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On Super Weak Compactness of Subsets and its Equivalences in Banach Spaces

Lixin Cheng[*]
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
lxcheng@xmu.edu.cn

Qingjin Cheng[†]
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
qjcheng@xmu.edu.cn

Sijie Luo
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

Kun Tu
School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

Jichao Zhang
School of Science, Hubei University of Technology, Wuhan 430068, China

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Analogous to weak compactness of subsets of Banach spaces and to property of subsets in super reflexive spaces, the purpose of this paper is to discuss super weak compactness of both convex and nonconvex subsets in Banach spaces. As a result, we give three characterizations of super weakly compact sets: The first one is Grothendiek's type theorem; the second one is James' type characterization and the last one is super Banach-Saks property. We also show that super weak compactness, finite index property and finite dual index property of a closed convex set are actually equivalent. Therefore, eleven notions and properties eventually coincide for a closed bounded convex set. We also present some characterizations of uniformly weakly null sequences. These are done by localizing some basic properties of ultrapowers and using some geometric procedures of Banach spaces.

Keywords: Super weakly compact set, super Banach-Saks set, finite index, finite dual index, ultraproduct, Banach space.

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[†]Corresponding author. Support in partial by NSFC, grant no. 11471271, and by NSFF, grant no. 2017J0101.

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1. Introduction

It is well known that, because of their nice properties, super reflexive Banach spaces have played a very important role in the geometry of Banach spaces and its applications. Recall that a Banach space is said to be super reflexive provided every Banach space finitely representable in it is reflexive. In 1972, Enflo [17] presented the following celebrated renorming theorem: Every super reflexive Banach space can be renormed to be uniformly convex. Pisier ([36], 1975) further showed that every super reflexive Banach space is renormed to have a convexity modulus of power type. Another remarkable result is due to James [26]: A Banach space is super reflexive if and only if it does not admit finite tree property, i.e. for every $\varepsilon > 0$ the closed unit ball does not admit (n, ε)-tree for all sufficiently large $n \in \mathbb{N}$.

As localizations of super reflexivity of Banach spaces, analogous to the weak compactness of subsets in Banach spaces, a number of notions have been introduced and studied since middle 70's of the last century. Beauzamy [4, 1976] introduced the notion of uniformly convexifying operators $T \in B(X, Y)$ for Banach spaces X and Y: $T(B_X)$ does not admit finite tree property. He showed that T is uniformly convexifying if and only if X admits an equivalent norm $\|\cdot\|$ such that for every $\varepsilon > 0$ there is $\delta > 0$ so that $\|T(u - v)\| < \varepsilon$ whenever $u, v \in X$ satisfy $\|u\| = \|v\| = 1$ and $\|u + v\| > 2 - \delta$. Since then, many different and generalized notions appeared (see, for instance, [6, 23, 33, 35]).

Raja [38, 2008] introduced the notion of finitely dentable map, which can be understood as a generalization of Beauzamy's uniformly convexifying operator to nonlinear case. He showed that an operator $T \in B(X, Y)$ is uniformly convexifying if and only if $T(B_X)$ is finitely dentable in Y; A bounded closed convex set $C \subset Y$ is finitely dentable if and only if $C \subset T(B_X)$ for some Banach space X and uniformly convexifying operator $T \in B(X, Y)$. Moreover, he gave Cepedello's theorem [9] (see, also, [7, p. 94]) a localization: A bounded closed convex set C is finitely dentable if and only if it has uniform Δ-convex approximating property, i.e. it satisfies that every Lipschitz function on it can be uniformly approximated by a sequence of differences of convex Lipschitz functions. We also call a finitely dentable set admitting finite-index property.

For the study of renorming properties of smoothness, Fabian, Montesinos and Zizler [19, 2009] introduced the notion of finite ε-dual index of bounded nonempty subsets. We also call a bounded subset admitting finite ε-dual index possessing finite-dual-index property. Among many other things, they showed a bounded subset M of a Banach space X has finite ε-dual index if and only if X possesses an equivalent M-uniformly Gateaux smooth norm.

In 2010, through a localized setting of the notion of finite representability, Cheng, Cheng, Wang and Zhang [10] introduced the notion of super weak compactness for closed bounded convex sets. Analogous to James' characterizations of super reflexive Banach spaces [26], they showed that a bounded closed convex subset C is not super weakly compact if and only if C admits finite tree property, which is equivalent to the following uniform finite separating property: There exists $\theta > 0$ satisfying that for all $n \in \mathbb{N}$, there is a sequence $\{x_i\}_{i=1}^n \subset C$ such that

$$\operatorname{dist}(\operatorname{co}\{x_1, \cdots, x_k\}, \operatorname{co}\{x_{k+1}, \cdots, x_n\}) \geq \theta, \quad \text{for all } 1 \leq k < n. \tag{1}$$

They finally gave Enflo-James' theorem a localized version: A bounded closed convex subset C of a Banach space X is super weakly compact if and only if there is an equivalent norm on X such that it is uniformly convex on C. Recently, Cheng, Luo and Zhou [12] studied properties of the normed semigroup generated by super weakly compact convex sets of a Banach space.

The purpose of this paper is to study properties of (non-convex) relatively super weakly compact sets, and to investigate the relationships of notions for convex sets, such as super weak compactness, uniform convexifiability, uniform smoothing, finite-index property, finite-dual-index property and super Banach-Saks property. As a result, we give three characterizations of relatively super weakly compact sets: The first one is Grothendiek's type theorem; the second one is James' type characterization; and the last one is super Banach-Saks property. We also show that the notions and properties previously mentioned are actually equivalent for a bounded closed convex set.

The paper is organized as follows. In the next section, we first give the notion of finite representability a localized setting. Because ultraproducts and ultrapowers of subsets play a crucial role in discussing relative super weak compactness of non-convex subsets, we then present some relationships between a subset and its ultrapower, which are generalizations of the corresponding properties of Banach spaces. In the third section, we show that relative super weak compactness of subsets is invariant under operations of product, continuous linear mapping (hence, linear isomorphism), ultrapower and n-convex combinations for all $n \in \mathbb{N}$. In the fourth section, we give two characterizations of relatively super weakly compact sets: One is Grothendiek's type theorem; and the other is James' type characterization of non super weakly compact sets. In the fifth section, we show that for a bounded closed convex set, super weak compactness, finite-index and finite-dual-index properties are actually equivalent. In summary of this section, we conclude that for a bounded closed convex set, the following ten notions and properties coincide: super weak compactness, finite-index property, finite-dual-index property, uniform convexifiability, uniform smoothing, non-uniform finite bi-orthogonality, non-uniform finite separating property, non-finite-tree property, Δ-convex function approximating property and being the closure of the image of the closed unit ball of a Banach space under a uniformly convexifying operator. In the sixth section, we discuss Banach-Saks subsets of Banach spaces. With the aim of Rosenthal's result [39] and Mercourakis' result [34], we show that super Banach-Saks property of a bounded set is equivalent to its relative super weak compactness; and give some characterizations of "uniformly weakly convergent sequences".

In this paper, the ultraproduct construction shall play a crucial role in proofs of properties of super weakly compact sets. It is a common thread linking properties of a subset A and subsets which are finitely representable in A. For more information on ultraproducts in Banach spaces, we refer the reader to Heinrich [24].

2. Localization of ultrapowers and finite representability

The notion of finite representability is localized to convex subsets in [10]. In this section, we first extend this notion to non-convex subsets in Banach spaces.

An *n-simplex* S of a linear space is the convex hull $\mathrm{co}\{x_0, x_1, \cdots, x_n\}$ of $n+1$ affinely independent vectors $x_0, x_1, \cdots, x_n \in X$, i. e. $x_1 - x_0, \cdots, x_n - x_0$ are linearly independent. A *0-simplex* is just a singleton $\{x_0\}$. If no confusion arises, we call an n-simplex simply “simplex”. We use $\mathrm{aff}(A)$ ($\mathrm{co}(A)$, resp.) to denote the affine (convex, reps.) hull of A, i.e.

$$\mathrm{aff}(A) = \left\{ \sum_{j=1}^{n} \alpha_j x_j : x_j \in A,\ \alpha_j \in \mathbb{R}, \sum_{j=1}^{n} \alpha_j = 1,\ 1 \le j \le n,\ n \in \mathbb{N} \right\}.$$

Definition 2.1. Let $A \subset X$ and $B \subset Y$ be two subsets of Banach spaces X and Y.

(i) We say that B is *(affinely) finitely representable in A* if for every $\varepsilon > 0$ and for every simplex S_B with vertices in B there exist a simplex S_A with vertices in A, and an affine mapping $T\colon \mathrm{aff}(S_B) \to \mathrm{aff}(S_A)$ such that $T(S_B) = S_A$ and such that

$$(1-\varepsilon)\|x-y\| \le \|Tx - Ty\| \le (1+\varepsilon)\|x-y\|,\ \forall x, y \in \mathrm{aff}(S_B). \tag{2}$$

(ii) B is said to be *linearly finitely representable in A* if for every $\varepsilon > 0$ and for every simplex S_B with vertices in B there exist a simplex S_A with vertices in A, and a linear mapping $T\colon \mathrm{span}(S_B) \to \mathrm{span}(S_A)$ such that $T(S_B) = S_A$ and such that (2) holds on $\mathrm{span}(S_B)$.

(iii) B is *crudely finitely representable in A* if there is a constant $\beta \ge 1$ so that for every simplex S_B with vertices in B there exist a simplex S_A with vertices in A, and an affine mapping $T\colon \mathrm{aff}(S_B) \to \mathrm{aff}(S_A)$ such that $T(S_B) = S_A$ and such that

$$\beta^{-1}\|x-y\| \le \|Tx - Ty\| \le \beta\|x-y\|,\ \forall x, y \in \mathrm{aff}(S_B). \tag{3}$$

We can analogously define crudely linear finite representability of a set B in a set A.

For an abstract set Γ, $c_0(\Gamma)$ will denote the Banach space of all scalar valued functions f on Γ which vanish at infinity (i.e., for which $\{\gamma \in \Gamma : |f(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$) with the maximum norm $\|f\|_\infty = \max\{|f(\gamma)| : \gamma \in \Gamma\}$.

Example 2.2. Let $A = \{e_j\}$ be the standard unit vector basis of c_0. Then every set B of a Banach space which is linearly finitely representable in A is equivalent to the standard unit vector basis of $c_0(B)$.

Proof. We first show that every nonempty linearly independent subset $F \subset B$ is equivalent to the standard unit vector basis of $c_0(F)$. Indeed, it suffices to show that for any subset $F = \{y_1, \cdots, y_n\} \subset B$ of n linearly independent elements, there is a linear isometry

$$T\colon V \equiv \text{span}\{y_1, \cdots, y_n\} \to U \equiv \text{span}\{e_1, \cdots, e_n\}$$

such that $T\{y_1, \cdots, y_n\} = \{e_1, \cdots, e_n\}$. We write $E = \{e_1, \cdots, e_n\}$. Note that $S_F \equiv \text{co}\{y_1, \cdots, y_n\}$ is an $(n-1)$-simplex with linearly independent vertices $y_1, \cdots, y_n \in B$. For every $\varepsilon > 0$, there exist an $(n-1)$-simplex S_ε with vertices in $\{e_j\}_{j\in\mathbb{N}}$ and a linear mapping $T_\varepsilon\colon V \to U_\varepsilon \equiv \text{span}(S_\varepsilon)$ such that $T_\varepsilon(S_F) = S_\varepsilon$ and

$$(1-\varepsilon)\|x\| \leq \|T_\varepsilon(x)\| \leq (1+\varepsilon)\|x\|, \ \forall x \in V. \tag{4}$$

Let T^ε be a linear isometry from the $(n-1)$-simplex S_ε to $\text{co}\{e_1, \cdots, e_n\}$. Then the linear operator $T^\varepsilon \circ T_\varepsilon\colon V \to U \equiv \text{span}\{e_1, \cdots, e_n\}$ also satisfies the inequality (4) on V. Let T be a cluster point of $\{T^\varepsilon \circ T_\varepsilon\}$ as $\varepsilon \to 0^+$. Then $T\colon V \to U$ is a linear isometry and with $T(F) = E$. In particular, this gives that $\|b\| = 1$ for all $b \in B$.

Next, we show that B is linearly independent. Suppose, to the contrary, that there exists $b_0 \in B$ such that $b_0 \in \text{span}(B \setminus \{b_0\})$. Then there is a minimal linearly independent (finite) set $F \subset B \setminus \{b_0\}$ such that $b_0 \in \text{span}(F)$. Enumerate $F = \{b_1, b_2, \cdots, b_n\}$. Then $n \geq 2$; otherwise, we have $b_0 = -b_1$. For the 1-simplex $\text{co}\{b_0, b_1\}$, by the fact we have just proven, there is a linear isometry $T\colon \text{span}\{b_0, b_1\} \to \text{span}\{e_1, e_2\}$ such that $T\{b_0, b_1\} = \{e_1, e_2\}$. Consequently, $T(0) = T(b_0 + b_1) = e_1 + e_2$, a contradiction. Let $b_0 = \sum_{j=1}^n \alpha_j b_j$. By the fact we have just proven again, there is a linear isometry $T\colon \text{span}(F) \to c_0$ with $T\{b_1, \cdots, b_n\} = \{e_1, e_2, \cdots, e_n\}$. Then

$$\min\{|\alpha_j|\colon 1 \leq j \leq n\} > 0, \tag{5}$$

and

$$1 = \|b_0\| = \|T(b_0)\| = \|\sum_{j=1}^n \alpha_j T(b_j)\| = \max\{|\alpha_j| : 1 \leq j \leq n\}. \tag{6}$$

Since for all $1 \leq j \leq n$ the set $\{b_0, b_j\}$ is linearly independent, there is a liner isometry $S_j\colon \text{span}\{b_0, b_j\} \to c_0$ with $S_j\{b_0, b_j\} = \{e_1, e_2\}$. Thus

$$\|b_j - b_0\| = \|S_j(b_j - b_0)\| = \|e_1 - e_2\| = 1,$$
$$\|b_j + b_0\| = \|S_j(b_j + b_0)\| = \|e_1 + e_2\| = 1$$

for all $1 \le j \le n$. Therefore

$$1 = \|b_j - b_0\| = \|T(b_j) - T(b_0)\| = \max\{|\alpha_j - 1|,\ |\alpha_i|, i \neq j\}. \tag{7}$$

This together with (5) gives that $0 < \alpha_j \le 1$ for all j, and hence $\alpha_{j_0} = 1$ for some j_0. This is impossible since

$$1 = \|b_0 + b_{j_0}\| = \|T(b_0 + b_{j_0})\| = \max\{2, \alpha_i, i \neq j_0\} = 2 \tag{8}$$

for all $1 \le j < i \le n$. □

A *filter* $\mathcal{F}$ is a collection of subsets of a set Ω satisfying (i) $\emptyset \notin \mathcal{F}$; (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$; (iii) $A \in \mathcal{F}$ and $A \subset B \subset \Omega$ entail $B \in \mathcal{F}$. A filter $\mathcal{F}$ is said to be *free* if $\cap\{F \in \mathcal{F}\} = \emptyset$. A filter $\mathcal{U}$ is called an *ultrafilter* if for any $A \subset \Omega$, either $A \in \mathcal{U}$, or, $\Omega \setminus A \in \mathcal{U}$. Let K be a topological space, and $f\colon \Omega \to K$ a function. We say f is *convergent* to some $k \in K$ with respect to a filter $\mathcal{F}$ if for every neighborhood U of k, we have $f^{-1}(U) \in \mathcal{F}$; in this case, we denote $\lim_{\mathcal{F}} f = k$.

We will recall the definition of an ultraproduct (ultrapower) of Banach spaces. For a nonempty set Ω, let $\{X_\omega : \omega \in \Omega\}$ be a collection of Banach spaces. Then their *ultraproduct* is defined by

$$\prod_{\mathcal{U}} X_\omega = (\bigoplus_{\omega \in \Omega} X_\omega)_{\ell_\infty} / \{(x_\omega) : \lim_{\mathcal{U}} \|x_\omega\| = 0\}. \tag{9}$$

$\lim_{\mathcal{U}} \|x_\omega\| = 0$ means for all $\varepsilon > 0$, $\{\omega \in \Omega : \|x_\omega\| < \varepsilon\} \in \mathcal{U}$. Please note that the ultraproduct is a quotient of the ℓ_∞-sum of $\{X_\omega\}$, so its elements are classes of the respective equivalences relation, not the generalized sequences itself. We will use in the sequel the notations $[(x_\omega)]$ (or, simply $\boldsymbol{x} = (\boldsymbol{x}(\omega))$, if no confusion arises) to denote the equivalence class of (x_ω). Thus, for a collection $\{A_\omega \subset X_\omega : \omega \in \Omega\}$ of subsets, its ultraproduct is

$$\prod_{\mathcal{U}} A_\omega = \{[(x_\omega)] : (x_\omega) \in (\bigoplus_{\omega \in \Omega} X_\omega)_{\ell_\infty} : x_\omega \in A_\omega \text{ for } \omega \in \Omega\}. \tag{10}$$

In particular, if $X_\omega = X$, and $A_\omega = A \subset X$ for all $\omega \in \Omega$, then we denote by $A_{\mathcal{U}} = \prod_{\mathcal{U}} A$, the $\mathcal{U}$*-ultrapower* of A. For distinction, we also use $(x, y)_{\mathcal{U}}$ to denote vectors in $(X \times Y)_{\mathcal{U}}$ and $(x_{\mathcal{U}}, y_{\mathcal{U}})$, vectors in $X_{\mathcal{U}} \times Y_{\mathcal{U}}$.

Proposition 2.3. *Let A be a nonempty subset of a Banach space X, and $\mathcal{U}$ a free ultrafilter on some set Ω. Then*

(i) *$A_{\mathcal{U}}$ is finitely representable in A.*

(ii) *Any linearly independent set $\mathbf{B} \subset A_{\mathcal{U}}$ is linearly finitely representable in A.*

Proof. Let $\boldsymbol{x}_0 = [(\boldsymbol{x}_0(\omega))], \cdots, \boldsymbol{x}_n = [(\boldsymbol{x}_n(\omega))] \in A_{\mathcal{U}}$ be $n+1$ vectors. For each j choose an arbitrary representant $(\boldsymbol{x}_j(\omega))$ with $\boldsymbol{x}_j(\omega) \in A$ for all $\omega \in \Omega$. We first show that if $\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n$ are linearly independent then the set

$$U = \{\omega \in \Omega : \boldsymbol{x}_0(\omega), \cdots, \boldsymbol{x}_n(\omega) \text{ are linearly independent}\} \tag{11}$$

belongs to $\mathcal{U}$. Otherwise, $V \equiv \Omega \setminus U \in \mathcal{U}$, and for each $\omega \in V$ the equation $\sum_{j=0}^n \alpha_j(\omega)\boldsymbol{x}_j(\omega) = 0 \in X$ has a nonzero solution $(\alpha_0(\omega), \cdots, \alpha_n(\omega)) \in \mathbb{R}^{n+1}$. Without loss of generality we assume $\sum_{j=0}^n |\alpha_j(\omega)| = 1$. Let $\alpha_j = \lim_{\mathcal{U}} \alpha_j(\omega)$ for $0 \leq j \leq n$. Then $\sum_{j=0}^n |\alpha_j| = 1$ and $\sum_{j=0}^n \alpha_j \boldsymbol{x}_j = \mathbf{0} \in X_{\mathcal{U}}$. This says that $\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n$ are linearly dependent, which contradicts to the hypothesis.

Analogously, we can show that if the vectors $\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n$ are affinely independent then the set

$$U' = \{\omega \in \Omega : \boldsymbol{x}_0(\omega), \cdots, \boldsymbol{x}_n(\omega) \text{ are affinely independent}\} \tag{12}$$

belongs to $\mathcal{U}$. Indeed, by the fact we have just proven, it suffices to note

$$U' = \{\omega \in \Omega : (\boldsymbol{x}_1 - \boldsymbol{x}_0)(\omega), \cdots, (\boldsymbol{x}_n - \boldsymbol{x}_0)(\omega) \text{ are linearly independent}\}.$$

We next prove that if $\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n$ are linearly independent then the set $\{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n\}$ is linearly finitely representable in A. Let U be defined by (11). Then for each $\omega \in U$, $F_\omega \equiv \operatorname{span}\{\boldsymbol{x}_0(\omega), \cdots, \boldsymbol{x}_n(\omega)\}$ is an $n+1$ dimensional subspace of X. By applying an Auerbach basis $\{e_0(\omega), \cdots, e_n(\omega)\}$ of F_ω (i.e. $e_j(\omega)$ $(j = 0, \cdots, n)$ are unit vectors in F_ω satisfying there exist $n+1$ unit vectors $e_j^*(\omega)$ $(j = 0, \cdots, n)$ in X^* such that $\langle e_j^*(\omega), e_i(\omega)\rangle = \delta_{ij}$. Here $\delta_{ij} = 0$, if $i \neq j$; $= 1$, otherwise), we can show that there is a norm $|\cdot|_\omega$ on $\mathbb{R}^{n+1}$ with

$$\|x\|_\infty \leq |x|_\omega \leq \|x\|_1, \; \forall x \in \mathbb{R}^{n+1} \tag{13}$$

such that F_ω is linearly isometric to $\mathbb{R}^{n+1}_\omega \equiv (\mathbb{R}^{n+1}, |\cdot|_\omega)$ (see [13, Lemma 3.3]). For each $\omega \in U$, let $L_\omega \colon F_\omega \to \mathbb{R}^{n+1}_\omega$ be a linear isometry. Since $U \in \mathcal{U}$, Arzelá-Ascoli theorem and (13) entail that there exists a norm $|\cdot|$ on $\mathbb{R}^{n+1}$ such that $\lim_{\mathcal{U}} |\cdot|_\omega = |\cdot|$ uniformly on each bounded subset of $\mathbb{R}^{n+1}$ (see also [13, Lemma 3.4]). Consequently,

$$\mathbf{F} \equiv \operatorname{span}\{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n\} \cong \Pi_{\mathcal{U}} E_\omega \cong \Pi_{\mathcal{U}} G_\omega \cong (\mathbb{R}^{n+1}, |\cdot|), \tag{14}$$

where $E_\omega = F_\omega$, if $\omega \in U$; $= \{0\}$, otherwise; and $G_\omega = \mathbb{R}^{n+1}_\omega$, if $\omega \in U$; $= \{0\}$, otherwise.

For each $\omega \in U$, let $P_\omega \colon \mathbf{F} \to F_\omega$ be defined by $P_\omega(\sum_{j=0}^n \lambda_j \boldsymbol{x}_j) = \sum_{j=0}^n \lambda \boldsymbol{x}_j(\omega)$ for all $\lambda_j \in \mathbb{R}$. It is clear that $\lim_{\mathcal{U}} \|P_\omega(\boldsymbol{x})\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbf{F}$, and $P_\omega(\mathbf{S}) = S_\omega$, where $\mathbf{S} = \operatorname{co}\{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n\}$ and $S_\omega = \operatorname{co}\{\boldsymbol{x}_0(\omega), \cdots, \boldsymbol{x}_n(\omega)\}$. Therefore, $\lim_{\mathcal{U}} |L_\omega \circ P_\omega(\boldsymbol{x})|_\omega = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbf{F}$, which implies that $L_\omega \circ P_\omega$ converges to some linear isometry $\mathbf{F} \to (\mathbb{R}^{n+1}, |\cdot|)$ along $\mathcal{U}$. Thus, P_ω converges to some linear isometry $P \colon \mathbf{F} \to X$ along $\mathcal{U}$. Consequently, for every $\varepsilon > 0$, there exists $\omega \in U$ such that

$$(1 - \varepsilon)\|\boldsymbol{x}\| \leq \|P_\omega(\boldsymbol{x})\| \leq (1 + \varepsilon)\|\boldsymbol{x}\|, \; \text{ for all } \boldsymbol{x} \in \mathbf{F} \tag{15}$$

with $P_\omega(\mathbf{S}) = S_\omega$. Therefore, we have shown that every linearly independent subset $\mathbf{B} \subset A_{\mathcal{U}}$ is linearly finitely representable in A, that is, (ii) holds.

To show (i), suppose that $\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n$ are affinely independent. We write $\boldsymbol{y}_j = \boldsymbol{x}_j - \boldsymbol{x}_0$. Then the n vectors $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n$ are linearly independent. Let

$$\mathbf{F} = \operatorname{span}\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n\} \text{ and } \mathbf{F}' = \operatorname{aff}\{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_n\}. \tag{16}$$

Then $\mathbf{F}' = \boldsymbol{x}_0 + \mathbf{F}$. For each $\omega \in U'$, let

$$F_\omega = \operatorname{span}\{\boldsymbol{y}_1(\omega), \cdots, \boldsymbol{y}_n(\omega)\},\ F'_\omega = \operatorname{aff}\{\boldsymbol{x}_0(\omega), \cdots, \boldsymbol{x}_n(\omega)\} \tag{17}$$

(where $\boldsymbol{y}_j(\omega) = \boldsymbol{x}_j(\omega) - \boldsymbol{x}_0(\omega)$) and define

$$P_\omega \colon \mathbf{F} \to F_\omega,\quad P'_\omega \colon \mathbf{F}' \to F'_\omega \tag{18}$$

by

$$P_\omega(\sum_{j=1}^n \lambda_j \boldsymbol{y}_j) = \sum_{j=1}^n \lambda_j \boldsymbol{y}_j(\omega) \text{ for all } \boldsymbol{y} = \sum_{j=1}^n \lambda_j \boldsymbol{y}_j \in \mathbf{F} \tag{19}$$

and

$$P'_\omega(\sum_{j=0}^n \lambda_j \boldsymbol{x}_j) = \sum_{j=0}^n \lambda_j \boldsymbol{x}_j(\omega) \text{ for all } \boldsymbol{x} = \sum_{j=0}^n \lambda_j \boldsymbol{x}_j \in \mathbf{F}'. \tag{20}$$

Then we have $\lim_{\mathcal{U}} \|P_\omega(\boldsymbol{y})\| = \|\boldsymbol{y}\|$ for all $\boldsymbol{y} \in F$. Note

$$P'_\omega(\sum_{j=0}^n \lambda_j \boldsymbol{x}_j) = \boldsymbol{x}_0(\omega) + P_\omega(\sum_{j=1}^n \lambda_j \boldsymbol{y}_j) \tag{21}$$

for all $\boldsymbol{x} = \sum_{j=0}^n \lambda_j \boldsymbol{x}_j \in \mathbf{F}'$. This implies

$$\lim_{\mathcal{U}} \|P'_\omega(\boldsymbol{x}) - P'_\omega(\boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\| \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbf{F}'. \tag{22}$$

Hence, we have shown that for every $\varepsilon > 0$ there exist $\omega \in U'$, n affinely independent vectors $x_0(\omega), \cdots, x_n(\omega) \in A$ and an affine mapping $P'_\omega \colon \mathbf{F}' \to F'_\omega$ such that

$$(1-\varepsilon)\|\boldsymbol{x} - \boldsymbol{y}\| \leq \|P'_\omega(\boldsymbol{x}) - P'_\omega(\boldsymbol{y})\| \leq (1+\varepsilon)\|\boldsymbol{x} - \boldsymbol{y}\| \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbf{F}'$$

and such that $P'_\omega(\boldsymbol{x}_j) = x_j(\omega)$ for $j = 0, \cdots, n$. This says that $A_{\mathcal{U}}$ is finitely representable in A, i.e. (i) holds. □

Proposition 2.4. *Let $A \subset X$ and $B \subset Y$ be two subsets of Banach spaces X and Y, and A bounded.*

(i) *If B is finitely representable in A, then for any affinely independent subset $B_0 \subset B$ there exist a free ultrafilter $\mathcal{U}$, and an affine isometry $T \colon \operatorname{aff}(B) \to \operatorname{aff}(A)_{\mathcal{U}}$ such that $T(B_0) \subset A_{\mathcal{U}}$.*

(ii) *If B is crudely finitely representable in A, then for any affinely independent subset $B_0 \subset B$ there exist a free ultrafilter $\mathcal{U}$, and an affine embedding $T \colon \operatorname{aff}(B) \to \operatorname{aff}(A)_{\mathcal{U}}$ such that $T(B_0) \subset A_{\mathcal{U}}$.*

Proof. (i) Suppose that B is finitely representable in A. For every family $C \subset B$ consisting of finitely many affinely independent vectors, $N \equiv \mathrm{co}(C)$ is a simplex contained in the finite dimensional affine space $F \equiv \mathrm{aff}(C)$. Since B is finitely representable in A, for each $m \in \mathbb{N}$, there exist a simplex $M_m \equiv \mathrm{co}(A_m)$ with vertex set $A_m \subset A$, and an affine mapping $\tilde{T}_{m,C} \colon F \to E_m \equiv \mathrm{aff}(A_m)$ with $\tilde{T}_{m,C}(N) = M_m$ such that

$$(1 - 1/m)\|x - y\| \leq \|\tilde{T}_{m,C}x - \tilde{T}_{m,C}y\| \leq (1 + 1/m)\|x - y\|, \quad \forall x, y \in F. \quad (23)$$

By Zorn's lemma, for each affinely independent subset B_0, there exists a maximal affinely independent subset of B containing B_0. Without loss of generality, we can assume B_0 itself is maximal. Clearly, $\mathrm{aff}(B_0) = \mathrm{aff}(B)$. Let $\Omega = \mathbb{N} \times \{C \subset B_0 : C \text{ is finite}\}$, and fix $z_0 \in A$. Further, let $T_{m,C} \colon \mathrm{aff}(B) \to \mathrm{aff}(A)$ be defined as $\tilde{T}_{m,C}$ on $\mathrm{aff}(C)$ and as z_0 on $\mathrm{aff}(B) \setminus \mathrm{aff}(C)$.

Now fix an ultrafilter $\mathcal{U}$ on Ω which contains the sets

$$\{(m', C') \in \Omega : m' \geq m, C' \supset C\} : \ (m, C) \in \Omega,$$

and define $T \colon \mathrm{aff}(B) \to \mathrm{aff}(A)_{\mathcal{U}}$ by $T(x) = [(T_{m,C}(x))_{(m,C)\in\Omega}]$. Since A is bounded, the generalized sequence $\{T_{m,C}(x)\}_{(m,C)\in\Omega}$ is bounded for each x. By the choice of $\mathcal{U}$, it is easy to check that T is an affine isometry and that $T(B_0) \subset A_{\mathcal{U}}$.

(ii) Suppose that B is crudely finitely representable in A. Let β be the constant from Definition 2.1(iii), let B_0 be as above and $z_0 \in A$ be fixed. Let Ω be the family of all the finite subsets of B_0. For $C \in \Omega$ let $\tilde{T}_C \colon \mathrm{aff}(C) \to \mathrm{aff}(A)$ be provided by the definition of "crudely finitely representable". Extend $\tilde{T}_C$ to T_C by using z_0 as above. Let $\mathcal{U}$ be an ultrafilter on Ω containing all the sets of the form $\{C' \in \Omega : C' \supset C\}$, where $C \in \Omega$. Finally, define $T \colon \mathrm{aff}(B) \to \mathrm{aff}(A)_{\mathcal{U}}$ by $T(x) = [(T_C(x))_{C\in\Omega}]$. Then T is an affine embedding of $\mathrm{aff}(B)$ into $\mathrm{aff}(A)_{\mathcal{U}}$ with $T(B_0) \subset A_{\mathcal{U}}$. $\square$

3. Some basic properties of non-convex super weakly compact sets

In this section, we shall show that super weak compactness is invariant under operations of continuous linear mapping, product and union of finitely many sets and n-convex combinations of a set.

Definition 3.1. A (weakly closed, resp.) subset A of a Banach space X is said to be *relatively super weakly compact* (*super weakly compact*, resp.) provided every subset B of a Banach space Y which is finitely representable in A is relatively weakly compact.

This concept used in [10, Corollary 2.15] entails that in a Banach space X every compact set is super weakly compact; X is super-reflexive if and only if its closed unit ball is super weakly compact; and if X is super reflexive then every bounded set is relatively super weakly compact.

As we well know, the weak closure of a subset in an infinite dimensional Banach space could be much larger than the subset. To avoid difficulty of "weak closure", we shall use "relative super weak compactness" in stead of "super weak compactness" in the following discussion. We shall see that a subset is relatively super weakly compact is equivalent to the weak closure of the subset is super weakly compact (Proposition 3.10).

Proposition 3.2. *Let Ω be a set, $\{X, X_\omega : \omega \in \Omega\}$ be a collection of Banach spaces, $A \subset X$, $A_\omega \subset X_\omega$ for all $\omega \in \Omega$, and let $\mathcal{U}$ be a free ultrafilter on Ω satisfying that there is a sequence $\{U_n\} \subset \mathcal{U}$ so that $\cap U_n = \emptyset$; in particular, a free ultrafilter on $\mathbb{N}$. Then*

(i) *$\prod_{\mathcal{U}} A_\omega$ is norm closed in $\prod_{\mathcal{U}} X_\omega$.*

(ii) *There is $T \in B(X_{\mathcal{U}}, X^{**})$ with $\|T\| = 1$ so that*

 (a) *$T \circ D = Id$, the identity;*

 (b) *$s\text{-}\overline{A}^w \subset T(A_{\mathcal{U}})$; In particular, if A is relatively weakly compact, then $\overline{A}^w \subset T(A_{\mathcal{U}})$, where $s\text{-}\overline{A}^w$ is the weakly sequential closure of A, and D is the natural diagonal identity embedding from X to $X_{\mathcal{U}}$.*

Proof. (i) Given $\boldsymbol{x} \in \overline{\prod_{\mathcal{U}} A_\omega}$, let $\{\boldsymbol{x}_n\} \subset \prod_{\mathcal{U}} A_\omega$ be a sequence with $\|\boldsymbol{x} - \boldsymbol{x}_n\| < \frac{1}{n}$. Fix some representants $(\boldsymbol{x}(\omega))$ and $(\boldsymbol{x}_n(\omega))$ of $\boldsymbol{x}$ and $\{\boldsymbol{x}_n\}$. Then $V_n \equiv \{\omega \in \Omega : \|\boldsymbol{x}(\omega) - \boldsymbol{x}_n(\omega)\| < \frac{1}{n}\} \in \mathcal{U}$. Set $W_n = U_n \cap V_1 \cap \cdots \cap V_n$. Then $W_n \in \mathcal{U}$. Fix any $x_0 \in A$ and let $\boldsymbol{y} = [(\boldsymbol{y}(\omega))_{\omega\in\Omega}] \in A_{\mathcal{U}}$ be defined by $\boldsymbol{y}(\omega) = \boldsymbol{x}_m(\omega)$, $\omega \in W_m \setminus W_{m+1}, m \in \mathbb{N}$; $= x_0$, otherwise. Clearly, $\boldsymbol{x} = \boldsymbol{y} \in \prod_{\mathcal{U}} A_\omega$.

(ii) Note that if $\mathcal{U}$ is an ultrafilter and if K is a compact Husdorff topological space then for any $f \colon \Omega \to K$ there is a unique $k \in K$ so that $\lim_{\mathcal{U}} f = k$. Let X be endowed with the w^*-topology of X^{**} and define $T(\boldsymbol{x}) = w^*\text{-}\lim_{\mathcal{U}} \boldsymbol{x}(\omega)$ for all $\boldsymbol{x} \in X_{\mathcal{U}}$. Then we see that $T \colon X_{\mathcal{U}} \to X^{**}$ is a linear operator of norm one, and satisfies $T \circ D = Id$, that is, (a) holds.

To show (b), let $\{x_n\} \subset A$ be a sequence such that $x_n \to x \in X$ in the weak topology of X, and let $\{U_n\} \subset \mathcal{U}$ be a monotone non-increasing sequence so that $\cap U_n = \emptyset$. Fix any $x_0 \in A$ and define $\boldsymbol{x} = [(\boldsymbol{x}(\omega))_{\omega\in\Omega}] \in A_{\mathcal{U}}$ by $\boldsymbol{x}(\omega) = x_m$, $\omega \in U_m \setminus U_{m+1}, m \in \mathbb{N}$; $= x_0$, otherwise. Then we have $x = T(\boldsymbol{x})$. Thus, $s\text{-}\overline{A}^w \subset T(A_{\mathcal{U}})$. Finally, it suffices to note $s\text{-}\overline{A}^w = \overline{A}^w$ whenever A is relatively weakly compact. □

Proposition 3.3. *A subset A of a Banach space X is relatively super weakly compact if and only if all of its countable subsets are relatively super weakly compact.*

Proof. It suffices to show sufficiency. Clearly, A is bounded. Suppose, to the contrary, that A is not relatively super weakly compact. Then there is a bounded non-relatively weakly compact set B in some Banach space Y, which is finitely representable in A. Thus, there is a linearly independent sequence $C \equiv \{x_n\} \subset B$ which does not admit any weak cluster point.

Let $S_n = \mathrm{co}\{x_1, x_2, \cdots, x_{n+1}\}$, and $k \in \mathbb{N}$, there exist a simplex $S_{n,k} = \mathrm{co}\{y_{k,1}, y_{k,2}, \cdots, y_{k,n+1}\}$ with vertices $y_{k,1}, y_{k,2}, \cdots, y_{k,n+1} \in A$, and an affine mapping T_k such that $T_k(S_n) = S_{n,k}$ satisfying

$$(1 - 1/k)\|x - y\| \leq \|T_k(x) - T_k(y)\| \leq (1 + 1/k)\|x - y\|, \text{ for all } x, y \in S_n.$$

Thus, $\{x_1, x_2, \cdots, x_{n+1}\}$ is finitely representable in $\cup_{k\in\mathbb{N}}\{y_{k,1}, y_{k,2}, \cdots, y_{k,n+1}\}$. Consequently, the non relatively weakly compact set $C = \{x_n\}$ is finitely representable in the countable subset $\cup_{k,n\in\mathbb{N}}\{y_{k,1}, y_{k,2}, \cdots, y_{k,n+1}\}$ of A. This is a contradiction. □

Proposition 3.4. *Let A be a nonempty subset of a Banach space X. Then the following statements are equivalent.*

(i) *A is relatively super weakly compact;*

(ii) *$A_{\mathcal{U}}$ is relatively super weakly compact for some free ultrafilter $\mathcal{U}$;*

(iii) *$A_{\mathcal{U}}$ is relatively super weakly compact for every free ultrafilter $\mathcal{U}$;*

(iv) *$A_{\mathcal{U}}$ is relatively weakly compact for every free ultrafilter $\mathcal{U}$.*

Proof. (i) $\Longrightarrow$ (iii). Since A is relatively super weakly compact, it is bounded. By Proposition 2.3, $A_{\mathcal{U}}$ is finitely representable in A. Thus, transitivity of finite representability implies that if a subset B of a Banach space is finitely representable in $A_{\mathcal{U}}$, then it is finitely representable in A. Relative super weak compactness of A entails that B is relatively weakly compact. Hence, $A_{\mathcal{U}}$ is relatively super weakly compact for every free ultrafilter $\mathcal{U}$.

(iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv) are trivial.

It remains to show (iv) $\Longrightarrow$ (i). Clearly, A is bounded. Suppose, to the contrary, that A is not relatively super weakly compact. Then, there is a bounded non-relatively weakly compact set B, which is finitely representable in A. Consequently, there is a sequence $\{x_n\} \subset B$ which has no weak cluster point. We can assume that $\{x_n\}$ is a linearly independent set. By Proposition 2.4, there exist a free ultrafilter $\mathcal{U}$ and an affine isometry $T\colon \mathrm{aff}(B) \to \mathrm{aff}(A)_{\mathcal{U}}$ such that $\{T(x_n)\} \subset A_{\mathcal{U}}$. Clearly, $\{T(x_n)\}$ is not relatively weakly compact. This is a contradiction to relative weak compactness of $A_{\mathcal{U}}$. □

Corollary 3.5. *The union of two relatively super weakly compact sets in a Banach space is again relatively super weakly compact.*

Proof. Let A, B be two relatively super weakly compact sets of a Banach space X. Suppose, to the contrary, that $C = A \cup B$ is not relatively super weakly compact. By Proposition 3.4, there is a free ultrafilter $\mathcal{U}$ such that $C_{\mathcal{U}}$ is not relatively weakly compact. Again by Proposition 3.4, both $A_{\mathcal{U}}$ and $B_{\mathcal{U}}$

are relatively (super) weakly compact. Consequently, $C_{\mathcal{U}}$ is relatively weakly compact since $C_{\mathcal{U}} = A_{\mathcal{U}} \cup B_{\mathcal{U}}$. This is a contradiction. □

Theorem 3.6. *Let $A \subset X$ and $B \subset Y$ be two bounded subsets of Banach spaces X and Y. Suppose that $T\colon X \to Y$ is a bounded linear operator with $T(A) = B$. If A is relatively super weakly compact, then B is also relatively super weakly compact.*

Proof. Suppose that B is not relatively super weakly compact. Then by Proposition 3.4, there is a free ultrafilter $\mathcal{U}$ such that $B_{\mathcal{U}}$ is not relatively weakly compact in $Y_{\mathcal{U}}$. Since A is relatively super weakly compact, again by Proposition 3.4, $A_{\mathcal{U}}$ is relatively super weakly compact. We define $S\colon X_{\mathcal{U}} \to Y_{\mathcal{U}}$ by $S(\boldsymbol{x}) = [(T\boldsymbol{x}(\omega))_{\omega\in\Omega}]$. Then S is a bounded linear operator with $S(A_{\mathcal{U}}) = B_{\mathcal{U}}$. This contradicts to that $S(A_{\mathcal{U}})$ is not relatively weakly compact. □

Corollary 3.7. (i) *Relative super weak compactness is invariant under linear isomorphisms;* (ii) *A bounded set crudely finitely representable in a relatively super weakly compact set is again relatively super weakly compact.*

Proof. (i) is just a direct consequence of Theorem 3.6. To show (ii), let X, Y be Banach spaces, $A \subset X$ be a relatively super weakly compact set, and $B \subset Y$ be crudely finitely representable in A. If B is not relatively super weakly compact, then by Proposition 3.3 there is a non-relatively super weakly compact sequence $B_0 \equiv \{x_n\} \subset B$ of linearly independent vectors. By Proposition 2.4, there is an affine embedding $T\colon \mathrm{aff}(B) \to \mathrm{aff}(A)_{\mathcal{U}}$ such that $T(B_0) \subset A_{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$, and $A_{\mathcal{U}}$ is relatively super weakly compact.

Fix $b_0 \in B$ and let $S(x) = T(b) - T(b_0)$ for each $x = b - b_0 \in \mathrm{aff}(B) - b_0 \equiv F$. Then S is a linear embedding from the linear space F to the linear space $E = \mathrm{aff}(A)_{\mathcal{U}} - T(b_0)$ satisfying $S(B_0 - b_0) \subset A_{\mathcal{U}} - T(b_0)$. Since $B_0 - b_0 = S^{-1}[S(B_0 - b_0)]$ is not relatively super weakly compact, by Theorem 3.6, $S(B_0 - b_0)$ is not relatively super weakly compact. This contradicts to that $S(B_0 - b_0)$ is a subset of the relatively super weakly compact set $A_{\mathcal{U}} - T(b_0)$. □

Theorem 3.8. *The product of two relatively super weakly compact sets is again relatively super weakly compact.*

Proof. Let $A \subset X$ and $B \subset Y$ be two relatively super weakly compact sets of Banach spaces X and Y. We put ℓ_1 norm on the product $Z \equiv X \times Y$, i.e. $\|(x, y\| = \|x\| + \|y\|$ for all $(x, y) \in Z$. Suppose, to the contrary, that $C \equiv A \times B$ is not relatively super weakly compact, then by Proposition 3.4 again, there is a free ultrafilter $\mathcal{U}$ such that $C_{\mathcal{U}}$ is not relatively weakly compact. Note that $T\colon Z_{\mathcal{U}} \to X_{\mathcal{U}} \times_{\ell_1} Y_{\mathcal{U}}$ defined by $T((x, y)_{\mathcal{U}}) = (x_{\mathcal{U}}, y_{\mathcal{U}})$ is a linear isometry, and note that both $A_{\mathcal{U}}$ and $B_{\mathcal{U}}$ are relatively weakly compact. We see $T(C_{\mathcal{U}}) = A_{\mathcal{U}} \times B_{\mathcal{U}}$ is relatively weakly compact. Thus, $C_{\mathcal{U}}$ is relatively weakly compact, and this is a contradiction. □

Corollary 3.9. *Let A and B be two relatively super weakly compact subsets of a Banach space X. Then $A + B$ is again relatively super weakly compact.*

Proof. By Theorem 3.8, $A \times B$ is relatively super weakly compact in X^2. Since $T\colon X^2 \to X$ defined by $T(x, y) = x + y$ is a bounded linear operator, due to Theorem 3.6, $T(A \times B) = A + B$ is relatively super weakly compact. $\square$

Proposition 3.10. *A subset A of a Banach space is relatively super weakly compact if and only if $\overline{A}^w$ is super weakly compact.*

Proof. It suffices to show necessity. Since A is relatively super weakly compact, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, by Proposition 3.4, $A_{\mathcal{U}}$ is again relatively super weakly compact. Making use of Proposition 3.2(ii)(b), there is a bounded linear operator T such that $\overline{A}^w \subset T(A_{\mathcal{U}})$. Theorem 3.6 entails $\overline{A}^w$ is super weakly compact. $\square$

4. Characterizations of super weakly compact sets

In this section, with the aim of James' characterization of non-weakly compact sets [25], and properties established in the previous sections, we shall show two characterizations of a relatively super weakly compact set. One is the following theorem of Grothendiek's type; and the other is James' type characterization for non-super weakly compact sets (Theorem 4.8).

Theorem 4.1. *A subset A of a Banach space X is relatively super weakly compact if (and only if) for every $\varepsilon > 0$ there is a relatively super weakly compact set B such that $A \subset B + \varepsilon B_X$.*

Proof. Suppose, to the contrary, that A is not relatively super weakly compact. Then by Proposition 3.4, there is a free ultrafilter $\mathcal{U}$ such that $A_{\mathcal{U}}$ is not relatively weakly compact, while $B_{\mathcal{U}}$ is relatively weakly compact. Note $A_{\mathcal{U}} \subset (B + \varepsilon B_X)_{\mathcal{U}} = B_{\mathcal{U}} + \varepsilon B_{X_{\mathcal{U}}}$. By Grothendieck's lemma ([15, Lemma 2 p. 227]), $A_{\mathcal{U}}$ is relatively weakly compact. This contradiction completes our proof. $\square$

A Banach space X is said to be strongly generated by a super reflexive Banach space Y if there exists $T \in B(Y, X)$ such that for every weakly compact set $A \subset X$ and for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ satisfying $A \subset n_\varepsilon T(B_Y) + \varepsilon B_X$ [40] (see, also, [19]).

Corollary 4.2. *Let A be a subset of a Banach space X. If there exist a super reflexive Banach space Y and $T \in B(Y, X)$ such that for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ satisfying $A \subset n_\varepsilon T(B_Y) + \varepsilon B_X$, then A is relatively super weakly compact. In particular, every weakly compact set in a strongly super reflexive generated Banach space is super weakly compact.*

Proof. It immediately follows from Theorem 4.1. Since Y is super reflexive, according to Theorem 3.6, $T(B_Y)$ is relatively super weakly compact in X. $\square$

Example 4.3. Let μ be a finite measure. Then every weakly compact set in $L_1(\mu)$ is super weakly compact. Indeed, it suffices to note that $L_1(\mu)$ is strongly generated by $L_2(\mu)$ (see, for instance, [20]).

For any subset A in a linear space, we denote by $\mathrm{co}_n(A)$, n-convex combination of A, i.e. $\mathrm{co}_n(A) = \{\sum_{j=1}^n \lambda_j x_j : x_j \in A,\ \lambda_j \geq 0,\ \sum_{j=1}^n \lambda_j = 1\}$. Clearly, $\mathrm{co}(A) = \cup_{n\in\mathbb{N}}\mathrm{co}_n(A)$.

Theorem 4.4. *Let A be a subset of a Banach space X. If A is relatively super weakly compact, then $\mathrm{co}_n(A), n = 1, 2, \cdots$ are again relatively super weakly compact.*

Proof. Since A is relatively super weakly compact, it is bounded. Let $M = \sup\{\|a\| : a \in A\}$, and let $K = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in [0, 1]^n, \sum_{j=1}^n \lambda_j = 1\} \subset \ell_1^n$. Since K is compact, for every $\varepsilon > 0$ there is a finite $\varepsilon' = \varepsilon/M$-net $N_{\varepsilon'}$. For each $\lambda = (\lambda_1, \cdots, \lambda_n) \in N_{\varepsilon'}$, we define a linear operator $T_\lambda \colon X^n \to X$ for $x = (x_1, \cdots, x_n) \in X^n$ by $T_\lambda(x) = \sum_{j=1}^n \lambda_j x_j$. Then T_λ is bounded. According to Theorems 3.6 and 3.8, $T_\lambda(A^n)$ is relatively super weakly compact. We use Corollary 3.5 to see that $C \equiv \cup_{\lambda\in N_{\varepsilon'}} T_\lambda(A^n)$ is again relatively super weakly compact. It is easy to check $\mathrm{co}_n(A) \subset C + \varepsilon B_X$. We finish the proof by Theorem 4.1. $\square$

Problem. For a relatively super weakly compact set A, though $\mathrm{co}_n(A)$ is relatively super weakly compact for each $n \in \mathbb{N}$, we do not know whether $\mathrm{co}(A)$ is again relatively super weakly compact.

Example 4.5. (i) The standard unit vector basis $\{e_j\}$ of c_0 is relatively super weakly compact. (ii) The absolute closed convex hull of $\{e_j\}$ is super weakly compact. (iii) Any bounded sequence of disjoint elements in c_0 is relatively super weakly compact.

Proof. (i) Let $\{h_j\}$ be the standard unit vector basis of ℓ_2. Since it is a bounded set of the super-reflexive space ℓ_2, it is necessarily relatively super weakly compact. Note that the identity $I \colon \ell_2 \to c_0$ is continuous, and $I(h_j) = e_j$. By Theorem 3.6, $\{e_j\} \subset c_0$ is relatively super weakly compact.

(ii) Let $A = \{e_j\}$. Then (i) and Theorem 4.4 entail that $\mathrm{co}_n(A)$ is again relatively super weakly compact. Given $x \in \mathrm{co}(A)$, for every $n \in \mathbb{N}$ we have $\mathrm{dist}(x, \mathrm{co}_n(A)) \leq 1/(n+1)$. Therefore, $\mathrm{co}(A) \subset \mathrm{co}_n(A) + (1/n)B_{c_0}$ for all $n \in \mathbb{N}$. Theorem 4.1 and Corollary 3.9 together entail that $\mathrm{co}(A \cup -A) \subset \mathrm{co}(A) - \mathrm{co}(A)$ is again relatively super weakly compact. By Proposition 3.10, $\overline{\mathrm{co}}(A \cup -A)$ is super weakly compact.

(iii) Let $\{x_n\} \subset c_0$ be a bounded sequence of disjoint elements, i.e. $\mathrm{supp}x_n \cap \mathrm{supp}x_m = \emptyset$ whenever $m \neq n$. Since the normalized sequence $\{z_n\}$ ($z_n =$

$x_n/\|x_n\|$) is equivalent to $\{e_n\}$, and since $x_n = T(e_n)$ for all n, where $T\colon c_0 \to c_0$ is defined for $x = (x(n))$ by $T(x) = \sum_n \alpha_n x(n) z_n$, and $\alpha_n = \|x_n\|$, due to Theorem 3.6, both $\{z_n\}$ and $\{x_n\}$ are relatively super weakly compact. □

Remark 4.6. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be the standard unit vector of $c_0(\Gamma)$. Then $\mathrm{co}\{e_\gamma\}_{\gamma\in\Gamma}$ is relatively super weakly compact in $c_0(\Gamma)$. Indeed, note the identity $I\colon l_2(\Gamma) \to c_0(\Gamma)$ is continuous, and $B_{l_2(\Gamma)}$ is super weakly compact in Hilbert space $l_2(\Gamma)$. Hence $\mathrm{co}\{e_\gamma\}_{\gamma\in\Gamma} \subset I(B_{l_2(\Gamma)})$ is relatively super weakly compact.

Theorem 4.7. *A bounded subset A in $c_0(\Gamma)$ is relatively super weakly compact if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that*

$$\sharp\{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\} \le n_\varepsilon, \quad \textit{for all}\, x \in A. \tag{24}$$

Proof. Let $\Theta = \overline{\mathrm{co}}\{\pm e_\gamma : \gamma \in \Gamma\}$ and $\beta = \sup\{\|x\| : x \in A\}$. Given $\varepsilon > 0$, since $\mathrm{card}\{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\} \le n_\varepsilon$ for all $x \in A$, we obtain $A \subset n_\varepsilon \beta\Theta + \varepsilon B_X$. Since Θ is super weakly compact, applying Theorem 4.1, we finish the proof. □

Theorem 4.8. *Let A be a nonempty bounded weakly closed subset of a Banach space X. Then it is not super weakly compact if and only if it has uniform finite bi-orthogonal property, i.e. there is $\theta > 0$ so that for all $n \in \mathbb{N}$ there exist n vectors $x_1, \cdots, x_n \in A$ and $x_1^*, \cdots, x_n^* \in B_{X^*}$ satisfying*

$$\langle x_i^*, x_j\rangle = \begin{cases} \theta\,, & 1 \le i \le j \le n; \\ 0\,, & 1 \le j < i \le n. \end{cases} \tag{25}$$

Proof. Sufficiency. According to (25), we can prove that for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there exist $\{\boldsymbol{x}_n\} \subset A_{\mathcal{U}}$ and $\{\boldsymbol{x}_n^*\} \subset B_{X^*_{\mathcal{U}}}$ such that

$$\langle \boldsymbol{x}^*{}_i, \boldsymbol{x}_j\rangle = \begin{cases} \theta\,, & 1 \le i \le j < \infty; \\ 0\,, & 1 \le j < i < \infty. \end{cases} \tag{26}$$

Indeed, for each $n \in \mathbb{N}$, let $\{x_{n,i}\}_{i=1}^n \subset A$ and $\{x_{n,j}^*\}_{j=1}^n \subset B_{X^*}$ satisfy (25). Then we set

$$\boldsymbol{x}_n = [(0, \cdots, 0, x_{n,n}, x_{n+1,n}, x_{n+2,n}, \cdots)], \quad n \in \mathbb{N}. \tag{27}$$

Further, the functionals $\{\boldsymbol{x}_n^*\}$ are defined by

$$\langle \boldsymbol{x}_n^*, [(z_k)]\rangle = \lim_{\mathcal{U}} \langle x_{k,n}^*, z_k\rangle, \quad [(z_k)] \in X_{\mathcal{U}}. \tag{28}$$

Clearly, the sequences $\{\boldsymbol{x}_n\} \subset A_{\mathcal{U}}$ and $\{\boldsymbol{x}_n^*\} \subset B_{X^*_{\mathcal{U}}}$ fulfil (26). This shows that $A_{\mathcal{U}}$ is not weakly compact by James's criteria for non-weakly compact subsets [25], and hence A is not super weakly compact from Proposition 3.4.

Necessity. Suppose that A is not super weakly compact. By Proposition 3.4, there is a free ultrafilter $\mathcal{U}$ such that $A_{\mathcal{U}}$ is not relatively weakly compact.

Applying James's criteria for non-weakly compact sets again, there exist $0 < \theta_1 < 1$, $\{\boldsymbol{x}_i\}_{i=1}^{\infty} \subset A_{\mathcal{U}}$ and $\{\boldsymbol{x}_j^*\}_{j=1}^{\infty} \subset B_{X_{\mathcal{U}}^*}$ such that

$$\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle = \begin{cases} \theta_1 , & 1 \le i \le j < \infty; \\ 0 , & 1 \le j < i < \infty. \end{cases} \tag{29}$$

Clearly, $\{\boldsymbol{x}_i\}_{i=1}^{\infty} \subset A_{\mathcal{U}}$ is a linearly independent set. By Proposition 2.3, it is linearly finitely representable in A. Therefore, for each $n \in \mathbb{N}$, there is a linear embedding T from $\mathbf{F} \equiv \operatorname{span}\{\boldsymbol{x}_j\}_{j=1}^n$ to $F \equiv \operatorname{span}\{x_j\}_{j=1}^n$ satisfying

$$\frac{1}{2}\|\boldsymbol{x}\| \le \|T(\boldsymbol{x})\| \le 2\|\boldsymbol{x}\|, \;\; \boldsymbol{x} \in \mathbf{F}. \tag{30}$$

with $x_j \equiv T(\boldsymbol{x}_j) \in A$.

Next, let y_j^* be a norm-preserved extension of $\boldsymbol{x}^*{}_j \circ T^{-1}$ for $j = 1, 2, \ldots, n$. Then $\|y_j^*\| = \|y_j^*\|_F \le \|\boldsymbol{x}_j^*\|\|T^{-1}\|_F \le 2$ for all $1 \le j \le n$. Therefore,

$$\langle y_i^*, x_j \rangle = \begin{cases} \theta_1 , & 1 \le i \le j \le n; \\ 0 , & 1 \le j < i \le n. \end{cases}$$

Now, we finish the proof by taking $x_i^* = (1/2)y_i^*$ and $\theta = (1/2)\theta_1$. □

Corollary 4.9. *Let A be a nonempty bounded subset of a Banach space X. Then it is not relatively super weakly compact if and only if it has uniform finite separating property, i.e. there is $\theta > 0$ so that for all $n \in \mathbb{N}$ there exist n vectors $x_1, \cdots, x_n \in A$ satisfying*

$$\operatorname{dist}(\operatorname{co}\{x_1, \cdots, x_j\}, \operatorname{co}\{x_{j+1}, \cdots, x_n\}) \ge \theta, \; \forall 1 \le j < n. \tag{31}$$

Proof. Sufficiency. Suppose, to the contrary, that A is relatively super weakly compact. Then by Proposition 3.4, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, $A_{\mathcal{U}}$ is again relatively super weakly compact. On the other hand, for each $n \in \mathbb{N}$, let $\{x_{n,i}\}_{i=1}^n \subset A$ satisfy (31), and let $\boldsymbol{x}_n \in A_{\mathcal{U}}$ be defined as (27). Then it is easy to see the sequence $\{\boldsymbol{x}_n\}$ satisfies the following inequalities.

$$\operatorname{dist}(\operatorname{co}\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_j\}, \operatorname{co}\{\boldsymbol{x}_{j+1}, \cdots, \boldsymbol{x}_n, \cdots\}) \ge \theta, \; \forall j \in \mathbb{N}. \tag{32}$$

This contradicts to relative weak compactness of $A_{\mathcal{U}}$ by [10, Theorem 2.14].

Necessity. This is an immediate consequence of Theorem 4.8. □

5. Equivalences of super weak compactness

The slicing indices, as variants of the Szlenk index have found many applications in the geometric theory of Banach spaces (see, e.g., [22, 21, 29, 30, 28, 37]). In this section, we shall show that for a bounded closed convex set the finite-index property [38], the finite-dual-index property [19] and the super weak compactness [10] are equivalent.

Let E be a normed space, $F \subset E^*$ be a subspace and let $B \subset E$ be a nonempty bounded set. A $\sigma(E,F)$-slice of B is a subset $S \subset B$ of the form:

$$S = S(B, x^*, \alpha) = \{x \in B : \langle x^*, x\rangle > \alpha\}, \text{ for some } x^* \in F,\ \alpha \in \mathbb{R}. \tag{33}$$

In particular, a $\sigma(E,E^*)$-slice is simply called a slice, and a $\sigma(E^*,E)$-slice is siad to be a w^*-slice whenever $B \subset E^*$.

Let ϱ be a seminorm defined on E. We denote by $d_\varrho(B)$ the ϱ-diameter of B. Given $\varepsilon > 0$, we define $\sigma(E,F)$-ϱ-ε ($\sigma(E,F)$-ε, resp.)-dentability derivative $[B]'_{(\sigma(E,F),\,\varrho,\,\varepsilon)}$ of B as follows.

$$[B]'_{(\sigma(E,F),\,\varrho,\,\varepsilon)} = \{x \in B : d_\varrho(S) > \varepsilon, \forall \sigma(E,F)\text{-slice } S \text{ of } B \text{ containing } x\}. \tag{34}$$

We often omit the letter ϱ in the notations above if it is the norm of E. Please note the three particular cases: (1) if ϱ is the norm of E, then we simply denote the $\sigma(E,F)$-ϱ-ε-dentability derivative $[B]'_{(\sigma(E,F),\ \varrho,\ \varepsilon)}$ by $[B]'_{(\sigma(E,F),\ \varepsilon)}$; In addition, if $F = E^*$, then we simply denote the $\sigma(E,E^*)$-ϱ-ε-dentability derivative $[B]'_{(\sigma(E,E^*),\ \varrho,\ \varepsilon)}$ by $[B]'_\varepsilon$, and call it ε-dentability derivative of B; (2) if $A \subset E^*$ and ϱ is the norm of E^*, then we simply denote the $\sigma(E^*,E)$-ϱ-ε-dentability derivative $[B]'_{(\sigma(E^*,E),\ \varrho,\ \varepsilon)}$ by $[B]'_{(w^*,\ \varepsilon)}$, and call it w^*-ε-dentability derivative of B; and (3) [Fabian et al. [19]] if $B \subset E^*$ and ϱ is on E^* generated by some bounded set $A \subset E$, i.e.

$$\varrho(x^*) = \sup\{|\langle x^*, x\rangle| : x \in A\}, \quad \text{for } x^* \in E^*, \tag{35}$$

then we simply denote the $\sigma(E^*,E)$-ϱ-ε-dentability derivative $[B]'_{(\sigma(E^*,E),\ \varrho,\ \varepsilon)}$ by $[B]'_{(w^*,\ A,\ \varepsilon)}$, and call it w^*-(A,ε)-dentability derivative of B.

Starting from $[B]'_{(\sigma(E,F),\ \varrho,\ \varepsilon)}$, we can successively define $[B]^{(n)}_{(\sigma(E,F),\,\varrho,\,\varepsilon)}$ for all $n \in \mathbb{N}$. B is said to have the *finite* $\sigma(E,F)$-ϱ*-index property* provided for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[B]^{(n)}_{(\sigma(E,F),\,\varrho,\,\varepsilon)} = \emptyset$. We should emphasis that if (in Case (1)) for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[B]^{(n)}_\varepsilon = \emptyset$, then we simply call the set B having the *finite-index property*; and if (in Case (3)) $B = B_{X^*}$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[B]^{(n)}_{(w^*,\ A,\ \varepsilon)} = \emptyset$, then we call the set A having the *finite-dual-index property*.

Clearly, the finite-index property and the finite-dual-index property of a bounded set are inherited by its subsets. The finite-dual-index property is also inherited by its absolute closed convex hull, but the finite-index property is not inherited by its convex hull (see, Example 5.5).

The following notations about localized uniform convexity were used in [10] and [14]. A convex function f defined on a convex set C in a Banach space X is said to be *uniformly convex* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in C,\ \|x - y\| \geq \varepsilon \implies f(x) + f(y) - 2f(\frac{x+y}{2}) \geq \delta. \tag{36}$$

The set C is called *uniformly convex* if for each $x_0 \in C$,

$$\text{the function } f = \|\cdot - x_0\|^2 \text{ is uniformly convex on } C. \tag{37}$$

We say that the set C is *uniformly convexifiable* if there is an equivalent norm on X such that C is uniformly convex with respect to the new norm. The main result in [10] is that "a bounded closed convex set is super weakly compact if and only if it is uniformly convexifiable."

A subset $Y \subset X^*$ is called 1-*norming* if $\|x\| = \sup\{|f(x)| : f \in Y, \|f\| \leq 1\}$ for all $x \in X$. The following lemma was motivated by Lemma 9 of Fabian et al. in [19].

Lemma 5.1. *Suppose that A is a closed bounded uniformly convex set in a Banach space X, and that $Y \subset X^*$ is a 1-norming subspace. Then for any continuous seminorm ϱ on X, A has the finite $\sigma(X,Y)$-ϱ-index property.*

Proof. Continuity of the seminorm ϱ entails that there is $M > 0$ such that $d_\varrho(B) \leq M\text{diam}(B)$ for every bounded set $B \subset X$. Thus, it suffices to show A has the finite $\sigma(X,Y)$-index property. Since A is uniformly convex, the definition of uniform convexity of a convex set always allows us to assume $0 \in A$. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two sequences $\{x_n\}, \{y_n\} \subset A$ with $\|x_n - y_n\| > \varepsilon$ we have $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 > \delta$. Let $0 < \sup_{x\in A} \|x\| \equiv r$. Without loss of generality, we assume $r \leq 1$; otherwise, we substitute $\frac{1}{r}A$ for A, since $S(\frac{1}{r}A, x^*, \frac{\alpha}{r}) = \frac{1}{r}S(A, x^*, \alpha)$.

Now, given $\varepsilon > 0$ and $x_0 \in A$, we consider $\sigma(X,Y)$-slices of A containing x_0 with radiuses greater than ε. Let $\{x_n^*\} \subset S_Y$ be such that $\langle x_n^*, x_0\rangle > \|x_0\| - 1/n$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$,

$$S_n \equiv S(A, x_n^*, \|x_0\| - 1/n) = \{x \in A : \langle x_n^*, x\rangle > \|x_0\| - 1/n\} \tag{38}$$

is a $\sigma(X,Y)$-slice of A containing x_0. Suppose $\text{diam}S_n > \varepsilon$ for all $n \in \mathbb{N}$. Then we can choose two sequences $\{x_n\}$ and $\{y_n\}$ with $x_n, y_n \in S_n$ such that $\|x_n - y_n\| > \varepsilon$. therefore, there is $\delta_0 > 0$ (depending only on ε) such that $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 > \delta_0$. Consequently,

$$\|(x_n + y_n)/2\| < \sqrt{(\|x_n\|^2 + \|y_n\|^2)/2 - \delta_0/4} \leq \sqrt{r^2 - \delta_0/4}. \tag{39}$$

On the other hand, since $(x_n + y_n)/2 \in S_n$, we obtain

$$\|x_0\| - 1/n < (\langle x_n^*, x_n\rangle + \langle x_n^*, y_n\rangle)/2 \leq \|(x_n + y_n)/2\|. \tag{40}$$

Let $n \longrightarrow \infty$. Then (39) and (40) together imply

$$\|x_0\| \leq \sqrt{r^2 - \delta_0/4} < r - \delta, \text{ for some } \delta > 0. \tag{41}$$

Therefore, $[A]'_{(\sigma(X,Y),\varepsilon)} \subset A \cap (r-\delta)B_X$. Let $n \in \mathbb{N}$ satisfy $(r-\delta)^{n-1} < \varepsilon/2$. Since δ does not depend on r, we can iterate this procedure $n-1$ times to get $[A]^{(n-1)}_{(\sigma(X,Y),\varepsilon)} \subset A \cap (r-\delta)^{n-1}B_X \subset A \cap (\varepsilon/2)B_X$. Hence, $[A]^{(n)}_{(\sigma(X,Y),\varepsilon)} = \emptyset$. $\square$

Theorem 5.2. *Let A be a nonempty closed bounded convex set of a Banach space X. Then A has finite-dual-index property if and only if it is super weakly compact.*

Proof. Sufficiency. Let $K = \overline{\text{co}}\{A \cup -A\}$ $(= \text{co}\{A \cup -A\})$. Then K is super weakly compact. In fact, since A is convex and super weakly compact, $\{A \cup -A\}$ is super weakly compact (Corollary 3.5). Consequently, $K = \overline{\text{co}}\{A \cup -A\} = \text{co}_2\{A \cup -A\}$ is again super weakly compact (Theorem 4.4). Without loss of generality, we assume that $K \subset B_X$ and that $X_K \equiv \text{span}(K)(= \cup_{n=1}^{\infty} nK)$ is dense in X; otherwise, we substitute $\overline{X}_K = \overline{\text{span}}(K)$ and $X^*/X_K{}^{\perp}$ for X and X^*, resp. Let p be the Minkowski functional generated by K, i.e.

$$p(x) = \inf\{\lambda > 0 : x \in \lambda K\}, \ x \in X_K. \tag{42}$$

It is easy to see that p is a norm with $p \geq \|\cdot\|$ (the original norm of X) on X_K, and that $X_p \equiv (X_K, p)$ is a Banach space with $B_{X_p} = K$. The dual norm of p is just ϱ defined by

$$\varrho(x^*) = \sup\{\langle x^*, x\rangle : x \in K\}, \ x^* \in X_p^*. \tag{43}$$

Note that the closed unit ball B_{X^*} of X^* is again a w^*-compact absolute convex set contained in $B_{X_p^*}$.

We first claim that B_{X^*} is super weakly compact in X_p^*. Otherwise, by Theorem 4.8, there exists $0 < \theta < 1$ such that for every $n \in \mathbb{N}$ there are $\{x_i^*\}_{i=1}^n \subset B_{X^*}$ and $\{x_j^{**}\}_{j=1}^n \subset B_{X_p^{**}}$ satisfying

$$\langle x_i^{**}, x_j^*\rangle = \begin{cases} \theta\ , & 1 \leq i \leq j \leq n; \\ 0\ , & 1 \leq j < i \leq n. \end{cases} \tag{44}$$

It follows from Goldstine's theorem, (43) and (44) that there are n vectors $x_1, \cdots, x_n \in K$ such that

$$\inf\{\langle x_j^*, x_i\rangle : j \leq i \leq n\} - \sup\{\langle x_j^*, x_i\rangle : 1 \leq i < j\} \geq \theta/2, \ 1 \leq j \leq n. \tag{45}$$

From the separation theorem of convex sets we know that (45) is equivalent to the following inequality:

$$\text{dist}(\text{co}\{x_1, \cdots, x_k\}, \text{co}\{x_{k+1}, \cdots, x_n\}) \geq \theta/2, \ \forall 1 \leq k < n. \tag{46}$$

By [10, Theorem 2.14], we see that K is not super weakly compact. This is a contradiction. Therefore, we have shown that B_{X^*} is super weakly compact in X_p^*.

Next, we claim that A has the finite-dual-index property. Let $Y = X_p$. Then $B_Y = K$. Since B_{X^*} is super weakly compact in $X_p^* = Y^*$, by Theorem 4.12 of Cheng et al. [10], there is an equivalent norm on Y^* which is uniformly convex on B_{X^*}. Due to Lemma 5.1, B_{X^*} regarded as a uniformly convex set of Y^* with respect to the new norm has the finite $\sigma(Y^*, Y)$-ϱ-index property. Note $B_{X^*} \subset X^* \subset Y^*$ and $K \subset Y \subset X$. We know that B_{X^*} has the finite $\sigma(X^*, X)$-ϱ-index property. or, equivalently, K has the finite-dual-index property.

Necessity. Let the set K, the norm p, the spaces X_K and X_p be as the same as in the proof of sufficiency. Let $I\colon X_p \to X$ be the identity mapping. Then $I^*\colon X^* \to X_p^*$ is one-to-one and $I^{**}\colon X_p^{**} \to X^{**}$ is norm one with $K \subset I^{**}(B_{X_p^{**}})$. Suppose that A has the finite-dual-index property, which is equivalent to that B_{X^*} has the finite-index property in the dual space X_p^*. Hence, by Theorem 1.2 of Raja [38], the mapping $I^*\colon X^* \to X_p^*$ is uniformly convexifying, i.e. B_{X^*} has no finite tree property in X_p^*. By [10, Theorem 2.14], it is super weakly compact in X_p^*. Thus, for every $\theta > 0$, there exists $n \in \mathbb{N}$ such that there do not exist $\{x_1^{**}, \cdots, x_n^{**}\} \subset B_{X_p^{**}}$ and $\{x_1^*, \cdots, x_n^*\} \subset B_{X^*}$ so that

$$\inf\{\langle x_j^*, x_i^{**}\rangle : j \le i \le n\} - \sup\{\langle x_j^*, x_i^{**}\rangle : 1 \le i < j\} \ge \theta,\ 1 \le j \le n. \tag{47}$$

This says that $B_{X_p^{**}}$ has no uniform finite separating property with respect to the original norm. Therefore, K has no uniform finite separating property by noticing $K \subset I(B_{X_p^{**}})$. We finish the proof by Corollary 4.9. □

Example 5.3. (i) c_0 has a weakly compact subset which is not super weakly compact. The following sets are defined in [19, Theorem 12]. For all $n \in \mathbb{N}$, let

$$W_n = \{\sum_{i=0}^{2^{n-1}-1} \varepsilon_i e_{2^n+i} : \varepsilon_i \in \{-1, 1\}\}, \text{ and } W = \cup_n W_n. \tag{48}$$

Then W is a weakly null sequence of c_0, which has no finite 1-dual index. By Theorem 5.2, $\overline{\mathrm{co}}(W)$ is not super weakly compact by [19, Theorem 12].

(ii) For any infinite compact set K, the Banach space $C(K)$ has a weakly compact subset which is not super weakly compact. Since c_0 is isomorphic to a subspace of $C(K)$ [18, Theorem 12.30], for any linear embedding $T\colon c_0 \to C(K)$, the set $T(W)$ is relatively weakly compact but non-relatively super weakly compact, where W is defined by (48).

(iii) For any reflexive but non-superreflexive Banach space X, B_X is weakly compact and non-super weakly compact.

Remark 5.4. For a closed bounded convex set A of a Banach space X, the three notions of super weak compactness, finite index property, finite dual index property are actually equivalent. But they do not coincide in general. The following example shows that super weak compactness and finite index property for a bounded non-convex set are not equivalent.

Example 5.5. Let X be a separable locally uniformly convex Banach space without the Radon-Nikodým property (say, c_0 with an equivalent locally uniformly convex norm), and let $A = S_X$. Since every point $x \in A$ is a strongly exposed point of B_X, for all $\varepsilon > 0$, we observe x is in some slice of B_X with diameter less than ε. Therefore, $[A]'_\varepsilon = \emptyset$. But A is not relatively weakly compact. Hence, A does not admit finite dual index property.

Let $(X, \|\cdot\|)$ be a Banach space and let $M \subset X$ be a subset. The norm $\|\cdot\|$ is called M-uniformly smooth [19] or M-uniformly Gâteaux smooth norm if $\lim_n \varrho(f_n - g_n) = 0$ whenever $f_n, g_n \in S_{X^*}$ are such that $\lim_n \|f_n + g_n\| = 2$. Here $\varrho(f) = \sup\{|f(x)| : x \in M\}$.

Now we conclude this section by the following summation.

Theorem 5.6. *Suppose that A is a closed bounded convex set. Then the following statements are equivalent.*

(i) *A is super weakly compact;*

(ii) *A has finite-index property;*

(iii) *A has finite-dual-index property;*

(iv) *A is uniformly convexifiable;*

(v) *A is uniformly smoothable;*

(vi) *A does not admit finite tree property;*

(vii) *A does not admits uniform finite bi-orthogonal property;*

(viii) *A does not admits uniform finite separating property;*

(ix) *A has Δ-convex function approximating property;*

(x) *A is contained in the image of a bounded set under a uniformly convexifying operator.*

Proof. By Lemma 5.1 and Theorem 5.2, we observe (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).
(ii) $\Longrightarrow$ (i) from Raja [38, Theorems 1.2]. (i) $\Longleftrightarrow$ (vii) is a counterpart of Theorem 4.8. Note definition of uniform convexity of a convex set (37), and note that a convex set C of a Banach space X is said to be uniformly convexifiable if there is an equivalent norm on X such that it is uniformly convex on C. The equivalences (i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (vi) $\Longleftrightarrow$ (viii) are just [10, Theorems 4.12 and 2.14].

We say that a convex set A has the Δ-*convex function approximating property* if every Lipschitz function defined on A is uniformly approximated by a sequence of Lipschitz convex functions. (ii) $\Longleftrightarrow$ (ix) $\Longleftrightarrow$ (x) hold according to Raja [38, Theorems 1.2 and 1.4]. While (iii) $\Longleftrightarrow$ (v), see Fabian, Montesinos and Zizler [19, Theorem 5], by noting "A is uniformly smoothable" means that there is an equivalent A-uniformly Gâteaux smooth norm on X. $\square$

6. On super Banach-Saks sets

The study of the Banach-Saks property of Banach spaces has continued for over 80 years since Banach and Saks' work [3] (see, for instance, [1, 2, 5, 16, 32, 34, 39, 41, 42]). A bounded set B of a Banach space X is said to be *Banach-Saks* (or, to have the Banach-Saks property) provided every sequence in B has an averaging convergent subsequence, i.e. for every sequence $\{x_n\} \subset B$ there is a subsequence $\{z_n\} \subset \{x_n\}$ so that $\{\frac{1}{n}\sum_{j=1}^{n} z_j\}_n$ converges in X. The set B is called *super Banach-Saks* if every set finitely representable in it is again Banach-Saks. Clearly, every Banach-Saks set is relatively weakly compact, and a Banach space has super Banach-Saks property if and only if it is super reflexive.

Let $\{x_n\}$ be a sequence of a Banach space X weakly convergent to some point $x \in X$. We say that it generates an ℓ_1*-spreading model* if there is $\alpha > 0$ so that for all finite subset $s \subset \mathbb{N}$ with cardinality $\sharp s \le \min s \equiv \min\{j \in s\}$ and for all sequence $\{a_n\} \subset \mathbb{R}$

$$\|\sum_{n\in s} a_n(x_n - x)\| \ge \alpha \sum_{n\in s} |a_n|. \tag{49}$$

A sequence $\{x_n\} \subset X$ is said to be *uniformly weakly convergent* to $x \in X$ provided $\{x_n - x\}$ is uniformly weakly null, i.e. for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ so that for every $x^* \in B_{X^*}$

$$\sharp\{n \in \mathbb{N} : |\langle x^*, x_n - x\rangle| \ge \varepsilon\} \le n_\varepsilon. \tag{50}$$

The following theorem, presented in Lopez-Abada, Ruizb and Tradacetec's recent paper [32], are due to Rosenthal [39] and Mercourakis [34].

Theorem 6.1. *Let B be bounded set of a Banach space X. Then the following statements are equivalent.*

(i) *B is Banach-Saks;*

(ii) *B is relatively weakly compact, and for every sequence $\{z_n\} \subset B$ weakly convergent to $z \in X$, the weakly null sequence $\{x_n\} \equiv \{z_n - z\}$ contains a uniformly weakly null subsequence;*

(iii) *B is relatively weakly compact and no weakly convergent sequence in B generates an ℓ_1-spreading model.*

We use the theorem above to show super Banach-Saks property is equivalent relative super weak compactness for a subset in a Banach space.

Theorem 6.2. *A bounded set B of a Banach space X is super Banach-Saks if and only if it is relatively super weakly compact.*

Proof. Sufficiency. Suppose B is relatively super weakly compact. It suffices to show that B is Banach-Saks since every set finitely representable in B is again relatively super weakly compact (Corollary 3.7). Otherwise, by Theorem 6.1, there is a sequence $\{x_n\} \subset B$ weakly convergent to some point x generates an ℓ_1-spreading model. Therefore, the standard unit vector basis $\{\delta_n\}$ of ℓ_1 is crudely finitely representable in $\{x_n - x\} \subset B$. This is a contradiction to Corollary 3.7, since $\{\delta_n\}$ is not relatively weakly compact.

Necessity. Suppose that B is super Banach-Saks. We need only to note that every set finitely representable in B is Banach-Saks, hence, relatively weakly compact. $\square$

The following result follows from Theorems 6.2, 3.6, 3.8, Proposition 3.4, Corollaries 3.5 and 3.9.

Corollary 6.3. *Super Banach-Saks property of subsets is invariant under operations of the product of finite many sets, addition, continuous linear transformation (hence, linear isomorphism), ultrapower under a free ultrafilter and n-convex combination of a set (for a set A for $n \in \mathbb{N}$, the n-combination $\mathrm{co}_n(A)$ of A is defined by*

$$\mathrm{co}_n(A) = \{\sum_{j=1}^{n} \lambda_j x_j : x_j \in A,\ 0 \le \lambda_j \le 1,\ \sum_{j=1}^{n} \lambda_j = 1\}). \tag{51}$$

Let $\{x_n\} \subset X$ be a bounded sequence. We define an operator $T \in B(X^*, \ell_\infty)$ by

$$Tx^* = (\langle x^*, x_n \rangle), \text{ for all } x^* \in X^*. \tag{52}$$

Thus, (50) is equivalent to the following condition if $\{x_n\}$ is weakly null.

$$\sharp\{n \in \mathbb{N} : |\langle \delta_n, Tx^* \rangle| \ge \varepsilon\} \le n_\varepsilon, \tag{53}$$

where $\{\delta_n\}$ is the standard unit vector basis of ℓ_1.

Apparently, the standard unit vector basis $\{e_j\}$ of c_0 is uniformly weakly null. Hence, every weakly null sequence in c_0 contains a uniformly weakly null subsequence, by a simple argument of block basis of $\{e_j\}$ (see, for instance, [31, Prop. 2.a.1]). Consequently, every relatively weakly compact set in c_0 is Banach-Saks. We denote by $\Theta = \overline{\mathrm{co}}\{\pm e_j\}$, where $\{e_j\}$ is the standard unit vector basis of c_0.

Theorem 6.4. *Let $\{x_n\}$ be a weakly null sequence of a Banach space X, and T be the operator associated with $\{x_n\}$ defined by (6.4). Then the following statements are equivalent.*

(i) *$\{x_n\}$ is uniformly weakly null;*

(ii) *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ so that*

$$T(B_{X^*}) \subset n_\varepsilon \Theta + \varepsilon B_{\ell_\infty}; \tag{54}$$

(iii) *Every weakly convergent sequence in $\mathrm{co}\{\pm x_n\}$ is uniformly weakly convergent.*

Proof. (i) $\Longrightarrow$ (ii). Given $x^* \in B_{X^*}$ and $\varepsilon > 0$, let

$$\Delta(x^*) = \{n \in \mathbb{N} : |\langle \delta_n, Tx^* \rangle| \ge \varepsilon\}. \tag{55}$$

Then (53) implies that there is $m_\varepsilon \in \mathbb{N}$ such that

$$Tx^* \in m_\varepsilon \beta \mathrm{co}\{\sigma_j e_j : j \in \Delta(x^*)\} + \varepsilon B_{c_0}, \tag{56}$$

where $\sigma_j = \operatorname{sgn}\langle \delta_j, Tx^* \rangle$ and $\beta = \sup\{|x_n| : n \in \mathbb{N}\}$. Put $n_\varepsilon = ([\beta]+1)m_\varepsilon$, and hence $T(B_{X^*}) \subset n_\varepsilon \Theta + \varepsilon B_{c_0}$. Here $[\cdot]\colon \mathbb{R} \to \mathbb{Z}$ denotes the floor function.

(ii) $\Longrightarrow$ (iii). Let $\{z_n\} \subset \operatorname{co}\{\pm x_n\}$ be such that $z_n \longrightarrow z \in X$, where $z_n = \sum_m \lambda_{n,m} x_m$, $\lambda_{n,m} \in \mathbb{R}$ with $\sum_m |\lambda_{n,m}| = 1$ for all $n \in \mathbb{N}$. We claim that $\{z_n\}$ is uniformly weakly convergent to z. Without loss of generality we can assume $z = 0$; Otherwise, we substitute $\{z_n - z\}$ for $\{z_n\}$ since $\{z_n\} \subset \operatorname{co}\{\pm x_n\}$ implies $\{z_n - z\} \subset 2\overline{\operatorname{co}}\{\pm x_n\}$. Let $S\colon B_{X^*} \to c_0$ be defined by

$$Sx^* = (\langle x^*, z_n \rangle), \text{ for all } x^* \in X^*. \tag{57}$$

Next, let $\Lambda = \{\lambda_{n,m}\}_{n,m\in\mathbb{N}}$. Then $\Lambda \in B(\ell_\infty)$ with $\|\Lambda\|_\infty = 1$ satisfies $S = \Lambda T$. This and (54) entail for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ so that

$$S(B_{X^*}) \subset n_\varepsilon \Lambda(\Theta) + \varepsilon B_{\ell_\infty} = n_\varepsilon \overline{\operatorname{co}}(\Lambda\{\pm e_n\}) + \varepsilon B_{\ell_\infty} \subset n_\varepsilon \Theta + \varepsilon B_{\ell_\infty}. \tag{58}$$

Since the ℓ_1-norm of every vector in Θ is less or equal to 1, every $u \in n_\varepsilon\Theta$ admits at most $k_\varepsilon \equiv [n_\varepsilon \|u\|_{\ell_1}/\varepsilon]$ coordinate components with absolute value great than or equal to ε. Therefore, by (58) every vector $x^* \in B_{X^*}$ satisfies

$$\sharp\{n \in \mathbb{N} : |\langle \delta_n, Sx^* \rangle| \geq 2\varepsilon\} \leq k_\varepsilon, \tag{59}$$

and which further entails that $\{z_n\}$ is uniformly weakly null.

(iii) $\Longrightarrow$ (i). This is trivial. □

Let Φ denote the set of all projections $P\colon c_0 \to c_0$ satisfying that there is a monotone-increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $P(e_{n_k}) = e_{n_k}$ for all $k \in \mathbb{N}$. The following property follows from Theorems 6.1 and 6.4.

Proposition 6.5. *A bounded subset A of a Banach space X is Banach-Saks if and only if for every sequence $\{y_n\} \subset A$ weakly convergent to some $y \in X$ there is a projection $P \in \Phi$ so that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ satisfying*

$$P(T(B_{X^*})) \subset n_\varepsilon \Theta + \varepsilon B_{c_0}, \tag{60}$$

where T is associated with $(x_n) \equiv (y_n - y)$ defined by (52).

The following result says that the absolute convex hull of a uniformly weakly null sequence is uniformly approximated by n-combinations of the sequence.

Proposition 6.6. *Let $\{x_n\} \subset X$ be a uniformly weakly null sequence. Then for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that*

$$\overline{\operatorname{co}}\{\pm x_n\} \subset \operatorname{co}_{n_\varepsilon}\{\pm x_n\} + \varepsilon B_X. \tag{61}$$

Proof. Given $\varepsilon > 0$, let $n_\varepsilon \in \mathbb{N}$ be such that

$$\sharp\Delta(x^*) \equiv \sharp\{n \in \mathbb{N} : |\langle x^*, x_n \rangle| \geq \varepsilon\} \leq n_\varepsilon, \text{for all } x^* \in B_{X^*}.$$

Suppose, to the contrary, that (61) does not hold. Let $x \in \mathrm{co}\{\pm x_n\}$ with $x = \sum_j \lambda_j \sigma_j x_j \notin \mathrm{co}_{n_\varepsilon}\{\pm x_n\} + \varepsilon B_X$, where $\sigma_j = \pm 1$ and $\lambda_j \geq 0$ with $\sum_j \lambda_j = 1$. Then, by separation theorem, there is $x^* \in S_{X^*}$ such that

$$\langle x^*, x\rangle = \sum_j \lambda_j \sigma_j \langle x^*, x_j\rangle > \sup\{\langle x^*, z\rangle : z \in \mathrm{co}_{n_\varepsilon}\{\pm x_n\}\} + \varepsilon. \tag{62}$$

Therefore,

$$\varepsilon > \sum_{j \in \mathbb{N}\setminus\Delta(x^*)} \lambda_j |\langle x^*, x_j\rangle| \geq \sum_{j \in \mathbb{N}\setminus\Delta(x^*)} \lambda_j \sigma_j \langle x^*, x_j\rangle > \varepsilon.$$

This is a contradiction. □

Theorem 6.7. *Let T be the operator associated with the weakly null sequence $\{x_n\}$ of X defined by* (52)*, and let $K = \overline{\mathrm{co}}\{\pm x_n\}$. Consider the following properties that the sequence $\{x_n\}$, the operator T and the set K may possess:*

(i) *$\{x_n\}$ is uniformly weakly null;*

(ii) *$T(B_{X^*})$ is super weakly compact in c_0;*

(iii) *K is super weakly compact;*

(iv) *K is super Banach-Saks.*

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

Proof. (i) $\Longrightarrow$ (ii). It is done by Theorems 4.7 and 6.4.

(ii) $\Longleftrightarrow$ (iii). As in Theorem 5.2, we may assume that $K \subset B_X$ and that X_K =span K is dense in X. Let p be the Minkowski functional generated by K. Then $X_p = (X_K, p)$ is a Banach space with $B_{X_p} = K$. The dual norm of p is just q defined by

$$q(x^*) = \sup\{|\langle x^*, x_n\rangle| : n \in \mathbb{N}\}, x^* \in X_p^*. \tag{63}$$

Thus $T\colon (B_{X^*}, q) \to T(B_{X^*})$ is an isometry. If $T(B_{X^*})$ is super weakly compact in c_0, then B_{X^*} is super weakly compact in (X^*, q). By the proof of the sufficiency part of Theorem 5.2, K has finite-dual-index property, and which in turn implies that K is super weakly compact. If K is super weakly compact, by the proof of the sufficiency part of Theorem 5.2 again, B_{X^*} is super weakly compact in X_p^*. Hence $T(B_{X^*})$ is super weakly compact in c_0.

(iii) $\Longleftrightarrow$ (iv). This is a consequence of Theorem 6.2. □

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