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The Farthest and the Nearest Points of Sets*

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In a reflexive strictly convex real Banach space we established relationship between the metric projection on some closed convex subset and the metric antiprojection on another ("dual") closed convex subset. Some applications of the result are considered.

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1. Introduction and main notations

In 1963 Stechkin proved [14] that in a real uniformly convex Banach space the set of points for each of which there exists a unique metric projection on a given closed subset is dense. This was one of the first papers about properties of the metric projection in a Banach space from geometrical point of view.

It is well known that in a Hilbert space the metric projection operator on a closed convex set is Lipschitz continuous with constant 1. In the paper [3] the result in a Hilbert space was refined for some class of closed convex subsets. In 1979 Bjornestal [6] proved that in a uniformly convex and uniformly smooth Banach space the metric projection operator on a closed convex subset is continuous and he found an estimate for the modulus of continuity of this operator. Other estimates of this modulus one can find in [13] for convex case and in [11] for nonconvex case, see also the references therein.

In 1966 Edelstein [7] considered an analog of the Stechkin's result for farthest points of a set. He proved that for each closed bounded subset A in a real

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uniformly convex Banach space the set of points for each of which there exists a unique metric antiprojection on (farthest point of) the subset A is dense, see also [1]. Ekeland and Temam [8] proved that under some additional assumptions this dense set can be chosen in such a way, that the metric antiprojection operator is (norm) continuous on this set.

In the papers [2, 4, 10] in various classes of Banach spaces we described the properties of a set which has a unique metric antiprojection for any sufficiently distant point. In particular, we proved that such a set should be a summand of the ball.

In the present article we establish the duality relationship between the metric projection and the metric antiprojection in a real reflexive strictly convex Banach space and obtain an estimate for the modulus of continuity of the metric antiprojection operator in the case of uniqueness of the metric antiprojection.

Throughout this paper E is a real Banach space. We denote by (f, x) the value of the functional $f \in E^*$ at the vector $x \in E$. For $r \geq 0$ and $c \in E$ let $B_r(c)$ be the closed ball of radius r with center c.

The *distance* from the point $x \in E$ to the subset $A \subset E$ is

$$\varrho_A(x) = \inf_{a \in A} \|x - a\|.$$

Define the *metric projection* of the point $x \in E$ on the set $A \subset E$ as

$$P_A(x) = \{a \in A : \ \|x - a\| = \varrho_A(x)\}.$$

The *antidistance* from the point $x \in E$ to the subset $A \subset E$ is

$$f_A(x) = \sup_{a \in A} \|x - a\|.$$

The *metric antiprojection* of the point $x \in E$ on the set $A \subset E$ is

$$F_A(x) = \{a \in A : \ \|x - a\| = f_A(x)\}.$$

For a set $A \subset E$ and $r > 0$ the r-*antineighborhood* of the set $A \subset E$ is defined as

$$T_A(r) = \{x \in E : \ f_A(x) > r\}.$$

2. The antiprojection on a summand of the ball

The *Minkowski sum* of two sets $A, C \subset E$ is $A + C = \{a + c : \ a \in A, \ c \in C\}$. A subset $A \subset E$ is called a *summand of the ball* $B_r(0)$ if there exists a subset $C \subset E$ with $A + C = B_r(0)$.

The next result characterizes a summand of the ball in terms of properties of the metric antiprojection.

Theorem 2.1. [10, Theorem 2]. *Let E be a uniformly convex space with Fréchet differentiable norm. Then a convex closed set $A \subset E$ is a summand of the ball $B_r(0)$ if and only if the metric antiprojection operator $x \mapsto F_A(x)$ is single-valued and continuous on $T_A(r)$.*

The *supporting function* of a set $A \subset E$ is defined by the formula

$$s(f, A) = \sup_{a \in A}(f, a), \quad f \in E^*.$$

Note that $s(f, A + C) = s(f, A) + s(f, C)$ for any subsets $A, C \subset E$.

Lemma 2.2. *Let $A \subset E$ be a summand of the ball $B_r(0)$. Then the closed convex set $A_1 = \bigcap_{a \in A} B_r(-a)$ satisfies the equation $A + A_1 = B_r(0)$. Moreover, if a closed convex set C satisfies the equation $A + C = B_r(0)$, then $C = A_1$.*

Proof. For any $a_1 \in A_1$ and $a \in A$ we have $a_1 \in B_r(-a)$, i.e. $a + a_1 \in B_r(0)$. Therefore $A + A_1 \subset B_r(0)$. Let a set $C \subset E$ be such that $A + C = B_r(0)$. Then any element $c \in C$ satisfies the inclusion $A + c \subset B_r(0)$ and hence belongs to the set $\bigcap_{a \in A} B_r(-a) = A_1$. Thus, $C \subset A_1$ and $B_r(0) = A + C \subset A + A_1$. Therefore, $A + A_1 = B_r(0)$.

Now assume additionally that the set C is closed and convex. From the equality $A + C = B_r(0) = A + A_1$, it follows that $s(f, A) + s(f, C) = s(f, A) + s(f, A_1)$ for all $f \in E^*$. Taking in mind that the set A is bounded and hence $s(f, A) < +\infty$, we get $s(f, C) = s(f, A_1)$ for all $f \in E^*$. Using closedness and convexity of the sets A_1 and C, we obtain that $C = A_1$. $\square$

According to Lemma 2.2 if the set $A \subset E$ is a summand of the ball $B_r(0)$, then there exists a unique convex closed set $A_1 \subset E$ such that $A + A_1 = B_r(0)$. We shall call this set A_1 the *dual* to the set A with respect to the ball $B_r(0)$.

We now consider some examples of summands of balls in Hilbert and Banach spaces.

A nonempty set $A \subset E$ is called *strongly convex with radius* $r > 0$ if it can be represented as an intersection of balls with radius r:

$$A = \bigcap_{x \in X} B_r(x) \ \text{ for some } X \subset E.$$

The properties of strongly convex sets were studied in [5, 9]. A ball $B_r(0)$ is called a *generating set* if any strongly convex with radius r is a summand of $B_r(0)$ [5]. Note that the balls $B_r(0)$ in a Hilbert space and in ℓ_∞ are generating sets [5, Theorem 2.6, Corollary 2.5]. The ball $B_r(0)$ in ℓ_p, $p \in (1, +\infty)$ is a generating set if and only if $p = 2$, that follows from an example constructed by G. E. Ivanov and E. I. Bekishev (see Appendix I).

If the unit ball $B_1(0)$ is a generating set, then (by the definition) any strongly convex with radius 1 is a summand of the unit ball. Let us consider an example of a summand of the unit ball in ℓ_p with $p \in (1, +\infty)\backslash\{2\}$. Note that in this case the ball is not a generating set. Fix $p \in (1, +\infty)\backslash\{2\}$ and positive numbers λ, μ with $\lambda^p + \mu^p = 1$. Consider the following closed convex sets

$$A = \Big\{ x = (x_1, x_2, \ldots) \in \ell_p : \ (|x_1| + \lambda)^p + \sum_{k=2}^{\infty} |x_k|^p \le 1 \Big\}, \tag{1}$$

$$A_1 = \Big\{ x = (x_1, x_2, \ldots) \in \ell_p : \ |x_1|^p + \Big(\Big(\sum_{k=2}^{\infty} |x_k|^p \Big)^{1/p} + \mu \Big)^p \le 1 \Big\}. \tag{2}$$

Let the number q is defined from $\frac{1}{p} + \frac{1}{q} = 1$. The direct calculations (see Appendix II) show that for any $f = (f_1, f_2, \ldots) \in \ell_q = (\ell_p)^*$ with $\|f\|_{\ell_q} = \left(\sum_{k=1}^{\infty} |f_k|^q\right)^{1/q} = 1$

$$s(f, A) = \begin{cases} 1 - \lambda |f_1|, & |f_1| \ge \lambda^{p-1}, \\ \mu \left(1 - |f_1|^q\right)^{1/q}, & |f_1| \le \lambda^{p-1}, \end{cases} \tag{3}$$

$$s(f, A_1) = \begin{cases} \lambda |f_1|, & |f_1| \ge \lambda^{p-1}, \\ 1 - \mu \left(1 - |f_1|^q\right)^{1/q}, & |f_1| \le \lambda^{p-1}. \end{cases} \tag{4}$$

Consequently, $s(f, A + A_1) = s(f, A) + s(f, A_1) = 1 = s(f, B_1(0))$. Since the set $A + A_1$ is convex and closed (due to weak compactness), we have $A + A_1 = B_1(0)$. Thus, A and A_1 are dual to each other with respect to the ball $B_1(0)$ and hence each of them is a summand of the unit ball.

3. Duality between metric projection and metric antiprojection

For a functional $f \in E^*$ and a set $A \subset E$ put

$$\operatorname{Exp}(f, A) = \operatorname*{Argmax}_{a \in A} (f, a) = \{ a \in A : \ (f, a) \ge (f, x) \quad \forall x \in A \}.$$

For any closed and convex $A, C \subset E$ and $f \in E^*$ one can easily see that

$$\operatorname{Exp}(f, A) + \operatorname{Exp}(f, C) = \operatorname{Exp}(f, A + C). \tag{5}$$

Note that if E is a reflexive and strictly convex Banach space, then for any nonzero functional $f \in E^*$ the set $\operatorname{Exp}(f, B_1(0))$ is a singleton. Hence, having in mind (5), if $A + C = B_r(0)$ and E is a reflexive strictly convex Banach space, then for any nonzero functional $f \in E^*$ the set $\operatorname{Exp}(f, A)$ is a singleton. In such a case we use e_f to denote the element of $\operatorname{Exp}(f, B_1(0))$ and e_f^A for the element of $\operatorname{Exp}(f, A)$. So, the equation (5) implies that

$$e_f^A + e_f^C = r e_f. \tag{6}$$

Lemma 3.1. *Let A be a summand of the ball $B_r(0)$ in a reflexive and strictly convex Banach space E. Then for any points $x, a \in E$ with $\|a - x\| > r$ the following assertions are equivalent:*

(i) $a \in F_A(x)$;

(ii) there exists a unit functional $f \in E^*$ such that $a = e_f^A$ and $\frac{a-x}{\|a-x\|} = e_f$.

Proof. Denote $R = \|a - x\|$. First of all note that the inclusions $a \in F_A(x)$ and $A \subset B_R(x)$ are equivalent provided that $a \in A$.

$(i) \Rightarrow (ii)$. According to the Hahn–Banach theorem there exists a unit functional $f \in E^*$ such that $\frac{a-x}{\|a-x\|} = e_f$. Since $A \subset B_R(x)$, then for any $b \in A$ we have $(f, b) \leq (f, x) + R = (f, a - Re_f) + R = (f, a)$. Hence, $a = e_f^A$.

$(ii) \Rightarrow (i)$. Let f be a unit functional from the assertion (ii). Since A is a summand of $B_r(0)$ (and hence of $B_R(0)$), there exists $C \subset E$ with $A + C = B_R(0)$. According to (6) we have $a + e_f^C = Re_f$. Consequently, $-x = Re_f - a = e_f^C \in C$ and therefore $A - x \subset A + C = B_R(0)$. Thus $A \subset B_R(x)$, i.e. $a \in F_A(x)$. $\square$

Lemma 3.2. *For a convex set $C \subset E$ and different points $x, z \in E$ the following assertions are equivalent:*

(i) $z \in P_C(x)$;

(ii) there exists a unit functional $f \in E^*$ such that $z = e_f^C$ and $\frac{x-z}{\|x-z\|} = e_f$.

Proof. Note first that the assertion (i) is equivalent to the equality $C \cap B = \emptyset$ with $B = \{b \in E : \ \|b - x\| < \|z - x\|\}$ (provided that $z \in C$).

$(i) \Rightarrow (ii)$. Due to the equality $C \cap B = \emptyset$ by the Hahn–Banach theorem there exists a unit functional $f \in E^*$ such that $z = e_f^C$ and $z - x = e_{-f}^B$. The latter formula means that $\frac{x-z}{\|x-z\|} = e_f$.

$(ii) \Rightarrow (i)$. Let f be a unit functional from the assertion (ii). Fix any $b \in B$ and $c \in C$. Then $(f, b) > (f, x) - \|z - x\| = (f, x - \|z - x\|e_f) = (f, z) = (f, e_f^C) \geq (f, c)$ and hence $b \neq c$. Thus, $C \cap B = \emptyset$. $\square$

Theorem 3.3. *Let A and C be closed convex sets in a reflexive and strictly convex Banach space E, $A + C = B_r(0)$. Let points $a, x, z \in E$ be such that $\|x - a\| = R > r$ and*

$$(R - r)a + rx = Rz. \tag{7}$$

Then the inclusions $a \in F_A(x)$ and $z \in P_{-C}(x)$ are equivalent.

Proof. Let us show that for any unit functional $f \in E^*$ the system of equations

$$a = e_f^A, \qquad \frac{a - x}{\|a - x\|} = e_f \tag{8}$$

is equivalent to the following system:

$$z = e_{-f}^{-C}, \qquad \frac{x-z}{\|x-z\|} = e_{-f}. \tag{9}$$

Indeed, by the equation $\|x-a\| = R$ the system (8) is equivalent to

$$Re_f = e_f^A - x, \qquad \frac{a-x}{\|a-x\|} = e_f.$$

Taking into account (6), we see that the equation $Re_f = e_f^A - x$ is equivalent to the equation $(R-r)e_f = -e_f^C - x$. Using (7), we have $\frac{a-x}{\|a-x\|} = -\frac{x-z}{\|x-z\|}$ and $\|x-z\| = R-r$. Finally, the system (8) is equivalent to

$$z - x = -e_f^C - x, \qquad \frac{x-z}{\|x-z\|} = -e_f.$$

Since $-e_f^C = e_{-f}^{-C}$ and $-e_f = e_{-f}$, the systems (8) and (9) are equivalent.

According to Lemma 3.1 the inclusion $a \in F_A(x)$ holds true if and only if there exists a unit functional $f \in E^*$ satisfying the system (8), which is equivalent to (9). Applying Lemma 3.2 for the set $-C$ and the functional $-f$, we complete the proof. $\square$

The first application of Theorem 3.3 is the following result which improves Theorem 1 from [4].

Theorem 3.4. Let $A \subset E$ be a closed convex summand of the ball $B_r(0)$ in a reflexive and strictly convex Banach space E. Then for any $x \in T_A(r)$ the set $F_A(x)$ is a singleton.

Proof. According to Lemma 2.2 there exists a convex closed set $C \subset E$ such that $A + C = B_r(0)$ ($C = A_1$ in terms of Lemma 2.2). Since $x \in T_A(r)$, there exists $a_0 \in A$ with $\|a_0 - x\| > r$, i.e. $a_0 - x \notin B_r(0) = A + C$. Consequently, $x \notin -C$. By reflexivity and strict convexity of the space E the set $P_{-C}(x)$ is a singleton, i.e. $P_{-C}(x) = \{z\}$ for some $z \in E$, $z \neq x$. Denote $R = r + \|z - x\|$, we get $R > r$. It's easy to see that the point $a = \frac{Rz - rx}{R-r}$ satisfies (7) and $\|a - x\| = \frac{R}{R-r}\|z - x\| = R$. From Theorem 3.3 we obtain that $F_A(x) = \{a\}$. $\square$

4. The moduli of continuity of the metric projection and the metric antiprojection

The *modulus of continuity of the metric projection* on the set $A \subset E$ is

$$\omega(t, R, A) = \sup_{\substack{x_1, x_2 \in E: \\ \|x_1 - x_2\| \le t, \\ \varrho_A(x_1) = R}} \ \sup_{\substack{z_1 \in P_A(x_1) \\ z_2 \in P_A(x_2)}} \|z_1 - z_2\|, \quad t \ge 0, \ R \ge 0. \tag{10}$$

The *modulus of continuity of the metric antiprojection* on the set $A \subset E$ is defined as

$$\beta(t,R,A) = \sup_{\substack{x_1,x_2\in E:\\ \|x_1-x_2\|\le t,\\ f_A(x_1)=R}} \ \sup_{\substack{a_1\in F_A(x_1)\\ a_2\in F_A(x_2)}} \|a_1-a_2\|, \qquad t\ge 0,\ R\ge 0. \tag{11}$$

Theorem 4.1. *Let A and C be closed convex sets in a reflexive and strictly convex Banach space E, $A+C=B_r(0)$. Then for any $R>r$ and $t\in(0,R-r)$*

$$|R\omega(t,R-r,-C)-(R-r)\beta(t,R,A)|\le 2rt.$$

Proof. Fix any $R>r$, $t\in(0,R-r)$ and $\varepsilon>0$. By (10) there exist $x_1,x_2,z_1,z_2\in E$ such that $\|x_1-x_2\|\le t$, $\varrho_{-C}(x_1)=R-r$, $z_i\in P_{-C}(x_i)$ $(i=1,2)$, $\omega(t,R-r,-C)\le\|z_1-z_2\|+\varepsilon$. Since the distance function $\varrho_{-C}(\cdot)$ is Lipschitz continuous with constant 1, it follows that $\varrho_{-C}(x_2)\ge\varrho_{-C}(x_1)-\|x_1-x_2\|\ge R-r-t>0$ and hence $z_2\ne x_2$. Define

$$R_i=r+\|z_i-x_i\|,\quad a_i=\frac{R_iz_i-rx_i}{R_i-r}\quad(i=1,2).$$

We have $\|x_i-a_i\|=R_i>r$ and by Theorem 3.3 it follows that $a_i\in F_A(x_i)$. Observe that $z_i=\left(1-\frac{r}{R_i}\right)a_i+\frac{r}{R_i}x_i$ and hence

$$z_1-z_2=\left(1-\frac{r}{R_1}\right)(a_1-a_2)+\frac{r}{R_1}(x_1-x_2)+\left(\frac{r}{R_2}-\frac{r}{R_1}\right)(a_2-x_2).$$

Consequently,

$$\|z_1-z_2\|\le\left(1-\frac{r}{R_1}\right)\|a_1-a_2\|+\frac{r}{R_1}\|x_1-x_2\|+rR_2\left|\frac{1}{R_2}-\frac{1}{R_1}\right|.$$

Taking into account that $|R_1-R_2|=|f_A(x_1)-f_A(x_2)|\le\|x_1-x_2\|\le t$ and $R_1=R$, we get

$$\omega(t,R-r,-C)\le\|z_1-z_2\|+\varepsilon\le\left(1-\frac{r}{R}\right)\|a_1-a_2\|+\frac{2rt}{R}+\varepsilon\le$$
$$\le\left(1-\frac{r}{R}\right)\beta(t,R,A)+\frac{2rt}{R}+\varepsilon.$$

Passing to the limit as $\varepsilon\to+0$, we get

$$R\omega(t,R-r,-C)-(R-r)\beta(t,R,A)\le 2rt.$$

Similar arguments lead to

$$(R-r)\beta(t,R,A)-R\omega(t,R-r,-C)\le 2rt. \qquad \square$$

We shall say that the space E has *the Mazur intersection property* if any closed convex bounded subset coincides with the intersection of *all closed balls,* each of which contains the set.

Corollary 4.2. *Suppose that E is a reflexive strictly convex Banach space with the Mazur intersection property and $B_1(0)$ is a generating set. The space E is isomorphic to a Hilbert space if and only if there exists a function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to+0} w(t) = 0$ such that for any $r > 0$, $R > r$ and any $A \subset E$ being a summand of the ball $B_r(0)$, one has for all $x_1, x_2 \in T_A(R)$*

$$\|a_1 - a_2\| \le \frac{r}{R-r} w(\|x_1 - x_2\|), \quad \{a_i\} = F_A(x_i),\ i = 1, 2. \tag{12}$$

Proof. If (12) holds, then from the property of the ball $B_1(0)$ (and hence of the ball $B_r(0)$) to be a generating set and by Theorem 4.1 the metric projection operator on any nonempty subset of the type $\bigcap_{x\in X} B_r(x)$ $(X \subset E)$ of the space E is uniformly continuous with a modulus, which does not depend on the radius r. By [3, Theorem 2.3] E is isomorphic to the Hilbert space.

If E is a Hilbert space, then (12) is satisfied with the function $w(t) = t$ [4, Theorem 2]. Consequently, if E is isomorphic to a Hilbert space, then (12) holds with the function $w(t) = Ct$ for some $C > 0$. $\square$

Remember that the *modulus of convexity* of the space E is defined for $\varepsilon \in [0, 2]$ as

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} \;\middle|\; x, y \in B_1(0),\ \|x - y\| \ge \varepsilon\right\}. \tag{13}$$

The *modulus of smoothness* of E is defined for $\varepsilon \ge 0$ as

$$\varrho_E(\varepsilon) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 \;\middle|\; x, y \in E : \|x\| = 1,\ \|y\| = \varepsilon\right\}. \tag{14}$$

Corollary 4.3. *Let A be a summand of the ball $B_r(0)$ in a uniformly convex Banach space E. Then the modulus of continuity of the metric antiprojection on A satisfies the estimate*

$$\beta(t, R, A) \le R\delta_E^{-1}\left(\varrho_E\left(\frac{2t}{R-r}\right)\right) + \frac{2(R+r)t}{R-r} \quad \forall R > r, \quad t \in (0, R-r),$$

where $\delta_E^{-1}(\cdot)$ is the inverse function of the modulus of convexity $\delta_E(\cdot)$.

In particular, for spaces ℓ_p and L_p we have

$$\beta(t, R, A) \le \begin{cases} R\left(\frac{8}{p(p-1)}\right)^{1/2}\left(\frac{2t}{R-r}\right)^{p/2} + o(t^{p/2}), & p \in (1, 2), \\ 2R\left(\frac{p(p-1)}{2}\right)^{1/p}\left(\frac{2t}{R-r}\right)^{2/p} + o(t^{2/p}), & p \in (2, +\infty), \end{cases} \qquad t \to +0.$$

Proof. According to Lemma 2.2 there exists a closed convex set $C \subset E$ such that $A + C = B_r(0)$. Using Theorem 3.1 and Lemma 3.2 from [11] we have

$$\omega(t, R - r, -C) \le (R - r)\delta_E^{-1}\left(\varrho_E\left(\frac{2t}{R-r}\right)\right) + 2t \quad \forall R > r, \quad t \in (0, R - r).$$

Applying Theorem 4.1, we obtain the first estimate. Using the asymptotical behavior of the moduli of convexity and smoothness of spaces ℓ_p and L_p [12, section 1.e], we get the second estimate. □

5. Appendix I: a ball in ℓ_p with $p \neq 2$ is not a generating set

We shall use the following proposition called the suppot principle.

Proposition 5.1. ([5, Theorem 1.1]) *Let B be a closed convex bounded subset in a reflexive Banach space E. The set B is a generating set if and only if for any nonempty subset $C \subset E$ of the form*

$$C = \bigcap_{x \in X} (B + x), \quad X \subset E$$

for any unit functional $x^* \in E^*$ and a point $x^C_{x^*} \in \operatorname*{Argmax}_{x \in C} (x^*, x)$ there exists a point $x^B_{x^*} \in \operatorname*{Argmax}_{x \in B} (x^*, x)$ with $C \subset B + x^C_{x^*} - x^B_{x^*}$.

Theorem 5.2. *Let $p \in (1, +\infty) \setminus \{2\}$. Then the unit ball*

$$B = \left\{ x = (x_1, x_2, \dots) : \sum_{k=1}^{\infty} |x_k|^p \leq 1 \right\}$$

in ℓ_p is not a generating set.

Proof. Assume that $p \in (1, 2)$. Consider the standard basis $e_1, e_2, ...$ in ℓ_p: $e_k = (\delta_{1k}, \delta_{2k}, ...)$, where δ_{nk} is the Kronecker delta. Let $e^*_1, e^*_2, ...$ be the dual functionals to $e_1, e_2, ...$ correspondingly, that is $\|e^*_n\|_* = 1$ and $(e^*_n, e_k) = \delta_{nk}$ for all $n, k \in \mathbb{N}$. Consider the set

$$C = (B - a_1) \cap (B - a_2),$$

where $a_1 = 3^{-1/p}(e_1 - e_2 + e_3)$, $a_2 = 3^{-1/p}(-e_1 + e_2 + e_3)$.

Observe that $x = (x_1, x_2, \dots)$ is an element of C iff

$$\begin{cases} |x_1 + 3^{-1/p}|^p + |x_2 - 3^{-1/p}|^p + |x_3 + 3^{-1/p}|^p + \sum\limits_{k=4}^{\infty} |x_k|^p \leq 1, \\ |x_1 - 3^{-1/p}|^p + |x_2 + 3^{-1/p}|^p + |x_3 + 3^{-1/p}|^p + \sum\limits_{k=4}^{\infty} |x_k|^p \leq 1. \end{cases} \tag{15}$$

Let us show that

$$x^C_{e^*_3} = 0. \tag{16}$$

Suppose that $x = (x_1, x_2, \ldots) \in C$. Adding the inequalities (15) together, we have

$$(|x_1 + 3^{-1/p}|^p + |x_1 - 3^{-1/p}|^p) + (|x_2 + 3^{-1/p}|^p + |x_2 - 3^{-1/p}|^p) + \\ + 2|x_3 + 3^{-1/p}|^p + 2\sum_{k=4}^{\infty} |x_k|^p \leq 2.$$

Since the function $\varphi(t) = |t|^p$ is strictly convex, then $|3^{-1/p}+t|^p+|3^{-1/p}-t|^p \geq \frac{2}{3}$ wherein the equality is realized iff $t = 0$. Consequently,

$$\frac{2}{3} + \frac{2}{3} + 2|x_3 + 3^{-1/p}|^p + 2\sum_{k=4}^{\infty} |x_k|^p \leq 2,$$

and hence

$$|x_3 + 3^{-1/p}|^p + \sum_{k=4}^{\infty} |x_k|^p \leq \frac{1}{3}.$$

This means that $x_3 \leq 0$, i.e. $(e_3^*, x) \leq 0$. So, $s(e_3^*, C) = 0$ and $x \in C$ satisfies the equality $(e_3^*, x) = s(e_3^*, C)$ if and only if $x = 0$. This proves (16).

Assume that B is a generating set. Since $x^B_{e_3^*} = e_3$, by Proposition 5.1 it follows that

$$C \subset B - e_3. \tag{17}$$

Fix $D > 3^{1/p}(p-1)$ and consider the following function for $t > 0$:

$$x(t) = (x_1(t), x_2(t), x_3(t), ...) = t(e_1 + e_2) - Dt^2 e_3 = (t, t, -Dt^2, 0, 0, 0, ...).$$

Observe that for sufficiently small $t > 0$

$$\begin{aligned}
&|x_1(t) + 3^{-1/p}|^p + |x_2(t) - 3^{-1/p}|^p + |x_3(t) + 3^{-1/p}|^p + \sum_{k=4}^{\infty} |x_k(t)|^p = \\
&= \frac{1}{3}\left(1 + 3^{1/p}t\right)^p + \frac{1}{3}\left(1 - 3^{1/p}t\right)^p + \frac{1}{3}\left(1 - 3^{1/p}Dt^2\right)^p \\
&= \frac{1}{3}\Big(1 + 3^{1/p}pt + \frac{3^{2/p}p(p-1)t^2}{2} + 1 - 3^{1/p}pt + \frac{3^{2/p}p(p-1)t^2}{2} + \\
&\qquad + 1 - 3^{1/p}pDt^2\Big) + o(t^2) \\
&= 1 + \frac{3^{2/p}p(p-1)t^2 - 3^{1/p}pD}{3}t^2 + o(t^2) \\
&= 1 - \frac{3^{1/p}p}{3}\left(D - 3^{1/p}(p-1)\right)t^2 + o(t^2) < 1.
\end{aligned}$$

Similarly,

$$|x_1(t) - 3^{-1/p}|^p + |x_2(t) + 3^{-1/p}|^p + |x_3(t) + 3^{-1/p}|^p + \sum_{k=4}^{\infty} |x_k(t)|^p < 1$$

for sufficiently small $t > 0$. It means that for sufficiently small $t > 0$ the vector $x = (x_1, x_2, ...) = (x_1(t), x_2(t), ...)$ satisfies (15) and hence $x(t) \in C$. Using (17) one has $x(t) \in B - e_3$, that is

$$|x_1(t)|^p + |x_2(t)|^p + |x_3(t) + 1|^p \leq 1 \tag{18}$$

for sufficiently small $t > 0$. On the other hand

$$|x_1(t)|^p + |x_2(t)|^p + |x_3(t) + 1|^p = 2t^p + 1 - Dpt^2 + o(t^2), \quad t \to +0.$$

In view of $p \in (1, 2)$ the latter equation contradicts (18). this means that B is not a generating set. Similar reasoning shows that in case $p \in (2, +\infty)$ the unit ball B in ℓ_p is not a generating set too. □

6. Appendix II: calculations of the supporting function

We prove Formulas (3) and (4).

For any $a, b \in \mathbb{R}$ such that $|a|^q + |b|^q = 1$ let us calculate

$$\sigma(a, b, \lambda) = \sup_{\substack{x,y\in\mathbb{R}: \\ (|x|+\lambda)^p+|y|^p\leq 1}} (ax + by). \tag{19}$$

We have

$$\sigma(a, b, \lambda) = \sup_{\substack{x\geq 0,\ y\geq 0: \\ (x+\lambda)^p+y^p\leq 1}} (|a|x + |b|y) = -\lambda|a| + \sup_{\substack{\xi\geq\lambda,\ y\geq 0: \\ \xi^p+y^p\leq 1}} (|a|\xi + |b|y). \tag{20}$$

Hence,

$$\sigma(a, b, \lambda) \leq -\lambda|a| + \sup_{\substack{\xi\geq 0,\ y\geq 0: \\ \xi^p+y^p\leq 1}} (|a|\xi + |b|y) = -\lambda|a| + (|a|^q + |b|^q)^{1/q} = -\lambda|a| + 1.$$

If $|a| \geq \lambda^{p-1}$, then, denoting $\xi_0 = |a|^{q/p}$, $y_0 = |b|^{q/p}$, we have $\xi_0 \geq \lambda$, $\xi_0^p + y_0^p = 1$. According to (20) we obtain

$$\sigma(a, b, \lambda) \geq -\lambda|a| + |a|\xi_0 + |b|y_0 = -\lambda|a| + |a|^q + |b|^q = -\lambda|a| + 1.$$

Thus, in the case $|a| \geq \lambda^{p-1}$ we have $\sigma(a, b, \lambda) = 1 - \lambda|a|$.

Now consider the case $|a| \leq \lambda^{p-1}$. In this case, putting $x_0 = 0$, $y_0 = \mu$, we have $(x_0 + \lambda)^p + y_0^p = 1$ and by (20)

$$\sigma(a, b, \lambda) \geq |a|x_0 + |b|y_0 = \mu|b|.$$

Assume that $x \geq 0$, $y \geq 0$, $(x+\lambda)^p+y^p \leq 1$. We have $\lambda^{p-1}(1-y^p)^{1/p}+\mu^{p-1}y \leq 1$. Hence, $\lambda^{p-1}(x + \lambda) + \mu^{p-1}y \leq 1$. Using the equality $\lambda^p + \mu^p = 1$, we get $\lambda^{p-1}x \leq \mu^{p-1}(\mu - y)$. Taking into account that $y \leq \mu$, $x \geq 0$, $|a| \leq \lambda^{p-1}$ and hence $|b| \geq \mu^{p-1}$, we have $|a|x \leq |b|(\mu - y)$. Thus, using (20), we obtain that $\sigma(a,b,\lambda) \leq \mu|b|$ and, in the case $|a| \leq \lambda^{p-1}$, we have $\sigma(a,b,\lambda) = \mu|b|$. Finally for any $a, b \in \mathbb{R}$ such that $|a|^q + |b|^q = 1$ we get

$$\sigma(a,b,\lambda) = \begin{cases} 1 - \lambda|a|, & |a| \geq \lambda^{p-1}, \\ \mu|b|, & |a| \leq \lambda^{p-1}. \end{cases} \tag{21}$$

Now, using (21), we calculate the supporting functions of sets A and A_1 defined in (1), (2). Let $f = (f_1, f_2, \ldots) \in \ell_q = (\ell_p)^*$ with $\|f\|_{\ell_q} = (\sum_{k=1}^{\infty} |f_k|^q)^{1/q} = 1$. We have

$$s(f,A) = \sup_{x=(x_1,x_2,\ldots)\in A} \sum_{k=1}^{\infty} f_k\, x_k = \sup_{\substack{x_1,y\in\mathbb{R}:\\ (|x_1|+\lambda)^p+|y|^p\leq 1}} \left(f_1 x_1 + \sup_{\substack{x_2,x_3,\ldots\in\mathbb{R}:\\ \sum_{k=2}^{\infty}|x_k|^p=|y|^p}} \sum_{k=2}^{\infty} f_k x_k \right),$$

$$s(f,A_1) = \sup_{x=(x_1,x_2,\ldots)\in A_1} \sum_{k=1}^{\infty} f_k x_k = \sup_{\substack{x_1,y\in\mathbb{R}:\\ |x_1|^p+(|y|+\mu)^p\leq 1}} \left(f_1 x_1 + \sup_{\substack{x_2,x_3,\ldots\in\mathbb{R}:\\ \sum_{k=2}^{\infty}|x_k|^p=|y|^p}} \sum_{k=2}^{\infty} f_k x_k \right).$$

Denoting $\bar{f} = (\sum_{k=2}^{\infty} |f_k|^q)^{1/q}$ and using (19), we find that

$$s(f,A) = \sup_{\substack{x_1,y\in\mathbb{R}:\\ (|x_1|+\lambda)^p+|y|^p\leq 1}} (f_1 x_1 + \bar{f} y) = \sigma(f_1, \bar{f}, \lambda),$$

$$s(f,A_1) = \sup_{\substack{x_1,y\in\mathbb{R}:\\ |x_1|^p+(|y|+\mu)^p\leq 1}} (f_1 x_1 + \bar{f} y) = \sigma(\bar{f}, f_1, \mu).$$

Using (21), we get

$$s(f,A) = \begin{cases} 1 - \lambda|f_1|, & |f_1| \geq \lambda^{p-1}, \\ \mu|\bar{f}|, & |f_1| \leq \lambda^{p-1}, \end{cases}$$

$$s(f,A_1) = \begin{cases} 1 - \mu|\bar{f}|, & |\bar{f}| \geq \mu^{p-1}, \\ \lambda|f_1|, & |\bar{f}| \leq \mu^{p-1}. \end{cases}$$

Since $|f_1|^q + |\bar{f}|^q = 1$ and $\lambda^p + \mu^p = 1$, it follows that the inequality $|\bar{f}| \leq \mu^{p-1}$ (i.e. $|\bar{f}|^q \leq \mu^{q(p-1)} = \mu^p$) is equivalent to the inequality $|f_1|^q \geq \lambda^p$ (i.e. $|f_1| \geq \lambda^{p-1}$). Taking into account that $|\bar{f}| = (1 - |f_1|^q)^{1/q}$, we obtain formulas (3) and (4).

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