Journal of Convex Analysis
Volume 25 (2018), No. 3, 957–982

Topological Derivative: Semidifferential via Minkowski Content

Michel C. Delfour*
Centre de Recherches Mathématiques et Dép. de Mathématiques et de Statistique, Université de Montréal, Montréal (Qc), Canada H3C 3J7
delfour@crm.umontreal.ca

Received: October 11, 2016
Revised manuscript received: July 13, 2017
Accepted: July 14, 2017

The generic notions of *shape* and *topological derivative* have proven to be both pertinent and useful from the theoretical and numerical points of view. The shape derivative is a differential while the topological derivative obtained by expansion methods is only a *semidifferential*. This arises from the fact that the tangent space to the underlying metric spaces of "geometries" is only a cone. We extend its definition to perturbations obtained by creating holes around curves, surfaces, and, potentially, microstructures. In that context, the *Hadamard semidifferential* that retains the advantages of the standard differential calculus including the *chain rule* and the fact that semiconvex functions are Hadamard semidifferentiable is a natural notion to study the semidifferentiability of objective functions with respect to the sets/geometries that belong to complete non-linear non-convex metric spaces. An important advantage for *state constrained objective functions* is that theorems on the one-sided differentiation of minimax of Lagrangians can be used to get the semidifferential. The same adjoint system will occur for shape and topological derivatives. The paper is specialized to the topological derivative on the metric space of characteristic functions.

Keywords: Topological derivative, semidifferential, Minkowski content, Lagrangian, minimax.

2010 Mathematics Subject Classification: 90C, 65K10

1. Introduction

In *Shape/Topological Analysis* and *Optimization*, the generic notions of *shape* and *topological derivatives* have proven pertinent and useful from the theoretical and numerical points of view. The *shape derivative* is a *differential* on some appropriate metric spaces while the *topological derivative* is not and is usually obtained by expansion methods. A *non-differentiability* can arise either from the non-differentiability of the function or the fact that the space on which the function is defined does not have a linear tangent space at each of its points. In the latter case, it is still possible to introduce a *semidifferential*, that is, a *one-*

*This research was supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada. The author wishes to thank the referee for her/his very pertinent comments on tangent cones.

ISSN 0944-6532 / $ 2.50

sided directional derivative, defined for the directions of the *adjacent tangent cone*[1] to the underlying space.

For instance, several direct constructions of complete *metric spaces* of *subsets of a hold-all* [15] and, more recently, additional new ones [11] have been introduced without appealing to the classical notions of atlases or smooth manifolds encountered in classical Differential Geometry. Since, at best, such spaces are groups, a semidifferential is the best that can be expected when the tangent space is not linear. In such cases, it is imperative to characterize the associated adjacent tangent cones.

In this general context, the definition of a differentiable function introduced by J. Hadamard [21] in 1923 is especially interesting since it implicitly involves the construction of trajectories (or paths) and tangent vectors to trajectories living in the space under investigation. His definition was relaxed by M. Fréchet [20] in 1937 by dropping the requirement that the differential be linear with respect to the directions or tangent vectors while preserving two important properties of the differential calculus: the continuity of the function and the chain rule. A vast literature on differentials on topological spaces followed (cf., for instance, the survey papers of V. I. Averbuh and O. G. Smoljanov [6], M. Z. Nashed [24]). The definition of Fréchet can be further relaxed to the one of *semidifferential* which nicely handles convex and semiconvex functions while preserving the above two properties.

Since the semidifferential is not required to be linear, it has far reaching consequences for a function $f\colon A \to B$ between two arbitrary sets A and B of some ambient topological vector spaces. De facto, the requirement that the tangent spaces in each points of the sets A and B be linear spaces can be dropped and it is sufficient to work with the adjacent tangent cones to A and B to make sense of a semidifferential. Shortcircuiting the requirement of a smooth manifold makes it sufficient to study the adjacent cones to metric spaces of sets/shapes/geometries.

In this paper, we review some classical results for the *shape derivative* which turns out to be a *differential* on the group of diffeomorphisms introduced by Micheletti [22] in 1972 endowed with her *Courant metric*. We show that the topological derivative rigorously introduced in 1999 by Sokołowski and Żochowski [27] is a *semidiferential* and extend its definition to perturbations by holes around curves, surfaces, and potentially microstructures. An important advantage of this point of view is that general theorems about the differentiation of minimax of Lagrangians can be used to compute and justify the final expression. In particular, the *adjoint system* associated with a given objective function and a state equation will be the same for both the shape and the topological derivatives.

[1]We use the terminology *adjacent tangent cone* of Aubin and Frankowska [3, pp. 126–128] for the Dubovitski-Milyutin tangent cone (see Definition 3.3).

To be specific and to illustrate the abstract notions, the paper is specialized to the metric group of characteristic functions of Lebesgue measurable subsets of $\mathbf{R}^N$ with a series of examples. We show that its tangent space is only a cone that contains distributions and measures associated with closed subsets of arbitrary dimension leading to general topological perturbations along curves and surfaces that can break the connectivity of the set. The more we know about the adjacent tangent cone, the more necessary optimality conditions we are likely to get to characterize an optimal set.

2. Group structures and analytical representations

2.1. Shape optimization

In *Shape Optimization* the variable sets are paramerized by functions or are the images of a fixed *reference set* by a family of homeomorphisms or diffeomorphisms. As a result, only the shape of the set is variable while the *topology* of the reference set is preserved by the images.

In 1972 Micheletti [22] introduced what she called the *Courant metric* on images of a smooth C^k open set by a family of C^k diffeomorphisms (Figure 2.1) and

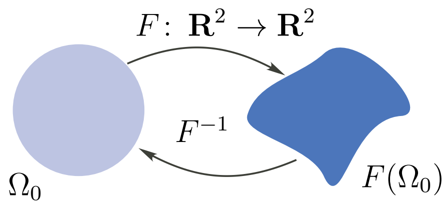


Figure 2.1: Image of a fixed set Ω_0 by a diffeomorphism $F\colon \mathbf{R}^2 \to \mathbf{R}^2$

showed that it was a complete metric group. Her construction was extended in the 1st edition of the book of Delfour and Zolésio [15] to other families of diffeomorphisms including the Lipschitzian case for which Murat and Simon [23] had only obtained a semi-metric in 1976. The construction was also extended to arbitrary closed sets and crack-free open sets (no smoothness required). Images of a segment, a circle, a surface yield families of curves, closed curves, surfaces (Figure 2.2). Such metric groups are infinite dimensional *Finsler manifolds*

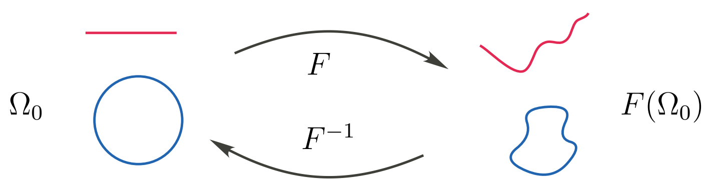


Figure 2.2: Image of a fixed line or circle by a diffeomorphism F

whose tangent space is the set of C^k or Lipschitz mappings from $\mathbf{R}^N$ to $\mathbf{R}^N$.

They are analogous to the group of invertible $N \times N$-matrices for which the tangent space is the whole space of $N \times N$-matrices. The differential calculus in Micheletti spaces is provided by the *Velocity Method* as a differential with respect to the elements (velocities) of the tangent space which generate the diffeomorphisms of the group: this is the *Eulerian differential* and the *Shape Gradient* (also called *shape derivative*) introduced in the 1979 thesis of Zolésio [33]. It is fair to say that the theory is complete for the spaces of Micheletti and the closely related spaces of Trouvé [30, 31] with applications to image processing (book of Younes [32]).

2.2. Optimization with respect to a family of subsets of a set

Ideally, it is desirable to free ourselves from images of a fixed set via diffeomorphisms and, more generally, from the notion of manifold with a smooth structure. For instance, choose as variables the subsets of a fixed non-empty *hold-all* D in the Euclidean N-dimensional space $\mathbf{R}^N$, $N \geq 1$. Consider the *symmetric difference* $\triangle$ of two sets A and B (see Figure 2.3) in the *power set* $\mathcal{P}(D) \stackrel{\text{def}}{=} \{A : A \subset D\}$:

$$A \triangle B \stackrel{\text{def}}{=} [A \cap \complement_D B] \cup [B \cap \complement_D A] = [A \cup B] \cap \complement_D [A \cap B].$$

This operation induces an *Abelian group* structure on $\mathcal{P}(\mathcal{D})$:

$$A \triangle B = B \triangle A, \quad (A \triangle B) \triangle C = A \triangle (B \triangle C),$$

$\varnothing$ is the *neutral element*, $A \triangle \varnothing = A$, and every element A of $\mathcal{P}(D)$ is its own inverse, $A \triangle A = \varnothing$. There is a one-to-one correspondence between subsets

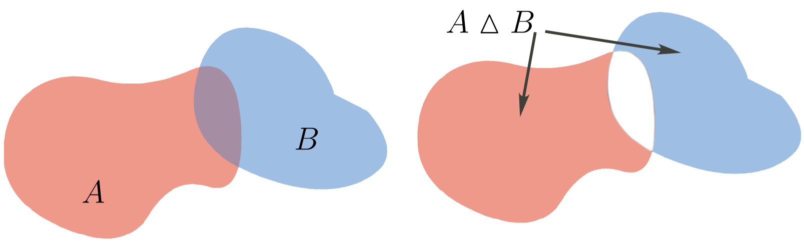


Figure 2.3: Symmetric difference of A and B

$A \in \mathcal{P}(D)$ and the set of *characteristic functions* $2^D \stackrel{\text{def}}{=} \{\chi \colon D \to \{0,1\}\}$:

$$A \mapsto \chi_A : \mathcal{P}(D) \to 2^D, \quad \chi_A(x) \stackrel{\text{def}}{=} \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{1}$$

This induces an Abelian group structure and a complete metric structure on 2^D:

$$(\chi_A \triangle \chi_B)(x) \stackrel{\text{def}}{=} \chi_{A \triangle B}(x) = |\chi_A(x) - \chi_B(x)|, \quad \rho(A,B) \stackrel{\text{def}}{=} \sup_{x \in D} |\chi_A(x) - \chi_B(x)|.$$

Given the N-dimensional Lebesgue measure m_N on $\mathbf{R}^N$, the construction extends to the characteristic functions of Lebesgue measurable subsets of a Lebesgue measurable hold-all $D \subset \mathbf{R}^N$:

$$[\Omega] \overset{\text{def}}{=} \{\Omega' : \Omega' = \Omega \quad \mathrm{m}_N\text{-a.e.}\}\,, \quad [\chi_\Omega] \overset{\text{def}}{=} \{\chi_{\Omega'} : \Omega' = \Omega \quad \mathrm{m}_N\text{-a.e.}\} \tag{2}$$
$$[\Omega] \longleftrightarrow [\chi_\Omega] \in \mathrm{X}(D) \overset{\text{def}}{=} \{[\chi_\Omega] : \Omega \subset D \text{ Lebesque measurable}\}\,.$$

$\mathrm{X}(D)$ is a closed subset of $L^\infty(D, m_N)$ and $L^p_{\text{loc}}(D, m_N)$, $1 \le p < \infty$, and this makes it a *complete metric Abelian group* in the norm topology of $L^\infty(D)$ and in the Fréchet topology of $L^p_{\text{loc}}(D, m_N)$, $1 \le p < \infty$.

Other *set parametrized functions* can be used to construct metric spaces of equivalence classes of sets, For instance,

► Identify a set with its *distance function*: Hausdorff (1914)-Pompéju (1905) metric, sets of positive reach (Federer 1959), set-valued analysis, ...

$$d_\Omega(x) \overset{\text{def}}{=} \begin{Bmatrix} \inf_{y \in \Omega} \|y - x\|, & \text{if } \Omega \neq \varnothing \\ +\infty, & \text{if } \Omega = \varnothing \end{Bmatrix}, \quad \begin{matrix} C_d(\mathbf{R}^N) \overset{\text{def}}{=} \left\{d_\Omega : \varnothing \neq \Omega \subset \mathbf{R}^N\right\} \\ [\Omega]_d \overset{\text{def}}{=} \left\{\Omega' : \overline{\Omega'} = \overline{\Omega}\right\} \end{matrix}$$
$$[\Omega]_d \mapsto d_\Omega \in C_d(\mathbf{R}^N) \subset C^0_{\text{loc}}(\mathbf{R}^N),\, C^{0,1}_{\text{loc}}(\mathbf{R}^N),\, W^{1,p}_{\text{loc}}(\mathbf{R}^N),\, 1 \le p < \infty.$$

$C_d(\mathbf{R}^N)$ is a closed subset of each of the above spaces endowed with their Fréchet topology. So it is a (non-linear, non-convex) complete metric space.

► Identify a set with its *oriented distance function*: geometry of hypersurfaces, tangential calculus on hypersurfaces...

$$b_\Omega(x) \overset{\text{def}}{=} d_\Omega(x) - d_{\complement\Omega}(x), \quad \begin{matrix} C_b(\mathbf{R}^N) \overset{\text{def}}{=} \left\{b_\Omega : \Omega \subset \mathbf{R}^N,\, \partial\Omega \neq \varnothing\right\} \\ [\Omega]_b \overset{\text{def}}{=} \left\{\Omega' : \overline{\Omega'} = \overline{\Omega} \text{ and } \partial\Omega' = \partial\Omega\right\} \end{matrix}$$
$$[\Omega]_b \longleftrightarrow b_\Omega \in C_b(\mathbf{R}^N) \subset C^0_{\text{loc}}(\mathbf{R}^N),\, C^{0,1}_{\text{loc}}(\mathbf{R}^N),\, W^{1,p}_{\text{loc}}(\mathbf{R}^N),\, 1 \le p < \infty.$$

$C_b(\mathbf{R}^N$ is a closed subset of each of the above spaces endowed with their Fréchet topology. So it is a (non-linear, non-convex) complete metric space.

Additional metric spaces can be found in [11] and the group structure can be restored in the cases of the distance functions d_Ω and b_Ω by introducing equivalent metrics whose completions include the missing empty set and $\mathbf{R}^N$. A common feature shared by those spaces is to have no interior in the surrounding vector space where they live. As a result, their tangent spaces will be more difficult to characterize and will generally not be vector spaces as for Riemannian or Finsler manifolds. Finite dimensional analogues would be the smooth sphere in $\mathbf{R}^3$ or a continuous curve in $\mathbf{R}^2$ with a kink where only two half tangents exist.

2.3. Topological derivative

We have seen at least two types of analytical representations of sets: parametrize sets by functions or parametrize functions by sets. The topological derivative turns out to be associated with the complete metric spaces of the second type described in the last section 2.2. Our main objective is to characterize the *tangent space* to such spaces and make sense of *semidifferentials* of objective functions with respect to general topological perturbations of sets. An example of an objective function which is a boundary integral will be given in section 5.

3. Hadamard differential and semidifferential

3.1. Hadamard differential

In 1923 J. Hadamard [21] gave a (geometrical) definition of a differential by introducing an *auxiliary function* $t \mapsto x(t)\colon \mathbf{R} \to \mathbf{R}^N$ of the *auxiliary variable* t such that

$$x(0) = a \quad \text{and} \quad x'(0) \overset{\text{def}}{=} \lim_{t\to 0} \frac{x(t)-a}{t} \text{ exists in } \mathbf{R}^N .$$

This auxiliary function defines a *path* that induces a perturbation or a variation of the point a. We shall use the terminology *time* for the auxiliary variable t and *admissible trajectory* for the auxiliary function x. Note that x need not be continuous or differentiable at $t \neq 0$. The *vector* $x'(0)$ *is the tangent to the trajectory* x at the point $x(0) = a$. Scaling t by an arbitrary non-zero real number generates a whole line tangent to x at a.

Definition 3.1. (Hadamard differentiability) A function $f\colon \mathbf{R}^N \to \mathbf{R}^K$ is *Hadamard differentiable* at $a \in \mathbf{R}^N$ if

(i) for all *admissible trajectories* x at a the limit

$$(f \circ x)'(0) \overset{\text{def}}{=} \lim_{t\to 0} \frac{f(x(t)) - f(a)}{t} \text{ exists in } \mathbf{R}^K$$

(ii) and there exists a *linear function* $Df(a)\colon \mathbf{R}^N \to \mathbf{R}^K$ such that for all admissible trajectories x at a: $(f \circ x)'(0) = Df(a)\,(x'(0))$.

$Df(a)$ can be identified with the *Jacobian matrix*. □

The definition of Hadamard differentiability is equivalent to the one of Fréchet differentiability in finite dimension (classical analysis). In Banach spaces, a Hadamard differentiable function at a point a is continuous at a and the chain rule is applicable.

In 1937, Fréchet [20] insisted on the fact that the definition of Hadamard is more general than his since it extends to functions $f\colon X \to \mathbf{R}^K$ defined on topological vector spaces X that are not normed vector spaces. For instance, in Banach spaces of functions, we can get the set of *tangent vectors (functions)* $x'(0)$ as *weak limits* ...and even as distributions.

There is an equivalent *analytical characterization* that is often used as a definition.

Theorem 3.2. (Analytical Definition) *A function* $f\colon \mathbf{R}^N \to \mathbf{R}^K$ *is Hadamard differentiable at* $a \in \mathbf{R}^N$ *if and only if*

(i) *for all* $v \in \mathbf{R}^N$ *the limit*

$$d_H f(a; v) \stackrel{\text{def}}{=} \lim_{\substack{t\to 0\\ w\to v}} \frac{f(a+tw)-f(a)}{t} \quad \textit{exists}$$

(ii) *and the function* $v \mapsto Df(a)v \stackrel{\text{def}}{=} d_H f(a;v)\colon \mathbf{R}^N \to \mathbf{R}^K$ *is linear.*

Fréchet [20] observed that, in function spaces, the Hadamard differentiability is not only a notion more general than the one [19] he introduced in 1911 but that the linearity in Definition 3.1 (ii) is not necessary to preserve the continuity of the function and the chain rule.

3.2. Hadamard semidifferential in linear spaces

However, relaxing the linearity was not sufficient for some convex continuous functions. For instance, the norm $x \mapsto f(x) = \|x\|\colon \mathbf{R}^N \to \mathbf{R}$ at $x = 0$ is not differentiable in the relaxed sense of Fréchet. To fix this, we need the notion of *semidifferential*.

Consider a *half trajectory* (or semi-trajectory) $x\colon [0,\delta) \to \mathbf{R}^N$, $\delta > 0$, such that $x(0) = 0 \in \mathbf{R}^N$ and the right-hand side tangent $x'(0^+)$ exists. At the origin 0, we now get

$$(f\circ x)'(0^+) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{f(x(t))-f(0)}{t} = \lim_{t\searrow 0} \left\| \frac{x(t)-x(0)}{t} \right\| = \|x'(0^+)\|,$$

where the notation $t \searrow 0$ means that t goes to 0 by strictly positive values. The same notion applies to *convex and semiconvex* continuous functions. When $(f\circ x)'(0^+)$ is a linear function of $x'(0^+)$ we recover a *differential*.

3.3. Hadamard semidifferential: from smooth to arbitrary sets

The *hypothesis of linearity of the differential* is also a severe restriction to define a differential of a function $f\colon A \subset \mathbf{R}^N \to B \subset \mathbf{R}^K$ since it requires that the *tangent space to* A at a and the *tangent space to* B at $f(a)$ be linear subspaces. The sets A and B need be *sufficiently smooth* in the sense that, at each point of A and B, the tangent spaces be linear subspaces.

Since the Hadamard semidifferential does not require the linearity of the tangent space, the *a priori smoothness assumption* of the sets A and B can *de* *facto* be dropped and, in view of the positive homogeneity, the semidifferential is well-defined on a tangent cone. Several tangent cones are available in the literature, but the *adjacent tangent cone* is especially well suited for semidifferentials. We give the following equivalent definition (see Aubin and Frankowska [3, pp. 126–128]).

Definition 3.3. The *adjacent tangent cone* to A at $a \in \overline{A}$ is the set

$$T_a^\flat A \stackrel{\text{def}}{=} \left\{ v \in \mathbf{R}^N : \forall \{t_n \searrow 0\},\, \exists \{x_n\} \subset A \text{ such that } \lim_{n\to\infty} \frac{x_n - a}{t_n} = v \right\}$$

($T_a^\flat A$ is a closed cone and $T_a^\flat \overline{A} = T_a^\flat A$).

Recall that a necessary optimality condition for a minimizer $a \in A$ of a Hadamard semidifferentiable function $f\colon A \to \mathbf{R}$ is $df(a;v) \geq 0$ for all $v \in T_a^\flat A$. So a complete knowledge of the tangent cone $T_a^\flat A$ is essential to fully extract information about the minimizer.

When the boundary ∂A of A is smooth, $T_a^\flat A$ is a linear subspace of $\mathbf{R}^N$ as in Figure 3.1, but, the linearity of $T_a^\flat A$ puts a severe restriction on the sets A.

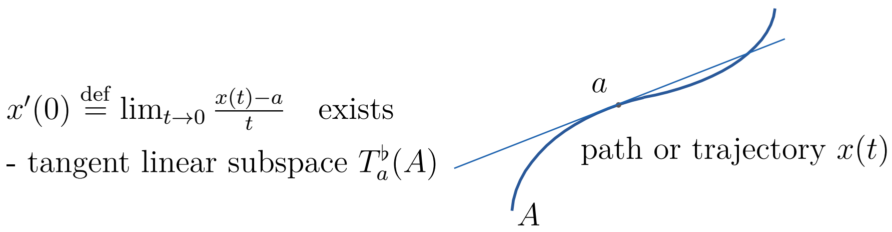


Figure 3.1: Tangent $x'(0))$ to the path $x(t)$ in A at the point $x(0) = a$.

For instance, the requirement that $T_a^\flat A$ be linear rules out a curve in $\mathbf{R}^2$ with kinks as shown in Figure 3.2.

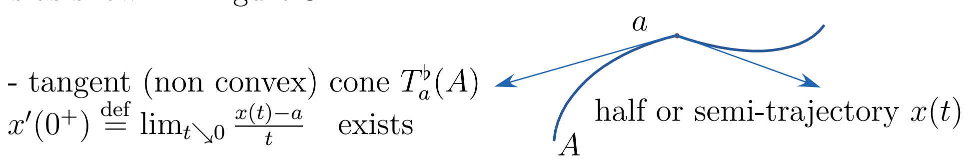


Figure 3.2: Half-tangent $x'(0^+))$ to the path $x(t)$ in A at the point $x(0) = a$.

This naturally leads to the following notions of admissible semi-trajectory.

Definition 3.4. Given $A \subset \mathbf{R}^N$, an *admissible semi-trajectory* (or *half trajectory*) in A at $a \in \overline{A}$ is a function $x\colon [0,\tau] \to A$, $\tau > 0$, such the *semi-tangent* (or *half tangent*) at a

$$x'(0^+) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{x(t) - a}{t}$$

exists (when the limit $x'(0^+)$ exists, it follows that $x(t) \to a$ as $t \searrow 0$).

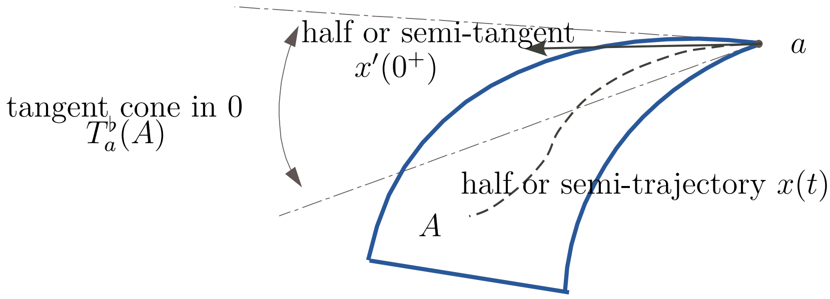


Figure 3.3: Half or semi-tangent $x'(0^+)$ to the semi-trajectory $x(t)$ in A at the point $x(0) = a$.

This gives an equivalent characterization of the adjacent tangent cone.

Theorem 3.5. $T^\flat_a A = \{x'(0^+) : x$ *is an admissible semi-trajectory in* A *at* $a\}$.

Following Fréchet, we now relax the linearity and formalize the notion of semi-differential for functions $f \colon A \to B$.

Definition 3.6. (Geometrical definition) Given $A \subset \mathbf{R}^N$ and $B \subset \mathbf{R}^K$, the function $f \colon A \to B$ is *Hadamard semidifferentiable at* $a \in A$ if

(i) for each *admissible semi-trajectory* x in A at a, the limit

$$(f \circ x)'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{f(x(t)) - f(a)}{t} \quad \text{exists, and}$$

(ii) there exists a (positively homogeneous) function $v \mapsto d_H f(a; v) \colon T^\flat_a A \to T^\flat_{f(a)} B$ such that for each *admissible semi-trajectory* x in A at a

$$(f \circ x)'(0^+) = d_H f(a; x'(0^+)).$$

The function $v \mapsto d_H f(a)(v) = d_H f(a; v)$ is referred to as the *(tangential) semidifferential of* f at $a \in A$. It can be shown that $d_H f(a)$ is continuous on $T^\flat_a A$. □

This definition has an equivalent analytical form.

Theorem 3.7. (Analytical definition) *Given* $A \subset \mathbf{R}^N$ *and* $B \subset \mathbf{R}^K$*, the function* $f \colon A \to B$ *is* Hadamard semidifferentiable at $a \in A$ *if and only if there exists a (positively homogeneous) function* $v \mapsto d_H f(a; v) \colon T^\flat_a A \to T^\flat_{f(a)} B$ *such that for all* $v \in T^\flat_a A$ *and all sequences* $\{x_n\} \subset A$ *and* $\{t_n \searrow 0\}$ *such that* $(x_n - a)/t_n \to v$

$$\lim_{n \to \infty} \frac{f(x_n) - f(a)}{t_n} = d_H f(a; v).$$

Remark 3.8. In the case of linear topological spaces A and B, the *geometrical Definition* 3.6 of the *semidifferential* appears explicitly in J. Durdil [16, p. 457] in 1973 and the *analytical definition* of Theorem 3.7 in J. P. Penot [26, p. 250] in 1978 under the name M-semi-differentiability. □

With the above definitions, the two important properties are preserved: continuity at a and the chain rule. In addition, we have a very flexible framework to do *Semidifferential Geometry* that reduces to *Differential Geometry* when linearity is added.

Definition 3.9. (Hadamard differentiability) Given $A \subset \mathbf{R}^N$ and $B \subset \mathbf{R}^K$, the function $f\colon A \to B$ is *Hadamard differentiable at* $a \in A$ if

(i) f is semidifferentiable at a,

(ii) $T^\flat_a A$ and $T^\flat_{f(a)} B$ are *linear* and $v \mapsto d_H f(a;v)\colon T^\flat_a A \to T_{f(a)} B$ is *linear*.

The semidifferential $d_H f(a)$ will be called the *differential* of f and denoted $Df(a)$. □

Remark 3.10. The previous definitions extend to subsets A of topological vector spaces X, but we have to be careful and retain the abstract notions that are really meaningful. There was an important activity on extensions to topological vector spaces in the sixties (cf., for instance, de Foglio [8] in 1959). Since then a considerable literature on differentials in topological vector spaces has appeared (see the well-documented 207-page paper of M. Z. Nashed [24] for a rather complete account until 1971, and V. I. Averbuh and O. G. Smoljanov [4, 5, 6] in 1987 and 1988). Yet, most of this material is not immediately relevant for our purposes.

3.4. Extension to Banach spaces

The geometrical Definition 3.6 of Hadamard semidifferentiability and the previous considerations extend to Banach and Fréchet spaces. Fréchet spaces with a countable family of semi-norms naturally occur to deal with convergence on compact subsets of $\mathbf{R}^N$ to allow unbounded sets as variables.

Theorem 3.11. *Let $f\colon A \to B$ be a function between subsets $A \subset X$ and $B \subset Y$ of two Banach spaces X and Y.*

(i) *If $d_H f(a,v)$ exists at $a \in A$ for all $v \in T^\flat_a A$, then $v \mapsto d_H f(a,v)\colon T^\flat_a A \to T^\flat_{f(a)} B$ is continuous, where $T^\flat_a A$ and $T^\flat_{f(a)} B$ are endowed with the relative topology induced by X and Y.*

(ii) *If $d_H f(a;0)$ exists at $a \in A$, then f is continuous at a.*

(iii) *Let $g\colon B \to C$ be a second function for a subset C of a third Banach space Z. Assume that f and g are respectively Hadamard semidifferentiable at a and $f(a)$. Then the composition $g \circ f$ is Hadamard semidifferentiable and $d_H(g \circ f)(a;v) = d_H g(f(a); d_H f(a;v))$.*

Remark 3.12. This is the version for the *strong* Hadamard semidifferential. There is a similar theorem for the *weak* Hadamard semidifferential

$$d_H^{\mathrm{w}} f(x;v) \overset{\text{def}}{=} \lim_{\substack{t\searrow 0\\ w \underset{\text{weakly}}{\rightharpoonup} v}} \frac{f(x+tw)-f(x)}{t}, \quad d_H^{\mathrm{s}} f(x;v) \overset{\text{def}}{=} \lim_{\substack{t\searrow 0\\ w \underset{\text{strongly}}{\longrightarrow} v}} \frac{f(x+tw)-f(x)}{t}.$$

In reflexive Banach spaces a function is Fréchet differentiable if and only it is weakly Hadamard semidifferentiable and $v \mapsto d_H^{\mathrm{w}} f(x;v)$ is linear [15, Thm. 2.1, p. 462].

4. Metric Group of Characteristic Functions

Consider the metric Abelian group of characteristic functions of Lebesgue measurable subsets in $\mathbf{R}^N$

$$\mathrm{X}(\mathbf{R}^N) \overset{\text{def}}{=} \left\{\chi_\Omega : \Omega \subset \mathbf{R}^N \text{ Lebesgue measurable}\right\} \subset L^\infty(\mathbf{R}^N).$$

It is a closed subset without interior of the Banach space $L^\infty(\mathbf{R}^N)$ and of the Fréchet spaces $L^p_{\text{loc}}(\mathbf{R}^N)$, $1 \le p < \infty$. The finite dimensional analog would be the sphere in $\mathbf{R}^3$. Similar considerations apply to the *oriented (signed) distance functions*, but this is beyond the scope of the present paper.

4.1. Velocity Method

Given $V \in C^0([0,\tau]; C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$ and the family of diffeomorphisms $\{T_t(V) : t \ge 0\}$ generated by V,

$$\frac{dT_t(V)}{dt} = V(t) \circ T_t(V),\; T_0(V) = I,$$

the *semitangent* at χ_Ω to the continuous trajectory $t \mapsto \chi_{T_t(V)(\Omega)} \colon [0,1] \to \mathrm{X}(\mathbf{R}^N)$ in $\mathrm{X}(\mathbf{R}^N)$ is obtained by considering the limit of the differential quotient

$$\frac{1}{t}\left(\chi_{T_t(V)(\Omega)} - \chi_\Omega\right) \in L^\infty(\mathbf{R}^N)$$

which does not exist in $L^\infty(\mathbf{R}^N)$ and $L^p_{\text{loc}}(\mathbf{R}^N)$, $1 \le p < \infty$. To get around that, consider the distribution associated with $\chi_{T_t(V)(\Omega)}$

$$\phi \mapsto \int_{\mathbf{R}^N} \chi_{T_t(V)(\Omega)}\, \phi\, dx = \int_{T_t(V)(\Omega)} \phi\, dx = \int_\Omega \phi \circ T_t\; \det DT_t\, dx \colon \mathcal{D}(\mathbf{R}^N) \to \mathbf{R}\,.$$

Since $V(0) \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, then

$$\left.\frac{d}{dt}\right|_{t=0^+} \int_\Omega \phi \circ T_t\; \det DT_t\, dx = \int_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx = \int_{\mathbf{R}^N} \chi_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx$$

(see, for instance, [15, Thm. 4.1, Chapter 9, p. 483]). The bilinear function

$$(\phi, V) \mapsto \int_{\mathbf{R}^N} \chi_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx : H^1_0(\mathbf{R}^N) \times C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N) \to \mathbf{R}$$

is continuous. This generates the continuous linear mapping

$$V \mapsto \nabla\chi_\Omega \cdot V : C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N) \to H^{-1}(\mathbf{R}^N),$$
$$(\nabla\chi_\Omega \cdot V)\,\phi \overset{\text{def}}{=} \int_{\mathbf{R}^N} \chi_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx,$$

where $\nabla\chi_\Omega$ is the distributional gradient of χ_Ω. The support of $\nabla\chi_\Omega \cdot V$ is in Γ. So, the tangent space to $\mathrm{X}(\mathbf{R}^N)$ (considered as a subset of the *space of distributions*) at χ_Ω contains the linear subspace

$$\left\{\nabla\chi_\Omega \cdot V : V \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)\right\} \subset H^{-1}(\mathbf{R}^N) \subset \mathcal{D}(D)'$$

of functions in $H^{-1}(\mathbf{R}^N)$.

4.2. Topological Derivative as a Dilatation

The rigorous introduction of the *topological derivative* in 1999 by Sokołowski and Zòchowski [27]) (also the book by Novotny-Sokołowski [25]) provided a broader spectrum of notions of *derivative with respect to a set.* In its initial form, the topological derivative is obtained by introducing a *small hole* around a point $a \in \Omega$ of an open domain Ω, that is, a dilatation of the point a. Mathematically speaking, it is the Lebesgue-Besicovitch Differentiation Theorem. This idea readily extends to families of closed subsets E of Ω of dimension d, $1 \le d \le N-1$, whose Hausdorff measure $H^d(E)$ is finite.

Consider several examples for an open subset $\Omega \subset$ of $\mathbf{R}^N$. Given $r > 0$ and the distance function d_E to a subset E of $\mathbf{R}^N$, introduce the *r-dilatation* E_r of E

$$E_r \overset{\text{def}}{=} \left\{x \in \mathbf{R}^N : d_E(x) \le r\right\}, \quad d_E(x) \overset{\text{def}}{=} \inf_{e \in E} |x - e|. \tag{3}$$

Example 4.1. Let $N = 3$, $a \in \mathbf{R}^3$, $E = \{a\}$, $\dim E = 0$, $E_r = \overline{B_r(a)}$ (see Figure 4.1).

$$E = \{a\}, \quad E_r = \overline{B_r(a)}$$
$$t = \text{ volume of } \overline{B_r(a)} = \tfrac{4}{3}\pi\, r^3$$

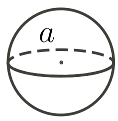


Figure 4.1: E_r is the closed ball of center a.

The function $\phi \mapsto \phi(a) : \mathcal{D}(\mathbf{R}^3) \to \mathbf{R}$ is a measure. The time t is chosen as the volume of $\overline{B_r(a)}$. Given an open subset Ω in $\mathbf{R}^3$ and $a \in \Omega$, the perturbed set will be

$$t \mapsto \Omega_t \overset{\text{def}}{=} \Omega \backslash \overline{B_r(a)} = \Omega \backslash B_{\sqrt[3]{t/\alpha_3}}(a).$$

The trajectory $t \mapsto \chi_{\Omega_t}$ is continuous in $X(\mathbf{R}^3)$. Given $\phi \in \mathcal{D}(\mathbf{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned}\frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right] &= -\frac{1}{\text{volume }(B_{\sqrt[3]{t/\alpha_3}}(a))} \int_{B_{\sqrt[3]{t/\alpha_3}}(a)} \chi_\Omega\, \phi\, dx \\ &= -\frac{1}{\text{volume }(B_r(a))} \int_{B_r(a)} \chi_\Omega\, \phi\, dx \to -\phi(a).\end{aligned}$$

The limit is a distribution (measure), that is $-\delta(a)$, the negative of the Dirac delta function at a. It generates a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_\Omega \phi\, dx\right] \to -\rho\phi(a),$$

but not a full tangent. Note that we can also create points by introducing the perturbed sets $\Omega_t = \Omega \cup B_{(t/\alpha_N)^{1/N}}(b)$ to get $+\phi(b)$, $b \in \mathbf{R}^N \setminus \overline{\Omega}$. $\square$

4.3. More topological perturbations and Minkowski content

4.3.1. Minkowski content

As was illustrated in Example 4.1, continuous trajectories in $X(\mathbf{R}^N)$ can be constructed by the r-dilatation of a point $a \in \operatorname{int}\Omega$ that creates a hole. In this process, the time t, the auxiliary variable in the Hadamard semidifferential, is the volume of the ball of radius r and not the dilatation parameter r itself. We now look at more general topological perturbations obtained by dilation of curves, hypersurfaces, or similar objects E in $\mathbf{R}^N$ or in submanifolds of $\mathbf{R}^N$. Saying that the limit of the dilatation of a set converges as the dilatation goes to zero is directly related to the existence of the notions of *Minkowski content* of the set E.

Definition 4.2. Given d, $0 \le d \le N$, the *d-dimensional upper and lower Minkowski contents* of a set E are defined through an r-dilatation of the set E as follows

$$M^{*d}(E) \overset{\text{def}}{=} \limsup_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}}, \quad M_*^d(E) \overset{\text{def}}{=} \liminf_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}}, \tag{4}$$

where m_N is the Lebesgue measure in $\mathbf{R}^N$ and α_{N-d} is the volume of the unit ball in $\mathbf{R}^{N-d}$. Since the dilatation E_r does not distinguish between E and its closure, it can be assumed that E is closed in $\mathbf{R}^N$. When the two limits exist and are equal, we say that E admits a d-dimensional *Minkowski content* and their common value will be denoted $M^d(E)$.

Intuitively, $M^d(E)$ is a measure of the d-dimensional "area" or "volume" of an object E in $\mathbf{R}^N$. It plays a role similar to the *d-dimensional Hausdorff or Radon measure* in $\mathbf{R}^N$, but it is generally not a measure.

In this paper, we are interested in closed subsets E of $\mathbf{R}^N$ such that

$$M^d(E) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{\mathrm{m}_N(E_r)}{\alpha_{N-d}\, r^{N-d}} \tag{5}$$

and for which M^d would be a *measure* such that the following distribution

$$\phi \mapsto \int_E \phi\, dM^d = \lim_{r \searrow 0} \frac{1}{\alpha_{N-d}\, r^{N-d}} \int_{\mathbf{R}^N} \chi_{E_r} \phi\, d\mathrm{m}_N : \mathcal{D}(\mathbf{R}^N) \to \mathbf{R}\,. \tag{6}$$

makes sense. Therefore, in applications to d-dimensional objects, $0 \le d \le N$, the appropriate choice of time, that is, the auxiliary variable, is the volume $t = \alpha_{N-d}\, r^{N-d}$ of the ball of radius r in $\mathbf{R}^{N-d}$, that is, $r = (t/\alpha_{N-d})^{1/(N-d)}$,

$$\phi \mapsto \int_E \phi\, dM^d = \lim_{t \searrow 0} \frac{1}{t} \int_{E_{(t/\alpha_{N-d})^{1/(N-d)}}} \phi\, d\mathrm{m}_N : \mathcal{D}(\mathbf{R}^N) \to \mathbf{R}\,. \tag{7}$$

Given a Lebesgue measurable set $\Omega \subset \mathbf{R}^N$, several perturbations can be introduced:

$$\Omega_t = \Omega \backslash E_r, \quad \Omega_t = \Omega \cup E_r, \text{ and } \Omega_t = \Omega \vartriangle E_r, \quad t = \alpha_{N-d}\, r^{N-d}, \tag{8}$$

depending on whether E_r is removed, added, or both removed and added. In each case a continuous trajectory $t \mapsto \chi_{\Omega_t}$ is obtained in $\mathrm{X}(\mathbf{R}^N)$ such that

$$\chi_{\Omega_t} \to \chi_{\Omega \backslash E}, \quad \chi_{\Omega_t} \to \chi_{\Omega \cup E}, \text{ and } \chi_{\Omega_t} \to \chi_{\Omega \vartriangle E} \text{ in } L^p_{\mathrm{loc}}(\mathbf{R}^N),\ 1 \le p < \infty.$$

If $\mathrm{m}_N(E) = 0$, then, in all cases, $\chi_{\Omega_t} \to \chi_\Omega$ in $L^p_{\mathrm{loc}}(\mathbf{R}^N)$, $1 \le p < \infty$. If $\Omega \subset \mathbf{R}^N$ is open and E is a closed subset of Ω, the expected semitangents are

$$-\int_E \chi_\Omega\, \phi\, dM^d, \quad 0, \quad -\int_E \chi_\Omega\, \phi\, dM^d; \tag{9}$$

if $\Omega \subset \mathbf{R}^N$ is open and E is a closed subset of $\mathbf{R}^N \backslash \overline{\Omega}$, the respective semitangents are

$$0, \quad \int_E (1 - \chi_\Omega)\, \phi\, dM^d, \quad \int_E (1 - \chi_\Omega)\, \phi\, dM^d. \tag{10}$$

4.3.2. Examples

We proceed with more examples. H^d will denote the d-dimensional Hausdorff measure.

Example 4.3. Let A be a bounded open subset of $\mathbf{R}^N$ of class $C^{1,1}$. Its boundary ∂A is a $C^{1,1}$ hypersurface in $\mathbf{R}^N$ of dimension $N-1$. Let b_A be the *oriented (signed) distance function.* If ∂A is compact, there exists $\varepsilon > 0$ such that $b_A \in$

$C^{1,1}(U_\varepsilon(\partial A))$ in the ε-*tubular neighbourhood* $U_\varepsilon(\partial A) = \{x \in \mathbf{R}^3 : |b_A(x)| < \varepsilon\}$ of ∂A. The *projection* of points of $U_\varepsilon(\partial A)$ onto ∂A is $p_{\partial A}(x) = x - b_A(x)\,\nabla b_A(x)$.

Choose $E = \partial A$ and consider the perturbations

$$\Omega_t = \Omega\backslash E_r,\ t = 2r = \alpha_1 r,\ \chi_{\Omega_t} \to \chi_{\Omega\backslash E} = \chi_\Omega \text{ in } L^p(\mathbf{R}^N)\text{-strong},\ 1 \le p < \infty,$$

since $\mathrm{m}_N(\partial A) = 0$. It is easy to check that

$$\begin{aligned}\lim_{t\searrow 0} \frac{1}{t}\int_{\mathbf{R}^N} \chi_{\Omega_t}\,\phi\,d\mathrm{m}_N &= \lim_{r\searrow 0} \frac{1}{\alpha_1 r}\int_{\mathbf{R}^N} \chi_{\Omega_{2r}}\,\phi\,d\mathrm{m}_N \\ &= -\int_E \chi_\Omega\,\phi\,dH^{N-1} = -\int_E \phi\,dH^{N-1}.\end{aligned}$$

Given $\phi \in \mathcal{D}(\mathbf{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned}\frac{1}{t}\left[\int_{\Omega_t} \phi\,d\mathrm{m}_N - \int_\Omega \phi\,d\mathrm{m}_N\right] &= -\frac{1}{t}\int_{U_{t/2}(E)} \chi_\Omega\,\phi\,d\mathrm{m}_N \\ &= -\frac{1}{\alpha_1 r}\int_{U_r(E)} \chi_\Omega\,\phi\,d\mathrm{m}_N \to -\int_E \phi\,dH^2.\end{aligned}$$

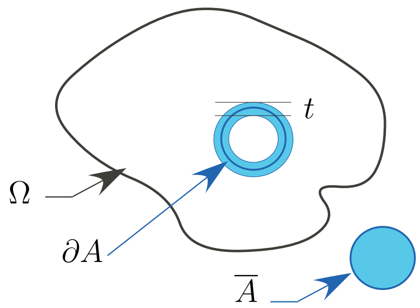


Figure 4.2: Ω_t has two disjoint components.

This distribution is a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\,d\mathrm{m}_N - \int_\Omega \phi\,d\mathrm{m}_N\right] \to -\rho\int_E \phi\,dH^2.$$

Note that when $E = \partial A \subset \Omega$, removing the $t/2$-dilated hypersurface ∂A creates two disjoint components $\Omega_t\backslash A_{t/2}$ and $\Omega_t \cap A$ (see Figure 4.2). □

Example 4.4. Let A be a compact C^2-submanifold of $\mathbf{R}^N$ of dimension d, $1 \le d \le N-1$ such that $H^d(A) < \infty$. In that case $\partial A = A$ and $\mathrm{m}_N(A) = 0$. For instance, A could be a closed non-intersecting C^2-curve in $\mathbf{R}^3$ without boundary. Assuming that there exists $\varepsilon > 0$ such that $d_A^2 \in C^2(U_\varepsilon(A))$.

$U_\varepsilon(A) = \{x \in \mathbf{R}^N : d_A(x) < \varepsilon\}$, the *projection* onto ∂A is $p_A(x) = x - \frac{1}{2}\nabla d_A^2(x)$, $Dp_A(x) = I - \frac{1}{2}D^2 d_A^2(x)$, $\operatorname{Im} Dp_A(x)$ is the *tangent space* at $x \in A$ to A, and $d = \dim A(x) = \dim(\operatorname{Im} Dp_A(x))$.

Let $E = A$, $r > 0$, $t = \alpha_{N-d}\, r^{N-d}$, and the r-*dilatation*

$$E_r \overset{\text{def}}{=} \{x \in \mathbf{R}^N : d_A(x) \le r\}$$

Given $U_\varepsilon(E) \subset \Omega$, the perturbed sets will be

$$t \mapsto \Omega_t \overset{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash E_{(t/\alpha_{N-d})^{N-d}}$$

Given $\phi \in \mathcal{D}(\mathbf{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned}\frac{1}{t}\left[\int_{\Omega_t} \phi\, d\mathrm{m}_N - \int_\Omega \phi\, d\mathrm{m}_N\right] &= -\frac{1}{t}\int_{E_{(t/\alpha_{N-d})^{N-d}}} \chi_\Omega\, \phi\, d\mathrm{m}_N \\ &= -\frac{1}{\alpha_{N-d}\, r^{N-d}}\int_{E_r} \chi_\Omega\, \phi\, d\mathrm{m}_N \to -\int_E \phi\, dH^d.\end{aligned}$$

This distribution (measure) is again a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, d\mathrm{m}_N - \int_\Omega \phi\, d\mathrm{m}_N\right] \to -\rho\int_E \phi\, dH^d.$$

4.4. Rectifiable sets: Dilatation and Minkowski content

4.4.1. Rectifiable sets and densities

Thanks to the pioneering and seminal work of Federer [18] and the extension of his work by Ambrosio et al [2], the previous constructions can be readily extended to dilation of some families of rectifiable sets for which the d-dimensional Minkowski content is equal to the d-dimensional Hausdorff measure:

$$M^d(E) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{\mathrm{m}_N(E_r)}{\alpha_{N-d}\, r^{N-d}} = H^d(E). \tag{11}$$

This property is related to the property of d-rectifiability in $\mathbf{R}^N$.

Definition 4.5. (Federer [18, pp. 251–252]) Let E be a subset of a metric space X. $E \subset X$ is d-rectifiable if it is the image of a compact subset K of $\mathbf{R}^d$ by a Lipschitz continuous function $f\colon\ \mathbf{R}^d \to X$. □

Theorem 4.6. ([18, p. 275]) *If $E \subset \mathbf{R}^N$ is compact and d-rectifiable, then $M^d(E) = H^d(E)$.*

There are several extensions of the notion of d rectifiability to non-compact sets.

Definition 4.7. (Rectifiable sets [1, Dfn. 2.57, p. 80]) Let $E \subset \mathbf{R}^N$ be an H^d-measurable set.

(i) E is *countably d-rectifiable* if there exist countably many Lipschitzian functions $f_i\colon\ \mathbf{R}^d \to \mathbf{R}^N$ such that

$$E \subset \cup_{i=0}^{\infty} f_i(\mathbf{R}^d). \tag{12}$$

(ii) E is *countably H^d-rectifiable* if there exist countably many Lipschitz functions $f_i\colon\ \mathbf{R}^d \to \mathbf{R}^N$ such that $E \backslash \cup_i f_i(\mathbf{R}^d)$ is H^d-negligible, that is,

$$H^d\left(E \backslash \cup_{i=0}^{\infty} f_i(\mathbf{R}^d)\right) = 0. \tag{13}$$

(iii) E is *H^d-rectifiable* if E is *countably H^d-rectifiable* and $H^d(E) < \infty$. $\square$

Ambrosio et al. [1] have remarked that any d-rectifiable compact set E satisfies the following property: there exists $\gamma > 0$ and a probability measure η in $\mathbf{R}^N$ satisfying

$$\eta(B_r(x)) \geq \gamma\, r^d, \quad \forall r \in (0,1), \forall x \in E. \tag{14}$$

If $f\colon\ \mathbf{R}^d \to \mathbf{R}^N$ is the Lipschitz map of constant $\ell > 0$, $E = f(K)$, $K_{1/\ell}$ is the $1/\ell$-dilatation of E, and $c = H^d(K_{1/\ell})$, the probability measure is

$$\eta(B) = \frac{1}{c} H^d(K_{1/\ell} \cap f^{-1}(B)) = 1$$

because

$$\eta(B_r(f(x))) \geq \frac{1}{c} H^d(K_{1/\ell}) = \frac{\alpha_d}{cL^d} r^d, \quad \forall x \in K, \forall r \in (0,1).$$

They refer to this property as a *density lower bound*. Under that hypothesis, they extend Theorem 4.6 to countably H^d-rectifiable compact sets.

Theorem 4.8. ([2, p. 110]) *Let $E \subset \mathbf{R}^N$ be a countably H^d-rectifiable compact set and assume that* (14) *holds for some $\gamma > 0$ and some Radon measure η in $\mathbf{R}^N$ which is absolutely continuous with respect to H^d. Then $M^d(E) = H^d(E)$.*

For the dilatation of a point $E = \{x\}$, we have used the Lebesgue-Besicovitch Differentiation Theorem for a Lebesgue measurable set A which involves the *N-dimensional density* of A in $\mathbf{R}^N$

$$\Theta_N(A,x) \stackrel{\text{def}}{=} \lim_{r \searrow 0} \frac{\mathrm{m}_N(B_r(x) \cap A)}{\mathrm{m}_N(B_r(x))} = \chi_A(x) \quad \mathrm{m}_N\text{-a.e. in } \mathbf{R}^N. \tag{15}$$

There is something similar for a H^d-rectifiable set E.

Definition 4.9. (d-dimensional densities [2, Dfn. 2.55, p. 78]) Given a Borel subset E of $\mathbf{R}^N$ such that $H^d(E) < +\infty$, the *upper* and *lower d-dimensional densities* of H^d at x are respectively defined as

$$\Theta_d^*(E,x) \stackrel{\text{def}}{=} \limsup_{r\searrow 0} \frac{H^d(B_r(x)\cap E)}{\alpha_d\, r^d}, \quad \Theta_{*d}(E,x) \stackrel{\text{def}}{=} \liminf_{r\searrow 0} \frac{H^d(B_r(x)\cap E)}{\alpha_d\, r^d},$$

and, if they agree, we denote the common value by $\Theta_d(E,x)$. □

Assume that $E \subset \mathbf{R}^N$ is an H^d-measurable set with $H^d(E) < \infty$. In the extreme cases $d = 0$ and $d = N$, we know that the density $\Theta_d(E,x)$ exists and coincides with the *characteristic function* χ_E for H^d-a.e. $x \in \mathbf{R}^N$. On the other hand, if $0 < d < N$, the only information can be summarized by (cf. [2, sec. 2.9, p. 79]])

$$\Theta_d(E,x) = 0 \text{ for } H^d\text{-a.e. } x \in \mathbf{R}^N \setminus E \tag{16}$$

$$2^{-d} \leq \Theta_d^*(E,x) \leq 1 \text{ for } H^d\text{-a.e. } x \in E. \tag{17}$$

Theorem 4.10. (Besicovitch-Marstrand-Mattila [2, Thm. 2.63, p. 83]) *Let E be a Borel set in $\mathbf{R}^N$ with $H^d(E) < \infty$. Then, E is H^d-rectifiable if and only if $\Theta_d(E,x) = 1$ for H^d-a.e. $x \in E$ (that is, $\Theta_d(E,x) = \chi_E(x)$ for H^d-a.e. $x \in \mathbf{R}^N$).*

4.5. Sets of positive reach: Dilatations and orthogonal dilatations

4.5.1. Sets of positive reach and Steiner formula

The reader will notice that in all the examples of section 4.3.2 the closed set E has positive reach.

We give the following definitions for an A which is not necessarily closed.

Definition 4.11. Let A, $\varnothing \neq A \subset \mathbf{R}^N$. Denote by d_A the distance function of x to A.

(i) The *set of all projections onto* $\overline{A}$ will be denoted by

$$\Pi_A(x) \stackrel{\text{def}}{=} \left\{p \in \overline{A} : |p - x| = d_A(x)\right\}. \tag{18}$$

When $\Pi_A(x)$ is a singleton, its element will be denoted by $p_A(x)$.

(ii) The *skeleton of A* is the set of all points of $\mathbf{R}^N$ whose projection onto $\overline{A}$ is not unique. It will be denoted by

$$\text{Sk}(A) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N : \Pi_A(x) \text{ is not a singleton}\right\}. \tag{19}$$

Note that $\text{Sk}(\overline{A}) = \text{Sk}(A)$ since $d_A = d_{\overline{A}}$. □

Definition 4.12. Given A, $\varnothing \neq A \subset \mathbf{R}^N$, and a point $a \in \overline{A}$

$$\operatorname{reach}(A,a) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } a \in \partial\overline{A} \cap \overline{\mathrm{Sk}(A)}, \\ \sup\left\{r > 0 : \mathrm{Sk}(A) \cap B_r(a) = \varnothing\right\}, & \text{otherwise,} \end{cases}$$

$$\operatorname{reach}(A) \overset{\text{def}}{=} \inf\left\{\operatorname{reach}(A,a) : a \in A\right\}.$$

The set A is said to have *positive reach* if $\operatorname{reach}(A) > 0$.

Remark 4.13. (1) Since $\mathrm{Sk}(\overline{A}) = \mathrm{Sk}(A)$, $\operatorname{reach}(\overline{A}) = \operatorname{reach}(A)$.

(2) For any set A, $\mathrm{Sk}(A) \subset \mathrm{Sk}(\partial A)$ and

$$\forall a \in \partial A, \quad 0 \leq \operatorname{reach}(\partial A, a) \leq \operatorname{reach}(A,a) \leq +\infty.$$

(3) For all $a \notin \partial\overline{A} \cap \overline{\mathrm{Sk}(A)}$, $\operatorname{reach}(A,a) > 0$.

For such sets Federer [17, Thm. 5.6, p. 455] has extended the *Steiner formula* and, as a consequence, they have a *Minkowski content* for some d, $0 \leq d \leq N$.

Theorem 4.14. *Given a closed subset $A \subset \mathbf{R}^N$ of positive reach, there exist unique* Radon measures $\psi_0, \psi_1, , \dots, \psi_N$ *over $\mathbf{R}^N$ such that, for all $0 \leq r <$* $\operatorname{reach}(A)$*,*

$$\mathrm{m}_N\left(\left\{x \in \mathbf{R}^N : d_A(x) \leq r \text{ and } p_A(x) \in B\right\}\right) = \sum_{m=0}^{N} \alpha_{N-m}\, r^{N-m}\, \psi_m(B), \quad (20)$$

whenever B is a Borel set of $\mathbf{R}^N$, and, consequently,

$$\int\limits_{\{x:\, d_A(x) \leq r,\, p_A(x) \in B\}} f \circ p_A \, d\mathrm{m}_N = \sum_{m=0}^{N} \alpha_{N-m}\, r^{N-m} \int_B f\, d\psi_m, \quad (21)$$

whenever f is a bounded real valued Baire function on $\mathbf{R}^N$ with bounded support.

4.5.2. Dilatation of A and normal dilatations of subsets E of ∂A

Setting $B = A$ in Theorem 4.14, then, for $0 \leq r < \operatorname{reach}(A)$, we get the r-dilatation of A

$$\left\{x \in \mathbf{R}^N : d_A(x) \leq r \text{ and } p_A(x) \in A\right\} = \left\{x \in \mathbf{R}^N : d_A(x) \leq r\right\} = A_r. \quad (22)$$

If A is compact

$$H^d(A_r) = \sum_{m=0}^{N} \alpha_{N-m}\, r^{N-m}\, \Phi_m(A), \quad (23)$$

where $\Phi_i(A) = \psi_i(A)$, $0 \le i \le N$, are called *total curvatures* of A, Moreover, the Minkowski content is the first of the $N+1$ Radon measures for which the limit is finite and non-zero:

$$\exists d,\ 0 \le d \le N, \quad M^d(A) \stackrel{\text{def}}{=} \lim_{r \searrow 0} \frac{\mathrm{m}_N(A_r)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(A). \tag{24}$$

In particular, if A is compact and reach$A > 0$, then ∂A is $(N-1)$-rectifiable and $M^{N-1}(A) = \Phi_{N-1}(A)$ (cf. [1, Prop. 1, p. 743, Cor. 3, p. 744]).

Another use of the theorem is for closed subsets E of ∂A. Here the set

$$E_r^A \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N : d_A(x) \le r \text{ and } p_A(x) \in E\right\}, \quad 0 \le r < \mathrm{reach}(A), \quad E \subset \partial A,$$

corresponds to an *orthogonal dilatation* with respect to A. So we can rewrite all the examples of the previous section with such closed subsets $E \subset \partial A$ and the orthogonal dilatation E_r^A. Compared to the section on rectifiable sets, it seems that the closed subset E of ∂A need not be a d-rectifiable set for some d.

4.5.3. Examples of orthogonal dilatations

We add the following examples to the ones of section 4.3.2.

Example 4.15. Let A be an open subset of $\mathbf{R}^N$ of class $C^{1,1}$ of Example 4.3 and ∂A the associated compact hypersurface of dimension $N-1$.

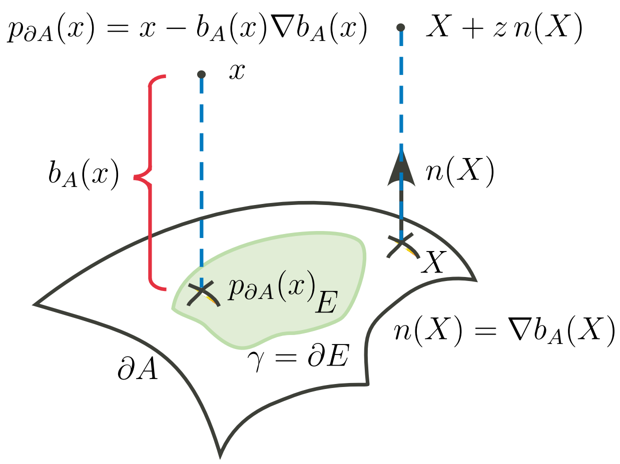
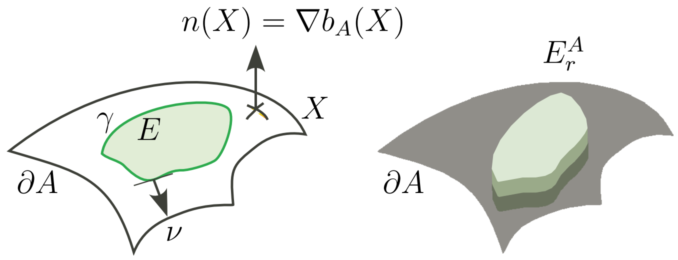


Figure 4.3: Mapping $x \mapsto (p_{\partial A}(x), b_A(x)) \colon U_\varepsilon(\partial A) \to \partial A \times (-\varepsilon, \varepsilon)$.

Let E be a closed subset of ∂A (Figure 4.3).

Consider E_r^A (see Figure 4.4) and the perturbed sets

$$\Omega_t = \Omega \backslash E_r^A, \quad t = 2r = \alpha_1 r, \quad \chi_{\Omega_t} \to \chi_{\Omega\backslash E} = \chi_\Omega \text{ a.e.} \tag{25}$$



(a) E with boundary γ (b) Sandwich E_r^A

Figure 4.4: Orthogonal dilatation (sandwich or thin shell) E_r^A

since $\mathrm{m}_N(\partial A) = 0$. It is easy to check that

$$\lim_{t \searrow 0} \frac{1}{t} \int_{\mathbf{R}^N} \chi_{\Omega_t} \, \phi \, d\mathrm{m}_N = \lim_{r \searrow 0} \frac{1}{\alpha_1 r} \int_{\mathbf{R}^N} \chi_{\Omega_{2r}} \, \phi \, d\mathrm{m}_N =$$
$$= -\int_E \chi_\Omega \, \phi \, d\psi_{N-1} = -\int_E \phi \, d\psi_{N-1}.$$

Given $\phi \in \mathcal{D}(\mathbf{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\frac{1}{t}\left[\int_{\Omega_t} \phi \, d\mathrm{m}_N - \int_\Omega \phi \, d\mathrm{m}_N\right] = -\frac{1}{t} \int_{E^A_{t/2}} \chi_\Omega \, \phi \, d\mathrm{m}_N =$$
$$= -\frac{1}{\alpha_1 r} \int_{E_r^A} \chi_\Omega \, \phi \, d\mathrm{m}_N \to -\int_E \phi \, d\psi_{N-1}.$$

This distribution is a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi \, d\mathrm{m}_N - \int_\Omega \phi \, d\mathrm{m}_N\right] \to -\rho \int_E \phi \, d\psi_{N-1}. \qquad \square$$

Example 4.16. Let A be a C^2-submanifold of $\mathbf{R}^N$ of dimension d, $1 \le d \le N-1$ such that A is locally H^d finite. In that case $\partial A = A$ and $\mathrm{m}_N(A) = 0$. For instance, A could be a closed non-intersecting C^2-curve in $\mathbf{R}^3$ without boundary. Here, there exists $\varepsilon > 0$ such that $d_A^2 \in C^2(U_\varepsilon(A))$. $U_\varepsilon(A) = \left\{x \in \mathbf{R}^N : d_A(x) < \varepsilon\right\}$, the *projection* onto ∂A is $p_A(x) = x - \frac{1}{2}\nabla d_A^2(x)$, $Dp_A(x) = I - \frac{1}{2}D^2 d_A^2(x)$, $\mathrm{Im}\, Dp_A(x)$ is the *tangent space* at $x \in A$ to A, and $d = \dim A(x) = \dim\left(\mathrm{Im}\, Dp_A(x)\right)$.

Let $E \subset \partial A$ be closed, $r > 0$, $t = \alpha_{N-d}\, r^{N-d}$, and the *orthogonal dilatation* (Figure 4.5)

$$E_r^A \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N : d_A(x) \le r, \quad p_A(x) \in E\right\}.$$

Assuming that $U_\varepsilon(\partial A) \subset \Omega$ for some $\varepsilon > 0$ and $0 < r < \varepsilon$, the perturbed set will be

$$t \mapsto \Omega_t \stackrel{\text{def}}{=} \Omega \backslash E_r^A = \Omega \backslash E^A_{(t/\alpha_{N-d})^{N-d}}.$$

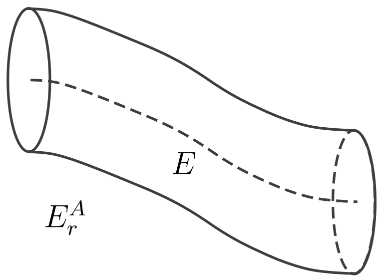


Figure 4.5: Orthogonal dilatation (or spaguetti) E_r^A around the curve E.

Given $\phi \in \mathcal{D}(\mathbf{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\frac{1}{t}\left[\int_{\Omega_t} \phi\, dm_N - \int_\Omega \phi\, dm_N\right] = -\frac{1}{t}\int_{E^A_{(t/\alpha_{N-d})^{N-d}}} \chi_\Omega\, \phi\, dm_N =$$
$$= -\frac{1}{\alpha_{N-d}\, r^{N-d}} \int_{E_r^A} \chi_\Omega\, \phi\, dm_N \to -\int_E \phi\, dH^d.$$

This distribution is again a *half tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dm_N - \int_\Omega \phi\, dm_N\right] \to -\rho \int_E \phi\, dH^d.$$

Another example where the connectivity could be checked is the following.

Example 4.17. Consider the square of side 1 in $\mathbf{R}^2$

$$\Omega \stackrel{\text{def}}{=} \left\{(x,y) \in \mathbf{R}^2 : |x| \le 1,\ |y| \le 1\right\}, \quad E \stackrel{\text{def}}{=} \left\{(x,y) \in \mathbf{R}^2 : |x| \le 1,\ y = 0\right\}.$$

E is a closed subset of the vertical line

$$A \stackrel{\text{def}}{=} \left\{(x,y) \in \mathbf{R}^2 : y = 0\right\}.$$

which has positive reach with $\partial A = A$. Given $0 < r < 1$, $t = 2r$, the perturbed domains $\Omega_t = \Omega \backslash E_r^A$ have two disjoint connected components. □

5. Semidifferential of boundary integrals

Given an open subset Ω of $\mathbf{R}^N$ with boundary $\Gamma = \partial\Omega$, consider a *boundary* objective function of the form

$$J(\Omega) \stackrel{\text{def}}{=} \int_\Gamma f\, dH^{N-1} \tag{26}$$

for some sufficiently smooth function $f\colon V(\Gamma) \to \mathbf{R}$ defined in a neighbourhood of Γ. we want to compute the topological derivative (semidifferential) with respect $E = \partial\Omega$

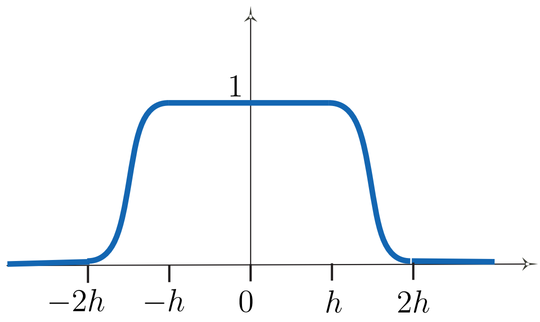


Figure 5.1: The function ρ_h with parameter h

For the boundary integral, it is useful to introduce the following technique which will also be used in the next section. Let $\rho_h\colon \mathbf{R} \to [0,1]$, $h > 0$, be the smooth *cut-off function* (see Figure 5.1) in $\mathcal{D}(\mathbf{R})$ such that

$$\rho_h(y) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } |y| \le h, \\ \in (0,1), & \text{if } h < |y| < 2h, \\ 0, & \text{if } |y| \ge 2h. \end{cases} \tag{27}$$

Let b_Ω be the *oriented (signed) distance function* defined as

$$b_\Omega(x) \stackrel{\text{def}}{=} d_\Omega(x) - d_{\mathbf{R}^N\setminus\Omega}(x). \tag{28}$$

Given $k > 0$ and $A \subset \mathbf{R}^N$, denote by $U_k(A) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : d_A(x) < k\}$ the *k-tubular neighbourhood* of A. Since $d_{\partial\Omega}(x) = |b_\Omega(x)|$, $U_k(\partial\Omega) = \{x \in \mathbf{R}^N : |b_\Omega(x)| < k\}$. If Ω is of class C^k, $k \ge 2$, then there exists $h > 0$ such that $b_\Omega \in C^k(U_{2h}(\partial\Omega))$.

Associate with any $g\colon U_{2h}(\partial\Omega) \to \mathbf{R}$ or $g\colon \mathbf{R}^N \to \mathbf{R}$, the *cut-off function* g_h defined as

$$g_h \stackrel{\text{def}}{=} (\rho_h \circ b_\Omega)\, g \tag{29}$$

which is equal to g in the *narrow band* $U_h(\partial\Omega)$ and is zero outside $U_{2h}(\partial\Omega)$. The smoothness of g_h in $\mathbf{R}^N$ is directly related to the combined smoothness of g and b_Ω in $U_{2h}(\partial\Omega)$.

If Ω is an open domain of class $C^{1,1}$, there exists $h > 0$ such that $b_\Omega \in C^{1,1}(U_{2h}(\partial\Omega))$ and the function (introduced in [13] in 2004)

$$b_\Omega^h(x) \stackrel{\text{def}}{=} \rho_h(b_\Omega(x))\, b_\Omega(x)$$

belongs to $C^{1,1}(\overline{\mathbf{R}})$. It coincides with b_Ω in the *narrow band* $U_h(\partial\Omega)$ around $\partial\Omega$ and is zero outside of $U_{2h}(\partial\Omega)$ where the skeleton might be present.

The boundary integral in (26) can now be rewritten as follows

$$\int_{\partial\Omega} f\, dH^{N-1} = \int_{\partial\Omega} f\, \nabla b_\Omega^h \cdot n\, dH^{N-1} = \int_\Omega \operatorname{div}(f\, \nabla b_\Omega^h)\, dm_N,$$

where $n = \nabla b_\Omega$ is the outward unit normal on $\partial\Omega$.

We are now back to an integral over Ω and the general theory applies. Assuming enough smoothness for f and Ω, the semidifferential at Ω for $E = \partial\Omega$ is

$$\begin{aligned}\int_{\partial\Omega} \nabla f \cdot \nabla b_\Omega^h + f\, \Delta b_\Omega^h\, dH^{N-1} &= \int_{\partial\Omega} \nabla f \cdot n + f\, \Delta b_\Omega\, dH^{N-1} \\ &= \int_{\partial\Omega} \frac{\partial f}{\partial n} + f\, \Delta b_\Omega\, dH^{N-1},\end{aligned}$$

where the restriction of ∇b_Ω to $\partial\Omega$ is the outward unit normal and the one of Δb_Ω to $\partial\Omega$ is the *mean curvature* of $\partial\Omega$. Observe that, as it should be, the parameter h disappears in the final expression. It is the same expression as the one of the shape derivative with respect to a velocity field V

$$\int_{\partial\Omega} \left(\frac{\partial f}{\partial n} + f\, \Delta b_\Omega\right) V \cdot n\, dH^{N-1}$$

where V would be the unit normal n since $V \cdot n = n \cdot n = 1$ in the formula.

References

[1] L. Ambrosio, A. Colesanti, E. Villa: Outer Minkowski content for some classes of closed sets, Math. Ann. 342(4) (2008) 727–748.

[2] L. Ambrosio, N. Fusco, D. Pallara: Functions of Bounded Variation and Free Discontinuity Problems, The Clarendon Press, Oxford University Press, New York (2000).

[3] J.-P. Aubin, H. Frankowska: Set-Valued Analysis, Birkhäuser, Boston et al. (1990).

[4] V. I. Averbuh, O. G. Smoljanov; Differentiation in linear topological spaces (Russian), Dokl. Akad. Nauk SSSR 173 (1967) 735–738; English translation: Soviet Math. Dokl. 8 (1967) 444–448.

[5] V. I. Averbuh, O. G. Smoljanov: The theory of differentiation in linear topological spaces (Russian), Uspehi Mat. Nauk 22 (1967), no. 6 (138), 201–260; English translation: Russian Math. Surveys 22(6) (1967) 201–258.

[6] V. I. Averbuh, O. G. Smoljanov: The various definitions of the derivative in linear topological spaces (Russian), Uspehi Mat. Nauk 23 (1968), no. 4 (142), 67–116; English translation: Russian Math. Surveys 23(4) (1968) 67–113.

[7] V. I. Averbuh, O. G. Smoljanov: An addendum to the article: "Different definitions of derivative in linear topological spaces" (Russian), Uspehi Mat. Nauk 23 (1968), no. 5 (143), 223–224.

[8] S. F. L. de Foglio: Extension de la différentielle d'Hadamard-Fréchet aux applications entre deux espaces vectoriels L (French), C. R. Acad. Sci. Paris 248 (1959) 1108–1110.

[9] E. De Giorgi: Su una teoria generale della misura, Ann. Mat. Pura Appl. (4), 36 (1954) 191–213.

[10] M. C. Delfour: Introduction to Optimization and Semidifferential Calculus, MOS-SIAM Series, Philadelphia (2011).

[11] M. C. Delfour: Metrics spaces of shapes and geometries from set parametrized functions, in: "New Trends in Shape Optimization", A. Pratelli and G. Leugering (eds.), Internat. Series of Numerical Mathematics, Birkhäuser, Basel (2015) 57–101.

[12] M. C. Delfour, J.-P. Zolésio: Velocity method and Lagrangian formulation for the computation of the shape Hessian, SIAM J. Control Optim. 29(6) (1991) 1414–1442.

[13] M. C. Delfour, J.-P. Zolésio: Oriented distance function and its evolution equation for initial sets with thin boundary, SIAM J. Control Optim. 42(6) (2004) 2286–2304.

[14] M. C. Delfour, J.-P. Zolésio: Evolution equations for shapes and geometries, J. Evol. Equ. 6(3) (2006) 399–417.

[15] M. C. Delfour, J.-P. Zolésio: Shapes and Geometries, Metrics, Analysis, Differential Calculus, and Optimization (second edition), SIAM Series on Advances in Design and Control, Philadelphia (2011).

[16] J. Durdil: On Hadamard differentiability, Com. Math, Univ. Carolinae 14(3) (1973) 457–470.

[17] H. Federer: Curvature measures, Trans. Amer. Math. Soc. 93 (1959) 418–419.

[18] H. Federer: Geometric Measure Theory, Grundlehren der math. Wissenschaften 153, Springer-Verlag, Berlin et al. (1969).

[19] M. Fréchet: Sur la notion de différentielle, C. R. Acad. Sci. Paris 152 (1911) 845–847.

[20] M. Fréchet: Sur la notion de différentielle, J. Math. Pures Appl. 16 (1937) 233–250.

[21] J. Hadamard: La notion de différentielle dans l'enseignement, Scripta Univ. Ab. Bib., Hierosolymitanarum, Jerusalem (1923); reprinted in Math. Gazette 19(236) (1935) 341–342.

[22] A. M. Micheletti: Metrica per famiglie di domini limitati e proprietà generiche degli autovalori, Ann. Scuola Norm. Sup. Pisa (3), 26 (1972) 683–694.

[23] F. Murat, J. Simon: Sur le contrôle par un domaine géométrique, Rapport 76015, Université Pierre et Marie Curie, Paris (1976).

[24] M. Z. Nashed: Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials in nonlinear functional analysis, in: "Nonlinear Functional Analysis and Applications", L. B. Rail (ed.)), Proc. Advanced Sem., Math. Res. Center, Univ. of Wisconsin, Madison 1970, Academic Press, New York (1971) 103–309.

[25] A. A. Novotny, J. Sokolowski: Topological Derivatives in Shape Optimization, Interaction of Mechanics and Mathematics, Springer, Berlin et al. (2013).

[26] J. P. Penot: Calcul sous-différentiel et optimisation, J. Functional Analysis 27 (1978) 248–276.

[27] J. Sokołowski, A. Żochowski: On the topological derivative in shape optimization, SIAM J. Control Optim. 37(4) (1999) 1251–1272.

[28] J. Sokołowski, A. Żochowski: Topological derivatives for elliptic problems, in: "Conference on inverse problems, control and shape optimization" (Carthage, 1998), Inverse Problems (1), 15 (1999) 123–134.

[29] J. Sokołowski, J.-P. Zolésio: Introduction to Shape Optimization. Shape Sensitivity Analysis, Springer Ser. Comput. Math. 16, Springer-Verlag, Berlin et al. (1992).

[30] A. Trouvé: Action de groupe de dimension infinie et reconnaissance de formes, C. R. Acad. Sci. Paris, Sér. I Math. 321(8) (1995) 1031–1034.

[31] A. Trouvé: An approach of pattern recognition through infinite dimensional group actions, Rapport de recherche du LMENS, France (1995).

[32] L. Younes: Shape and Diffeomorphisms, Appl. Math. Sci. 171, Springer-Verlag, Berlin et al. (2010).

[33] J. P. Zolésio: Identification de domaines par déformation, Thèse de doctorat d'état, Université de Nice, Nice (1979).