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Property (α) on Locally Convex Spaces

Ignacio Monterde
Inst. de Matemática Pura y Aplicada, Universitat Politècnica de València,
Camino de Vera, 46022 Valencia, Spain
nachomonterde@gmail.com

Vicente Montesinos*
Inst. de Matemática Pura y Aplicada, Universitat Politècnica de València,
Camino de Vera, 46022 Valencia, Spain
vmontesinos@mat.upv.es

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We study property (α) for closed convex subsets of locally convex spaces and we prove that the quasi drop property implies property (α). Moreover, if the locally convex space is complete, property (α) implies weak compactness. Thus, if every closed, convex and bounded subset of a complete locally convex spaces E has property (α), then E is semi-reflexive.

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1. Introduction

Given a non-void closed convex subset B of a Banach space X, let

$$F(B) := \{f \in X^*, f \text{ is bounded above on } B\}.$$

For $f \in F(B)$, put $\sup f[B] := \sup\{f(x) : x \in B\}$. Given $\delta > 0$ and $f \in F(B)$, the *δ-slice* determined by f on B is defined as

$$S(f, B, \delta) := \{x \in B : f(x) \geq \sup f[B] - \delta\}.$$

The *Kuratowski index of non-compactness* $\alpha(M)$ of a subset M of X is the infimum of all $\varepsilon > 0$ such that M can be covered by a finite number of subsets of X with diameter less than ε. The set B is said to have *property (α)* if, for every $f \in F(B)$,

$$\lim_{\delta \to 0^+} \alpha[S(f, B, \delta)] = 0.$$

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B is said to have the *drop property* if for any non-empty closed subset A disjoint from B, there exists a point $a \in A$, such that $D(a,B) \cap A = \{a\}$, where $D(a,B) = \mathrm{co}(\{a\} \cup B)$ is called the *drop* defined by B and $a \notin B$.

Several authors have studied property (α) in Banach spaces (see [2], [3], [5], [6] and [8]). In [3], Rolewicz and Kutzarova proved that *in a Banach space X, every convex closed subset with drop property has property (α).* The second named author of this paper proved in [5] that *the closed unit ball of a Banach space X has property (α) if, and only if, it has drop property, if, and only if, X is reflexive and has property (KK).*

The purpose of this paper is to study property (α) in locally convex spaces. As it is customary, E' denotes the topological dual of E. Given a closed convex subset B of a locally convex space E, $F(B)$ and $S(f,B,\delta)$ are defined as above for $\delta > 0$ and $f \in F(B) \subset E'$.

2. Main results

Definition 2.1. Let B be a closed convex subset of a locally convex space $(E,\mathcal{T})$. B is said to have *property (α)* is for every $f \in F(B)$ and for every neighbourhood U of the origin, there exists $\delta > 0$ such that $S(f,B,\delta)$ can be covered by a finite number of translates of U.

Property (α) depends on the topology on E and not only on the dual pair $\langle E, E' \rangle$. Actually, if B is bounded then B has property (α) for the weak topology, since every bounded set is weakly totally bounded. However, we cannot expect to have property (α) for an arbitrary convex bounded subset of a locally convex space (see Theorem 2.6 below). There is a close connection between property (α) and another property which has been widely studied in the context of Banach and, more generally, locally convex spaces, the so-called *drop property*. There is no complete agreement on what the right extension of the Banach space concept to the locally convex case should be, and so, several definitions (with an increasing level of generality) have been proposed (see, for example, [4]):

Definition 2.2. A closed convex subset B of a locally convex space $(E,\mathcal{T})$ is said to have the *drop* (resp. *countably drop*, resp. *quasi drop*) *property* if for every nonempty sequentially closed (resp. countably closed, resp. closed) subset A of E disjoint from B there exists $a \in A$ such that $D(a,B) \cap A = \{a\}$.

B is said to have the *weak drop* (resp. *countably weak drop*, resp. *quasi weak drop*) property if it has the drop (resp. countably drop, resp. quasi drop) property for the weak topology.

The following implications are clear:

$$\text{drop} \Rightarrow \text{countably drop} \Rightarrow \text{quasi drop}.$$

Property (α) is more general than any of the previous concepts, as it is implied by the quasi drop property. This is the content of the following result.

Theorem 2.3. *Let B be a closed convex bounded subset of a locally convex space $(E, \mathcal{T})$ having the quasi drop property. Then, it has property (α).*

Proof. Suppose B does not have property (α). Then, there exists $f \in F(B)$ and an absolutely convex closed neighbourhood U of the origin, such that no $S(f, B, \delta)$ can be covered by a finite number of translates of U. Let $\|\cdot\|$ be the seminorm defined by the Minkowski gauge of U. We shall prove the following

$$\textbf{Claim:} \qquad \sup_{x \in S(f,B,\delta)} \inf_{y \in L} \|x - y\| \geq 1, \tag{1}$$

for every $\delta > 0$ and every finite-dimensional subspace $L \subset E$.

In order to prove the Claim, we proceed by contraposition; assume that for some finite-dimensional subspace L and some $\delta > 0$ we have

$$\sup_{x \in S(f,B,\delta)} \inf_{y \in L} \|x - y\| := \gamma < 1.$$

Take $1 > \beta > \gamma$. For every $x \in S(f, B, \delta)$ the set $(x + U) \cap L$ is non-empty. The set $K := \{l \in L : \exists x \in S(f, B, \delta) \text{ such that } \|x - l\| < \beta\}$ is a bounded (hence relatively compact) subset of $(L, \|\cdot\|)$; therefore, for $0 < \varepsilon < (1 - \beta)$, we can find an ε-net $\{l_1, l_2, \ldots, l_n\}$ in $(K, \|\cdot\|)$. It follows that the set $\bigcup_{i=1}^{n} l_i + U$ covers $S(f, B, \delta)$, a contradiction, and the Claim is proved.

The rest of the proof is analogous to the proof provided in [8]: let $M := \sup f[B]$. We shall construct by induction a sequence (x_n) such that $x_n \in D(x_{n-1}, B)$, $f(x_n) > M$ and $\|x_n - z\| > 1/3$ for every n and every $z \in^{(} x_i)_1^{n-1}$. As the starting point, choose any $x_1 \in E$ such that $f(x_1) > M$. Suppose we have already obtained $x_1, \ldots, x_n$ and take $0 < \delta < f(x_n) - M$. Consider the finite dimensional subspace $L :=^{\{} x_i\}_1^n$. Then, by the `Claim`, there is an element $\overline{x_{n+1}} \in S(f, B, \delta)$, such that $\|\overline{x_{n+1}} - z\| > 2/3$ for every $z \in L$. It is easy to see that

$$x_{n+1} := \frac{\overline{x_{n+1}} + x_n}{2}$$

satisfies the requirements. The set $\{x_n; n \in \mathbb{N}\}$ is closed and $x_{n+1} \in D(x_n, B)$ for all n, so B does not have the quasi drop property. $\square$

However, property (α) is strictly more general than the quasi drop property: let $(X, \|\cdot\|)$ be a Banach space such that $\|\cdot\|$ in X^* is Fréchet differentiable (the space X is then reflexive) and let $E \neq X$ be a dense subspace of X. We `claim` that the closed unit ball B_E of the normed space $(E, \|\cdot\|)$ has property (α) although it has not drop property. That it has property (α) follows from the Šmulyan criterium of Fréchet differentiability. In order to see that it has not drop property, let $x_0 \in S_X$ such that $x_0 \notin E$. There exists a continuous linear functional f which attains its maximum in B_X on x_0. Let

$H := \{x \in E : f(x) > f(x_0)\}$ be an open half-space defined by f. Given two vectors $a \neq b$ in X we denote by $]a,b[$ the "open" interval defined by a and b, i.e., $]a,b[:= \{\lambda a + (1-\lambda)b : \ \lambda \in]0,1[\}$. Take $x_1 \in H$ such that $\|x_1 - x_0\| < 1$. It is easy to see that we can find $b_1 \in B_E$ such that there exists $x_2 \in]b_1, x_1[\cap H$ with $\|x_2 - x_0\| < 1/2$. Next, choose $b_2 \in B_E$ such that there exists $x_3 \in]b_2, x_2[\cap H$ with $\|x_3 - x_0\| < 1/4$. Proceed in this way to define the sequence (x_n). No subsequence of (x_n) converges in $(E, \|\cdot\|)$, so $\{x_n : \ n \in \mathbb{N}\}$ is a closed subset disjoint from B_E, and there is no $x \in \{x_n : \ n \in \mathbb{N}\}$ such that $D(x, B_E) \cap \{x_n \ n \in \mathbb{N}\} = \{x\}$.

A similar example for the weak topology is even easier. Qiu proved the following result:

Theorem 2.4. [7, Th. 3.4] *Let $(X, \mathcal{T})$ be a locally convex space and B a Mackey complete closed bounded convex subset of X. Then B has the quasi weak drop property if and only if B is weakly compact.*

We already mentioned that every closed convex and bounded subset B of a locally convex space $(E, \sigma(E, E'))$ has property (α). However, according to Theorem 2.4, if B is Mackey complete and not weakly compact it cannot have the quasi weak drop property.

In [4] we proved that every closed convex and countably compact subset of a locally convex space has the countably drop property. Thus, we get:

Corollary 2.5. *Every closed convex and countably compact subset of a locally convex space has property (α).*

By using James' theorem, it is easy to see that every closed convex bounded set B with property (α) in a Banach space is weakly compact (see [2]). We prove next that the same is true in quasicomplete locally convex spaces:

Theorem 2.6. *Let B be a closed convex bounded subset of a locally convex space $(E, \mathcal{T})$. Suppose that B has property (α) and $(E, \mathcal{T})$ is quasicomplete. Then, B is weakly compact.*

Proof. Let $\mathcal{S}$ be the family of all continuous seminorms defined in E. Take $f \in F(B)$, $p \in \mathcal{S}$ and $\varepsilon > 0$, and define $U = \{x \in E : p(x) < \varepsilon\}$. Let (x_μ) be a net in B such that $f(x_\mu) \to \sup f[B]$. We can find $\delta > 0$ such that $S(f, B, \delta)$ can be covered by a finite number of translates of U, i.e., there exist $y_1, \ldots, y_k$ in E such that $S(f, B, \delta) \subset \bigcup_{i=1}^k y_i + U$. There exists μ_0 such that for all $\mu \geq \mu_0$, $x_\mu \in S(f, B, \delta)$. This defines a subnet $\{x_\mu : \mu \geq \mu_0\}$ in $S(f, B, \delta)$. Assume that for all $\mu_1 \geq \mu_0$, there exists $\mu \geq \mu_1$ such that $x_\mu \in y_1 + U$. Then there exists a subnet in $y_1 + U$. If not, there exists $\mu_1 \geq \mu_0$ such that $x_\mu \notin y_1 + U$ for all $\mu \geq \mu_1$. In such a case, assume that for all $\mu_2 \geq \mu_1$ there exists $\mu \geq \mu_2$ such that $x_\mu \in y_2 + U$. Therefore there exists a subnet in $y_2 + U$. Otherwise, there exists $\mu_2 \geq \mu_1$ such that $x_\mu \notin y_2 + U$ for every $\mu \geq \mu_2$. Following this argument we conclude that, for some $i \in \{1, 2, \ldots, n\}$, there is a subnet in $y_i + U$.

Define a mapping $E \xrightarrow{\varphi} \mathbb{R}^{E\times\mathcal{S}}$, by $\varphi(x)(y,p) = p(x-y)$ for $y \in E$ and $p \in \mathcal{S}$. It is easy to see that φ is a homeomorphic embedding when E is endowed with the $\mathcal{T}$-topology and $\mathbb{R}^{E\times\mathcal{S}}$ with the product topology. Moreover, $\varphi(A)$ is bounded for every bounded set A in E. Let (x_μ) be a net in B such that $f(x_\mu) \to \sup f[B]$. By Tychonoff's theorem there exists a subnet (denoted again by (x_μ)) such that $\varphi(x_\mu)(y,p)$ $(= p(x_\mu - y))$ converges for every $y \in E$ and $p \in \mathcal{S}$. Fix $p \in \mathcal{S}$ and $\varepsilon > 0$. By the previous argument we can find y_j and a subnet (x_α) of (x_μ) such that $p(x_\alpha - y_j) < \varepsilon$ for all α. Therefore, $(p(x_\alpha - y_j))_\alpha$ converges to some $l \leq \varepsilon$. But $(p(x_\alpha - y_j))_\alpha$ and $(p(x_\mu - y_j))_\mu$ converge to the same limit l, so $\lim_\mu p(x_\mu - y_j) \leq \varepsilon$. By the triangle inequality, and having in mind that $p \in \mathcal{S}$ and $\varepsilon > 0$ are arbitrary, we obtain that (x_μ) is Cauchy, so it converges to some $x \in B$. We have that $f(x) = \sup f[B]$. It is enough to apply now James' theorem to conclude that B is weakly compact. □

In [7, Th.3.5], it is proved that *a locally convex space E quasicomplete in its Mackey topology is semi-reflexive if every closed convex bounded set has the quasi drop property.* As a consequence of Theorem 2.6 we can obtain a similar statement for property (α):

Corollary 2.7. *Let $(E, \mathcal{T})$ be a locally convex space, such that every closed convex bounded set has property (α). If $(E, \mathcal{T})$ is quasicomplete, then E is semireflexive.*

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