

Directional Convexity and Characterizations of Beta and Gamma Functions

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The logarithmic convexity of restrictions of the Beta function to rays parallel to the main diagonal and the functional equation

$$\varphi(x+1) = \frac{x(x+k)}{(2x+k+1)(2x+k)} \phi(x), \quad x > 0,$$

for $k > 0$ allow to get a characterization of the Beta function. This fact and the notion of the beta-type function lead to a new characterization of the Gamma function.

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1. Introduction

The Euler Gamma function and its two-variable companion, the Beta function, both playing important role in applications, have found much attention in real and complex analysis. The Gamma function, actually its restriction to the set of positive reals, is logarithmically convex and, as an extension of the $(n-1)!$, satisfies the functional equation

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1.$$

Bohr and Møllerup [3], 1922, proved that these two properties characterize the Gamma function. The class of logarithmically convex functions is rather narrow – for instance no power function has this property. Gronau and Matkowski [6]

improved the Bohr-Mollerup theorem showing that the Gamma function is the only solution of this equation that is geometrically convex in a vicinity of $+\infty$. Applying this result, Alzer and Matkowski [2] gave a characterization of the Gamma function based on the convexity with respect to power means.

It is known [Dragomir, Agarwal, Barnett [4]] that the Beta function is logarithmically convex as a two-variable function in $(0, \infty)^2$. It follows that the Beta function is *logarithmically convex in the direction of the main diagonal*, that is, for every $k > 0$, the restriction of the Beta function to the ray $\{(x, x+k) : x > 0\}$ is logarithmically convex, and, moreover, solves the functional equation

$$\phi(x+1) = \frac{x(x+k)}{(2x+k+1)(2x+k)} \phi(x), \quad x > 0.$$

In section 3, applying the Krull theorem [5], 1949, we prove that this functional equation, the logarithmic convexity in the direction of the main diagonal (much weaker than the logarithmic convexity), and a natural initial condition characterize the Beta function (Theorem 3.1). Employing the method of Gronau-Matkowski [7], we show that the same functional equation and the geometrical convexity also allow to get a characterization of the Beta function (Theorem 3.5). In section 4, given a function $\gamma: (a, \infty) \rightarrow (0, \infty)$, $(a \geq 0)$, we define the two-place function $B_\gamma: (a, \infty)^2 \rightarrow (0, \infty)$, referred to as *beta-type function of a generator γ* , by

$$B_\gamma(x, y) := \frac{\gamma(x)\gamma(y)}{\gamma(x+y)}, \quad x, y > a,$$

and we observe that the equality $B_{\gamma_2} = B_{\gamma_1}$ holds iff $\frac{\gamma_2}{\gamma_1}$ is an exponential function. It turns out that this fact and the above mentioned results about the Beta function allow to obtain new characterizations of the Gamma function. In the final section the directional convexity is briefly discussed.

2. Preliminaries and auxiliary results

Recall that the Euler Gamma function $\Gamma: (0, \infty) \rightarrow (0, \infty)$ defined by

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0,$$

satisfies the functional equation

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1. \quad (1)$$

The Beta function, $\mathcal{B}: (0, \infty)^2 \rightarrow (0, \infty)$ defined by

$$\mathcal{B}(x, y) := \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0, \quad (2)$$

plays an important role in applications, is symmetric, that is

$$\mathcal{B}(x, y) = \mathcal{B}(y, x), \quad x, y > 0,$$

and has the following integral representation

$$\mathcal{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0. \quad (3)$$

Krull ([5], cf. also Kuczma [10], pp. 114–118) proved the following

Theorem 2.1. *Let $a \geq -\infty$ be arbitrarily fixed. Suppose that $F: (a, \infty) \rightarrow \mathbb{R}$ is convex (resp. concave), and*

$$\lim_{x \rightarrow \infty} [F(x+1) - F(x)] = 0.$$

Then, for every fixed $(x_0, y_0) \in (a, \infty) \times \mathbb{R}$, there exists exactly one concave (resp. convex) function $\varphi: (a, \infty) \rightarrow \mathbb{R}$ satisfying the functional equation

$$\varphi(x+1) = \varphi(x) + F(x), \quad x > a, \quad (4)$$

such that $\varphi(x_0) = y_0$; moreover, for all $x > a$,

$$\begin{aligned} \varphi(x) = y_0 + (x - x_0) F(x_0) - \\ - \sum_{n=0}^{\infty} \{F(x+n) - F(x_0+n) - (x-x_0)[F(x_0+n+1) - F(x_0+n)]\}. \end{aligned} \quad (5)$$

Since for any $c > a$ and any function $\varphi_0: [c, c+1] \rightarrow \mathbb{R}$ satisfying the initial condition $\varphi_0(c+1) = \varphi_0(c) + F(c)$ there exists a unique solution $\varphi: (a, \infty) \rightarrow \mathbb{R}$ of the functional equation (4) such that $\varphi|_{[c, c+1]} = \varphi_0$ (cf. Kuczma [10], p. 70), the following meaningful improvement of Krull's theorem holds true.

Theorem 2.2. *Let $a \geq -\infty$ be arbitrarily fixed. Suppose that $F: (a, \infty) \rightarrow \mathbb{R}$ is convex (resp. concave), and*

$$\lim_{x \rightarrow \infty} [F(x+1) - F(x)] = 0.$$

Then, for every fixed $(x_0, y_0) \in (a, \infty) \times \mathbb{R}$, there exists exactly one function $\varphi: (a, \infty) \rightarrow \mathbb{R}$ satisfying the functional equation

$$\varphi(x+1) = \varphi(x) + F(x), \quad x > a,$$

that is concave (resp. convex) in some interval (b, ∞) , $b \geq a$, and such that $\varphi(x_0) = y_0$; moreover, (5) holds for all $x > a$.

Let $I \subset (0, \infty)$ be an interval. A function $\phi: I \rightarrow (0, \infty)$ is called *geometrically convex*, if

$$\phi(x^t y^{1-t}) \leq [\phi(x)]^t [\phi(y)]^{1-t} \quad x, y \in I, \quad t \in (0, 1);$$

geometrically concave, if the reversed inequality is satisfied; and *geometrically affine*, if the equality holds.

It is easy to check that a function $\phi: (0, \infty) \rightarrow (0, \infty)$ is geometrically affine iff $\phi(x) = ax^p$ for some $a > 0$, and $p \in \mathbb{R}$.

Remark 2.3. If $\phi: I \rightarrow (0, \infty)$ is Jensen geometrically convex, that is

$$\phi(\sqrt{xy}) \leq \sqrt{\phi(x)\phi(y)} \quad x, y \in I,$$

and ϕ satisfies one of the following conditions: ϕ is continuous at a point; or ϕ is bounded from above in a neighborhood of a point, or ϕ is Lebesgue measurable, then ϕ is geometrically convex.

A function $\phi: I \rightarrow (0, \infty)$ is geometrically convex (Jensen geometrically convex) if, and only if, the function $\log \circ \phi \circ \exp$ is convex (Jensen convex) in the interval $\log(I)$. $\square$

From Gronau and Matkowski [7], (Theorems 2 and 3), we have the following

Theorem 2.4. Let $G: (0, \infty) \rightarrow (0, \infty)$ be such that $\log \circ G$ is concave on some vicinity of ∞ , and

$$\lim_{x \rightarrow \infty} \frac{G(x + \delta)}{G(x)} = 1 \quad \text{for some } \delta > 0.$$

Then, for every $c > 0$, there exists exactly one solution $\phi: (0, \infty) \rightarrow (0, \infty)$ of the functional equation

$$\phi(x + 1) = G(x) \phi(x), \quad x > 0,$$

such that ϕ is geometrically convex on some vicinity of ∞ and $\phi(1) = c$. Moreover, for all $x > 0$,

$$\phi(x) = c \lim_{n \rightarrow \infty} G(n)^{-\frac{\log(n+1+x) - \log(n+1)}{\log(n+1) - \log n}} \frac{1}{G(x)} \prod_{j=1}^n \frac{G(j)}{G(j+x)}.$$

3. Characterization of the Beta function

The main result of this section is the following

Theorem 3.1. Let $\mathcal{B}$ denote the Beta function and let $k \geq 0$ be arbitrarily fixed. If $\phi: (0, \infty) \rightarrow (0, \infty)$ is such that

$$\phi(x+1) = \frac{x(x+k)}{(2x+k+1)(2x+k)} \phi(x), \quad x > 0; \quad (6)$$

$$\phi(1) = \mathcal{B}(1, 1+k), \quad (7)$$

and $\log \circ \phi$ is convex, then

$$\phi(x) = \mathcal{B}(x, x+k), \quad x > 0. \quad (8)$$

Proof. Let us assume that $\phi: (0, \infty) \rightarrow (0, \infty)$ satisfies equation (6) and condition (7). Put $\varphi = \log \circ \phi$, and define the function $F: (0, \infty) \rightarrow (0, \infty)$ by the formula

$$F(x) = \log \frac{x(x+k)}{(2x+k+1)(2x+k)}, \quad x > 0. \quad (9)$$

Hence, taking the logarithm of both sides of equation (6), we get

$$\varphi(x+1) = \varphi(x) + F(x), \quad x > 0, \quad (10)$$

and, from (7),

$$\varphi(1) = \log \mathcal{B}(1, 1+k). \quad (11)$$

Calculating the first and the second derivatives of F we get

$$F'(x) = \frac{1}{x} + \frac{1}{x+k} - \frac{2}{2x+k} - \frac{2}{2x+k+1}, \quad x > 0,$$

and, for every $x > 0$,

$$\begin{aligned} F''(x) &= -\frac{1}{x^2} - \frac{1}{(x+k)^2} + \frac{4}{(2x+k)^2} + \frac{4}{(2x+k+1)^2} \\ &= -\frac{P(x)}{x^2(x+k)^2(2x+k)^2(2x+k+1)^2}, \end{aligned}$$

where

$$\begin{aligned} P(x) &= 16x^5 + 4x^4(6k^2 + 10k + 1) + 8kx^3(k+1)(6k+1) + \\ &\quad + 2k^2x^2(k+1)(17k+5) + 2k^3x(k+1)(5k+3) + k^4(k+1)^2. \end{aligned}$$

Since k together with all occurring coefficients in the numerator are non-negative, the second derivative F'' is non-positive and, consequently, F is concave on $(0, \infty)$. Moreover, from the definition of F , we get

$$\begin{aligned} \lim_{x \rightarrow \infty} [F(x+1) - F(x)] &= \\ &= \lim_{x \rightarrow \infty} \left[\log \frac{(x+1)(x+k+1)}{(2x+k+2)(2x+k+3)} - \log \frac{x(x+k)}{(2x+k+1)(2x+k)} \right] = \\ &= \lim_{x \rightarrow \infty} \log \frac{(x+1)(x+k+1)}{(2x+k+2)(2x+k+3)} \frac{(2x+k)(2x+k+1)}{x(x+k)} = \log 1 = 0. \end{aligned}$$

Thus, the function F in equation (10) satisfies the suitable conditions of Theorem 2.1. Taking into account that the function φ is convex and is a solution of equation (10), applying the Theorem 2.1 with

$$a = 0, \quad x_0 = 1, \quad y_0 = \log \mathcal{B}(1, 1 + k),$$

we conclude that φ is the only function satisfying equation (10) such that $\varphi(1) = \log \mathcal{B}(1, 1 + k)$, i.e. satisfying condition (11).

To finish the proof it is enough to show that, for a fixed $k \geq 0$, the function $f: (0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) := \log \mathcal{B}(x, x + k), \quad x > 0, \quad (12)$$

is convex and satisfies equation (10).

Though the convexity of f follows from the convexity of $\log \circ \mathcal{B}$ ([4]), for the sake of completeness, we present short argument. Note that, by (3),

$$\mathcal{B}(x, x + k) = \int_0^1 t^{x-1} (1 - t)^{x+k-1} dt, \quad x > 0.$$

Hence, making use of (12), we have

$$f'(x) = \frac{\int_0^1 t^{x-1} (1 - t)^{x+k-1} (\log t + \log(1 - t)) dt}{\mathcal{B}(x, x + k)}, \quad x > 0,$$

and, for all $x > 0$,

$$f''(x) [\mathcal{B}(x, x + k)]^2 = \left(\int_0^1 [g_x(t)]^2 dt \right) \left(\int_0^1 [h_x(t)]^2 dt \right) - \left(\int_0^1 g_x(t) h_x(t) dt \right)^2,$$

where the functions $g_x, h_x: (0, 1) \rightarrow \mathbb{R}$ are defined by

$$g_x(t) := \left(t^{x-1} (1 - t)^{x+k-1} \right)^{\frac{1}{2}} \log(t(1 - t)) \quad h_x(t) := \left(t^{x-1} (1 - t)^{x+k-1} \right)^{\frac{1}{2}}.$$

By the Cauchy-Schwarz inequality we have

$$\left(\int_0^1 g_x(t) h_x(t) dt \right)^2 \leq \left(\int_0^1 [g_x(t)]^2 dt \right) \left(\int_0^1 [h_x(t)]^2 dt \right),$$

which together with the above equality implies that f is convex.

From (2), taking into account that

$$\Gamma(x + 1) = x\Gamma(x), \quad x > 0,$$

we have, for all $x, y > 0$,

$$\begin{aligned}\mathcal{B}(x+1, y+1) &= \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma((x+y+1)+1)} = \frac{xy\Gamma(x)\Gamma(y)}{(x+y+1)\Gamma(x+y+1)} = \\ &= \frac{xy\Gamma(x)\Gamma(y)}{(x+y+1)(x+y)\Gamma(x+y)} = \frac{xy}{(x+y+1)(x+y)} \mathcal{B}(x, y).\end{aligned}$$

Setting here $y = x + k$ gives

$$\mathcal{B}(x+1, x+k+1) = \frac{x(x+k)}{(2x+k+1)(2x+k)} \mathcal{B}(x, x+k), \quad x > 0,$$

whence, by (12), the function $\varphi := f$ satisfies the functional equation

$$\varphi(x+1) = \varphi(x) + F(x) \quad x > 0,$$

where F is defined by (9). Thus, the function $\varphi = f$ satisfies equation (10) and, clearly, condition (11). This completes the proof. $\square$

Remark 3.2. Since, by (1) and (2), we have

$$\mathcal{B}(1, 1+k) = \frac{\Gamma(1)\Gamma(1+k)}{\Gamma(1+(1+k))} = \frac{k\Gamma(k)}{(1+k)\Gamma(k+1)} = \frac{k\Gamma(k)}{(1+k)k\Gamma(k)} = \frac{1}{1+k},$$

condition (7) is equivalent to $\phi(1) = \frac{1}{k+1}$.

Remark 3.3. Since the Beta function is symmetric, the above theorem remains true on replacing the function $(0, \infty) \ni x \mapsto \log \circ \mathcal{B}(x, x+k)$ by $(0, \infty) \ni x \mapsto \log \circ \mathcal{B}(x+k, x)$.

From Theorem 2.2 and Theorem 3.1 we obtain the following

Corollary 3.4. Let $\mathcal{B}$ denote the Beta function and let $k \geq 0$ be arbitrarily fixed. If $\phi: (0, \infty) \rightarrow (0, \infty)$ satisfies equation (6), condition (7), and ϕ is logarithmically convex in an interval (a, ∞) for some $a > 0$, then (8) holds true.

This corollary can substantially be strengthened with the aid of geometrical convexity:

Theorem 3.5. Let $\mathcal{B}$ denote the Beta function and let $k \geq 0$ be arbitrarily fixed. If $\phi: (0, \infty) \rightarrow (0, \infty)$ satisfies equation (6), condition (7), and ϕ is geometrically convex in an interval (a, ∞) for some $a > 0$, then (8) holds true. Moreover, for all $x > 0$,

$$\phi(x) = c \lim_{n \rightarrow \infty} \left(\frac{n(n+k)}{(2n+k+1)(2n+k)} \right)^{-\frac{\log(n+1+x) - \log(n+1)}{\log(n+1) - \log n}} \frac{(2x+k+1)(2x+k)}{x(x+k)} \prod_{j=1}^n \frac{j(j+k)(2(j+x)+k+1)(2(j+x)+1)}{(j+x)(j+x+k)(2j+k+1)(2j+k)},$$

where $c = \frac{1}{(k+1)}.$

Proof. Define $G: (0, \infty) \rightarrow (0, \infty)$ by

$$G(x) = \frac{x(x+k)}{(2x+k+1)(2x+k)}.$$

In the proof of Theorem 3.1 we have shown the concavity of the function $F := \log \circ G$, and that

$$\lim_{x \rightarrow \infty} [\log G(x+1) - \log G(x)] = 0,$$

which implies

$$\lim_{x \rightarrow \infty} \frac{G(x+\delta)}{G(x)} = 1$$

with $\delta = 1$. Thus all assumptions of Theorem 2.4 are satisfied. Consequently, for every $c > 0$, there exists exactly one solution $\phi: (0, \infty) \rightarrow (0, \infty)$ of the functional equation

$$\phi(x+1) = G(x)\phi(x), \quad x > 0,$$

such that ϕ is geometrically convex on (a, ∞) for some $a \geq 0$ and $\phi(1) = c$. The only thing left that has to be verified is the validity of

$$\phi(x) = \mathcal{B}(x, x+k), \quad x > 0.$$

Since $x \mapsto \mathcal{B}(x, x+k)$ is an increasing function and, by Theorem 3.1, logarithmically convex, the mapping $x \mapsto \mathcal{B}(x, x+k)$ is also geometrically convex, see Lemma 1 in [6]. Now clearly, putting $c = \mathcal{B}(1, 1+k)$, yields the claim. $\square$

4. Beta-type functions, their equality and a characterization of the Gamma function

Taking into account the specific form (2) of the Beta function, we introduce a broader class of two-variable *beta-type functions* and we examine their properties.

Definition 4.1. Let $a \geq 0$ be arbitrary fixed, and let $\gamma: (a, \infty) \rightarrow (0, \infty)$. The function $B_\gamma: (a, \infty)^2 \rightarrow (0, \infty)$,

$$B_\gamma(x, y) := \frac{\gamma(x)\gamma(y)}{\gamma(x+y)}, \quad x, y > a,$$

is called *beta-type function*, and the function γ is said to be its *generator*.

Remark 4.2. Taking here $a = 0$ and $\gamma = \Gamma$ where Γ is the Euler Gamma function we obtain

$$\mathcal{B} = B_\Gamma.$$

Thus the Beta function, denoted by $\mathcal{B}$, is a beta-type function, and the Euler Gamma function is its generator.

Theorem 4.3. Let $a \geq 0$ be fixed, and let $\gamma_1, \gamma_2: (a, \infty) \rightarrow (0, \infty)$ be arbitrary functions.

(i) The equality $B_{\gamma_1} = B_{\gamma_2}$ holds if, and only if, there exists a (unique) exponential function $E: \mathbb{R} \rightarrow (0, \infty)$, i.e. satisfying the functional equation

$$E(x+y) = E(x)E(y), \quad x, y \in \mathbb{R},$$

such that $\gamma_2(x) = E(x)\gamma_1(x)$ if $x > a$.

(ii) Assume that the function $\frac{\gamma_2}{\gamma_1}$ is Lebesgue measurable, or continuous at a point, or its graph is not dense in $(a, \infty) \times (0, \infty)$. Then $B_{\gamma_1} = B_{\gamma_2}$ if, and only if, there is (a unique) $c \in \mathbb{R}$ such that

$$\gamma_2(x) = e^{cx}\gamma_1(x), \quad x > a.$$

Proof. Assume that $B_{\gamma_1} = B_{\gamma_2}$ i.e., by Definition 4.1, we have

$$\frac{\gamma_2(x)\gamma_2(y)}{\gamma_2(x+y)} = \frac{\gamma_1(x)\gamma_1(y)}{\gamma_1(x+y)}, \quad x, y > a,$$

or, equivalently,

$$\frac{\gamma_2(x+y)}{\gamma_1(x+y)} = \frac{\gamma_2(x)\gamma_2(y)}{\gamma_1(x)\gamma_1(y)}, \quad x, y > a,$$

that is

$$\frac{\gamma_2}{\gamma_1}(x+y) = \frac{\gamma_2}{\gamma_1}(x) \frac{\gamma_2}{\gamma_1}(y), \quad x, y > a.$$

Clearly, the function $\frac{\gamma_2}{\gamma_1}$ can be extended to a unique exponential function $E: \mathbb{R} \rightarrow (0, \infty)$, which proves (i).

Part (ii) is a consequence of the suitable classical results (Aczél [1], Chapter II; Kuczma [11], Chapter XIII). $\square$

Hence, taking into account the last remark, we obtain

Corollary 4.4. *Let $\gamma: (0, \infty) \rightarrow (0, \infty)$ be an arbitrary function. Then*

(i) *the equality $B_\gamma = \mathcal{B}$ holds if, and only if, there is an exponential function $E: \mathbb{R} \rightarrow (0, \infty)$ such that $\gamma(x) = E(x)\Gamma(x)$ for $x > 0$.*

(ii) *Assume, in addition, that γ is Lebesgue measurable, or continuous at a point, or its graph is not dense in $(0, \infty)^2$; then $B_\gamma = \mathcal{B}$ if, and only if, there is $c \in \mathbb{R}$ such that $\gamma(x) = e^{cx}\Gamma(x)$ for $x > 0$.*

5. Directional convexity and concluding corollary

This section is motivated by the main result of section 3.

Let $C \subset \mathbb{R}^k$ be an open and convex set and let a vector $\mathbf{h} \in \mathbb{R}^k$, $\mathbf{h} \neq \mathbf{0}$, be fixed. Then, for every $\mathbf{x} \in C$, the set

$$I_{\mathbf{x}, \mathbf{h}} := \{t \in \mathbb{R} : \mathbf{x} + t\mathbf{h} \in C\}$$

is a nonempty open interval and $0 \in I_{\mathbf{x}, \mathbf{h}}$.

We say that a function $f: C \rightarrow \mathbb{R}$ is *convex (resp. concave, affine) in the direction of the vector $\mathbf{h}$* , if for every $\mathbf{x} \in C$, the function $\varphi_{\mathbf{x}, \mathbf{h}}: I_{\mathbf{x}, \mathbf{h}} \rightarrow \mathbb{R}$ defined by

$$\varphi_{\mathbf{x}, \mathbf{h}}(t) := f(\mathbf{x} + t\mathbf{h}), \quad t \in I_{\mathbf{x}, \mathbf{h}},$$

is convex (resp. concave, affine).

Of course, $f: C \rightarrow \mathbb{R}$ is *convex in the direction of the vector $\mathbf{h}$* , if and only if for every $r \in \mathbb{R}$, $r \neq 0$, the function $f: C \rightarrow \mathbb{R}$ is *convex in the direction of the vector $r\mathbf{h}$* .

Remark 5.1. Let $C \subset \mathbb{R}^k$ be an open convex set and $\mathbf{h} \in \mathbb{R}^k$, and suppose that $f: C \rightarrow \mathbb{R}$ is twice differentiable. Then

(i) f is *convex in the direction of the vector $\mathbf{h}$* , iff for every $\mathbf{x} \in C$,

$$(\varphi_{\mathbf{x}, \mathbf{h}})''(0) \geq 0;$$

(ii) if $k = 2$ and $\mathbf{h} = (u, v)$, $u^2 + v^2 \neq 0$, then f is convex in the direction of the vector (u, v) , iff for every $(x, y) \in C$,

$$(\varphi_{(x,y), (u,v)})''(0) = \frac{\partial^2 f}{\partial x^2}(x, y) u^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(x, y) uv + \frac{\partial^2 f}{\partial y^2}(x, y) v^2 \geq 0;$$

moreover, f is convex, iff for every vector $(u, v) \in \mathbb{R}^2$, the function f is convex in the direction of the vector (u, v) , that is iff, (cf. [9] pp. 78–81), for all $(x, y) \in C$ and for all $(u, v) \in \mathbb{R}^2$,

$$(\varphi_{(x,y), (u,v)})''(0) = \frac{\partial^2 f}{\partial x^2}(x, y) u^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(x, y) uv + \frac{\partial^2 f}{\partial y^2}(x, y) v^2 \geq 0.$$

In particular, in view of the second part of this remark, a twice differentiable function f defined on a convex set $C \subset \mathbb{R}^2$ is convex in the direction of the vector $(1, 1)$ (or in the direction of the main diagonal), iff

$$\frac{\partial^2 f}{\partial x^2}(x, y) + 2\frac{\partial^2 f}{\partial x \partial y}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) \geq 0, \quad (x, y) \in C.$$

Example 5.2. The function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x^2 + 5xy + y^2$, is not convex in the direction of the vector $(1, -1)$, as $(\varphi_{(x,y),(1,-1)})''(0) = -1$ for every $(x, y) \in \mathbb{R}^2$; in particular, f is not convex. Since $(\varphi_{(x,y),(1,1)})''(0) = 9$ for all $(x, y) \in \mathbb{R}^2$, the function f is convex in the direction of the vector $(1, 1)$.

Applying Theorem 4, we obtain the following

Corollary 5.3. *A function $B: (0, \infty)^2 \rightarrow (0, \infty)$ is symmetric, logarithmically convex in the direction of the vector $(1, 1)$, and*

$$B(x+1, x+y+1) = \frac{x(x+y)}{(2x+y+1)(2x+y)} B(x, y), \quad x, y > 0;$$

$$B(1, 1+y) = \mathcal{B}(1, 1+y), \quad y > 0;$$

if, and only if, $B = \mathcal{B}$.

Let us mention that some notions of directional convexity appeared in real analysis [12] as well as in complex analysis [8].

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