4, 9]). A $cwk(\mathbf{E})$-valued random variable $X\colon \Omega \rightrightarrows \mathbf{E}$ is *integrable* if $|X|$ is integrable. According to [4], a $cwk(\mathbf{E})$-valued mapping $X\colon \Omega \rightrightarrows \mathbf{E}$ is *measurable* iff for every $x^* \in \mathbf{E}^*$, the mapping $\delta^*(x^*, X)$ is measurable, where $\delta^*(x^*, X)$ denotes the support function of X. The *Hausdorff distance* between $A, B \in cwk(\mathbf{E})$ is given by the formula $d_H(A, B) = \sup_{x^* \in \overline{B}_{\mathbf{E}^*}} |\delta^*(x^*, A) - \delta^*(x^*, B)|$, where $\overline{B}_{\mathbf{E}^*}$ denotes the closed unit ball in $\mathbf{E}^*$. For every $cwk(\mathbf{E})$-valued integrable random set X, the *expectation of* X, EX, is defined by the Aumann integral of X: $EX = \{\int_\Omega f dP : f \in S^1_X\}$, where $\int_\Omega f dP$ is the usual Bochner integral and S^1_X is the set of all integrable selections of X. A double array $\{X_{mn}\}$ of random elements is said to be *uniformly integrable* in the Cesàro sense if

$$\lim_{x \to \infty} \sup_{m,n \geq 1} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} E(\|X_{kl}\| I_{[\|X_{kl}\| > x]}) = 0.$$

A double array $\{X_{mn}\}$ of random elements is said to be *compactly uniformly integrable* in the Cesàro sense if for every $\varepsilon > 0$, there exists a compact subset K_ε of $\mathbf{E}$ such that

$$\sup_{m,n \geq 1} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} E(\|X_{kl}\| I_{[X_{kl} \notin K_\varepsilon]}) < \varepsilon.$$

In general, the compactly uniform integrability in the Cesàro sense is stronger than the uniform integrability in the Cesàro sense. But, they are equivalent for real or identically distributed case. For more informations, see [16].

Throughout this paper, the symbol C will denote a generic constant ($0 < C < +\infty$) which is not necessarily the same one in each appearance. For $x > 0$, the symbol $[x]$ denotes the greatest integer not exceeding x, log stands for $\log_2$.

3. Complete convergence for double arrays of random variables

We first claim the following key theorem.

Theorem 3.1. *Let $\{X_{mn}\}$ be a double array of pairwise independent, uniformly integrable in the Cesàro sense random variables satisfying*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E|X_{mn}|^p}{(mn)^p} < \infty \quad \text{for some} \quad 1 \leq p \leq 2. \tag{1}$$

Then for all $\varepsilon > 0$,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} P\Big(\max_{\substack{1 \leq k \leq m \\ 1 \leq l \leq n}} \Big|\sum_{i=1}^{k} \sum_{j=1}^{l} (X_{ij} - EX_{ij})\Big| > \varepsilon mn\Big) < \infty. \tag{2}$$

Proof. For given $\varepsilon > 0$, there exists a constant $x = x(\varepsilon) > 0$ such that for all $k, l \geq 1$,

$$\frac{1}{2^k 2^l} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E(|X_{ij}| I_{[|X_{ij}|>x]}) \leq \frac{\varepsilon}{4}. \tag{3}$$

We have the estimation

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\max_{\substack{1\leq i\leq 2^k \\ 1\leq j\leq 2^l}} |\sum_{t=1}^{i} \sum_{h=1}^{j} (X_{th} - EX_{th})| > \varepsilon 2^k 2^l \Big) \\
&\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\max_{\substack{1\leq i\leq 2^k \\ 1\leq j\leq 2^l}} |\sum_{t=1}^{i} \sum_{h=1}^{j} (X_{th} I_{[|X_{th}|\leq x]} - E(X_{th} I_{[|X_{th}|\leq x]}))| > \frac{\varepsilon 2^k 2^l}{4} \Big) \\
&\quad + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\max_{\substack{1\leq i\leq 2^k \\ 1\leq j\leq 2^l}} |\sum_{t=1}^{i} \sum_{h=1}^{j} (X_{th} I_{[|X_{th}|> x]} - E(X_{th} I_{[|X_{th}|> x]}))| > \frac{3\varepsilon 2^k 2^l}{4} \Big) \\
&:= I_1 + I_2.
\end{aligned}$$

For I_1, combining the Markov's inequality with Corollary 2 in [12], we obtain

$$\begin{aligned}
I_1 &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\max_{\substack{1\leq i\leq 2^k \\ 1\leq j\leq 2^l}} |\sum_{t=1}^{i} \sum_{h=1}^{j} (X_{th} I_{[|X_{th}|\leq x]} - E(X_{th} I_{[|X_{th}|\leq x]}))| > \frac{\varepsilon 2^k 2^l}{4} \Big) \\
&\leq C \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2^k 2^l)^2} E\Big(\max_{\substack{1\leq i\leq 2^k \\ 1\leq j\leq 2^l}} |\sum_{t=1}^{i} \sum_{h=1}^{j} (X_{th} I_{[|X_{th}|\leq x]} - E(X_{th} I_{[|X_{th}|\leq x]}))| \Big)^2 \\
&\leq C \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\log^2(2^{k+1})}{(2^k)^2} \frac{\log^2(2^{l+1})}{(2^l)^2} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E(X_{ij}^2 I_{[|X_{ij}|\leq x]}) \\
&\leq C \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(k+1)^2}{2^k} \frac{(l+1)^2}{2^l} < \infty.
\end{aligned}$$

For I_2, by setting $Y_{ij} = |X_{ij}| I_{[|X_{ij}|>x]}$ and using (3) we have

$$\begin{aligned}
I_2 &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\max_{\substack{1\leq i\leq 2^k \\ 1\leq j\leq 2^l}} |\sum_{t=1}^{i} \sum_{h=1}^{j} (X_{th} I_{[|X_{th}|> x]} - E(X_{th} I_{[|X_{th}|> x]}))| > \frac{3\varepsilon 2^k 2^l}{4} \Big) \\
&\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} Y_{ij} > \frac{2\varepsilon 2^k 2^l}{4} \Big) \\
&\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\Big(\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} (Y_{ij} - EY_{ij}) > \frac{\varepsilon 2^k 2^l}{4} \Big)
\end{aligned}$$

$$= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\Big(\big(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(Y_{ij} - EY_{ij}) > \frac{\varepsilon 2^k 2^l}{4} \big) \big(\bigcap_{i=1}^{2^k}\bigcap_{j=1}^{2^l}(Y_{ij} \le 2^k 2^l) \big) \Big)$$

$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\Big(\big(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(Y_{ij} - EY_{ij}) > \frac{\varepsilon 2^k 2^l}{4} \big) \big(\bigcup_{i=1}^{2^k}\bigcup_{j=1}^{2^l}(Y_{ij} > 2^k 2^l) \big) \Big)$$

$$\le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\Big(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(Y_{ij} I_{[Y_{ij} \le 2^k 2^l]} - EY_{ij}) > \frac{\varepsilon 2^k 2^l}{4} \Big)$$

$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} P(Y_{ij} > 2^k 2^l).$$

Since $E(Y_{ij} I_{[Y_{ij} \le 2^k 2^l]}) \le EY_{ij}$ for all $1 \le i \le 2^k$, $1 \le j \le 2^l$ then we have

$$I_2 \le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P(|\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(Y_{ij} I_{[Y_{ij} \le 2^k 2^l]} - E(Y_{ij} I_{[Y_{ij} \le 2^k 2^l]}))| > \frac{\varepsilon 2^k 2^l}{4})$$

$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} P(Y_{ij} > 2^k 2^l)$$

$$\le C \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(2^k 2^l)^{-2}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E(Y_{ij}^2 I_{[Y_{ij} \le 2^k 2^l]}) + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(2^k 2^l)^{-p}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} EY_{ij}^p$$

$$\le C \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(2^k 2^l)^{-p} EY_{ij}^p \ \le \ C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{-p} E|X_{ij}|^p < \infty.$$

Therefore

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P(\max_{\substack{1 \le i \le 2^k \\ 1 \le j \le 2^l}} |\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th} - EX_{th})| > \varepsilon 2^k 2^l) < \infty.$$

For any positive integers m, n, there exist integers k, l such that $2^k \le m < 2^{k+1}$ and $2^l \le n < 2^{l+1}$. Then for each $1 \le i \le m$, $1 \le j \le n$ we note that

$$\frac{1}{mn}\max_{\substack{1 \le i \le m \\ 1 \le j \le n}} |\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th} - EX_{th})| \le \frac{4}{2^{k+1}2^{l+1}}\max_{\substack{1 \le i \le 2^{k+1} \\ 1 \le j \le 2^{l+1}}} |\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th} - EX_{th})|,$$

and so
$$\frac{1}{mn} P\Big(\frac{1}{mn}\max_{\substack{1 \le i \le m \\ 1 \le j \le n}} |\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th} - EX_{th})| > \varepsilon\Big)$$

$$\le \frac{4}{2^{k+1}2^{l+1}} P\Big(\frac{1}{2^{k+1}2^{l+1}}\max_{\substack{1 \le i \le 2^{k+1} \\ 1 \le j \le 2^{l+1}}} |\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th} - EX_{th})| > \frac{\varepsilon}{4}\Big).$$

Thus, setting $\varepsilon^* := (\varepsilon 2^{k+1}2^{l+1})/4$,

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P\Big(\max_{\substack{1\le i\le m\\1\le j\le n}}|\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th}-EX_{th})|>\varepsilon mn\Big)\\
=\ &\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}\frac{1}{mn}P\Big(\frac{1}{mn}\max_{\substack{1\le i\le m\\1\le j\le n}}|\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th}-EX_{th})|>\varepsilon\Big)\\
\le\ &\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{4(2^{k+1}-2^k)(2^{l+1}-2^l)}{2^{k+1}2^{l+1}}P\Big(\max_{\substack{1\le i\le 2^{k+1}\\1\le j\le 2^{l+1}}}|\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th}-EX_{th})|>\varepsilon^*\Big)\\
=\ &\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}P\Big(\max_{\substack{1\le i\le 2^{k+1}\\1\le j\le 2^{l+1}}}|\sum_{t=1}^{i}\sum_{h=1}^{j}(X_{th}-EX_{th})|>\varepsilon^*\Big)<\infty,
\end{aligned}
$$

and the proof is complete. □

Remark 3.2. The above theorem extends Theorem 1.1 in [1] to double arrays of random variables by the other proof. Here using subsequences $\{2^k\}$ and $\{2^l\}$, the last inequality in the proof of I_2 is verified.

The key tool in the proof of Theorem 3.1 is the two-parameter version of the Menschoff-Rademacher maximal inequality for orthogonal random variables proved by Móricz [12] as follows:

$$E(\max_{\substack{1\le k\le m\\1\le l\le n}}|\sum_{i=1}^{k}\sum_{j=1}^{l}Y_{ij}|)^2\le \log^2(2m)\log^2(2n)\sum_{i=1}^{m}\sum_{j=1}^{n}EY_{ij}^2,$$

where $Y_{ij}, i\ge 1, j\ge 1$ are orthogonal random variables. Note that, the assumption of pairwise independence in Theorem 3.1 ensures that the array $\{X_{ij}I_{[|X_{ij}|\le x]}-E(X_{ij}I_{[|X_{ij}|\le x]})\}$ is orthogonal. □

The following theorem generalizes Theorem 3.1 in [1] to double arrays of random elements.

Theorem 3.3. *Let $\{X_{mn}\}$ be a double array of pairwise independent, compactly uniformly integrable in the Cesàro sense random elements such that*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|X_{mn}\|^p}{(mn)^p}<\infty \ for\ some\ 1\le p\le 2.$$

Then for all $\varepsilon>0$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P(\max_{\substack{1\le k\le m\\1\le l\le n}}\|\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij}-EX_{ij})\|>\varepsilon mn)<\infty.$$

In particular, the Kolmogorov's SLLN holds, i.e.,

$$\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}(X_{kl}-EX_{kl})\to 0 \quad a.s., \quad m\vee n\to\infty.$$

Proof. For each $\varepsilon>0$, a compact subset K_ε of $\mathbf{E}$ can be chosen so that for all $m,n\geq 1$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]})<\frac{\varepsilon}{6}.$$

By the compactness of K_ε, there exists $\{x_1,...,x_p\}\subset K_\varepsilon$ such that $K_\varepsilon\subset\bigcup_{t=1}^{p}B(x_t,\varepsilon)$. For every $i,j\geq 1$, we define the $\mathbf{E}$-random variables as follows:

$$Z_{ij}=X_{ij}I_{[X_{ij}\in K_\varepsilon]} \quad \text{and} \quad Y_{ij}:=\begin{cases}0, & X_{ij}\notin K_\varepsilon,\\ x_1, & X_{ij}\in B(x_1,\frac{\varepsilon}{6})\cap K_\varepsilon,\\ x_t, & X_{ij}\in B(x_t,\frac{\varepsilon}{6})\cap[\bigcup_{j=1}^{t-1}B(x_j,\frac{\varepsilon}{6})]^c\cap K_\varepsilon,\\ & \quad t=2,\dots,p.\end{cases}$$

Then Y_{ij} can be written as

$$Y_{ij}=\sum_{t=0}^{p}x_tI_{[Y_{ij}=x_t]}=\sum_{t=0}^{p}x_tI_{[X_{ij}\in A_t]}\in\sigma(X_{ij}),$$

where $x_0=0$, $A_0=K_\varepsilon^c$, $A_1=B(x_1,\frac{\varepsilon}{6})\cap K_\varepsilon$, and

$$A_t=B(x_t,\frac{\varepsilon}{6})\cap[\bigcup_{j=1}^{t-1}B(x_j,\frac{\varepsilon}{6})]^c\cap K_\varepsilon.$$

For each $1\leq k\leq m$, $1\leq l\leq n$, using the triangle inequality gives

$$\begin{aligned}\|\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij}-EX_{ij})\|&\leq\|\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij}-Z_{ij})\|+\|\sum_{i=1}^{k}\sum_{j=1}^{l}(Z_{ij}-Y_{ij})\|\\ &+\|\sum_{i=1}^{k}\sum_{j=1}^{l}(Y_{ij}-EY_{ij})\|+\|\sum_{i=1}^{k}\sum_{j=1}^{l}(EY_{ij}-EZ_{ij})\|\\ &+\|\sum_{i=1}^{k}\sum_{j=1}^{l}(EZ_{ij}-EX_{ij})\|:=I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

For I_1, we have

$$I_1 \leq \sum_{i=1}^{k}\sum_{j=1}^{l}\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]} \leq \max_{\substack{1\leq k\leq m\\1\leq l\leq n}} |\sum_{i=1}^{k}\sum_{j=1}^{l}(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]} - E(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}))| + \sum_{i=1}^{m}\sum_{j=1}^{n}E(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}),$$

and so

$$\max_{\substack{1\leq k\leq m\\1\leq l\leq n}} I_1 \leq \max_{\substack{1\leq k\leq m\\1\leq l\leq n}} |\sum_{i=1}^{k}\sum_{j=1}^{l}(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]} - E(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}))| + \frac{\varepsilon mn}{6}.$$

For I_2, by the definition of Z_{ij}, we have

$$\|\sum_{i=1}^{k}\sum_{j=1}^{l}(Z_{ij}-Y_{ij})\| \leq \sum_{i=1}^{k}\sum_{j=1}^{l}\|Z_{ij}-Y_{ij}\| \leq \sum_{k=1}^{m}\sum_{l=1}^{n}\|Z_{kl}-Y_{kl}\| < \frac{\varepsilon mn}{6}.$$

Then $\max_{\substack{1\leq k\leq m\\1\leq l\leq n}} I_2 < \frac{\varepsilon mn}{6}$. For I_3,

$$\begin{aligned}
\|\sum_{i=1}^{k}\sum_{j=1}^{l}(Y_{ij}-EY_{ij})\| &= \|\sum_{i=1}^{k}\sum_{j=1}^{l}(\sum_{t=0}^{p}x_tI_{[Y_{ij}=x_t]} - \sum_{t=0}^{p}x_tP(Y_{ij}=x_t))\| \\
&\leq \sum_{t=0}^{p}\|x_t\||\sum_{i=1}^{k}\sum_{j=1}^{l}(I_{[Y_{ij}=x_t]} - P(Y_{ij}=x_t))| \\
&\leq \sum_{t=0}^{p}\|x_t\| \max_{\substack{1\leq k\leq m\\1\leq l\leq n}} |\sum_{i=1}^{k}\sum_{j=1}^{l}(I_{[Y_{ij}=x_t]} - P(Y_{ij}=x_t))|,
\end{aligned}$$

so that

$$\max_{\substack{1\leq k\leq m\\1\leq l\leq n}} I_3 \leq \sum_{t=0}^{p}\|x_t\| \max_{\substack{1\leq k\leq m\\1\leq l\leq n}} |\sum_{i=1}^{k}\sum_{j=1}^{l}(I_{[Y_{ij}=x_t]} - P(Y_{ij}=x_t))|.$$

For I_4, since

$$\|\sum_{i=1}^{k}\sum_{j=1}^{l}E(Y_{ij}-Z_{ij})\| \leq \sum_{k=1}^{m}\sum_{l=1}^{n}E\|Y_{kl}-Z_{kl}\| < \frac{\varepsilon mn}{6}$$

then $\max_{\substack{1\leq k\leq m\\1\leq l\leq n}} I_4 < \frac{\varepsilon mn}{6}$.

Finally, for I_5, by the definition of Z_{ij} again, $\max\limits_{\substack{1\le k\le m\\1\le l\le n}} I_5 < \dfrac{\varepsilon mn}{6}$ since

$$\|\sum_{i=1}^{k}\sum_{j=1}^{l}E(Z_{ij}-X_{ij})\| \le \sum_{k=1}^{m}\sum_{l=1}^{n}E\|Z_{kl}-X_{kl}\| < \frac{\varepsilon mn}{6}.$$

Combining the above parts, we have

$$\begin{aligned}
&P\Big(\max_{\substack{1\le k\le m\\1\le l\le n}}\|\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij}-EX_{ij})\| > \varepsilon mn\Big)\\
\le\ & P\Big(\max_{\substack{1\le k\le m\\1\le l\le n}}|\sum_{i=1}^{k}\sum_{j=1}^{l}(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}-E(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}))| > \frac{\varepsilon mn}{6}\Big)\\
+\ & P\Big(\max_{\substack{1\le k\le m\\1\le l\le n}}|\sum_{i=1}^{k}\sum_{j=1}^{l}(I_{[Y_{ij}=x_t]}-P(Y_{ij}=x_t))| > \frac{C\varepsilon mn}{6}\Big).
\end{aligned}$$

Moreover, Theorem 3.1 implies that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P(\max_{\substack{1\le k\le m\\1\le l\le n}}|\sum_{i=1}^{k}\sum_{j=1}^{l}(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}-E(\|X_{ij}\|I_{[X_{ij}\notin K_\varepsilon]}))| > \frac{mn\varepsilon}{6}) < \infty,$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P(\max_{\substack{1\le k\le m\\1\le l\le n}}|\sum_{i=1}^{k}\sum_{j=1}^{l}(I_{[Y_{ij}=x_t]}-P(Y_{ij}=x_t))| > \frac{Cmn\varepsilon}{6}) < \infty.$$

Consequently,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P(\max_{\substack{1\le k\le m\\1\le l\le n}}\|\sum_{i=1}^{k}\sum_{j=1}^{l}(X_{ij}-EX_{ij})\| > \varepsilon mn) < \infty.$$

The rest of the theorem follows from Theorem 1 in [6]. $\square$

4. Complete convergence for double arrays of random sets

In this section, $\overline{B}_{\mathbf{E}^*}$ denotes the closed unit ball of the dual space endowed with the topology of compact convergence, so that $(\overline{B}_{\mathbf{E}^*}, d)$ is a compact metrizable space, for instance, by the metric

$$d(x^*,y^*) := \sum_{n=1}^{\infty}\frac{1}{2^n}|x^*(e_n)-y^*(e_n)|,\ x^*,y^*\in\overline{B}_{\mathbf{E}^*}, \tag{4}$$

where $D = \{e_n\}$ is a dense sequence in $\overline{B}_{\mathbf{E}}$. Let $C(\overline{B}_{\mathbf{E}^*})$ be the separable Banach space of all continuous functions φ defined on $\overline{B}_{\mathbf{E}^*}$, endowed with the norm of uniform convergence $\|\varphi\|_\infty = \sup_{x^* \in \overline{B}_{\mathbf{E}^*}} |\varphi(x^*)|$, and let $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$ be the space of all convex compact valued integrable random sets. We first recall some needed results. We refer to [14].

Theorem 4.1. *Let X be a $cwk(\mathbf{E})$-valued integrable random set. The following properties hold:*

(a) *The set S^1_X of all integrable selections of X is convex $\sigma(L^1_{\mathbf{E}}, L^\infty_{\mathbf{E}^*})$-compact,*

(b) *EX is convex weakly compact,*

(c) $\delta^*(x^*, EX) = E(\delta^*(x^*, X))$ *for all* $x^* \in \mathbf{E}^*$,

(d) *if X is a $ck(\mathbf{E})$-valued integrable random set, then EX is convex compact.*

We aim to present the relationships between the space $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$ and the space $L^1_{\mathcal{C}(\overline{B}_{\mathbf{E}^*})}(\Omega, \mathcal{F}, P)$. Given $X \in \mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$, the function $\delta^*(., X)$ belongs to $L^1_{\mathcal{C}(\overline{B}_{\mathbf{E}^*})}(\Omega, \mathcal{F}, P)$ since the function $\delta^*(x^*, X(\omega))$ defined on $\Omega \times \overline{B}_{\mathbf{E}^*}$ is a Caratheodory integrable mapping, that is, $\delta^*(., X(\omega))$ is continuous on $\overline{B}_{\mathbf{E}^*}$ for every $\omega \in \Omega$, $\delta^*(x^*, X(.))$ is measurable on Ω for every $x^* \in \overline{B}_{\mathbf{E}^*}$, and $|\delta^*(x^*, X(\omega))| \leq |X(\omega)|$ for all $(\omega, x^*) \in \Omega \times \overline{B}_{\mathbf{E}^*}$ with $|X| \in L^1(\Omega, \mathcal{F}, P)$. More generally, the space $\mathcal{G}$ of all Caratheodory integrable integrands on $\Omega \times S$, where S is a compact metric plays a central role in the convergence of Young measures (see [15]), and this space can be identified with the space $L^1_{\mathcal{C}(S)}(\Omega, \mathcal{F}, P)$, where $\mathcal{C}(S)$ is the space of all continuous functions on S endowed with the norm of uniform convergence. Similarly, in the present situation, it turns out that the space $L^1_{\mathcal{C}(\overline{B}_{\mathbf{E}^*})}(\Omega, \mathcal{F}, P)$ associated with the space $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$ also plays a crucial role in the convergence problem in $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$ with respect to the Hausdorff distance. This fact is illustrated by establishing the relationships between these spaces that are summarized in the following.

Theorem 4.2. *Let $X \in \mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$ be given. Then the following properties hold:*

(a) $\delta^*(., X) \in L^1_{\mathcal{C}(\overline{B}_{\mathbf{E}^*})}(\Omega, \mathcal{F}, P)$,

(b) $E(\delta^*(., X)) \in \mathcal{C}(\overline{B}_{\mathbf{E}^*})$ *and the value of* $E(\delta^*(., X))$ *at the point* $x^* \in \overline{B}_{\mathbf{E}^*}$ *is given by*

$$E(\delta^*(., X))(x^*) = \langle E(\delta^*(., X)), \varepsilon_{x^*} \rangle = \int_\Omega \delta^*(x^*, X(\omega)) dP = \delta^*(x^*, EX),$$

where ε_{x^*} is the Dirac mass at the point $x^* \in \overline{B}_{\mathbf{E}^*}$.

(c) *Let $X, Y \in \mathcal{L}^1_{ck(\mathbf{E})}(\Omega, \mathcal{F}, P)$ be given. Then*

$$\|\delta^*(., X) - E(\delta^*(., Y))\|_\infty = \sup_{x^* \in \overline{B}_{\mathbf{E}^*}} |\delta^*(x^*, X) - E(\delta^*(., Y))(x^*)| = d_H(X, EY).$$

Thus, by the above considerations and Theorems 4.1–4.2, we note that the space $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P)$ is embedded via the mapping $j\colon X\mapsto \delta^*(.,X)$ as a convex cone into the Banach space $L^1_{C(\overline{B}_{\mathbf{E}^*})}(\Omega,\mathcal{F},P)$ so that:

$$E(j(X)) = j(EX) \quad \text{for all } X\in \mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P), \tag{5}$$

$$\|j(X)-E(j(Y))\|_\infty = d_H(X,EY) \quad \text{for all } X,Y\in \mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P). \tag{6}$$

Remark 4.3. Actually (5) and (6) are the main properties of this embedding we use in further applications. That is why we need to provide a concise presentation of these properties.

Let $\{X_{mn}\}$ be a double array of random sets in $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P)$. Then the array $\{\delta^*(.,X_{mn})\}$ is a double array of random elements in $L^1_{C(\overline{B}_{\mathbf{E}^*})}(\Omega,\mathcal{F},P)$. By Theorem 4.2, for all $m,n\geq 1$ and for all $1\leq k\leq m, 1\leq l\leq n$,

$$\begin{aligned} d_H\Big(\frac{1}{mn}\sum_{i=1}^{k}\sum_{j=1}^{l}X_{ij},\frac{1}{mn}\sum_{i=1}^{k}\sum_{j=1}^{l}EX_{ij}\Big) \\ = \Big\|\frac{1}{mn}\sum_{i=1}^{k}\sum_{j=1}^{l}(\delta^*(.,X_{ij})-E(\delta^*(.,X_{ij})))\Big\|_\infty. \end{aligned} \tag{7}$$

In particular, $|X_{mn}| = \|\delta^*(.,X_{mn})\|_\infty$ for all $m,n\geq 1$. □

Now, using the above embedding, some general results of the complete convergence for random sets are obtained under new conditions focus on the random elements $\{\delta^*(.,X_{mn})\}$ as follows.

Theorem 4.4. *Let $\{X_{mn}\}$ be a double array of random sets in $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P)$. Assume that:*

(i) $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\dfrac{E|X_{mn}|^p}{(mn)^p}<\infty$ *for some* $1\leq p\leq 2$,

(ii) *the double array $\{\delta^*(.,X_{mn})\}$ is pairwise independent and compactly uniformly integrable in the Cesàro sense in $\mathcal{C}(\overline{B}_{\mathbf{E}^*})$.*

Then for all $\varepsilon>0$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P\Big(\max_{\substack{1\leq k\leq m\\ 1\leq l\leq n}} d_H\Big(\sum_{i=1}^{k}\sum_{j=1}^{l}X_{ij},\sum_{i=1}^{k}\sum_{j=1}^{l}EX_{ij}\Big)>\varepsilon mn\Big)<\infty. \tag{8}$$

In particular, the Kolmogorov's SLLN holds, i.e.,

$$d_H\Big(\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}X_{kl},\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}EX_{kl}\Big)\to 0 \quad a.s., \quad m\vee n\to\infty. \tag{9}$$

Proof. As a consequence of our assumption, the double array $\{\delta^*(.,X_{mn})\}$ in the space $L^1_{C(\overline{B}_{\mathbf{E}^*})}(\Omega,\mathcal{F},P)$ satisfies the conditions of Theorem 3.3. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}P\Big(\max_{\substack{1\le k\le m\\ 1\le l\le n}}\|\sum_{i=1}^{k}\sum_{j=1}^{l}(\delta^*(.,X_{ij})-E(\delta^*(.,X_{ij})))\|_\infty>\varepsilon mn\Big)<\infty \tag{10}$$

and

$$\|\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}(\delta^*(.,X_{kl})-E(\delta^*(.,X_{kl})))\|_\infty\to 0 \quad \text{a.s.,}\quad m\vee n\to\infty. \tag{11}$$

Combining (7) and (10) (resp. (7) and (11)) we get (8) (resp. (9)) and the proof is complete. □

Remark 4.5. (i) If $\{X_{mn}\}$ is pairwise independent and compactly uniformly integrable in the Cesàro sense in $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P)$, that is, for each $\varepsilon>0$, we can choose a compact subset $\mathcal{K}_\varepsilon$ of $ck(\mathbf{E})$ such that for all $m,n\ge 1$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E(|X_{ij}|I_{[X_{ij}\notin\mathcal{K}_\varepsilon]})<\varepsilon.$$

Then $\{\delta^*(.,X_{mn})\}$ is pairwise independent, and compactly uniformly integrable in the Cesàro sense in $\mathcal{C}(\overline{B}_{\mathbf{E}^*})$. Indeed, it is easy to see that $\{\delta^*(.,X_{mn})\}$ is pairwise independent, and the set $M_\varepsilon=\{\delta^*(.,K_\varepsilon):K_\varepsilon\in\mathcal{K}_\varepsilon\}$ is a compact subset of $\mathcal{C}(\overline{B}_{\mathbf{E}^*})$. Moreover, for each $m,n\ge 1$, $[\delta^*(.,X_{mn})\notin M_\varepsilon]\subset[X_{mn}\notin\mathcal{K}_\varepsilon]$ and $\|\delta^*(.,X_{mn})\|_\infty=|X_{mn}|$. Therefore for all $m,n\ge 1$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E(\|\delta^*(.,X_{ij})\|_\infty I_{[\delta^*(.,X_{ij})\notin M_\varepsilon]})<\varepsilon.$$

This establishes the claim.

(ii) The below example will show that the compactly uniform integrability in the Cesàro sense of $\{\delta^*(.,X_{mn})\}$ is really weaker than the compactly uniform integrability in the Cesàro sense of $\{X_{mn}\}$. To construct this example we need following lemmas. □

Lemma 4.6. *Let c_0 be the space of all sequences that convergence to zero with the norm $\|x\|_{c_0}=\sup_{n\ge 1}|x_n|$. If $x=\{x_n\}\in c_0$ then*

$$B_x=\{y=\{y_n\}\subset\mathbb{R}:|y_n|\le|x_n| \textit{ for all } n\ge 1\}$$

is a nonempty convex compact subset of c_0.

Proof. First, since $x \in B_x$ then B_x is a nonempty subset of c_0. For all $y = \{y_n\}$, $z = \{z_n\} \in B_x$, and for all $\lambda \in [0,1]$, we have

$$|\lambda y_n + (1-\lambda)z_n| \le \lambda|y_n| + (1-\lambda)|z_n| \le |x_n| \text{ for all } n \ge 1,$$

so that $\lambda y + (1-\lambda)z \in B_x$. Thus B_x is convex. Let $\{y^m\} \subset B_x$, where $y^m = \{y^m_n\}_n$ and $y^m \to y^0$ as $m \to \infty$ with $y^0 = \{y^0_n\}_n \in c_0$. We have

$$|y^0_n| \le \sup_n |y^0_n - y^m_n| + |y^m_n| \le \|y^0 - y^m\|_{c_0} + |x_n| \text{ for all } n \ge 1.$$

Letting $m \to \infty$ in the above inequalities we obtain $|y^0_n| \le |x_n|$ for all $n \ge 1$. Therefore $y^0 \in B_x$, and so B_x is closed. Since c_0 is complete, it follows that B_x is complete. Hence we need only prove that B_x is totally bounded. Consider any $\varepsilon > 0$ and define $\delta := \varepsilon/2$. Since $\lim_{n\to\infty} |x_n| = 0$ then there exists an integer N such that $|x_n| < \delta$ for all $n > N$. Now consider the subset $B^N_x \subset B_x$ defined by

$$B^N_x = \{y = \{y_n\} \in B_x : y_n = 0 \text{ for all } n > N\}.$$

We note that the set B^N_x is isometric to the closed bounded subset B of the space $\mathbb{R}^N$

$$B = \{(y_1, ..., y_N) \in \mathbb{R}^N : |y_i| \le |x_i|,\ i = 1, ..., N\},$$

and so is compact. In particular it is totally bounded, so we can choose points $u^1, ..., u^r \in c_0$ such that $B^N_x \subset \bigcup_{i=1}^r N(u^i, \delta)$, where $N(u^i, \delta) = \{v \in c_0 : \|v - u^i\|_{c_0} < \delta\}$. Now let $y = \{y_n\} \in B_x$ be given. We have $|y_n| \le |x_n| < \delta$ for all $n > N$ and $y = (y_1, ..., y_N, 0, ...) + (0, ..., 0, y_{N+1}, ...)$. Since we have $z = (y_1, ..., y_N, 0, ...) \in B^N_x$, there exists $p \in \{1, ..., r\}$ such that $\|z - u^p\|_{c_0} < \delta$ and $\|t\|_{c_0} < \delta$ with $t = (0, ..., 0, y_{N+1}, ...)$. Hence $\|y - u^p\|_{c_0} \le \|z - u^p\|_{c_0} + \|t\|_{c_0} < \varepsilon$. Thus, $B_x \subset \bigcup_{i=1}^r N(u^i, \varepsilon)$ which implies that B_x is a compact subset of c_0. The proof is completed. □

Next, applying Arzelà-Ascoli's theorem, the useful lemma is obtained.

Lemma 4.7. *Let $(\overline{B}_{c_0^*}, d)$ be compact metrizable space with the metric d given by* (4)*. Then*

$$\mathcal{K} = \{f \in C(\overline{B}_{c_0^*}) : \|f\|_\infty \le 2, |f(x^*) - f(y^*)| \le 2d(x^*, y^*) \textit{ for all } x^*, y^* \in \overline{B}_{c_0^*}\}$$

is a nonempty compact subset of $C(\overline{B}_{c_0^*})$.

Proof. We first note that $\mathcal{K}$ is a nonempty subset of $C(\overline{B}_{c_0^*})$ because the norm $\|.\|_{c_0^*}$ belongs to $\mathcal{K}$. Now, let $\{f_n\} \subset \mathcal{K}$ and $f_n \to f$ uniformly. Then $\|f\|_\infty = \lim_{n\to\infty} \|f_n\|_\infty \le 2$. Since $f_n \in \mathcal{K}$ then

$$|f_n(x^*) - f_n(y^*)| \le 2d(x^*, y^*) \text{ for all } x^*, y^* \in \overline{B}_{c_0^*}.$$

Passing to the limit when n goes to ∞, by using the facts that $f_n \to f$ pointwise, and

$$|f(x^*) - f(y^*)| \leq |f(x^*) - f_n(x^*)| + |f_n(x^*) - f_n(y^*)| + |f_n(y^*) - f(y^*)|,$$

we obtain $|f(x^*) - f(y^*)| \leq 2d(x^*, y^*)$ for all $x^*, y^* \in \overline{B}_{c_0^*}$, and so $\mathcal{K}$ is closed. It is clear that $\mathcal{K}$ is uniformly bounded since $\|f\|_\infty \leq 2$ for all $f \in \mathcal{K}$. Next consider $\varepsilon > 0$, and choose $\delta = \dfrac{\varepsilon}{3}$. Then for all $x^*, y^* \in \overline{B}_{c_0^*}$, $d(x^*, y^*) < \delta$, and for all $f \in \mathcal{K}$ we have $|f(x^*) - f(y^*)| \leq 2d(x^*, y^*) < 2\delta < \varepsilon$, which shows that $\mathcal{K}$ is equicontinuous. Hence, Arzelà-Ascoli theorem implies the conclusion. $\square$

Example 4.8. Let $r^{kl} = \{r_n^{kl}\}_n$ be the sequence of pairwise independent random variables defined as follows:

$$P(r_n^{kl} = \frac{1}{n^{kl}}) = P(r_n^{kl} = -\frac{1}{n^{kl}}) = \frac{1}{2} \text{ for all } k, l \geq 1.$$

We note that $r^{kl}(\omega)$ belongs to c_0 for all $k, l \geq 1$ and for all $\omega \in \Omega$. For each $\omega \in \Omega$, we set

$$X_{kl}(\omega) = \{x = \{x_n\} \subset \mathbb{R} : |x_n| \leq |r_n^{kl}(\omega)| \text{ for all } n \geq 1\}.$$

Then we have $\{X_{kl}\}$ is the array of random sets in $\mathcal{L}^1_{ck(c_0)}(\Omega, \mathcal{F}, P)$. For each $x = \{x_n\} \in X_{k_1 l_1}(\omega)$, we need to show that $d(x, X_{k_2 l_2}(\omega)) = \|x\|_{c_0}$ with the two distinct sets $X_{k_1 l_1}(\omega)$ and $X_{k_2 l_2}(\omega)$. Indeed, for all $y = \{y_n\} \in X_{k_2 l_2}(\omega)$, we have

$$\|x - y\|_{c_0} \geq |x_n - y_n| \geq |x_n| - |y_n| \geq |x_n| - \frac{1}{n^{k_2 l_2}} \text{ for all } n \geq 1,$$

and so

$$\|x - y\|_{c_0} \geq \sup_n (|x_n| - \frac{1}{n^{k_2 l_2}}) = \sup_n |x_n| - \inf_n \frac{1}{n^{k_2 l_2}} = \sup_n |x_n| = \|x\|_{c_0}.$$

Moreover, for all $\varepsilon > 0$, by letting $u = (0, 0, ...) \in X_{k_2 l_2}(\omega)$ we obtain $\|x\|_{c_0} + \varepsilon > \|x - u\|_{c_0}$ and $d(x, X_{k_2 l_2}(\omega)) = \inf\limits_{y \in X_{k_2 l_2}(\omega)} \|x - y\|_{c_0} = \|x\|_{c_0}$ by definition of the infimum. This gives $\sup\limits_{x \in X_{k_1 l_1}(\omega)} d(x, X_{k_2 l_2}(\omega)) = \sup\limits_{x \in X_{k_1 l_1}(\omega)} \|x\|_{c_0} = 1$, so that $d_H(X_{k_1 l_1}(\omega), X_{k_2 l_2}(\omega)) = 1$. Therefore, if $\mathcal{K}$ is any compact subset of $ck(c_0)$ then $\mathcal{K}$ contains at most finitely many elements of the array $\{X_{kl}\}$. Hence

$$\sup_{m,n \geq 1} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} E(|X_{kl}| I_{[X_{kl} \notin \mathcal{K}]}) = 1,$$

which implies that the array $\{X_{kl}\}$ is not compactly uniformly integrable in the Cesàro sense. $\square$

We next prove that the array $\{\delta^*(.,X_{kl})\}$ is compactly uniformly integrable in the Cesàro sense in the separable Banach space $(C(\overline{B}_{c_0^*}),\|.\|_\infty)$. Indeed, consider the compact subset $\mathcal{K}$ of $C(\overline{B}_{c_0^*})$ in Lemma 4.7

$$\mathcal{K}=\{f\in C(\overline{B}_{c_0^*}):\|f\|_\infty\le 2,|f(x^*)-f(y^*)|\le 2d(x^*,y^*)\text{ for all }x^*,y^*\in\overline{B}_{c_0^*}\},$$

where the metric d given by (4). By Theorem 4.2, we have $\delta^*(.,X_{kl}(\omega))\in C(\overline{B}_{c_0^*})$ for all $k,l\ge 1$, and for all $\omega\in\Omega$. Moreover, $\|\delta^*(.,X_{kl}(\omega))\|_\infty=|X_{kl}(\omega)|=1$, and for all $x^*,y^*\in\overline{B}_{c_0^*}$ we have

$$|\delta^*(x^*,X_{kl}(\omega))-\delta^*(y^*,X_{kl}(\omega))|\le\|x^*-y^*\|_{c_0^*}=\sup_{n\ge 1}|x^*(e_n)-y^*(e_n)|\le 2d(x^*,y^*)$$

for all $k,l\ge 1$, for all $\omega\in\Omega$. By definition of $\mathcal{K}$, $\delta^*(.,X_{kl}(\omega))\in\mathcal{K}$ for all $k,l\ge 1$ and for all $\omega\in\Omega$. Therefore $I_{[\omega:\delta^*(.,X_{kl}(\omega))\notin\mathcal{K}]}=0$ for all $k,l\ge 1$. Hence the double array $\{\delta^*(.,X_{kl})\}$ is compactly uniformly integrable in the Cesàro sense in $C(\overline{B}_{c_0^*})$. In addition, it is easy to see that $\{\delta^*(.,X_{kl})\}$ is pairwise independent, and $\{X_{kl}\}$ satisfies the condition (i) in Theorem 4.4, then the conclusions of Theorem 4.4 are verified.

We finish this section with an important result generalizing Theorem 2.1 in [13] to double arrays of pairwise i.i.d. random sets.

Theorem 4.9. *Let $\{X_{mn}\}$ be a double array of random sets in $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P)$ such that $E(|X_{11}|\log^+|X_{11}|)<\infty$ and $\{\delta^*(.,X_{mn})\}$ is a double array of pairwise i.i.d. random elements in $L^1_{C(\overline{B}_{\mathbf{E}^*})}(\Omega,\mathcal{F},P)$. Then*

$$\lim_{m\vee n\to\infty}d_H(\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij},EX_{11})=0\quad a.s.$$

Proof. We have $|X_{11}|=\|\delta^*(.,X_{11})\|_\infty$. Applying Theorem 2 in [7], we get

$$\lim_{m\vee n\to\infty}\|\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n\delta^*(.,X_{ij})-E(\delta^*(.,X_{11}))\|_\infty=0\quad\text{a.s.}$$

Therefore by (7) we give the desired conclusion. □

Open problem. Find examples of double arrays $\{X_{mn}\}$ of random sets in $\mathcal{L}^1_{ck(\mathbf{E})}(\Omega,\mathcal{F},P)$ such that $E(|X_{11}|\log^+|X_{11}|)<\infty$ and $\{X_{mn}\}$ is not pairwise i.i.d., but the double array $\{\delta^*(.,X_{mn})\}$ is pairwise i.i.d. in $L^1_{C(\overline{B}_{\mathbf{E}^*})}(\Omega,\mathcal{F},P)$.

5. Some applications for random fuzzy variables

In this section, we provide some applications to the SLLN for double arrays of random fuzzy variables $\{X^{mn}\}$ with respect to a generalized Hausdorff distance.

The concept of random fuzzy variables defined by Castaing et al. (see [3]) as follows. A fuzzy convex upper semicontinuous variable is a mapping $X\colon \mathbf{E} \to [0,1]$ such that

(i) X is upper semicontinuous,
(ii) $\{x \in \mathbf{E} : X(x) = 1\} \neq \emptyset$,
(iii) X is fuzzy convex, that is, $X(\lambda x + (1-\lambda)y) \geq \min(X(x), X(y))$ for all $\lambda \in [0,1]$ and for all $x, y \in \mathbf{E}$.

A *random fuzzy convex upper semicontinuous variable* is an $\mathcal{F}\otimes\mathcal{B}(\mathbf{E})$-measurable mapping $X : \Omega \times \mathbf{E} \to [0,1]$ such that for each $\omega \in \Omega$ the mapping $X_\omega\colon \mathbf{E} \to [0,1]$ is a fuzzy convex upper semicontinuous variable.

By upper semicontinuity and fuzzy convexity, for each $\omega \in \Omega$ and for each $\alpha \in]0,1]$, the level set $X_\alpha(\omega) := \{x \in \mathbf{E} : X_\omega(x) \geq \alpha\}$ is closed convex. Moreover, it is measurable by the projection theorem (see Theorem III. 23, [4]). Similarly, the multifunction $\omega \mapsto \overline{\{x \in \mathbf{E} : X(\omega,x) > \alpha\}}$ is measurable, and so is the mapping $X_{\alpha+}(\omega) := \overline{\{x \in \mathbf{E} : X(\omega,x) > \alpha\}}$ for each $\alpha \in [0,1[$. Further, for any $\mathcal{F}\otimes\mathcal{B}(\mathbf{E})$-measurable mapping $\varphi\colon \Omega \times \mathbf{E} \to [0,+\infty]$, the function $m(\omega) := \sup\{\varphi(\omega,x) : x \in X_\alpha(\omega)\}$ is measurable by Lemma III.39 in [4], and so is the function $m_+(\omega) := \sup\{\varphi(\omega,x) : x \in X_{\alpha+}(\omega)\}$. In particular, X_{0+} is measurable, where $X_{0+}(\omega) = \overline{\{x \in \mathbf{E} : X(\omega,x) > 0\}}$, and $|X_{0+}|(\omega) := \sup\{\|x\| : x \in X_{0+}(\omega)\}$ is measurable. Further assume that $X_{0+}(\omega)$ is compact and $|X_{0+}| \in L^1$, then X_α is convex compact valued and integrably bounded: $|X_\alpha| \leq |X_{0+}|$ for all $\alpha \in]0,1]$, here the inclusion $|X_{0+}| \in L^1$ makes sense because of the above measurable properties.

Given a random fuzzy convex upper semicontinuous variable X, the fuzzy expectation of X is a fuzzy convex upper semicontinuous variable $\tilde{E}X$ such that

$$[\tilde{E}X]_\alpha = EX_\alpha \quad \text{for all } \alpha \in]0,1].$$

The first is the SLLN for the level sets associated with random fuzzy convex upper semicontinuous variables under a new condition focus on double arrays of random elements $\{\delta^*(.,X_\alpha^{mn})\}$ as follows.

Theorem 5.1. *Let $\{X^{mn}\}$ be a double array of random fuzzy convex upper semicontinuous variables defined on $\Omega \times \mathbf{E}$ satisfying:*

(i) $X_{0+}^{mn}(\omega)$ *is compact for every $\omega \in \Omega$ and for every $m,n \geq 1$,*

(ii) $g := \sup\limits_{m,n} |X_{0+}^{mn}| \in L^2(\Omega,\mathcal{F},P)$.

For each $\alpha \in]0,1]$, we assume:

(iii) $\{X_\alpha^{mn}\}$ *and* $\{X_{\alpha^+}^{mn}\}$ *are independent,*

(iv) $\{\delta^*(.,X_\alpha^{mn})\}$ *is compactly uniformly integrable in the Cesàro sense in* $L^1_{\mathcal{C}(\overline{B}_{\mathbf{E}^*})}(\Omega,\mathcal{F},P)$,

(v) *For every $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_r = 1$ such that for all $m, n \geq 1$,* $\max_{1\leq k\leq r} Ed_H(X^{mn}_{\alpha^+_{k-1}}, X^{mn}_{\alpha_k}) < \varepsilon$.

Then we have

$$\lim_{m\vee n\to\infty} \sup_{\alpha\in]0,1]} d_H(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X^{ij}_{\alpha}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} EX^{ij}_{\alpha}) = 0 \quad a.s.$$

Proof. Let $\alpha \in]0, 1]$ be given. Then there exists k (depending on α) such that $\alpha_{k-1} < \alpha \leq \alpha_k$, $k = 1, ..., r$. We have the estimation

$$\begin{aligned}
&\frac{1}{mn} d_H(\sum_{i=1}^{m}\sum_{j=1}^{n} X^{ij}_{\alpha}, \sum_{i=1}^{m}\sum_{j=1}^{n} EX^{ij}_{\alpha}) \\
&\leq \frac{1}{mn}\max_{1\leq k\leq r} \sum_{i=1}^{m}\sum_{j=1}^{n}(d_H(X^{ij}_{\alpha^+_{k-1}}, X^{ij}_{\alpha_k}) - Ed_H(X^{ij}_{\alpha^+_{k-1}}, X^{ij}_{\alpha_k})) \\
&\quad + \frac{1}{mn}\max_{1\leq k\leq r} d_H(\sum_{i=1}^{m}\sum_{j=1}^{n} X^{ij}_{\alpha_k}, \sum_{i=1}^{m}\sum_{j=1}^{n} EX^{ij}_{\alpha_k}) \\
&\quad + \frac{2}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\max_{1\leq k\leq r} Ed_H(X^{ij}_{\alpha_k}, X^{ij}_{\alpha^+_{k-1}}) := I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, we note that for every $k = 1, ..., r$ and for every i, j,

$$E(d_H(X^{ij}_{\alpha^+_{k-1}}, X^{ij}_{\alpha_k})^2) \leq 2^2 E(|X^{ij}_{0+}|^2) \leq 2^2 E(g^2).$$

Whence by our assumption,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E(d_H(X^{mn}_{\alpha^+_{k-1}}, X^{mn}_{\alpha_k})^2)}{(mn)^2} \leq 2^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E(g^2)}{(mn)^2} < \infty.$$

Applying the Kolmogorov's SLLN to the array $\{d_H(X^{mn}_{\alpha^+_{k-1}}, X^{mn}_{\alpha_k}) : m, n \geq 1\}$ yields

$$\lim_{m\vee n\to\infty} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(d_H(X^{ij}_{\alpha^+_{k-1}}, X^{ij}_{\alpha_k}) - Ed_H(X^{ij}_{\alpha^+_{k-1}}, X^{ij}_{\alpha_k})) = 0 \quad \text{a.s.},$$

and so $I_1 \to 0$ a.s. For I_2, we note that $\{\delta^*(., X^{mn}_{\alpha})\}$ is compactly uniformly integrable in the Cesàro sense in $\mathcal{C}(\overline{B}_{\mathbf{E}^*})$ by (iv), and verifies

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E(|X^{mn}_{\alpha}|^2)}{(mn)^2} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E(|X^{mn}_{0+}|^2)}{(mn)^2} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E(g^2)}{(mn)^2} < \infty$$

by using (ii). It follows from Theorem 4.4 that

$$\lim_{m\vee n\to\infty} d_H(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{\alpha_k}^{ij}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} EX_{\alpha_k}^{ij}) = 0 \quad \text{a.s.},$$

so that $I_2 \to 0$ a.s. For I_3, we also have $I_3 < 2\varepsilon$ for m, n large. Now by combining the above arguments we obtain

$$\frac{1}{mn} d_H(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{\alpha}^{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} EX_{\alpha}^{ij}) < I_1 + I_2 + 2\varepsilon \quad \text{for all } \alpha \in]0,1].$$

Since I_1, I_2 and $\varepsilon > 0$ do not depend on α, then from the above inequality it follows that

$$\sup_{\alpha\in]0,1]} \frac{1}{mn} d_H(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{\alpha}^{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} EX_{\alpha}^{ij}) \leq I_1 + I_2 + 2\varepsilon.$$

Passing to the limit when $m \vee n$ goes to ∞ in the preceding inequality yields

$$\lim_{m\vee n\to\infty} \sup_{\alpha\in]0,1]} \frac{1}{mn} d_H(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{\alpha}^{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} EX_{\alpha}^{ij}) \leq 2\varepsilon \quad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary, this gives the desired conclusion. □

The next theorem generalizes Theorem 5.4 in [14] to double arrays of random fuzzy convex upper semicontinuous variables.

Theorem 5.2. *Assume that* $\mathbf{E}$ *is a separable Banach space. Let* $\{X^{mn}\}$ *be a double array of random fuzzy convex upper semicontinuous variables defined on* $\Omega \times \mathbf{E}$ *satisfying:*

(i) $X_{0+}^{mn}(\omega)$ *is compact for every* $\omega \in \Omega$ *and for every* $m, n \geq 1$,

(ii) $g := \sup_{m,n} |X_{0+}^{mn}| \in L^1(\Omega, \mathcal{F}, P)$.

Assume that $\{\delta^*(., X_{\alpha}^{mn})\}$ *is pairwise i.i.d. such that* $E(|X_{\alpha}^{11}| \log^+ |X_{\alpha}^{11}|) < \infty$ *for every* $\alpha \in]0,1]$, *then we have*

$$\lim_{m\vee n\to\infty} d_H(\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} X_{\alpha}^{kl}, EX_{\alpha}^{11}) = 0 \quad a.s. \tag{12}$$

for every $\alpha \in]0,1]$. *In particular, if* $\mathbf{E}$ *is reflexive and* $\alpha \mapsto X_{\alpha}^{1}$ *is scalarly left continuous, then we have*

$$\lim_{m\vee n\to\infty} \int_0^1 d_H([\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} X^{kl}]_{\alpha}, [\tilde{E}X^{11}]_{\alpha}) d\alpha = 0 \quad a.s., \tag{13}$$

where $\tilde{E}X^{11}$ *denotes the fuzzy expectation of* X^{11}.

Proof. For each $\alpha \in]0,1]$, by our hypothesis, $\{\delta^*(.,X_\alpha^{mn})\}$ is a double array of pairwise i.i.d. convex compact valued random sets, applying Theorem 4.9 gives (12). Moreover, we note that

$$d_H(\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}X_\alpha^{kl}, EX_\alpha^{11}) \leq g + Eg$$

for all $m,n \geq 1$ and for all $\alpha \in]0,1]$. Then by the Lebesgue dominated convergence theorem, it is immediate that

$$\lim_{m\vee n\to\infty}\int_0^1 d_H(\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}X_\alpha^{kl}, EX_\alpha^{11})d\alpha = 0 \quad \text{a.s.} \tag{14}$$

Now if $\mathbf{E}$ is reflexive (or more generally is a dual of a WCG Banach space) and $\alpha \mapsto X_\alpha^{11}$ is scalarly left continuous, then by virtue of a recent result of Castaing et al. (see Theorem 4.7, [3]) on the integration of fuzzy level sets, we may consider the fuzzy expectation $\tilde{E}X^{11}$ of X^{11} that is a fuzzy convex upper semicontinuous variable defined on $\mathbf{E}$ so that $[\tilde{E}X^{11}]_\alpha = EX_\alpha^{11}$ for every $\alpha \in]0,1]$. Thus, combining this with (14), we get (13). The proof is completed. □

A variant of the preceding result for sequences is provided by Inoue ([11], Theorem 3.2) in which a tightness condition is imposed.

Acknowledgements. This work was supported by the Department of Science and Technology (HCMC-DOST) and the Institute for Computational Science and Technology (ICST) at Ho Chi Minh City, Vietnam, under grant number 14/2017/HD-KHCNTT, and by the Vietnam National Foundation for Science and Technology Development (NAFOSTED), under grant number 101.03-2017.24. A part of this work was done when the authors were working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). They would like to express their sincere thanks to VIASM for providing a fruitful research environment and hospitality.

References

[1] P. Bai, P.-Y. Chen, S. H. Sung: On complete convergence and the strong law of large numbers for pairwise independent random variables, Acta Math. Hungar. 142 (2014) 502–518.

[2] L. E. Baum, M. Katz: Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965) 108–123.

[3] C. Castaing, C. Godet-Thobie, T. Duyen Hoang, P. Raynaud de Fitte: On the integration of fuzzy level sets, Adv. Math. Econ. 19 (2015) 1–32.

[4] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Lecture Notes Math. 580, Springer-Verlag, Berlin (1977).

[5] P. Chen, T. C. Hu, X. Liu, A. Volodin: On complete convergence for arrays of rowwise negatively associated random variables, Theory Probab. Appl. 52 (2008) 323–328.

[6] L. V. Dung, N. D. Tien: Strong laws of large numbers for random fields in martingale type p Banach spaces, Statist. Probab. Lett. 80 (2010) 756–763.

[7] N. Etemadi: An elementary proof of the strong law of large numbers, Z. Wahr. Verw. Geb. 55 (1981) 119–122.

[8] A. Gut: On complete convergence in the law of large numbers for subsequences, Ann. Probab. 13 (1985) 1286–1291.

[9] F. Hiai, H. Umegaki: Integrals, conditional expectations and martingales of multivalued functions, J. Multi. Anal. 7 (1977) 149–182.

[10] P. Hsu, H. Robbins: Complete convergence and the law of large numbers, Proc. Natl. Acad. Sci. USA 33 (1947) 25–31.

[11] H. Inoue: A strong law of large numbers for fuzzy random sets, Fuzzy Sets and Systems 41 (1991) 285–291.

[12] F. Móricz: Moment inequalities for the maximum of partial sums of random fields, Acta Sci. Math. 39 (1977) 353–366.

[13] M. L. Puri, D. Ralescu: Strong law of large numbers for Banach space valued random sets, Ann. Probab. 11 (1983) 222–224.

[14] N. V. Quang, H. T. Duyen: Convergence of weighted sums and strong law of large numbers for convex compact integrable random sets and fuzzy random sets, J. Convex Analysis 24 (2017) 213–238.

[15] M. Valadier: Young measures, in: Methods of Nonconvex Analysis, Lecture Notes Math. 1446, Springer-Verlag, Berlin (1990) 152–188.

[16] X. C. Wang, M. B. Rao: Some results on the convergence of weighted sums of random elements in separable Banach spaces, Studia Math. 86 (1987) 131–153.

[17] L. A. Zadeh: Fuzzy sets, Inform. Control 8 (1965) 338–353.