

Elliptic-Regularization of Nonpotential Perturbations of Doubly-Nonlinear Flows of Nonconvex Energies: A Variational Approach

Goro Akagi*

*Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578 Japan;
Helmholtz Zentrum München, 85764 Neuherberg, Germany;
Technische Universität München, 85748 Garching bei München, Germany
akagi@m.tohoku.ac.jp*

Stefano Melchionna[†]

*Univ. of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
stefano.melchionna@univie.ac.at*

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This paper presents a variational approach to doubly-nonlinear (gradient) flows (P) of nonconvex energies along with nonpotential perturbations (i.e., perturbation terms without any potential structure). An elliptic-in-time regularization $(P)_\varepsilon$ of the original equation is introduced, and then, a variational approach and a fixed-point argument are employed to prove existence of strong solutions to $(P)_\varepsilon$. More precisely, we introduce a family of functionals (defined over entire trajectories) parametrized by a small parameter ε , whose Euler-Lagrange equation corresponds to the elliptic-in-time regularization of an unperturbed (i.e. without nonpotential perturbations) doubly-nonlinear flow. Secondly, due to the presence of nonpotential perturbation, a fixed-point argument is performed to construct strong solutions u_ε to the elliptic-in-time regularized equations $(P)_\varepsilon$. Finally, a strong solution to the original equation (P) is obtained by passing to the limit of u_ε as $\varepsilon \rightarrow 0$. Applications of the abstract theory developed in the present paper to concrete PDEs are also exhibited.

Keywords: Elliptic-in-time regularization, variational method, nonpotential perturbation, nonlinear evolution equation.

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1. Introduction

In this paper, we deal with a *nonpotential perturbation* problem $(P) = \{(1), (2)\}$ of a doubly-nonlinear (gradient) flow driven by a *dissipation potential* ψ and a (possibly) nonconvex *energy functional* ϕ defined on a uniformly convex Banach space V ,

$$d_V\psi(u') + \partial\phi(u) - f(u) \ni 0 \quad \text{a.e. in } (0, T), \quad (1)$$

$$u(0) = u_0 \quad (2)$$

where u' denotes the time derivative of the unknown $u : [0, T] \rightarrow V$, $\psi : V \rightarrow [0, \infty)$ is supposed to be convex and Gâteaux differentiable and its (Gâteaux) derivative is denoted by $d_V\psi$, $\partial\phi$ is a derivative of ϕ (in a proper sense) and $f(\cdot)$ is a nonpotential mapping from V into its dual space V^* (see below for more details). We assume that ϕ can be decomposed into the difference of two convex functionals,

$$\phi = \varphi^1 - \varphi^2$$

where $\varphi^1, \varphi^2 : V \rightarrow [0, \infty]$ are proper, lower semicontinuous, and convex functionals. We assume that φ^1 dominates φ^2 in a suitable sense (cf. (7)). This ensures that the difference ϕ is well defined. Here and henceforth, we simply write $\partial\phi = \partial\varphi^1 - \partial\varphi^2$, where the symbol ∂ in the right-hand side denotes the subdifferential of convex analysis. Although we impose the continuity as well as some growth condition on the map f , let us emphasize that we do not assume any potential structure on f (i.e. $f = \partial F$). Hence, throughout the paper, the perturbation term f is said to be *nonpotential*. Finally, we set $u_0 \in D(\varphi^1)$.

The study of doubly-nonlinear evolution equations of the form (1) with $f \equiv 0$ was initiated by Barbu [14], Arai [12], Senba [47], Colli-Visintin [21], and Colli [20], and then, they have been vigorously studied by many authors (see, e.g., [43] for an introduction, [2] for existence results, [5] for time dependent operators, [6] for problems with variable exponents, [1] and [36] for convergence of approximations and discretizations, [46] and [48] for long time behavior, [34] and [35] for rate-independent problems, i.e., problems with 1-homogeneous dissipation potential, and [51] for a variational approach based on the Brézis-Ekeland principle). They are mostly motivated in view of applications to physics and engineering. Indeed, equation (1) is extremely general and covers an extensive class of nonlinear PDEs and evolution equations used to describe a variety of phenomena such as phase transition [17, 18], [52], friction, and damage [16], [41]. The formulation (1) covers standard gradient flows (i.e., the case $d_V\psi(u') = u'$ which corresponds to a quadratic dissipation potential ψ on a Hilbert space), doubly-nonlinear flows (for general dissipation potentials defined on Banach spaces), and moreover, it can also comply with nonpotential perturbations. In the study of PDEs, one can often find out (possibly partial) gradient structures concealed in many PDEs describing dissipation phenomena, and such a gradient structure induces a leading feature of each equation; however, in most of cases, equations do not have full

gradient structure and they also include some nonpotential remainder terms. Typical examples may be Navier-Stokes equation and systems of PDEs. Other examples are illustrated in Section 7. See also [31] and Ôtani [39, 40]. One of the main purposes of the present paper is to develop a nonpotential perturbation theory for doubly-nonlinear flows capable of covering such a wider class of equations. Let us mention that in the case of quadratic dissipation potentials defined over a Hilbert space, such perturbation problems have been already studied by many authors (e.g., [53], [39, 40], and [49]).

The variational approach we apply to (1) is based on the so-called *Weighted Energy-Dissipation* (WED) functional. This procedure consists of introducing a one-parameter family of WED functionals I_ε defined over entire trajectories, and proving that their minimizers converge, up to subsequences, to *strong* solutions of the target problem, as the approximation parameter ε approaches to zero. Our interest in the WED formalism lies in the fact that it paves the way to the application of general techniques of the Calculus of Variation (e.g., Direct Method, relaxation, the Γ -convergence) in the evolutionary setting. Moreover, the WED procedure also brings a new tool to reveal qualitative properties of solutions and comparison principles for evolutionary problems [32]. Furthermore, we note that the uniqueness of solutions to both doubly-nonlinear flows and their elliptic regularization is delicate even in the unperturbed case. Indeed, in some cases, it is false (see, e.g., [20] and [3]). However, the strict convexity of the WED functional ensures the uniqueness of its minimizer. As a consequence, the WED procedure provides a way to select one within the (possibly many) solutions to the elliptic regularization of gradient flows.

In this paper, the uniqueness of the WED minimizer is used to define a (single-valued) *solution operator*, which maps a prescribed function v to the corresponding solution u . Moreover, it gives us an effective tool to check the structural stability and the continuity of the solution operator. These conditions will be required to apply a fixed-point theorem (see Section 3 below for more details) to construct a solution to the elliptic-in-time regularization of the target problem. Furthermore, the minimization problem provides more regular (in time) solutions, for the Euler-Lagrange equation associated with the WED functional corresponds to an elliptic-in-time regularization of the target problem. The elliptic-regularization approach to evolution equations has to be traced back at least to [29] and [38] (see also [30]). See also [14], where elliptic-in-time regularization of doubly-nonlinear problems is used to construct approximate solutions. The idea of the WED functional approach has already been used in [24] and [25]. Later, it has been reconsidered by Mielke and Ortiz [33] for rate-independent equations, by Mielke and Stefanelli [37] for gradient flows with λ -convex potentials, and by Akagi and Stefanelli for non(λ -)convex gradient flows [7] and doubly-nonlinear problems [4, 11, 10, 8]. Finally, the WED approach to nonpotential perturbations of gradient flows of nonconvex energies in Hilbert spaces was recently developed in [31].

Elliptic-in-time regularization is a well-established method and has been widely employed in various fields including numerical analysis and control theory. Indeed, it provides us more regular approximation of solutions and more choices of methods to tackle target equations (for instance, methods for elliptic PDEs are also applicable to parabolic and hyperbolic PDEs). On the other hand, for severely nonlinear evolution equations, there still remain many fundamental open issues such as existence of *strong* (i.e., twice differentiable) solutions for elliptic-in-time regularized equations and the convergence of such approximate solutions to a solution of the target equation.

One of main purposes of this paper is to propose a general theory to guarantee the existence of strong solutions to the elliptic-in-time regularization and the convergence of these solutions for a wider class of dissipative evolution equations as well as to extend the theory of [7], [10], [8] and [31] to nonpotential perturbation problems for doubly-nonlinear flows of nonconvex energies.

The main result of this paper ensures that a solution to (1)–(2) is obtained as the limit of solutions u_ε to an elliptic-in-time regularization $(P)_\varepsilon$ given by

$$\begin{aligned} -\varepsilon \frac{d}{dt} d_V \psi(u') + d_V \psi(u') + \partial \phi(u) - f(u) &\ni 0 \quad \text{a.e. in } (0, T), \\ u(0) = u_0, \quad d_V \psi(u'(T)) &= 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$ (to be precise, we shall treat a *weak formulation* of $(P)_\varepsilon$, see Definition 2.3 below). Problems $(P)_\varepsilon$ will be tackled by combining the minimization of WED functionals and a fixed-point argument. In particular, if $f(u)$ is replaced by $w := f(v)$ with a prescribed function v , we prove existence of solutions to the corresponding problem by minimization of a suitably defined WED functional (cf. Section 5).

We also note that, although uniqueness for the regularized unperturbed problem is not known, the minimization problem features uniqueness of solutions due to the strict convexity of the WED functional, and moreover, it also guarantees a continuous dependence of the solution $u = u(t)$ on the prescribed function $v = v(t)$.

The paper is organized as follows. In Section 2, we set up notation, enlist our assumptions and state our main results. In order to simplify the argument, we first prove our results only for convex energy functionals (namely, the case $\varphi^2 = 0$) in Section 3. Secondly, we extend the result of Section 3 to general nonconvex energies in Section 4. In Section 5, as a by-product, we shall develop a variational characterization based on WED functionals for doubly-nonlinear flows of nonconvex energies (i.e., (P) with $f = 0$). In Section 6, we briefly sketch a second fixed-point argument, which allows us to work under slightly different assumptions on the nonpotential term f . Finally, Section 7 concerns some applications of the abstract theory to concrete PDEs.

2. Main results

Let V be a uniformly convex Banach space with norm $|\cdot|_V$ and duality pairing $\langle \cdot, \cdot \rangle_V$, and let $(V^*, |\cdot|_{V^*})$ be its dual space such that V^* is also uniformly convex. Note that we do not assume the presence of a pivot (Hilbert) space $H \equiv H^*$ between V and V^* due to our variational approach. The pivot space H is often required, if (1) is treated by virtue of approximation techniques (in H) such as Yosida approximation. Moreover, in applications to nonlinear PDEs, such an abstract framework based on the Gel'fand triplet $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ may impose additional restrictions on the application to PDEs (cf. see [2] and also Remark 7.2).

Let X be a reflexive Banach space with norm $|\cdot|_X$ and duality pairing $\langle \cdot, \cdot \rangle_X$. Suppose that

$$X \hookrightarrow V \text{ and } V^* \hookrightarrow X^*$$

with compact densely-defined canonical injections. Let $\psi: V \rightarrow [0, \infty)$ be a Gâteaux differentiable convex functional and let $\varphi^1, \varphi^2: V \rightarrow [0, \infty]$ be two proper, lower semicontinuous, and convex functionals whose *effective domains* (i.e., the set of u for which $\varphi^i(u) < +\infty$) are denoted by $D(\varphi^1)$ and $D(\varphi^2)$, respectively. Let $p \in (1, \infty)$ and $m \in (1, \infty)$ be fixed and assume the following:

(A1) There exists $C_1 > 0$ such that, for all $u \in V$,

$$|u|_V^p \leq C_1(\psi(u) + 1); \quad (3)$$

(A2) There exists $C_2 > 0$ such that, for all $u \in V$,

$$|d_V \psi(u)|_{V^*}^{p'} \leq C_2(|u|_V^p + 1), \quad (4)$$

where $d_V \psi: V \rightarrow V^*$ denotes the Gâteaux derivative of ψ ;

(A3) There exists a positive and nondecreasing function ℓ_3 on $[0, \infty)$ such that, for all $u \in D(\varphi^1)$,

$$|u|_X^m \leq \ell_3(|u|_V)(\varphi^1(u) + 1); \quad (5)$$

(A4) Let $\varphi_X^1: X \rightarrow [0, \infty)$ be the restriction of $\varphi^1: V \rightarrow [0, \infty]$ onto X and let $\partial_X \varphi_X^1: X \rightarrow X^*$ be its subdifferential. There exists a nondecreasing function $\ell_4(\cdot)$ on $[0, \infty)$ such that, for all $u \in D(\varphi_X^1)$, $\eta^1 \in \partial_X \varphi_X^1(u)$,

$$|\eta^1|_{X^*}^{m'} \leq \ell_4(|u|_V)(|u|_X^m + 1);$$

(A5) Let $f: V \rightarrow V^*$ be continuous and such that

$$|f(u)|_{V^*}^{p'} \leq C_5(|u|_V^p + 1) \quad (6)$$

for all $u \in V$ and for some positive constant C_5 (independent of u).

(A6) $D(\varphi^1) \subset D(\partial_V \varphi^2)$. Moreover, there exist constants $k \in [0, 1)$, $C_6 > 0$, and a nondecreasing function ℓ_7 on $[0, \infty)$ such that

$$\varphi^2(u) \leq k\varphi^1(u) + C_6(|u|_V^p + 1) \quad (7)$$

for all $u \in D(\varphi^1)$, and

$$|\eta^2|_{V^*}^{p'} \leq \ell_7(|u|_V)(\varphi^1(u) + 1) \quad (8)$$

for all $u \in D(\varphi^1)$, $\eta^2 \in \partial_V \varphi^2(u)$. We now define the energy potential $\phi: V \rightarrow (-\infty, \infty]$ by $\phi(u) = \varphi^1(u) - \varphi^2(u)$ for all $u \in D(\varphi^1)$ and $\phi(u) = \infty$ for all $u \in V \setminus D(\varphi^1)$.

Moreover, we assume φ^1 to be strictly convex. Note that this is not restrictive. Indeed, given φ^1 and φ^2 satisfying assumptions **(A3)**–**(A4)**, **(A6)**, we define $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ by $\tilde{\varphi}^1(u) = \varphi^1(u) + |u|_V^r$, $\tilde{\varphi}^2(u) = \varphi^2(u) + |u|_V^r$ for all $u \in V$ and some $r \in (0, \min\{m, p\})$. Note that $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ satisfy assumptions **(A3)**–**(A4)**, **(A6)** and $\phi = \varphi^1 - \varphi^2 = \tilde{\varphi}^1 - \tilde{\varphi}^2$. Moreover, $\tilde{\varphi}^1$ is strictly convex.

As a consequence of **(A1)**–**(A4)** there exist constants C_i , $i \in \{8, 9\}$ and non-decreasing functions $\ell_{10}(\cdot)$, $\ell_{11}(\cdot)$ in $\mathbb{R}$ such that

$$|u|_V^p \leq C_8(\langle d_V \psi(u), u \rangle_V + 1) \text{ for all } u \in V, \quad (9)$$

$$\psi(u) \leq C_9(|u|_V^p + 1) \text{ for all } u \in V, \quad (10)$$

$$|u|_X^m \leq \ell_{10}(|u|_V)(\langle \eta^1, u \rangle_X + 1) \text{ for all } u \in D(\partial_X \varphi_X^1), \eta^1 \in \partial_X \varphi_X^1(u), \quad (11)$$

$$\varphi^1(u) \leq \ell_{11}(|u|_V)(|u|_X^m + 1) \text{ for all } u \in D(\varphi^1). \quad (12)$$

Finally, we assume $u_0 \in D(\varphi^1)$. (13)

Remark 2.1. (i) Under **(A3)** and **(A4)**, one can check $D(\varphi^1) = D(\varphi^1|_X) = X$.

Hence due to [13, Proposition 2.2], $\varphi^1|_X$ turns out to be continuous from X to $[0, \infty)$ (see Lemma A.3 in Appendix). Hence, (13) is equivalent to $u_0 \in X$.

(ii) By **(A5)**, the mapping $u \mapsto f(u)$ from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$ turns out to be continuous (see, e.g., [43]).

(iii) The main results of this paper can be extended to more general perturbations. More precisely, instead of **(A5)**, we may assume that f satisfies (6) and that f is demicontinuous (i.e. strongly-weakly continuous) from $L^p(0, T; V)$ to $L^{p'}(0, T; V^*)$, i.e.,

$$\begin{cases} \text{if } u_n \rightarrow u \text{ strongly in } L^p(0, T; V), \\ \text{then } f(u_n) \rightarrow f(u) \text{ weakly in } L^{p'}(0, T; V^*), \end{cases} \quad (14)$$

which is equivalent to the demiclosedness under (6), and the following compactness condition of the mapping f from $W^{1,p}(0, T; V) \cap L^m(0, T; X)$ to $L^{p'}(0, T; V^*)$:

$$\left\{ \begin{array}{l} \text{if } (u_n) \text{ is bounded in } L^m(0, T; X) \text{ and in } W^{1,p}(0, T; V), \\ \text{and } (f(u_n)) \text{ is bounded in } L^{p'}(0, T; V^*), \text{ then there} \\ \text{exists } f^* \in L^{p'}(0, T; V^*) \text{ such that, up to a subsequence,} \\ f(u_n) \rightarrow f^* \text{ strongly in } L^{p'}(0, T; V^*). \end{array} \right. \quad (15)$$

By (14) as well as the Aubin-Lions-Simon compactness lemma, the limit f^* is identified with $f(u)$, whenever u_n is a (not relabeled) subsequence strongly convergent to u in $C([0, T]; V)$.

- (iv) Note that (constant-in-time) external forces can be considered by choosing the term f to be independent of the variable u . We remark that our results can be also extended to problems with time dependent external forces, $g: (0, T) \rightarrow V^*$, with only minor modification. More precisely, we can relax assumption **(A5)** by requiring that $f(u) = \tilde{f}(u) + g$, where $\tilde{f}$ satisfies **(A5)** and $g: (0, T) \rightarrow V^*$ belongs to the space $L^{p'}(0, T; V^*)$. We refer the reader to [8] for the WED approach to (unperturbed) doubly-nonlinear systems with time-dependent external forces. $\square$

We now focus on the Cauchy problem

$$d_V \psi(u') + \eta^1 - \eta^2 - f(u) = 0 \text{ in } V^* \text{ a.e. in } (0, T), \quad (16)$$

$$\eta^1 \in \partial_V \varphi^1(u), \quad \eta^2 \in \partial_V \varphi^2(u), \quad (17)$$

$$u(0) = u_0. \quad (18)$$

Before stating the main results, we give a definition of strong solution.

Definition 2.2. A function $u \in C([0, T]; V)$ is said to be a *strong solution* of (16)–(18) if the following conditions are satisfied:

- (i) $u \in L^m(0, T; X) \cap W^{1,p}(0, T; V)$, $d_V \psi(u') \in L^{p'}(0, T; V^*)$,
- (ii) there exist $\eta^1, \eta^2 \in L^{p'}(0, T; V^*)$ such that $\eta^1 \in \partial_V \varphi^1(u)$ and $\eta^2 \in \partial_V \varphi^2(u)$ a.e. in $(0, T)$,
- (iii) $d_V \psi(u') + \eta^1 - \eta^2 - f(u) = 0$ in V^* a.e. in $(0, T)$, and $u(0) = u_0$.

Concerning the elliptic-in-time regularization of (16)–(18), we shall treat the following weak formulation,

$$-\varepsilon \xi' + \xi + \eta^1 - \eta^2 - f(u) = 0 \text{ in } X^* \text{ a.e. in } (0, T), \quad (19)$$

$$\xi = d_V \psi(u'), \quad \eta^1 \in \partial_X \varphi_X^1(u), \quad \eta^2 \in \partial_V \varphi^2(u), \quad (20)$$

$$u(0) = u_0, \quad \xi(T) = 0. \quad (21)$$

Here, we are concerned with strong solutions of (19)–(21) defined as follows:

Definition 2.3. A function $u \in C([0, T]; V)$ is said to be a *strong solution* for (19)–(21) if it satisfies the following conditions:

- (i) $u \in L^m(0, T; X) \cap W^{1,p}(0, T; V)$, $\xi = d_V \psi(u) \in L^{p'}(0, T; V^*)$, and $\xi' \in L^{m'}(0, T; X^*) + L^{p'}(0, T; V^*)$,
- (ii) there exist $\eta^1 \in L^{m'}(0, T; X^*)$, $\eta^2 \in L^{p'}(0, T; V^*)$ such that $\eta^1 \in \partial_X \varphi_X^1(u)$, $\eta^2 \in \partial_V \varphi^2(u)$ and the following holds:

$$\begin{aligned} -\varepsilon \xi' + \xi + \eta^1 - \eta^2 - f(u) &= 0 \text{ in } X^* \text{ a.e. in } (0, T), \\ u(0) &= u_0, \quad \xi(T) = 0. \end{aligned}$$

We now state our main result.

Theorem 2.4. *Let the assumptions (A1)–(A6) and (13) be satisfied. Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the elliptic-in-time regularization (19)–(21) admits strong solutions u_ε in the sense of Definition 2.3. Moreover, there exists a sequence $\varepsilon_n \rightarrow 0$ such that $u_{\varepsilon_n} \rightarrow u$ weakly in $L^m(0, T; X) \cap W^{1,p}(0, T; V)$ and strongly in $C([0, T]; V)$ and the limit u solves the target equation (16)–(18) in the sense of Definition 2.2.*

A proof is given first in the case of convex energy functionals (i.e., $\varphi^2 = 0$) in Section 3. A proof for the nonconvex energy case, i.e., the case $\phi = \varphi^1 - \varphi^2$ for φ^2 satisfying (A6), is then presented in Section 4. It is based on an approximation of subdifferential operators to reduce the problem into the convex energy case and on an application of the results obtained in Section 3 (to be more precise, Proposition 3.2 below).

3. Convex energies: Proof of Theorem 2.4

In this section, we treat only convex energies, i.e., $\varphi^2 = 0$, and prove Theorem 2.4 in this simpler case. We start by showing existence of strong solutions to the elliptic-in-time regularized equation,

$$-\varepsilon \xi' + \xi + \eta = f(u), \tag{22}$$

$$\xi = d_V \psi(u'), \quad \eta \in \partial_X \phi_X(u), \tag{23}$$

$$u(0) = u_0, \quad \xi(T) = 0 \tag{24}$$

for $\varepsilon > 0$ small enough. The strategy of our proof relies on a variational technique based on the minimization of WED functionals (see [8] and [10]) as well as a fixed-point argument.

3.1. A fixed-point argument

Let us define the map $S: L^p(0, T; V) \rightarrow L^p(0, T; V)$ by

$$S: v \mapsto w := f(v) \mapsto u,$$

where u is the unique global minimizer of the WED functional $I_{\varepsilon, w}: L^p(0, T; V) \rightarrow (-\infty, \infty]$ defined by

$$I_{\varepsilon,w}(u) = \int_0^T \exp(-t/\varepsilon)(\varepsilon\psi(u') + \phi(u) - \langle w, u \rangle_V), \quad (25)$$

if $u \in K(u_0) \cap L^m(0, T; X)$, and by $I_{\varepsilon,w}(u) = \infty$ otherwise, over the set $K(u_0) := \{u \in W^{1,p}(0, T; V) : u(0) = u_0\}$. For the well-posedness of the map S (namely, existence and uniqueness of a minimizer $u = \arg \min I_{\varepsilon,w}$), we employ the following fact (see [8, Theorem 5.1]):

Theorem 3.1. *Let $w \in L^{p'}(0, T; V^*)$ and (A1)–(A4) and (13) be satisfied with $\varphi^2 \equiv 0$. Then, for all $\varepsilon > 0$, the WED functional $I_{\varepsilon,w}$ defined by (25) admits at least one minimizer u_ε such that*

$$u_\varepsilon \in L^m(0, T; X) \cap W^{1,p}(0, T; V),$$

$$\xi_\varepsilon = d_V \psi(u_\varepsilon) \in L^{p'}(0, T; V^*) \text{ and } \xi'_\varepsilon \in L^{m'}(0, T; X^*) + L^{p'}(0, T; V^*).$$

Furthermore, there exists $\eta_\varepsilon \in L^{m'}(0, T; X^*)$ such that $\eta_\varepsilon \in \partial_X \phi_X(u_\varepsilon)$ and $(u_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)$ satisfies

$$-\varepsilon \xi'_\varepsilon + \xi_\varepsilon + \eta_\varepsilon = w \text{ in } X^* \text{ a.e. in } (0, T), \quad (26)$$

$$u_\varepsilon(0) = u_0, \quad \xi_\varepsilon(T) = 0. \quad (27)$$

In addition, if ϕ is strictly convex, then the minimizer of $I_{\varepsilon,w}$ is unique.

The goal of this subsection is now to prove that S has a fixed point. More precisely, we shall prove the following:

Proposition 3.2. *Let assumptions (A1)–(A4) and (13) be satisfied with $\varphi^2 \equiv 0$. Let f satisfy condition (6) and assume that f is demicontinuous from $L^p(0, T; V)$ to $L^{p'}(0, T; V^*)$, i.e.,*

$$\left. \begin{array}{l} \text{if } u_n \rightarrow u \text{ strongly in } L^p(0, T; V), \\ \text{then } f(u_n) \rightarrow f(u) \text{ weakly in } L^{p'}(0, T; V^*). \end{array} \right\} \quad (28)$$

Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the map S has at least one fixed point u_ε . Moreover, such a fixed point is a strong solution to the elliptic-in-time regularized equation (22)–(24).

Although we assumed $f: V \rightarrow V^*$ to be continuous in (A5), we have formulated Proposition 3.2 in the more general case of f satisfying (28), i.e., the demicontinuity. This will allow us to apply Proposition 3.2 to the map $f + \partial_V \varphi_\lambda^2$ (here φ_λ^2 is the Moreau-Yosida regularization of φ^2) in Section 4.

To prove Proposition 3.2, we simply check several assumptions to apply the Schaefer fixed-point theorem (see Theorem A.4 in Appendix) to the map S . Our proof is divided into several steps.

3.1.1. A priori estimates

We now derive some (uniform in ε) estimates for the solution of (26)–(27). Throughout this section, the symbols C and c will denote some positive constants independent of ε which may vary even within the same line.

Fix $\varepsilon > 0$, $v \in L^p(0, T; V)$, and $w = f(v)$ and let $u := u_\varepsilon$ be the solution to (26)–(27) given by Theorem 3.1. Since $u' \in D(\partial_V \psi)$ and $\xi \in \partial_V \psi(u')$ (indeed $D(\partial_V \psi) = V$ and $\partial_V \psi(v) = \{d_V \psi(v)\}$ for all $v \in V$), by defining the *Fenchel conjugate* ψ^* of ψ by

$$\psi^*(v) = \sup_{w \in V} \{\langle v, w \rangle_V - \psi(w)\} \quad \text{for } v \in V^*$$

and by using the *Fenchel identity*,

$$\psi(w) + \psi^*(v) = \langle v, w \rangle_V \Leftrightarrow w \in \partial_{V^*} \psi^*(v) \Leftrightarrow v \in \partial_V \psi(w),$$

we have $u' \in \partial_{V^*} \psi^*(\xi)$. Thus, $\langle \xi', u' \rangle_V = \frac{d}{dt} \psi^*(\xi)$ by a chain-rule for subdifferentials (see, e.g., [10]). Testing equation (26) with u' and integrating both sides over $(0, t)$, one gets

$$-\varepsilon \int_0^t \frac{d}{dt} \psi^*(\xi) + \int_0^t \langle \xi, u' \rangle_V + \int_0^t \frac{d}{dt} \phi(u) = \int_0^t \langle f(v), u' \rangle_V. \quad (29)$$

Remark 3.3. As equation (26) holds in X^* and $u'(t)$ belongs to V for a.e. $t \in (0, T)$, the argument above is formal. However, it can be justified through by an approximation argument based on the Moreau-Yosida regularization of convex potentials. More details on the derivation of (29) can be found in [11, 10, 8]. $\square$

As a consequence of the assumptions (6) and (9), it follows that

$$\begin{aligned} -\varepsilon \psi^*(\xi(t)) + \varepsilon \psi^*(\xi(0)) + c \int_0^t |u'|_V^p + \phi(u(t)) - \phi(u_0) &\leq \\ &\leq C + \frac{c}{2} \int_0^t |u'|_V^p + C \int_0^t |v|_V^p. \end{aligned} \quad (30)$$

As ϕ and ψ^* are bounded from below, by $u_0 \in D(\phi)$, we have

$$\frac{c}{2} \int_0^t |u'|_V^p + \phi(u(t)) \leq C + C \int_0^t |v|_V^p + \varepsilon \psi^*(\xi(t)) \quad (31)$$

and, recalling $\xi(T) = 0$ and $\psi^*(0) \leq 0$,

$$\frac{c}{2} \int_0^T |u'|_V^p + \phi(u(T)) \leq C + C \int_0^T |v|_V^p. \quad (32)$$

Note that

$$\int_0^t \frac{d}{dt} |u|_V^p = \int_0^t p |u|_V^{p-2} \langle F_V u, u' \rangle_V \leq \frac{c}{2} \int_0^t |u'|_V^p + C \int_0^t |u|_V^p,$$

where $F_V: V \rightarrow V^*$ denotes the duality mapping between V and V^* . Hence, substituting it into (31), we obtain

$$|u(t)|_V^p + \phi(u(t)) \leq C + C \int_0^t |v|_V^p + C \int_0^t |u|_V^p + \varepsilon \psi^*(\xi(t)). \quad (33)$$

Applying Gronwall's lemma (cf. Lemma A.2 in Appendix), one gets

$$|u(t)|_V^p \leq C + C \int_0^t \left(|v|_V^p + \int_0^t |v|_V^p \right) + \varepsilon \psi^*(\xi(t)) + C\varepsilon \int_0^t \psi^*(\xi).$$

By substituting it into (33), integrating both sides over $[0, T]$ and taking the sum with (32), we get

$$\begin{aligned} \frac{c}{2} \int_0^T |u'|_V^p + \int_0^T |u|_V^p + \phi(u(T)) + \int_0^T \phi(u) &\leq \\ &\leq C + C \int_0^T |v|_V^p + C\varepsilon \int_0^T \psi^*(\xi). \end{aligned} \quad (34)$$

We now show that

$$\psi^*(\xi) \leq C |u'|_V^p + C. \quad (35)$$

Indeed, by definition $\psi^*(\xi) = \sup_{w \in V} (\langle \xi, w \rangle_V - \psi(w))$ and by using assumption **(A1)**, for any $\delta > 0$, one can take a constant $C_\delta > 0$ such that

$$\begin{aligned} \psi^*(\xi) &\leq \sup_{w \in V} \left\{ |\xi|_{V^*} |w|_V + 1 - \frac{1}{C_1} |w|_V^p \right\} \\ &\leq \sup_{w \in V} \left\{ C_\delta |\xi|_{V^*}^{p'} + \delta |w|_V^p + 1 - \frac{1}{C_1} |w|_V^p \right\}. \end{aligned}$$

Choosing $\delta = \frac{1}{C_1}$ and using assumption **(A2)**, we get

$$\psi^*(\xi) \leq \sup_{w \in V} \{C |\xi|_{V^*}^{p'} + 1\} = C |\xi|_{V^*}^{p'} + 1 \leq C |u'|_V^p + C.$$

Thus, substituting it into (34), we can choose $\varepsilon = \varepsilon(T, \psi)$ (depending on ψ and T , but not on ϕ) sufficiently small to obtain

$$\int_0^T |u|_V^p + \int_0^T |u'|_V^p + \int_0^T \phi(u) \leq C + C \int_0^T |v|_V^p. \quad (36)$$

Therefore, u is uniformly bounded in $W^{1,p}(0, T; V)$ and hence in $C([0, T]; V)$ by $C + C \int_0^T |v|_V^p$. As a consequence of assumption **(A3)**, we obtain the following estimate:

$$\|u\|_{W^{1,p}(0,T;V)}^p + \|u\|_{L^m(0,T;X)}^m \leq C(1 + \ell_3(C + C\|v\|_{L^p(0,T;V)})) \left(1 + \int_0^T |v|_V^p\right). \quad (37)$$

Remark 3.4. We can prove estimate (37) in an easier way. Indeed, using the nonnegativity of ϕ and $u(0) = u_0 \in V$, we can deduce from (32) that $\|u'\|_{L^p(0,T;V)}^p \leq C + C \int_0^T |v|_V^p$. By substituting (35) into (31) and by integrating both sides over $(0, T)$, we get $\int_0^T \phi(u) \leq C + C \int_0^T |v|_V^p$ and thus (37) by virtue of **(A3)**. However, the argument starting from (33) to derive estimate (37) will be used later (see (39) and (47) below). $\square$

3.1.2. The map $S: L^p(0, T; V) \rightarrow L^p(0, T; V)$ is continuous

We recall that S is the composition of two maps: $S: v \mapsto w := f(v) \mapsto u$. We notice that, as a consequence of (28), the map $v \mapsto f(v)$ is demicontinuous from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$. Thus, we are only left to prove that the solution operator $w \mapsto u$ is weakly-strongly continuous from $L^{p'}(0, T; V^*)$ into $L^p(0, T; V)$. Let $\{w_h\}$ be a sequence in $L^{p'}(0, T; V^*)$ such that $w_h \rightarrow w$ weakly in $L^{p'}(0, T; V^*)$ as $h \rightarrow 0$. Then, there exists C independent of h such that $\|w_h\|_{L^{p'}(0,T;V^*)} \leq C$ and estimate (37) implies that the family $\{u_h = \operatorname{argmin} I_{\varepsilon, w_h}\}$ of minimizers is uniformly bounded in $W^{1,p}(0, T; V) \cap L^m(0, T; X) \hookrightarrow L^p(0, T; V)$, and hence, we deduce that, up to a subsequence k_h , $u_{k_h} \rightarrow u$ strongly in $L^p(0, T; V)$. In order to identify the limit u , we prove that the corresponding WED functionals I_{ε, w_h} converge to $I_{\varepsilon, w}$ in the sense of Γ -convergence in the strong topology of $L^p(0, T; V)$ (see, e.g., [22]). Indeed let $\{u_h\} \in L^p(0, T; V)$ be such that $u_h \rightarrow u$ strongly in $L^p(0, T; V)$. Then, as ψ and ϕ are convex and l.s.c., we find that

$$\begin{aligned} \liminf_{h \rightarrow 0} I_{\varepsilon, w_h}(u_h) &= \liminf_{h \rightarrow 0} \int_0^T \exp(-t/\varepsilon) (\varepsilon \psi(u'_h) + \phi(u_h) - \langle w_h, u_h \rangle_V) \\ &\geq I_{\varepsilon, w}(u). \end{aligned}$$

For each $u \in K(u_0) \cap L^m(0, T; X)$, we simply set $u_h \equiv u$ as a recovering sequence. Then, one can immediately check that

$$\lim_{h \rightarrow 0} I_{\varepsilon, w_h}(u) = I_{\varepsilon, w}(u).$$

As a consequence of the Γ -convergence of the WED functionals along with the convergence $u_{k_h} \rightarrow u$ strongly in $L^p(0, T; V)$, we deduce that u minimizes $I_{\varepsilon, w}$. We recall that for every w the minimizer of $I_{\varepsilon, w}$ is unique (due to the strict convexity of $I_{\varepsilon, w}$). Thus, the convergence holds for the whole sequence $\{u_h\}$. This proves continuity of S .

3.1.3. Compactness

We here prove the compactness of the map $S: L^p(0, T; V) \rightarrow L^p(0, T; V)$. Let $\{v_h\} \subset L^p(0, T; V)$ be a bounded sequence and let u_h be the minimizer of $I_{\varepsilon, f(v_h)}$. Then, as a consequence of estimate (37), the family $\{u_h\}$ is uniformly bounded in $W^{1,p}(0, T; V) \cap L^m(0, T; X) \hookrightarrow L^p(0, T; V)$, and hence, up to a subsequence, $u_h \rightarrow u$ strongly in $L^p(0, T; V)$.

3.1.4. Boundedness of $\{\bar{v} \in L^p(0, T; V) : \bar{v} = \alpha S(\bar{v}) \text{ for } \alpha \in [0, 1]\}$

In order to apply the Schaefer fixed-point theorem (see Theorem A.4 in Appendix), we are only left to prove that the set $A := \{\bar{v} \in L^p(0, T; V) : \bar{v} = \alpha S(\bar{v}) \text{ for } \alpha \in [0, 1]\}$ is bounded. Note that A is bounded if and only if $\{\bar{v} \in L^p(0, T; V) : \bar{v}/\alpha = S(\bar{v}) \text{ for } \alpha \in (0, 1]\}$ is bounded. Hence, we shall prove that $B = \{v \in L^p(0, T; V) : v = S(\alpha v) \text{ for } \alpha \in (0, 1]\}$ is bounded. This yields the boundedness of A .

In case $B = \emptyset$, we immediately find that $A = \{0\}$, and hence, nothing remains to be proved. In case $B \neq \emptyset$, let $v \in B$. Then, there exists $\alpha \in (0, 1]$ such that $v = S(\alpha v)$. Let u be the minimizer of $I_{\varepsilon, f(\alpha v)}$, i.e. $u = S(\alpha v)$ ($= v$). Then, u solves

$$-\varepsilon \xi' + \xi + \eta = f(\alpha v) = f(\alpha u) \text{ in } X^* \text{ a.e. in } (0, T), \quad (38)$$

where $\xi = d_V \psi(u)$ and $\eta \in \partial_X \phi_X(u)$ a.e. in $(0, T)$. We shall prove that solutions to (38) are bounded in $L^p(0, T; V)$ uniformly in α for ε small enough. Testing (38) with u' and integrating both sides over $(0, t)$, we get

$$-\varepsilon \int_0^t \frac{d}{dt} \psi^*(\xi) + \int_0^t \langle \xi, u' \rangle_V + \int_0^t \frac{d}{dt} \phi(u) = \int_0^t \langle f(\alpha u), u' \rangle_V$$

(see also Remark 3.3). As a consequence of (9) and **(A5)**, we get

$$\begin{aligned} & -\varepsilon \psi^*(\xi(t)) + \varepsilon \psi^*(\xi(0)) + c \int_0^t |u'|_V^p + \phi(u(t)) - \phi(u_0) \leq \\ & \leq C + \frac{c}{2} \int_0^t |u'|_V^p + C \int_0^t |f(\alpha u)|_{V^*}^{p'} \leq C + \frac{c}{2} \int_0^t |u'|_V^p + C \int_0^t |u|_V^p. \end{aligned} \quad (39)$$

Proceeding as in Section 3.1.1, one can derive formula (33) with $v = u$ and thus (for $\varepsilon > 0$ small enough) formula (36)–(37) with $v = 0$. In particular, $\|u\|_{L^p(0, T; V)} \leq C$, where C does not depend on ε and α .

Thanks to the Schaefer fixed-point theorem (see Theorem A.4 in Appendix), the map S has a fixed point u_ε for $\varepsilon > 0$ small enough. Furthermore, the fixed point u_ε of S solves (22)–(24) and satisfies the relation,

$$u_\varepsilon = \operatorname{argmin}_{\tilde{u} \in K(u_0)} \int_0^T \exp(-t/\varepsilon) (\varepsilon \psi(\tilde{u}') + \phi(\tilde{u}) - \langle f(u_\varepsilon), \tilde{u} \rangle_V).$$

This completes the proof of Proposition 3.2.

3.2. The causal limit

We shall now prove that solutions u_ε of the elliptic-in-time regularized equations (22)–(24) converge, up to a subsequence, to a solution of the target equation (16)–(18), in case ϕ is convex (i.e. $\varphi^2 = 0$). More precisely, we prove the following proposition:

Proposition 3.5. *Assume (A1)–(A5), (13) and $\varphi^2 = 0$. For each $\varepsilon > 0$ small enough, let u_ε be a solution of the elliptic-in-time regularized equation (22)–(24). Then, there exist a sequence $\varepsilon_n \rightarrow 0$ and a limit $u \in L^m(0, T; X) \cap W^{1,p}(0, T; V)$ such that $u_{\varepsilon_n} \rightarrow u$ weakly in $L^m(0, T; X) \cap W^{1,p}(0, T; V)$ and strongly in $C([0, T]; V)$ and the limit u solves (16)–(18).*

Our proof is divided into two steps.

3.2.1. A priori uniform estimates

Testing equation (22) with $u' = u'_\varepsilon$ and integrating it over $(0, t)$, we get

$$-\varepsilon \int_0^t \frac{d}{dt} \psi^*(\xi_\varepsilon) + \int_0^t \langle \xi_\varepsilon, u'_\varepsilon \rangle_V + \int_0^t \frac{d}{dt} \phi(u_\varepsilon) = \int_0^t \langle f(u_\varepsilon), u'_\varepsilon \rangle_V$$

(see also Remark 3.3). As a consequence of assumption (9), we obtain

$$\begin{aligned} -\varepsilon \psi^*(\xi_\varepsilon(t)) + \varepsilon \psi^*(\xi_\varepsilon(0)) + \int_0^t |u'_\varepsilon|_V^p + \phi(u_\varepsilon(t)) - \phi(u_0) &\leq \\ &\leq C + \frac{1}{2} \int_0^t |u'_\varepsilon|_V^p + C \int_0^t |f(u_\varepsilon)|_{V^*}^{p'}. \end{aligned}$$

By repeating the same argument as in Section 3.1.4 (with $\alpha = 1$), we can obtain, for $\varepsilon > 0$ sufficiently small, the estimate,

$$\|u_\varepsilon\|_{W^{1,p}(0,T;V)} + \|u_\varepsilon\|_{L^m(0,T;X)} \leq C.$$

Due to the assumptions (A1)–(A5), we have

$$\|\eta_\varepsilon\|_{L^{m'}(0,T;X^*)} \leq C, \quad \|\xi_\varepsilon\|_{L^{p'}(0,T;V^*)} \leq C, \quad \|f(u_\varepsilon)\|_{L^{p'}(0,T;V^*)} \leq C,$$

and, by comparison of each term in (22),

$$\|\varepsilon \xi'_\varepsilon\|_{L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*)} \leq C.$$

3.2.2. The passage to the limit

From the a priori estimate obtained above, we can derive, along some not-reabeled subsequence, the following convergences:

$$\begin{aligned}\eta_\varepsilon &\rightarrow \eta \text{ weakly in } L^{m'}(0, T; X^*), \\ \xi_\varepsilon &\rightarrow \xi \text{ weakly in } L^{p'}(0, T; V^*), \\ \varepsilon \xi'_\varepsilon &\rightarrow 0 \text{ weakly in } L^{p'}(0, T; V^*) + L^{m'}(0, T; X^*), \\ u_\varepsilon &\rightarrow u \text{ weakly in } W^{1,p}(0, T; V) \text{ and in } L^m(0, T; X), \\ f(u_\varepsilon) &\rightarrow f^* \text{ weakly in } L^{p'}(0, T; V^*),\end{aligned}$$

and hence, thanks to Aubin-Lions-Simon's compactness lemma (see [50]),

$$u_\varepsilon \rightarrow u \text{ strongly in } C([0, T]; V). \quad (40)$$

In particular, $u_\varepsilon(t) \rightarrow u(t)$ strongly in V for all $t \in [0, T]$ and $u(0) = u_0$. As a consequence of the continuity of $f: V \rightarrow V^*$ (see also Remark 2.1) and the strong convergence (40), we have $f^* = f(u)$ and

$$f(u_\varepsilon) \rightarrow f(u) \text{ strongly in } L^{p'}(0, T; V^*). \quad (41)$$

Thus,

$$\xi + \eta - f(u) = 0 \text{ in } X^* \text{ a.e. in } [0, T]. \quad (42)$$

From the final condition $\xi_\varepsilon(T) = 0$ and the convergences above, it follows that

$$-\langle \varepsilon \xi_\varepsilon(t), v \rangle_X = \left\langle \int_t^T \varepsilon \xi'_\varepsilon(s) ds, v \right\rangle_X = \int_t^T \langle \varepsilon \xi'_\varepsilon(s), v \rangle_X ds \rightarrow 0$$

for all $v \in X$, which yields $\varepsilon \xi_\varepsilon(t) \rightarrow 0$ weakly in X^* for each $t \in [0, T]$.

We now check the inclusion $\eta(t) \in \partial_V \phi(u(t))$ for a.e. $t \in (0, T)$. Since η and u have sufficient regularity, thanks to [10, Proposition 2.1], it is sufficient to show a (weak) relation $\eta(t) \in \partial_X \phi_X(u(t))$ for a.e. $t \in (0, T)$. By comparison in (22), integrating by parts (cf. [10] for a rigorous proof of the integration-by-parts formula) and using the convergences obtained so far, we have

$$\begin{aligned}\int_0^T \langle \eta_\varepsilon, u_\varepsilon \rangle_X &= \int_0^T \langle \varepsilon \xi'_\varepsilon, u_\varepsilon \rangle_X - \int_0^T \langle \xi_\varepsilon, u_\varepsilon \rangle_V + \int_0^T \langle f(u_\varepsilon), u_\varepsilon \rangle_V = \\ &= -\varepsilon \langle \xi_\varepsilon(0), u_0 \rangle_X - \int_0^T \langle \varepsilon \xi_\varepsilon, u'_\varepsilon \rangle_V - \int_0^T \langle \xi_\varepsilon, u_\varepsilon \rangle_V + \int_0^T \langle f(u_\varepsilon), u_\varepsilon \rangle_V \\ &\longrightarrow -\int_0^T \langle \xi, u \rangle_V + \int_0^T \langle f(u), u \rangle_V.\end{aligned}$$

Hence, we particularly get

$$\limsup_{\varepsilon \rightarrow 0} \int_0^T \langle \eta_\varepsilon, u_\varepsilon \rangle_X \leq -\int_0^T \langle \xi, u \rangle_V + \int_0^T \langle f(u), u \rangle_V = \int_0^T \langle \eta, u \rangle_X.$$

By the demiclosedness of the maximal monotone operator $\partial_X \phi_X$ and by applying [27, Proposition 1.1], we conclude that $\eta(t) \in \partial_X \phi_X(u(t))$ for a.e. $t \in (0, T)$. We finally show $\xi(t) = d_V \psi(u'(t))$ for almost all $t \in (0, T)$. Combining [8, Theorem 5.1] and [10, Theorem 3.3, Lemma A.1], we get the following inequality:

$$\begin{aligned} \int_0^t \langle \xi_\varepsilon, u'_\varepsilon \rangle_V &\leq \varepsilon \langle \xi_\varepsilon(t), u'_\varepsilon(t) \rangle_V + \varepsilon \psi(0) - \phi(u_\varepsilon(t)) + \\ &\quad + \phi(u_0) + \int_0^t \langle f(u_\varepsilon), u'_\varepsilon \rangle_V, \end{aligned} \quad (43)$$

which can be formally obtained by substituting identity (22) into the left-hand side of (43) and by integrating it by parts. Thus, from the convergences above (in particular, (41)), using the lower semicontinuity of ϕ and recalling that $\eta \in L^p(0, T; V)$ (by comparison of each term in (42)) and that $\xi_\varepsilon(T) = 0$, we deduce that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_0^T \langle \xi_\varepsilon, u'_\varepsilon \rangle_V &\leq \lim_{\varepsilon \rightarrow 0} \varepsilon \psi(0) - \liminf_{\varepsilon \rightarrow 0} \phi(u_\varepsilon(T)) + \phi(u_0) + \lim_{\varepsilon \rightarrow 0} \int_0^T \langle f(u_\varepsilon), u'_\varepsilon \rangle_V \\ &\leq -\phi(u(T)) + \phi(u_0) + \int_0^T \langle f(u), u' \rangle_V \\ &= -\int_0^T \langle \eta, u' \rangle_V + \int_0^T \langle f(u), u' \rangle_V = \int_0^T \langle \xi, u' \rangle_V. \end{aligned}$$

Thus, $\xi(t) = d_V \psi(u'(t))$ for a.e. $t \in (0, T)$, and hence, u solves (16)–(18) with $\eta^2 = 0$.

4. Nonconvex energies: Proof of Theorem 2.4

This section is devoted to the proof of Theorem 2.4 for general nonconvex energy functionals. To this end, we shall employ the *Moreau-Yosida regularization* for convex functionals (equivalently, the *Yosida approximation* for subdifferentials) to approximate the target equation in order to reduce the problem to the convex energy setting of Proposition 3.2. Finally, we shall pass to the limit of approximated solutions and obtain a solution to the target equation.

4.1. The Moreau-Yosida regularization.

We first regularize equation (19): for every $\lambda > 0$, we define the Moreau-Yosida regularization of φ^2 by

$$\varphi_\lambda^2(u) = \min_{v \in V} \left(\frac{\lambda}{p} \left| \frac{u - v}{\lambda} \right|_V^p + \varphi^2(v) \right) = \frac{\lambda}{p} \left| \frac{u - J_\lambda u}{\lambda} \right|_V^p + \varphi^2(J_\lambda u),$$

where J_λ denotes the resolvent for $\partial_V \varphi^2$ (see (79) in Appendix), and consider the following approximate equations for (19)–(21):

$$-\varepsilon \xi'_{\varepsilon,\lambda} + \xi_{\varepsilon,\lambda} + \eta_{\varepsilon,\lambda}^1 - \eta_{\varepsilon,\lambda}^2 - f(u_{\varepsilon,\lambda}) = 0 \text{ in } X^* \text{ a.e. in } (0, T), \quad (44)$$

$$\xi_{\varepsilon,\lambda} = d_V \psi(u'_{\varepsilon,\lambda}), \quad \eta_{\varepsilon,\lambda}^1 \in \partial_X \varphi_X^1(u_{\varepsilon,\lambda}), \quad \eta_{\varepsilon,\lambda}^2 = \partial_V \varphi_\lambda^2(u_{\varepsilon,\lambda}), \quad (45)$$

$$u_{\varepsilon,\lambda}(0) = u_0, \quad \xi_{\varepsilon,\lambda}(T) = 0. \quad (46)$$

Note that $\partial_V \varphi_\lambda^2(u): V \rightarrow V^*$ is single-valued and demicontinuous, and it satisfies assumption (6) (see Section A.1 in Appendix for details). Moreover, the mapping $Q_\lambda: u \mapsto \partial_V \varphi_\lambda^2(u)$ from $L^p(0, T; V)$ to $L^{p'}(0, T; V^*)$ also satisfies the demicontinuity (28). Indeed, let $u_n \rightarrow u$ strongly in $L^p(0, T; V)$ and fix $v \in L^p(0, T; V)$. Then, by demicontinuity of $\partial_V \varphi_\lambda^2(\cdot)$ from V into V^* , we see that $\langle \partial_V \varphi_\lambda^2(u_n(t)), v(t) \rangle_V \rightarrow \langle \partial_V \varphi_\lambda^2(u(t)), v(t) \rangle_V$ for a.e. $t \in (0, T)$. Thus, the family of functions $t \mapsto \langle \partial_V \varphi_\lambda^2(u_n(t)), v(t) \rangle_V$ is pointwise convergent to $t \mapsto \langle \partial_V \varphi_\lambda^2(u(t)), v(t) \rangle_V$ on $(0, T)$. Moreover, by virtue of (6) with $f = \partial_V \varphi_\lambda^2$, it is uniformly integrable. Thanks to Vitali's convergence theorem (see, e.g., [44, page 133]), the function $t \mapsto \langle \partial_V \varphi_\lambda^2(u(t)), v(t) \rangle_V$ turns out to be integrable on $(0, T)$, and moreover, $\langle \partial_V \varphi_\lambda^2(u_n(\cdot)), v(\cdot) \rangle_V \rightarrow \langle \partial_V \varphi_\lambda^2(u(\cdot)), v(\cdot) \rangle_V$ strongly in $L^1(0, T)$. Therefore, the mapping Q_λ is demicontinuous from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$. Applying Proposition 3.2 with f replaced by $f + \partial_V \varphi_\lambda^2$, we deduce existence of (at least) a solution $u_{\varepsilon,\lambda}$ to (44)–(46).

4.2. Uniform estimates

In order to pass to the limit of solutions as $\lambda \rightarrow 0$, and then, as $\varepsilon \rightarrow 0$, we first establish some (uniform in λ and ε) estimates. Hereafter, C and c will denote positive constants not depending on ε and λ which may vary even within the same line. Testing equation (44) with $u'_{\varepsilon,\lambda}$ and integrating it over $(0, t)$, we get

$$\begin{aligned} -\varepsilon \int_0^t \frac{d}{dt} \psi^*(\xi_{\varepsilon,\lambda}) + \int_0^t \langle \xi_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda} \rangle_V + \int_0^t \frac{d}{dt} \varphi^1(u_{\varepsilon,\lambda}) - \int_0^t \frac{d}{dt} \varphi_\lambda^2(u_{\varepsilon,\lambda}) &= \\ &= \int_0^t \langle f(u_{\varepsilon,\lambda}), u'_{\varepsilon,\lambda} \rangle_V \end{aligned}$$

(see Remark 3.3). By virtue of (9), we estimate

$$\begin{aligned} -\varepsilon \psi^*(\xi_{\varepsilon,\lambda}(t)) + \varepsilon \psi^*(\xi_{\varepsilon,\lambda}(0)) + c \int_0^t |u'_{\varepsilon,\lambda}|_V^p + \varphi^1(u_{\varepsilon,\lambda}(t)) - \varphi_\lambda^2(u_{\varepsilon,\lambda}(t)) &\leq \\ &\leq \varphi^1(u_0) - \varphi_\lambda^2(u_0) + C + \frac{c}{2} \int_0^t |u'_{\varepsilon,\lambda}|_V^p + C \int_0^t |f(u_{\varepsilon,\lambda})|_{V^*}^{p'}, \end{aligned}$$

and thus, by using assumption **(A6)**,

$$\begin{aligned} -\varepsilon \psi^*(\xi_{\varepsilon,\lambda}(t)) + \varepsilon \psi^*(\xi_{\varepsilon,\lambda}(0)) + c \int_0^t |u'_{\varepsilon,\lambda}|_V^p + (1-k) \varphi^1(u_{\varepsilon,\lambda}(t)) &\leq \\ &\leq \varphi^1(u_0) - \varphi_\lambda^2(u_0) + C(|u_{\varepsilon,\lambda}|_V^p + 1) + \frac{c}{2} \int_0^t |u'_{\varepsilon,\lambda}|_V^p + C \int_0^t |f(u_{\varepsilon,\lambda})|_{V^*}^{p'}. \end{aligned} \quad (47)$$

Here we note that, for any $\mu > 0$ and $w \in W^{1,p}(0, T; V)$,

$$\begin{aligned} \mu \frac{d}{dt} |w|_V^p &= p\mu |w|_V^{p-1} \frac{d}{dt} |w|_V \\ &\leq p\mu |w|_V^{p-1} |w'|_V \leq (p-1)\mu^{p'} |w|_V^p + |w'|_V^p, \end{aligned}$$

which yields, by integrating over $(0, t)$,

$$\mu |w(t)|_V^p \leq \mu |w(0)|_V^p + (p-1)\mu^{p'} \int_0^t |w|_V^p + \int_0^t |w'|_V^p \quad \text{for } t \in [0, T].$$

Substituting the inequality above into (47) with $\mu > 0$ large enough, we obtain

$$\begin{aligned} & -\varepsilon \psi^*(\xi_{\varepsilon, \lambda}(t)) + \varepsilon \psi^*(\xi_{\varepsilon, \lambda}(0)) + \frac{c}{4} \int_0^t |u'_{\varepsilon, \lambda}|_V^p + (1-k) \varphi^1(u_{\varepsilon, \lambda}(t)) \leq \\ & \leq \varphi^1(u_0) + C(|u_0|_V^p + 1) + C \int_0^t |u_{\varepsilon, \lambda}|_V^p + C \int_0^t |f(u_{\varepsilon, \lambda})|_{V^*}^{p'} \stackrel{(6)}{\leq} \\ & \stackrel{(6)}{\leq} \varphi^1(u_0) + C(|u_0|_V^p + 1) + C \left(\int_0^t |u_{\varepsilon, \lambda}|_V^p + 1 \right). \end{aligned} \quad (48)$$

Repeating the argument of the last section, we obtain for ε small enough

$$\int_0^T |u_{\varepsilon, \lambda}|_V^p + \int_0^T |u'_{\varepsilon, \lambda}|_V^p + \int_0^T \varphi^1(u_{\varepsilon, \lambda}) \leq C,$$

and thus

$$\|u_{\varepsilon, \lambda}\|_{W^{1,p}(0, T; V)} + \|u_{\varepsilon, \lambda}\|_{L^m(0, T; X)} \leq C, \quad (49)$$

$$\|\eta_{\varepsilon, \lambda}^1\|_{L^{m'}(0, T; X^*)} \leq C, \quad (50)$$

$$\|\xi_{\varepsilon, \lambda}\|_{L^{p'}(0, T; V^*)} \leq C, \quad (51)$$

$$\|f(u_{\varepsilon, \lambda})\|_{L^{p'}(0, T; V^*)} \leq C. \quad (52)$$

In particular, $\|u_{\varepsilon, \lambda}\|_{C([0, T]; V)} \leq C$. Substituting $t = T$ into estimate (48) and recalling $\xi_{\varepsilon, \lambda}(T) = 0$, we get

$$\varphi^1(u_{\varepsilon, \lambda}(T)) \leq C,$$

which along with the assumptions **(A3)**, **(A4)**, and **(A6)** implies

$$|u_{\varepsilon, \lambda}(T)|_X \leq C, \quad |\eta_{\varepsilon, \lambda}^1(T)|_{X^*} \leq C, \quad |\eta_{\varepsilon, \lambda}^2(T)|_{V^*} \leq C.$$

By using assumption **(A6)** again, we get

$$\|\eta_{\varepsilon, \lambda}^2\|_{L^{p'}(0, T; V^*)} \leq C, \quad (53)$$

$$\int_0^T \varphi^2(u_{\varepsilon, \lambda}) \leq C. \quad (54)$$

Finally, by comparison in equation (44),

$$\|\varepsilon \xi'_{\varepsilon, \lambda}\|_{L^{p'}(0, T; V^*) + L^{m'}(0, T; X^*)} \leq C.$$

4.3. The passage to the limit as $\lambda \rightarrow 0$.

Owing to the obtained uniform estimates, up to some (not relabeled) subsequence $\lambda \rightarrow 0$, we have

$$\begin{aligned} \eta_{\varepsilon,\lambda}^1 &\rightarrow \eta_\varepsilon^1 \text{ weakly in } L^{m'}(0, T; X^*), \\ \eta_{\varepsilon,\lambda}^2 &\rightarrow \eta_\varepsilon^2 \text{ weakly in } L^{p'}(0, T; V^*), \\ \xi_{\varepsilon,\lambda} &\rightarrow \xi_\varepsilon \text{ weakly in } L^{p'}(0, T; V^*), \\ \xi'_{\varepsilon,\lambda} &\rightarrow \xi'_\varepsilon \text{ weakly in } L^{p'}(0, T; V^*) + L^{m'}(0, T; X^*), \\ u_{\varepsilon,\lambda} &\rightarrow u_\varepsilon \text{ weakly in } W^{1,p}(0, T; V) \text{ and in } L^m(0, T; X), \\ u_{\varepsilon,\lambda}(T) &\rightarrow v_\varepsilon \text{ weakly in } X, \\ \eta_{\varepsilon,\lambda}^2(T) &\rightarrow q_\varepsilon^2 \text{ weakly in } V^*, \\ f(u_{\varepsilon,\lambda}) &\rightarrow f_\varepsilon^* \text{ weakly in } L^{p'}(0, T; V^*), \end{aligned}$$

and hence, thanks to Aubin-Lions-Simon's compactness lemma (see [50]),

$$u_{\varepsilon,\lambda} \rightarrow u_\varepsilon \text{ strongly in } C([0, T]; V), \quad (55)$$

$$\xi_{\varepsilon,\lambda} \rightarrow \xi_\varepsilon \text{ strongly in } C([0, T]; X^*). \quad (56)$$

In particular, $u_{\varepsilon,\lambda}(t) \rightarrow u_\varepsilon(t)$ strongly in V for all $t \in [0, T]$, which also yields $v_\varepsilon = u_\varepsilon(T)$, $u_\varepsilon(0) = u_0$, and $\xi_\varepsilon(T) = 0$. By virtue of the continuity of $f: V \rightarrow V^*$ and the convergence (55) (see also Remark 2.1), we get $f_\varepsilon^* = f(u_\varepsilon)$ and

$$f(u_{\varepsilon,\lambda}) \rightarrow f(u_\varepsilon) \text{ strongly in } L^{p'}(0, T; V^*). \quad (57)$$

Thus, we assure that

$$-\varepsilon \xi'_\varepsilon + \xi_\varepsilon + \eta_\varepsilon^1 - \eta_\varepsilon^2 - f(u_\varepsilon) = 0. \quad (58)$$

The inclusions $\eta_\varepsilon^2 \in \partial_V \varphi^2(u_\varepsilon)$ and $q_\varepsilon^2 \in \partial_V \varphi^2(u_\varepsilon(T))$ follow by a standard monotonicity argument (see, e.g. [13, Chap. II, Section 1.2]) as a consequence of the strong convergence (55). We shall now identify the limit η_ε^1 as $\eta_\varepsilon^1 \in \partial_X \varphi_X^1(u_\varepsilon)$ a.e. in $(0, T)$. By a standard argument for monotone operators, from the weak convergences obtained above it follows that

$$\liminf_{\lambda \rightarrow 0} \int_s^t \langle \eta_{\varepsilon,\lambda}^1, u_{\varepsilon,\lambda} \rangle_X \geq \int_s^t \langle \eta_\varepsilon^1, u_\varepsilon \rangle_X, \quad (59)$$

$$\liminf_{\lambda \rightarrow 0} \int_s^t \langle \xi_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda} \rangle_V \geq \int_s^t \langle \xi_\varepsilon, u'_\varepsilon \rangle_V \quad (60)$$

for all $0 \leq s \leq t \leq T$. Let us note that the quantity $a(t) = \liminf_{\lambda \rightarrow 0} |\xi_{\varepsilon,\lambda}(t)|_{V^*}^{p'}$, belongs to the space $L^1(0, T)$ by Fatou's Lemma and estimate (51). In particular, $a(t) < \infty$ for a.a. $t \in (0, T)$, and for such t , we can take a subsequence λ_n^t (possibly depending on t) such that

$$\xi_{\varepsilon,\lambda_n^t}(t) \rightarrow \xi_\varepsilon(t) \text{ weakly in } V^*.$$

Thus, thanks to the convergence (55), we observe that the set $\mathcal{L} \subset (0, T)$ defined by

$$\mathcal{L} := \left\{ t \in (0, T) \left| \begin{array}{l} t \text{ is a Lebesgue point for } t \mapsto \langle \xi_\varepsilon(t), u_\varepsilon(t) \rangle_V, \text{ and} \\ \text{for any sequence } \lambda_n \rightarrow 0 \text{ there exists a subsequence} \\ \lambda'_n \text{ such that } \langle \xi_{\varepsilon, \lambda'_n}(t), u_{\varepsilon, \lambda'_n}(t) \rangle_V \rightarrow \langle \xi_\varepsilon(t), u_\varepsilon(t) \rangle_V \end{array} \right. \right\}$$

has the full (Lebesgue) measure. Thus, by virtue of the convergences above and of (60), we deduce, for arbitrary $t_1, t_2 \in \mathcal{L}$, $t_2 > t_1$ and a (not-relabelled) subsequence $\lambda \rightarrow 0$ (possibly depending on the choice of t_1, t_2), that

$$\begin{aligned} & \limsup_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \eta_{\varepsilon, \lambda}^1, u_{\varepsilon, \lambda} \rangle_X = \\ &= \limsup_{\lambda \rightarrow 0} \left\{ - \int_{t_1}^{t_2} \langle \xi_{\varepsilon, \lambda}, u_{\varepsilon, \lambda} \rangle_V + \int_{t_1}^{t_2} \langle \varepsilon \xi'_{\varepsilon, \lambda}, u_{\varepsilon, \lambda} \rangle_X + \right. \\ & \quad \left. + \int_{t_1}^{t_2} \langle f(u_{\varepsilon, \lambda}), u_{\varepsilon, \lambda} \rangle_V + \int_{t_1}^{t_2} \langle \eta_{\varepsilon, \lambda}^2, u_{\varepsilon, \lambda} \rangle_V \right\} \leq \\ & \leq \lim_{\lambda \rightarrow 0} \left\{ - \int_{t_1}^{t_2} \langle \xi_{\varepsilon, \lambda}, u_{\varepsilon, \lambda} \rangle_V + \int_{t_1}^{t_2} \langle f(u_{\varepsilon, \lambda}), u_{\varepsilon, \lambda} \rangle_V + \int_{t_1}^{t_2} \langle \eta_{\varepsilon, \lambda}^2, u_{\varepsilon, \lambda} \rangle_V \right\} + \\ & \quad + \lim_{\lambda \rightarrow 0} \langle \varepsilon \xi_{\varepsilon, \lambda}, u_{\varepsilon, \lambda} \rangle_V \Big|_{t_1}^{t_2} - \liminf_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \varepsilon \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V \leq \\ & \leq - \int_{t_1}^{t_2} \langle \xi_\varepsilon, u_\varepsilon \rangle_V + \int_{t_1}^{t_2} \langle f(u_\varepsilon), u_\varepsilon \rangle_V + \int_{t_1}^{t_2} \langle \eta_\varepsilon^2, u_\varepsilon \rangle_V + \\ & \quad + \langle \varepsilon \xi_\varepsilon, u_\varepsilon \rangle_V \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \langle \varepsilon \xi_\varepsilon, u'_\varepsilon \rangle_V \leq \\ & \leq \int_{t_1}^{t_2} \langle \eta_\varepsilon^1, u_\varepsilon \rangle_X. \end{aligned}$$

Here, we also used the integration-by-parts formula

$$\langle \varepsilon \xi_\varepsilon, u_\varepsilon \rangle_V \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \langle \varepsilon \xi_\varepsilon, u'_\varepsilon \rangle_V = \int_{t_1}^{t_2} \langle \varepsilon \xi'_\varepsilon, u_\varepsilon \rangle_X \quad (61)$$

derived in [10]. This fact together with (59) yields

$$\lim_{\lambda \rightarrow 0} \int_s^t \langle \eta_{\varepsilon, \lambda}^1, u_{\varepsilon, \lambda} \rangle_X = \int_s^t \langle \eta_\varepsilon^1, u_\varepsilon \rangle_X \quad \text{for a.e. } 0 \leq s \leq t \leq T,$$

and hence, $\eta_\varepsilon^1 \in \partial_X \varphi_X^1(u_\varepsilon)$ a.e in $(0, T)$.

Let us now show the inclusion $\xi = d_V \psi(u')$. For arbitrary $t_1, t_2 \in \mathcal{L}$, $t_2 > t_1$, by integration by parts, we compute, up to a (not-relabeled) subsequence,

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V &= \lim_{\lambda \rightarrow 0} \langle \xi_{\varepsilon, \lambda}, u_{\varepsilon, \lambda} \rangle_V \Big|_{t_1}^{t_2} - \liminf_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \xi'_{\varepsilon, \lambda}, u_{\varepsilon, \lambda} \rangle_X = \\ &= \langle \xi_{\varepsilon}, u_{\varepsilon} \rangle_V \Big|_{t_1}^{t_2} - \frac{1}{\varepsilon} \lim_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \xi_{\varepsilon, \lambda} - \eta_{\varepsilon, \lambda}^2 - f(u_{\varepsilon, \lambda}), u_{\varepsilon, \lambda} \rangle_V - \frac{1}{\varepsilon} \lim_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \eta_{\varepsilon, \lambda}^1, u_{\varepsilon, \lambda} \rangle_X. \end{aligned}$$

As a consequence of the convergence above and of identity (58), we get

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V &= \\ &= \langle \xi_{\varepsilon}, u_{\varepsilon} \rangle_V \Big|_{t_1}^{t_2} - \frac{1}{\varepsilon} \int_{t_1}^{t_2} \langle \xi_{\varepsilon} - \eta_{\varepsilon}^2 - f(u_{\varepsilon}), u_{\varepsilon} \rangle_V - \frac{1}{\varepsilon} \int_{t_1}^{t_2} \langle \eta_{\varepsilon}^1, u_{\varepsilon} \rangle_X \\ &= \langle \xi_{\varepsilon}, u_{\varepsilon} \rangle_V \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \langle \xi'_{\varepsilon}, u_{\varepsilon} \rangle_X \stackrel{(61)}{=} \int_{t_1}^{t_2} \langle \xi_{\varepsilon}, u'_{\varepsilon} \rangle_V. \end{aligned}$$

In particular, by virtue of (60),

$$\lim_{\lambda \rightarrow 0} \int_{t_1}^{t_2} \langle \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V = \int_{t_1}^{t_2} \langle \xi_{\varepsilon}, u'_{\varepsilon} \rangle_V.$$

From the arbitrariness of $t_1, t_2 \in \mathcal{L}$, we conclude that $\xi_{\varepsilon} = d_V \psi(u'_{\varepsilon})$ a.e. in $(0, T)$. This proves the first half of Theorem 2.4.

Before moving on to the causal limit as $\varepsilon \rightarrow 0$, let us derive an energy inequality for later use. Repeating the same argument as in (43), we have

$$\begin{aligned} \int_0^t \langle \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V &\leq \varepsilon \langle \xi_{\varepsilon, \lambda}(t), u'_{\varepsilon, \lambda}(t) \rangle_V + \varepsilon \psi(0) - \\ &\quad - \varphi^1(u_{\varepsilon, \lambda}(t)) + \varphi^1(u_0) + \int_0^t \langle \eta_{\varepsilon, \lambda}^2 + f(u_{\varepsilon, \lambda}), u'_{\varepsilon, \lambda} \rangle_V. \end{aligned}$$

Thus, thanks to the convergences obtained above, to the lower semicontinuity of φ^1 , and to identity (58), we obtain (recalling that $\xi_{\varepsilon, \lambda}(T) = 0$)

$$\begin{aligned} \int_0^T \langle \xi_{\varepsilon}, u'_{\varepsilon} \rangle_V &\leq \liminf_{\lambda \rightarrow 0} \int_0^T \langle \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V \leq \limsup_{\lambda \rightarrow 0} \int_0^T \langle \xi_{\varepsilon, \lambda}, u'_{\varepsilon, \lambda} \rangle_V \quad (62) \\ &\leq \varepsilon \psi(0) - \liminf_{\lambda \rightarrow 0} \varphi^1(u_{\varepsilon, \lambda}(T)) + \varphi^1(u_0) \\ &\quad + \limsup_{\lambda \rightarrow 0} \varphi_{\lambda}^2(u_{\varepsilon, \lambda}(T)) - \varphi^2(u_0) + \lim_{\lambda \rightarrow 0} \int_0^T \langle f(u_{\varepsilon, \lambda}), u'_{\varepsilon, \lambda} \rangle_V. \end{aligned}$$

Note that, by definition of subdifferential and by using the convergences obtained above, we have, as $\lambda \rightarrow 0$,

$$\varphi_{\lambda}^2(u_{\varepsilon, \lambda}(T)) \leq \varphi_{\lambda}^2(u_{\varepsilon}(T)) + \langle \eta_{\varepsilon, \lambda}^2(T), u_{\varepsilon, \lambda}(T) - u_{\varepsilon}(T) \rangle_V \rightarrow \varphi^2(u_{\varepsilon}(T))$$

(we also used the fact that $u_{\varepsilon,\lambda}(T) \in D(\varphi^1) \subset D(\varphi^2)$ by (A6)), i.e.,

$$\limsup_{\lambda \rightarrow 0} \varphi_{\lambda}^2(u_{\varepsilon,\lambda}(T)) \leq \varphi^2(u_{\varepsilon}(T)).$$

Thus, substituting the above into (62), we get

$$\begin{aligned} \int_0^T \langle \xi_{\varepsilon}, u'_{\varepsilon} \rangle_V &\leq \varepsilon \psi(0) - \varphi^1(u_{\varepsilon}(T)) + \\ &+ \varphi^1(u_0) + \varphi^2(u_{\varepsilon}(T)) - \varphi^2(u_0) + \int_0^T \langle f(u_{\varepsilon}), u'_{\varepsilon} \rangle_V. \end{aligned} \quad (63)$$

4.4. The causal limit as $\varepsilon \rightarrow 0$

We now deal with the passage to the limit as $\varepsilon \rightarrow 0$. We first note that, thanks to the estimates mentioned above and the weak lower semicontinuity of norms, we have

$$\begin{aligned} \|u_{\varepsilon}\|_{W^{1,p}(0,T;V)} + \|u_{\varepsilon}\|_{L^m(0,T;X)} &\leq C, & \|\eta_{\varepsilon}^1\|_{L^{m'}(0,T;X^*)} &\leq C, \\ \|\xi_{\varepsilon}\|_{L^{p'}(0,T;V^*)} &\leq C, & \|f(u_{\varepsilon})\|_{L^{p'}(0,T;V^*)} &\leq C, \\ \|\eta_{\varepsilon}^2\|_{L^{p'}(0,T;V^*)} &\leq C, & \|\varepsilon \xi'_{\varepsilon}\|_{L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*)} &\leq C, \\ \|u_{\varepsilon}(T)\|_X &\leq C, & \|q_{\varepsilon}^2\|_{V^*} &\leq C, \end{aligned}$$

where $q_{\varepsilon}^2 \in \partial_V \varphi^2(u_{\varepsilon}(T))$ (see Section 4.3). Up to a (not relabeled) subsequence, we get the following convergence results

$$\begin{aligned} \eta_{\varepsilon}^1 &\rightarrow \eta^1 && \text{weakly in } L^{m'}(0,T;X^*), \\ \eta_{\varepsilon}^2 &\rightarrow \eta^2 && \text{weakly in } L^{p'}(0,T;V^*), \\ \xi_{\varepsilon} &\rightarrow \xi && \text{weakly in } L^{p'}(0,T;V^*), \\ \varepsilon \xi'_{\varepsilon} &\rightarrow 0 && \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*), \\ \varepsilon \xi_{\varepsilon} &\rightarrow 0 && \text{strongly in } C([0,T];X^*), \\ u_{\varepsilon} &\rightarrow u && \text{weakly in } W^{1,p}(0,T;V) \text{ and in } L^m(0,T;X), \\ &&& \text{strongly in } C([0,T];V), \\ u_{\varepsilon}(T) &\rightarrow v && \text{weakly in } X, \\ q_{\varepsilon}^2 &\rightarrow q^2 && \text{weakly in } V^*, \\ f(u_{\varepsilon}) &\rightarrow f(u) && \text{strongly in } L^{p'}(0,T;V^*). \end{aligned}$$

In particular, $v = u(T)$, and $u(0) = u_0$, and moreover,

$$\xi + \eta^1 - \eta^2 - f(u) = 0 \quad \text{in } X^*, \quad 0 < t < T. \quad (64)$$

As a consequence of the strong convergence $u_{\varepsilon} \rightarrow u$ in $C([0,T];V)$ and the weak convergence of η_{ε}^2 (respectively, q_{ε}^2), by the demiclosedness of maximal

monotone operators (see also [27, Proposition 1.1]), we obtain the relation $\eta^2(t) \in \partial_V \varphi^2(u(t))$ for a.e. $t \in (0, T)$ (respectively, $q^2 \in \partial_V \varphi^2(u(T))$). Repeating the same argument as in Section 3, we can obtain

$$\limsup_{\varepsilon \rightarrow 0} \int_0^T \langle \eta_\varepsilon^1, u_\varepsilon \rangle_X \leq \int_0^T \langle \eta^1, u \rangle_X,$$

which proves $\eta^1 \in \partial_X \varphi_X^1(u)$, and hence, $\eta^1 \in \partial_V \varphi^1(u)$ by $\eta^1 = f(u) + \eta^2 - \xi \in L^{p'}(0, T; V^*)$ (see [10]). In order to prove the relation $\xi = d_V \psi(u')$, we proceed as follows. By definition of subdifferential and by using the convergences obtained so far, we have, as $\varepsilon \rightarrow 0$,

$$\varphi^2(u_\varepsilon(T)) \leq \varphi^2(u(T)) + \langle q_\varepsilon^2, u_\varepsilon(T) - u(T) \rangle_V \rightarrow \varphi^2(u(T)),$$

i.e., $\limsup_{\varepsilon \rightarrow 0} \varphi^2(u_\varepsilon(T)) \leq \varphi^2(u(T))$. Thus, by the convergences above along with the semicontinuity of φ^1 , estimate (63) and identity (64), we derive

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_0^T \langle \xi_\varepsilon, u'_\varepsilon \rangle_V &\leq \lim_{\varepsilon \rightarrow 0} \varepsilon \psi(0) - \liminf_{\varepsilon \rightarrow 0} \varphi^1(u_\varepsilon(T)) + \varphi^1(u_0) + \\ &\quad + \limsup_{\varepsilon \rightarrow 0} \varphi^2(u_\varepsilon(T)) - \varphi^2(u_0) + \lim_{\varepsilon \rightarrow 0} \int_0^T \langle f(u_\varepsilon), u'_\varepsilon \rangle_V \\ &\leq -\varphi^1(u(T)) + \varphi^1(u_0) + \varphi^2(u(T)) - \varphi^2(u_0) + \int_0^T \langle f(u), u' \rangle_V = \int_0^T \langle \xi, u' \rangle_V. \end{aligned}$$

Here we also used the chain rule developed in [10] (cf. (61)). Thus, $\xi(t) = d_V \psi(u'(t))$ for a.a. $t \in (0, T)$.

Remark 4.1. Thanks to the results of Section 3.2 and to the uniform estimates obtained in Section 4, one can pass to the limit also in the opposite order: first let $\varepsilon \rightarrow 0$ and then let $\lambda \rightarrow 0$. $\square$

5. WED principle for doubly-nonlinear flows of nonconvex energies

In this short section, as a by-product, we develop a variational characterization based on WED functionals for doubly-nonlinear flows of nonconvex energy functionals (i.e., the case $f = 0$). The WED variational principle has been mainly studied for convex (at most λ -convex) energy functionals. Nonconvex energy functionals have not yet been treated except in [7], where standard gradient flows (i.e., with quadratic dissipation potential ψ) of non(λ -)convex energy functionals are considered. In particular, doubly-nonlinear flows with nonconvex energies have never been studied so far. The following theorem provides a generalization of the results in [7] to doubly-nonlinear flows of nonconvex energies.

Theorem 5.1. *Assume (A1)–(A6) and let $f = 0$. Then, for every $\varepsilon > 0$, the WED functional W_ε defined by*

$$W_\varepsilon(u) = \begin{cases} \int_0^T e^{-t/\varepsilon} (\varepsilon\psi(u') + \varphi^1(u) - \varphi^2(u)) & \text{if } u \in K(u_0) \cap L^m(0, T; X), \\ \infty & \text{otherwise,} \end{cases}$$

admits at least one global minimizer over the set $K(u_0) := \{u \in W^{1,p}(0, T; V) : u(0) = u_0\}$. Furthermore, every local minimizer u_ε solves (19)–(21), and moreover, $u_\varepsilon \rightarrow u$ weakly in $W^{1,p}(0, T; V) \cap L^m(0, T; X)$ and strongly in $C([0, T]; V)$, where the limit u solves (16)–(18).

Proof. We shall follow the strategy for proving [7, Theorem 4.1]. For all $\lambda > 0$, let us define regularized WED functionals $W_{\varepsilon,\lambda} : L^p(0, T; V) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$W_{\varepsilon,\lambda}(u) = \begin{cases} \int_0^T e^{-t/\varepsilon} (\varepsilon\psi(u') + \varphi^1(u) - \varphi_\lambda^2(u)) & \text{if } u \in K(u_0) \cap L^m(0, T; X), \\ \infty & \text{otherwise,} \end{cases}$$

and decompose them as the difference $W_{\varepsilon,\lambda}(u) = C_\varepsilon^1(u) - C_{\varepsilon,\lambda}^2(u)$ of convex functionals

$$C_\varepsilon^1(u) := \begin{cases} \int_0^T e^{-t/\varepsilon} (\varepsilon\psi(u') + \varphi^1(u)) & \text{if } u \in K(u_0) \cap L^m(0, T; X), \\ \infty & \text{otherwise,} \end{cases}$$

$$C_{\varepsilon,\lambda}^2(u) := \int_0^T e^{-t/\varepsilon} \varphi_\lambda^2(u),$$

where φ_λ^2 denotes the Moreau-Yosida regularization of φ^2 as in Section 4. Note that C_ε^1 is lower semicontinuous in $L^p(0, T; V)$. Moreover, for all sequences $u_n \rightarrow u$ strongly in $L^p(0, T; V)$, we have $C_{\varepsilon,\lambda}^2(u_n) \rightarrow C_{\varepsilon,\lambda}^2(u)$. Indeed, by convexity and lower semicontinuity of φ_λ^2 , one has

$$\liminf_{n \rightarrow \infty} C_{\varepsilon,\lambda}^2(u_n) \geq C_{\varepsilon,\lambda}^2(u).$$

Furthermore, by definition of subdifferential, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} C_{\varepsilon,\lambda}^2(u_n) &\leq C_{\varepsilon,\lambda}^2(u) + \lim_{n \rightarrow \infty} \langle \partial_{L^p(0,T;V)} C_{\varepsilon,\lambda}(u_n), u_n - u \rangle_{L^p(0,T;V)} = \\ &= C_{\varepsilon,\lambda}^2(u) + \lim_{n \rightarrow \infty} \int_0^T \exp(-t/\varepsilon) \langle \partial_V \varphi_\lambda^2(u_n), u_n - u \rangle_V = C_{\varepsilon,\lambda}^2(u) \end{aligned} \quad (65)$$

(see also Section A.1 in Appendix). Thanks to (7) and the fact that $\varphi_\lambda^2(u) \leq \varphi^2(u)$ for all $u \in D(\varphi^2)$, $W_{\varepsilon,\lambda}$ is bounded from below. Let $\{u_n\}$ be a minimizing sequence of $W_{\varepsilon,\lambda}$. Thus, by using the coercivity assumptions (3), (5), and assumption (7) (see also [7]), we can deduce that the sequence $\{u_n\}$ is uniformly bounded in $W^{1,p}(0, T; V) \cap L^m(0, T; X)$, and up to a (not relabeled) subsequence, $u_n \rightarrow u_{\varepsilon,\lambda}$ strongly in $C([0, T]; V)$ for some $u_{\varepsilon,\lambda} \in W^{1,p}(0, T; V) \cap$

$L^m(0, T; X)$. Thus, by using the lower semicontinuity of C_ε^1 and (65), we conclude that $u_{\varepsilon, \lambda}$ is a minimizer of $W_{\varepsilon, \lambda}$. In particular, $u_{\varepsilon, \lambda}$ solves

$$0 \in \partial_{L^p(0, T; V)} W_{\varepsilon, \lambda}(u_{\varepsilon, \lambda}) = \partial_{L^p(0, T; V)} C_\varepsilon^1(u_{\varepsilon, \lambda}) - d_{L^p(0, T; V)} C_{\varepsilon, \lambda}^2(u_{\varepsilon, \lambda}),$$

which is equivalent to (44)–(46) with $f = 0$ (see [10]). Thus, as in Section 4, we are ready to prove that (up to subsequences) $u_{\varepsilon, \lambda} \rightarrow u_\varepsilon$ weakly in $W^{1,p}(0, T; V) \cap L^m(0, T; X)$ for $\lambda \rightarrow 0$ and that the limit u_ε solves (19)–(21). Furthermore, by lower semicontinuity, we have $\liminf_{\lambda \rightarrow 0} C_\varepsilon^1(u_{\varepsilon, \lambda}) \geq C_\varepsilon^1(u_\varepsilon)$, and as in [7, Theorem 4.1],

$$\limsup_{\lambda \rightarrow 0} C_{\varepsilon, \lambda}^2(u_{\varepsilon, \lambda}) \leq \limsup_{\lambda \rightarrow 0} \int_0^T e^{-t/\varepsilon} \varphi^2(u_{\varepsilon, \lambda}) \leq \int_0^T e^{-t/\varepsilon} \varphi^2(u_\varepsilon).$$

Moreover, $W_{\varepsilon, \lambda}(v) \rightarrow W_\varepsilon(v)$ for all $v \in D(W_\varepsilon)$ as $\lambda \rightarrow 0$. Thus, u_ε minimizes W_ε . Indeed, let $v \in D(W_\varepsilon)$. Then, it follows that

$$W_\varepsilon(u_\varepsilon) = C_\varepsilon^1(u_\varepsilon) - \int_0^T e^{-t/\varepsilon} \varphi^2(u_\varepsilon) \leq \liminf_{\lambda \rightarrow 0} W_{\varepsilon, \lambda}(u_{\varepsilon, \lambda}) \leq \lim_{\lambda \rightarrow 0} W_{\varepsilon, \lambda}(v) = W_\varepsilon(v).$$

This proves existence of a minimizer u_ε of the WED functional W_ε such that u_ε solves the Euler-Lagrange equation (19)–(21) (with $f = 0$) of W_ε (or the elliptic-in-time regularized equation).

We next prove that every local minimizer of W_ε solves (19)–(21) by using a penalization argument (cf. [7]). To this aim, let $\hat{u}_\varepsilon$ be a local minimizer of W_ε and define the penalized functional,

$$\hat{W}_\varepsilon(u) = \int_0^T e^{-t/\varepsilon} (\varepsilon \psi(u') + \hat{\varphi}^1(u) - \varphi^2(u))$$

with

$$\hat{\varphi}^1(u) = \varphi^1(u) + \frac{\alpha}{p} |u - \hat{u}_\varepsilon|_V^p$$

for some $\alpha > 0$. Then, arguing as in [7, Theorem 4.2] one can check that, for $\alpha = \alpha(\hat{u}_\varepsilon)$ sufficiently big, the functional $\hat{W}_\varepsilon$ admits a unique global minimizer $\hat{u}_\varepsilon$. Note that $\hat{\varphi}^1$ satisfies assumption **(A3)**, **(A4)**, and **(A6)**. Thus, by applying the results obtained so far, for each $\lambda > 0$, there exists a minimizer $\tilde{u}_{\varepsilon, \lambda}$ of the regularized WED functional,

$$\hat{W}_{\varepsilon, \lambda}(u) = \int_0^T e^{-t/\varepsilon} (\varepsilon \psi(u') + \hat{\varphi}^1(u) - \varphi_\lambda^2(u)),$$

such that $\tilde{u}_{\varepsilon, \lambda}$ solves the Euler-Lagrange equation,

$$\begin{aligned} -\varepsilon \tilde{\xi}'_{\varepsilon, \lambda} + \tilde{\xi}_{\varepsilon, \lambda} + \tilde{\eta}_{\varepsilon, \lambda}^1 + \alpha \tilde{\gamma}_{\varepsilon, \lambda} - \tilde{\eta}_{\varepsilon, \lambda}^2 &= 0 \text{ in } X^* \text{ a.e. in } (0, T), \\ \tilde{\xi}_{\varepsilon, \lambda} &= d_V \psi(\tilde{u}'_{\varepsilon, \lambda}), \quad \tilde{\eta}_{\varepsilon, \lambda}^1 \in \partial_X \varphi_X^1(\tilde{u}_{\varepsilon, \lambda}), \quad \tilde{\eta}_{\varepsilon, \lambda}^2 = \partial_V \varphi_\lambda^2(\tilde{u}_{\varepsilon, \lambda}), \\ \tilde{\gamma}_{\varepsilon, \lambda} &= |\tilde{u}_{\varepsilon, \lambda} - \hat{u}_\varepsilon|_V^{p-2} F_V(\tilde{u}_{\varepsilon, \lambda} - \hat{u}_\varepsilon), \\ \tilde{u}_{\varepsilon, \lambda}(0) &= u_0, \quad \tilde{\xi}_{\varepsilon, \lambda}(T) = 0. \end{aligned}$$

Indeed, the functional $u \mapsto \int_0^T e^{-t/\varepsilon} \frac{1}{p} |u(t) - \hat{u}_\varepsilon(t)|_V^p$ is Fréchet differentiable in $L^p(0, T; V)$ and its Fréchet derivative at $\tilde{u}_{\varepsilon, \lambda}$ is given as $e^{-t/\varepsilon} \tilde{\gamma}_{\varepsilon, \lambda}$. We can now derive uniform estimates for $\tilde{u}_{\varepsilon, \lambda}$ and prove convergence of $\tilde{u}_{\varepsilon, \lambda}$ to a limit $\tilde{u}_\varepsilon$ which minimizes $\hat{W}_\varepsilon$ and solves

$$\begin{aligned} -\varepsilon \tilde{\xi}'_\varepsilon + \tilde{\xi}_\varepsilon + \tilde{\eta}_\varepsilon^1 - \tilde{\eta}_\varepsilon^2 + \alpha \tilde{\gamma}_\varepsilon &= 0 \text{ in } X^* \text{ a.e. in } (0, T), \\ \tilde{\xi}_\varepsilon &= d_V \psi(\tilde{u}'_\varepsilon), \quad \tilde{\eta}_\varepsilon^1 \in \partial_X \varphi_X^1(\tilde{u}_\varepsilon), \quad \tilde{\eta}_\varepsilon^2 \in \partial_V \varphi^2(\tilde{u}_\varepsilon), \\ \tilde{\gamma}_\varepsilon &= |\tilde{u}_\varepsilon - \hat{u}_\varepsilon|_V^{p-2} F_V(\tilde{u}_\varepsilon - \hat{u}_\varepsilon), \\ \tilde{u}_\varepsilon(0) &= u_0, \quad \tilde{\xi}_\varepsilon(T) = 0. \end{aligned}$$

Recall that, by penalization, the unique minimizer of $\hat{W}_\varepsilon$ is $\hat{u}_\varepsilon$ (see also [8]). Thus, $\hat{u}_\varepsilon = \tilde{u}_\varepsilon$ and, as a consequence, we have that $\tilde{\gamma}_\varepsilon = 0$. Hence, $\hat{u}_\varepsilon$ solves (19)–(21).

Finally, the limiting procedure as $\varepsilon \rightarrow 0$ has already been proved in Section 4. This completes the proof. $\square$

6. An alternative approach

In this section, we exhibit a slightly different approach. More precisely, we shall prove the assertion of Theorem 2.4 under different assumptions on f and φ^2 .

We recall that, in Section 3, we introduced the map S defined by

$$\begin{aligned} S: L^p(0, T; V) &\rightarrow L^p(0, T; V), \\ S: v &\mapsto w := f(v) \mapsto u, \end{aligned}$$

where u is the global minimizer of $I_{\varepsilon, w}$, and looked for a fixed point of S to construct a solution of the elliptic-in-time regularization (26)–(27). Note that the map S is the composition of two different maps: $v \mapsto f(v)$ and $w \mapsto u := \operatorname{argmin} I_{\varepsilon, w}$. Alternatively, one may consider a map $\tilde{S}$ which is the composition of the same maps in the opposite order, namely

$$\begin{aligned} \tilde{S}: L^{p'}(0, T; V^*) &\rightarrow L^{p'}(0, T; V^*), \\ \tilde{S}: v &\mapsto u \mapsto f(u), \end{aligned}$$

where u is the global minimizer of $I_{\varepsilon, v}$. Note that if $\tilde{S}$ has a fixed point v , then the minimizer of $I_{\varepsilon, v}$ solves (26)–(27). In order to apply the Schaefer fixed-point theorem to $\tilde{S}$, one has to check the (strong) continuity of $\tilde{S}$ in $L^{p'}(0, T; V^*)$ as in Section 3.1.2. Furthermore, as for nonconvex energies, f is replaced by $f + \partial_V \varphi_\lambda^2$ above. In case V is a Hilbert space, one can prove the (strong) continuity of $\tilde{S}$ by employing the Lipschitz continuity of Yosida approximations in the Hilbert space setting. On the other hand, in case V is a Banach space, it seems somewhat difficult to prove the (strong) continuity of $\tilde{S}$, due to the lack of

(strong) continuity of the Yosida approximation $\partial_V \varphi_\lambda^2: V \rightarrow V^*$ in a Banach space setting (it is only demicontinuous. See Appendix and [13]). Hence in order to recover the continuity of $\tilde{S}$, we assume that

$$\varphi^2 \text{ is of class } C^1(V; \mathbb{R})$$

(in the sense of Fréchet derivative). Then, $\partial_V \varphi^2$ is single-valued and continuous from V into V^* . Hence, it is no longer necessary to employ the Yosida approximation of $\partial_V \varphi^2$. On the other hand, the growth and the continuity conditions in **(A5)** can be relaxed as follows:

(A5') $f: X \rightarrow V^*$ satisfies

$$|f(u)|_{V^*}^{p'} \leq C(1 + \varphi^1(u) + |u|_V^p) \text{ for all } u \in X \quad (66)$$

and some positive constant C . Moreover, if $u \in L^m(0, T; X) \cap W^{1,p}(0, T; V)$, then $f(u) \in L^{p'}(0, T; V^*)$. Furthermore, if $u_n \rightarrow u$ weakly in $L^m(0, T; X)$ and in $W^{1,p}(0, T; V)$, and $(f(u_n))$ is bounded in $L^{p'}(0, T; V^*)$, then, up to a (not relabeled) subsequence, $f(u_n) \rightarrow f(u)$ strongly in $L^{p'}(0, T; V^*)$.

Note that these assumptions guarantee the continuity of $\tilde{S}$ corresponding to nonconvex energies. In particular, one may prove the following:

Theorem 6.1. *Let assumptions **(A1)**–**(A4)**, **(A5')**, and **(A6)** be satisfied. Assume either that φ^2 is of class C^1 in V or that V is a Hilbert space. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the system (19)–(21) admits a solution u_ε . Moreover, there exists a sequence $\varepsilon_n \rightarrow 0$ such that $u_{\varepsilon_n} \rightarrow u$ weakly in $L^m(0, T; X) \cap W^{1,p}(0, T; V)$ and strongly in $C([0, T]; V)$ and the limit u solves (16)–(18).*

Proof. Let us start with the case $\varphi^2 = 0$, i.e., $\phi = \varphi^1$. Our proof follows the scheme of the proof of Theorem 2.4 for convex energies given in Section 3. Let $\tilde{S}: L^{p'}(0, T; V^*) \rightarrow L^{p'}(0, T; V^*)$ be defined as above. Fix $v \in L^{p'}(0, T; V^*)$ and let u be the global minimizer of $I_{\varepsilon, v}$. Then, $f(u) = \tilde{S}(v) \in L^{p'}(0, T; V^*)$. Following the procedure of Section 3.1.1, one can prove that u is uniformly bounded in $W^{1,p}(0, T; V) \cap L^m(0, T; X)$ in terms of $\|v\|_{L^{p'}(0, T; V^*)}$. This fact together with the (weak-strong) continuity assumption on f in **(A5')** ensures both continuity and compactness of the map $\tilde{S}$. Let now $v \in L^{p'}(0, T; V^*)$, $\alpha \in (0, 1]$, and $u = \operatorname{argmin} I_{\varepsilon, v}$ be such that

$$v = \alpha \tilde{S}(v) (= \alpha f(u)).$$

Then, u solves system (22)–(24) with $f(u)$ replaced by $\alpha f(u)$. By testing equation (22) with u' and proceeding as for (32)–(33), we get, for $t \in (0, T]$

$$\begin{aligned} \int_0^T |u'|_V^p &\leq C + C \int_0^T |\alpha f(u)|_{V^*}^{p'}, \\ |u(t)|_V^p + \phi(u(t)) &\leq C + C \int_0^t |\alpha f(u)|_{V^*}^{p'} + C \int_0^t |u|_V^p + \varepsilon \psi^*(\xi(t)). \end{aligned}$$

Using the growth assumption on f in **(A5')**, we obtain

$$\int_0^T |u'|_V^p \leq C + C \int_0^T (|u|_V^p + \phi(u)) , \quad (67)$$

$$|u(t)|_V^p + \phi(u(t)) \leq C + C \int_0^t (|u|_V^p + \phi(u)) + \varepsilon \psi^*(\xi(t)). \quad (68)$$

Thanks to Gronwall's Lemma, we get

$$|u(t)|_V^p + \phi(u(t)) \leq C + \varepsilon \psi^*(\xi(t)) + C\varepsilon \int_0^t \varepsilon \psi^*(\xi).$$

By substituting the latter into (68), integrating both sides over $[0, T]$, taking the sum with (67), and substituting again the latter into the right hand side, we obtain

$$\int_0^T |u'|_V^p + \int_0^T |u|_V^p + \int_0^T \phi(u) \leq C + C\varepsilon \int_0^T \varepsilon \psi^*(\xi) \leq C + C\varepsilon \int_0^T |u'|_V^p.$$

Here we used estimate (35). For ε sufficiently small this yields a bound of $\|u\|_{W^{1,p}(0,T;V) \cap L^m(0,T;X)}$ uniform in ε and α and hence an uniform bound on $\|f(u)\|_{L^{p'}(0,T;V^*)}$ thanks to **(A5')**. Thus, the set

$$\left\{ v \in L^{p'}(0, T; V^*) : v = \alpha \tilde{S}(v) \text{ for some } \alpha \in [0, 1] \right\}$$

is bounded. The Schaefer's fixed-point theorem allows us to conclude that $\tilde{S}$ has a fixed point v . As a consequence $u = \operatorname{argmin} I_{\varepsilon, f(u)}$ solves system (22)–(24). Being independent of ε the above estimates, together with the continuity assumption in **(A5')**, suffice also to pass to the causal limit $\varepsilon \rightarrow 0$ by following the scheme presented in Section 3.2. The case $\varphi^2 \neq 0$ follows from an argument analogous to the one presented in Section 4. $\square$

7. Applications to nonlinear PDEs

In this section, we apply our abstract theory to a couple of concrete nonlinear PDEs.

7.1. System of doubly-nonlinear parabolic equations

We emphasize again that systems of PDEs may not have any full gradient-flow structure; however, some of them can be reduced to a nonpotential perturbation problem for a (doubly-nonlinear) gradient flow. The following system of doubly-nonlinear differential equations falls within the scope of the abstract theory developed in the present paper (see also [31]). Let Ω be a bounded domain of

$\mathbb{R}^d$ with a smooth boundary $\partial\Omega$ and consider

$$\alpha_i(u'_i) - \Delta_m^{a_i} u_i = g_i(u_1, \dots, u_k) \quad \text{in } \Omega \times (0, T] \quad \text{for } i = 1, \dots, k, \quad (69)$$

$$\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T] \quad \text{for } i = 1, \dots, k, \quad (70)$$

$$u_i|_{t=0} = u_{0i} \quad \text{in } \Omega \quad \text{for } i = 1, \dots, k, \quad (71)$$

where n denotes the outward unit normal vector on $\partial\Omega$ and $\alpha_i: \mathbb{R} \rightarrow \mathbb{R}$ are maximal monotone operators such that there exist $p > 1$ and a positive constant C such that

$$\frac{1}{C}|s|^p - C \leq A_i(s) := \int_0^s \alpha_i(r) dr$$

$$\text{and } |\alpha_i(s)|^{p'} \leq C(|s|^p + 1) \quad \text{for all } s \in \mathbb{R}, \quad i = 1, \dots, k,$$

and $\Delta_m^{a_i}$ is the so-called m -Laplace operator with a coefficient function $a_i: \Omega \rightarrow \mathbb{R}$ given by

$$\Delta_m^{a_i} v = \nabla \cdot (a_i(x) |\nabla v|^{m-2} \nabla v), \quad 1 < m < \infty.$$

Here we also assume $u_{0i} \in W^{1,m}(\Omega)$, $a_i \in L^\infty(\Omega)$ and $\bar{a}_1 \leq a_i(x) \leq \bar{a}_2$ a.e. in Ω for some $\bar{a}_1, \bar{a}_2 > 0$, for all $i = 1, \dots, k$. Finally, we assume $g_i: \mathbb{R}^k \rightarrow \mathbb{R}^k$ to be continuous and to satisfy

$$|g_i(u_1, \dots, u_k)|^{p'} \leq C \left(1 + \sum_{i=1}^k |u_i|^p \right)$$

for some positive constant C . In order to apply the abstract theory, we set $V = (L^p(\Omega))^k$, $X = (W^{1,m}(\Omega))^k$, and note that X is compactly embedded in V for all $p < m^* := md/(d-m)^+ \leq \infty$. We define

$$\psi(u) = \sum_{i=1}^k \int_{\Omega} A_i(u_i) \quad \text{for } u \in V$$

and

$$\phi(u) = \frac{1}{m} \sum_{i=1}^k \int_{\Omega} a_i |\nabla u_i|^m \quad \text{for all } u \in X. \quad (72)$$

Then **(A3)** holds. Indeed, we find that $|u|_X^m \leq C\phi(u) + |u|_{L^m(\Omega)}^m$.

In case $p \geq m$, **(A3)** follows immediately by using $|\cdot|_{L^m(\Omega)} \leq C|\cdot|_{L^p(\Omega)} = C|\cdot|_V$.

In case $p < m$, by interpolation and the Sobolev inequality, we have that

$$|u|_{L^m(\Omega)} \leq C|u|_{L^p(\Omega)}^\theta |u|_{L^{m^*}(\Omega)}^{1-\theta} \leq |u|_V^\theta |u|_X^{1-\theta}$$

for some $\theta \in (0, 1)$ and $C \geq 0$. Hence, by Young's inequality,

$$\frac{1}{2}|u|_X^m \leq C\phi(u) + C|u|_V^m,$$

and hence, (A3) follows. On the other hand, one can easily check (A4). Set

$$f_i(u) = g_i(u_1, \dots, u_k) \quad \text{for all } i = 1, \dots, k.$$

Then, $f := (f_1, \dots, f_k)$ satisfies assumption (A5).

We refer the reader to Section 7 of [10] for checking that assumptions (A1)–(A4) are satisfied. Thus, by applying Theorem 2.4, we prove the following:

Theorem 7.1. *Let $1 < m < \infty$, $1 < p < m^* := md/(d-m)^+ \leq \infty$, and let the above assumptions be satisfied. Then, for every $\varepsilon > 0$ sufficiently small there exists (at least) one solution $\{u_{i\varepsilon}\}_i$ to the elliptic-in-time regularized equation*

$$\begin{aligned} -\varepsilon(\alpha_i(u'_i))' + \alpha_i(u'_i) - \Delta_m^{a_i} u_i &= g_i(u_1, \dots, u_k) && \text{in } \Omega \times (0, T] && \text{for } i = 1, \dots, k, \\ \frac{\partial u_i}{\partial n} &= 0 && \text{on } \partial\Omega \times (0, T] && \text{for } i = 1, \dots, k, \\ u_i(0) &= u_{0i} && \text{in } \Omega && \text{for } i = 1, \dots, k, \\ \varepsilon\alpha_i(u'_i(T)) &= 0 && \text{in } \Omega && \text{for } i = 1, \dots, k. \end{aligned}$$

Moreover, $u_{i\varepsilon} \rightarrow u_i$ strongly in $C([0, T]; L^p(\Omega))$ for each $i = 1, 2, \dots, k$, and the limit $\{u_i\}_i$ solves system (69)–(71). As for the Dirichlet problem (namely, the Neumann boundary condition is replaced by $u_i|_{\partial\Omega} = 0$), all the assertions above hold true.

Remark 7.2. Nonlinear equations such as (69)–(71) are also treated only for $p \geq 2$ in [2], where a perturbation theory for doubly-nonlinear abstract equation is developed in a framework based on the Gel'fand triplet, $V \hookrightarrow H \equiv H^* \subset V^*$. Indeed, the assumption $p \geq 2$ stems from the triplet, and it cannot be removed in the framework. On the other hand, our abstract theory is developed without using such a triplet, and therefore, the case $1 < p < 2$ can also be handled. $\square$

7.2. Biharmonic equation

The abstract theory developed in the current paper can be also applied to the quadratic dissipation potential $\psi(u) = \frac{1}{2}|u|_V^2$ in a Hilbert space V , and then, (1) reads

$$u' + \partial\phi(u) = f(u). \quad (73)$$

The WED approach to nonpotential perturbation problems (73) has been developed in [31], where equation (73) is formulated in a (single) Hilbert space setting. On the other hand, the following example does not fall within the scope of the theory of [31]:

$$u' + (\Delta)^2 u = \beta \cdot \nabla u \quad \text{in } \Omega \times (0, T], \quad (74)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (75)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (76)$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad (77)$$

where n denotes the outward normal on $\partial\Omega$. Indeed, the nonpotential term $f(u) := \beta \cdot \nabla u$ in the right-hand side is not well-defined on the whole of the Hilbert space $H = L^2(\Omega)$. However, this obstacle can be overcome in the current setting. Indeed, by following the approach presented in Section 6, we just require f to be defined over the effective domain of the energy potential $X = D(\varphi^1)$ (cf. **(A5')**). To apply our theory, we assume that Ω is a bounded subset of $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ with $\frac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$, and $\beta \in L^\infty(\Omega, \mathbb{R}^d)$. Moreover, we set two spaces $V = L^2(\Omega)$, $X = H^2(\Omega) \cap H_0^1(\Omega)$ and the energy functional

$$\phi(u) = \begin{cases} \int_{\Omega} \frac{|\Delta u|^2}{2} & \text{if } u \in H^2(\Omega) \cap H_0^1(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, set $\psi(u) = \frac{1}{2}|u|_V^2$, and $f: X \rightarrow V^*$ defined by $f(u) = \beta \cdot \nabla u$ and $p = m = 2$. Note that the map f satisfies assumption **(A5')**. Indeed it is straightforward to check the growth condition (66). Furthermore, thanks to the compact embeddings $X \hookrightarrow H^1(\Omega) \hookrightarrow V$ and to Aubin-Lions-Simon's compactness lemma (see [50]), we have that the space $L^2(0, T; X) \cap H^1(0, T; V)$ is compactly embedded into $L^2(0, T; H^1(\Omega))$. Thus, for all $u_n \rightarrow u$ weakly in $L^2(0, T; X) \cap H^1(0, T; V)$ there exists a (not relabeled) subsequence such that $\nabla u_n \rightarrow \nabla u$ strongly in $L^2(0, T; L^2(\Omega))$. Recalling that $V = L^2(\Omega) = V^*$ we then have that $f(u_n) = \beta \cdot \nabla u_n \rightarrow \beta \cdot \nabla u = f(u)$ strongly in $L^2(0, T; V^*)$. Finally, by uniqueness of the limit the convergence holds true for the whole sequence. Thus, assumption **(A5')** is satisfied. By applying the preceding abstract theory, and more precisely Theorem 6.1, we obtain the following result:

Theorem 7.3. *Let the assumptions mentioned above be satisfied. Then, for every $\varepsilon > 0$ sufficiently small, there exists (at least) one solution u_ε to equation*

$$\begin{aligned} -\varepsilon u'' + u' + (\Delta)^2 u &= \beta \cdot \nabla u && \text{in } \Omega \times (0, T], \\ u &= 0 && \text{on } \partial\Omega \times (0, T], \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega \times (0, T], \\ u(0) &= u_0 && \text{in } \Omega, \\ \varepsilon u'(T) &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, $u_\varepsilon \rightarrow u$ strongly in $C([0, T]; L^2(\Omega))$ and the limit u is a solution of equation (74)–(77).

Here, we dealt with a linear equation just for simplicity. However, we remark that doubly-nonlinear variants of (74)–(77) and the case where $\beta \cdot \nabla$ is replaced

by a nonlinear gradient term also fall within the framework of the preceding abstract results.

A. Appendix

A.1. Moreau-Yosida regularization with p -modulus duality mappings

In this section, we collect the definition and some properties of the Moreau-Yosida regularization of convex functionals defined on a Banach space. To this aim, we refer to [26], [19], [13].

Let V be a strictly convex, reflexive, and separable Banach space such that its dual space V^* is strictly convex. For every $p \in (1, \infty)$, we define the p -modulus duality mapping $F: V \rightarrow V^*$ by [19]

$$F(\cdot) := \partial_V \left(\frac{|\cdot|_V^p}{p} \right).$$

Note that, as $|\cdot|_V^p$ is strictly convex, $F(0) = \{0\}$. Since V^* is strictly convex, we have that F is single-valued (see, e.g., [26], [28]). Moreover,

$$\langle F(u), u \rangle_V = |u|_V^p = |F(u)|_{V^*}^{p'}. \quad (78)$$

Given a maximal monotone graph A of $V \times V^*$, we define the *resolvent* $J_\lambda: V \rightarrow D(A)$ (with respect to F) by

$$J_\lambda u = u_\lambda \quad \stackrel{\text{def}}{\iff} \quad F\left(\frac{u_\lambda - u}{\lambda}\right) + A(u_\lambda) \ni 0 \quad (79)$$

for each $u \in V$ and the *Yosida approximation* $A_\lambda: V \rightarrow V^*$ (with respect to F) by

$$A_\lambda(u) := F\left(\frac{u - J_\lambda u}{\lambda}\right).$$

Hence, by (78),

$$|A_\lambda(u)|_{V^*}^{p'} = \left| \frac{u - J_\lambda u}{\lambda} \right|_V^p. \quad (80)$$

Let now $\phi: V \rightarrow [0, \infty]$ be a proper, lower semicontinuous, and convex functional. For simplicity, we assume $0 \in D(\phi)$. Define the *Moreau-Yosida regularization* of ϕ , also known as *Moreau envelope* (see [26] for an analogous definition with p replaced by 2), by

$$\phi_\lambda(u) = \inf_{v \in V} \left\{ \frac{\lambda}{p} \left| \frac{u - v}{\lambda} \right|_V^p + \phi(v) \right\} \quad \text{for } u \in V. \quad (81)$$

Note that, for every $u \in V$, the subdifferential of the convex functional $v \mapsto \frac{\lambda}{p} \left| \frac{u - v}{\lambda} \right|_V^p + \phi(v)$ is the operator $v \mapsto F\left(\frac{u - v}{\lambda}\right) + \partial_V \phi(v)$ from V into V^* . Then,

the infimum in (81) is achieved at $J_\lambda u$, where J_λ is the resolvent for $\partial_V \phi$, thus, the following relation is satisfied

$$F\left(\frac{J_\lambda u - u}{\lambda}\right) + \partial_V \phi(J_\lambda u) \ni 0. \quad (82)$$

In particular, the subdifferential of the Moreau-Yosida regularization of ϕ corresponds to the Yosida approximation of $\partial_V \phi$. Moreover, by testing relation (82) with $J_\lambda u$, we see that

$$\left|\frac{u - J_\lambda u}{\lambda}\right|_V^p \leq \frac{2}{\lambda} \phi(0) + 2C \frac{|u|_V^p}{\lambda^p}. \quad (83)$$

Thus, thanks to identity (80),

$$|\partial_V \phi_\lambda(u)|_{V^*}^{p'} \leq \frac{2}{\lambda} \phi(0) + 2C \frac{|u|_V^p}{\lambda^p}. \quad (84)$$

Hence $J_\lambda: V \rightarrow V$ and $\partial_V \phi_\lambda: V \rightarrow V^*$ turn out to be bounded operators (for each λ fixed).

We are now ready to prove demicontinuity of $\partial_V \phi_\lambda$.

Lemma A.1. *For every fixed $\lambda > 0$, $\partial_V \phi_\lambda$ is demicontinuous, i.e., for every sequence $u_n \rightarrow u$ strongly in V , it holds that $\partial_V \phi_\lambda(u_n) \rightarrow \partial_V \phi_\lambda(u)$ weakly in V^* .*

Proof. Let $u_n \rightarrow u$ in V . Let $v_n = \partial_V \phi_\lambda(u_n)$. Since $F((J_\lambda u_n - u_n)/\lambda) + v_n = 0$, it follows that

$$\begin{aligned} & \left\langle (J_\lambda u_n - u_n) - (J_\lambda u_m - u_m), F\left(\frac{J_\lambda u_n - u_n}{\lambda}\right) - F\left(\frac{J_\lambda u_m - u_m}{\lambda}\right) \right\rangle_V + \\ & + \langle J_\lambda u_n - J_\lambda u_m, v_n - v_m \rangle_V = \left\langle u_m - u_n, F\left(\frac{J_\lambda u_n - u_n}{\lambda}\right) - F\left(\frac{J_\lambda u_m - u_m}{\lambda}\right) \right\rangle_V. \end{aligned}$$

As a consequence of the strong convergence $u_n \rightarrow u$ and of the boundedness $|x - J_\lambda x|_V^p \leq C(\lambda)(1 + |x|_V^p)$, we observe that

$$\lim_{m,n \rightarrow \infty} \left\langle u_m - u_n, F\left(\frac{J_\lambda u_n - u_n}{\lambda}\right) - F\left(\frac{J_\lambda u_m - u_m}{\lambda}\right) \right\rangle_V = 0.$$

Thus, as $\partial_V \phi_\lambda$ and F are monotone and $\partial_V \phi_\lambda(u) \in \partial_V \phi(J_\lambda u)$, we get

$$\lim_{m,n \rightarrow \infty} \left\langle (J_\lambda u_n - u_n) - (J_\lambda u_m - u_m), F\left(\frac{J_\lambda u_n - u_n}{\lambda}\right) - F\left(\frac{J_\lambda u_m - u_m}{\lambda}\right) \right\rangle_V = 0$$

and $\lim_{m,n \rightarrow \infty} \langle J_\lambda u_n - J_\lambda u_m, v_n - v_m \rangle_V = 0$. From estimates (83) and (84), there exists a (not-reabeled) subsequence such that $J_\lambda u_n \rightarrow \tilde{u}$ weakly in V ,

$v_n \rightarrow v$ weakly in V^* and $F((J_\lambda u_n - u_n)/\lambda) \rightarrow w$ weakly in V^* for some $\tilde{u} \in V$ and $v, w \in V^*$. Then, by [13, Lemma 1.3, pp. 42], one can conclude that $v \in \partial_V \phi(\tilde{u})$ and $F((\tilde{u} - u)/\lambda) + v = 0$. Thus, $\tilde{u} = J_\lambda u$, and hence, $v = \partial_V \phi_\lambda(u)$. Since the limits are unique, $J_\lambda u_n \rightarrow J_\lambda u$ weakly in V and $\partial_V \phi_\lambda(u_n) \rightarrow \partial_V \phi_\lambda(u)$ weakly in V^* along the whole sequences u_n and $\partial_V \phi_\lambda(u_n)$, respectively. Thus, $\partial_V \phi_\lambda$ turns out to be demicontinuous. $\square$

A.2. Auxiliary theorems

For the reader's convenience, we collect here some known results which we used in the analysis.

Theorem A.2. (Gronwall's Lemma) *Let $\alpha, u \in L^1(0, T)$ and $B > 0$. Assume that*

$$u(t) \leq \alpha(t) + \int_0^t B u(s) ds \quad \text{for a.a. } t \in (0, T). \quad (85)$$

Then

$$u(t) \leq \alpha(t) + \int_0^t B \alpha(s) \exp(B(t-s)) ds \quad \text{for all } t \in [0, T]. \quad (86)$$

Proof. Define $v(t) := \exp(-Bt) \int_0^t B u(s) ds$. Then, $v \in W^{1,1}(0, T)$, $v(0) = 0$, and

$$v'(t) = B \exp(-Bt) \left(u(t) - \int_0^t B u(s) ds \right) \leq B \exp(-Bt) \alpha(t) \quad \text{for a.a. } t \in (0, T).$$

Thus, integrating over $(0, t)$, we get

$$\exp(-Bt) \int_0^t B u(s) ds = v(t) \leq \int_0^t B \exp(-Bs) \alpha(s) ds,$$

which yields

$$\int_0^t B u(s) ds \leq \int_0^t B \exp(B(t-s)) \alpha(s) ds. \quad (87)$$

By substituting (87) into (85), we get (86). $\square$

We now give a proof of

Lemma A.3. *Under (A3) and (A4), it holds that $D(\varphi^1) = D(\partial_X \varphi_X^1) = X$. Moreover, φ_X^1 is continuous in X .*

Proof. One readily observes that φ_X^1 is proper and convex. We first show the lower semicontinuity of φ_X^1 in X . Let $\lambda \in \mathbb{R}$ and let $u_n \in [\varphi_X^1 \leq \lambda] := \{w \in X : \varphi_X^1(w) \leq \lambda\}$ be such that $u_n \rightarrow u$ strongly in X . Then, it follows that

$$\lambda \geq \liminf_{n \rightarrow \infty} \varphi_X^1(u_n) = \liminf_{n \rightarrow \infty} \varphi^1(u_n) \geq \varphi^1(u) = \varphi_X^1(u)$$

by the lower semicontinuity of φ^1 in V and the continuous embedding $X \hookrightarrow V$. Hence, we have $u \in [\varphi_X^1 \leq \lambda]$. Therefore, φ_X^1 is lower semicontinuous in X .

We next claim that $D(\partial_X \varphi_X^1)$ is closed in X . Indeed, let $u_n \in D(\partial_X \varphi_X^1)$ and $u \in X$ be such that $u_n \rightarrow u$ strongly in X . Then, there exists a sequence $\{\eta_n\}$ in X^* such that $\eta_n \in \partial_X \varphi_X^1(u_n)$, and moreover, **(A4)** implies $|\eta_n|_{X^*} \leq C$. Hence, we deduce that $\eta_n \rightarrow \eta$ weakly in X^* up to a (not-relabeled) subsequence. From the demicontinuity of $\partial_X \varphi_X^1$, we obtain $u \in D(\partial_X \varphi_X^1)$. Therefore, $D(\partial_X \varphi_X^1)$ is closed.

Now, we are ready to show that $D(\varphi^1) = X$. First, note that **(A3)** means $D(\varphi^1) \subset X$. Hence it suffices to show the inverse relation. Let $u \in X$. Then, it holds that $F_X(J_\lambda u - u) + \lambda \partial_X \varphi_X^1(J_\lambda u) \ni 0$, where $J_\lambda: X \rightarrow D(\partial_X \varphi_X^1)$ is the resolvent of $\partial_X \varphi_X^1$ and F_X is the duality pairing between X and X^* . Fix $v_0 \in D(\varphi_X^1)$. Test the equation above by $J_\lambda u - v_0$ and use the definition of subdifferential to get $|J_\lambda u|_X \leq C$. Moreover, test the same equation by $J_\lambda u - u$ and apply **(A4)** to derive

$$|J_\lambda u - u|_X \leq \lambda \ell_4 (|J_\lambda u|_V)^{1/m'} (|J_\lambda u|_X^m + 1)^{1/m'} \leq C\lambda \rightarrow 0,$$

which implies that $u \in D(\partial_X \varphi_X^1)$ by the closedness. Thus, we conclude that $X \subset D(\partial_X \varphi_X^1)$, which also yields $D(\varphi^1) = X$.

Finally, combining the lower semicontinuity of φ_X^1 in X and the fact that the interior of $D(\varphi_X^1)$ coincides with X , we deduce by [13, Proposition 2.2] that φ_X^1 is continuous in X . $\square$

Finally, we recall Schaefer's fixed-point theorem (see, e.g., [23]) below.

Theorem A.4. (Schaefer's fixed-point theorem) [23, Thrm. 4, Chap. 9]

Let B be a Banach space and let $S: B \rightarrow B$ be continuous and compact. Suppose that $\{u \in B : u = \alpha S(u) \text{ for } \alpha \in [0, 1]\}$ is bounded. Then, S has a fixed point.

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