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Separation Problems in the Context of h-Convexity

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The concept of h-convexity extends the notion of classical convexity, using some nonnegative function h of the weights on the right-hand side in the defining inequality. The aim of the present note is to show that nonconstant h-affine functions appear only in the classical convexity case. Affine and convex separation problems are also studied, without any further restrictions on h. The obtained results suggest that, in view of Convex Geometry or Convex Analysis, the notion of h-convexity may have only particular importance.

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1. Introduction

In his fundamental papers [11] and [12], Jensen declares: *"It seems to me that the notion of convex function is just as fundamental as positive function or increasing function. If I am not mistaken in this, the notion ought to find its place elementary expositions of the theory of real functions."* Definitely, he was not mistaken. Convexity has found its place in introductory courses. Moreover, it has initiated independent branches like Convex Geometry [2] or Convex Analysis [13]. In the field of Real Analysis, convexity still motivates many investigations and generalizations. Among the various extensions, the notion of h-convexity is in our focus:

Definition 1.1. Let D be a convex subset of a real vector space X and let $h\colon [0,1] \to \mathbb{R}$ be nonnegative. A function $f\colon D \to \mathbb{R}$ is said to be *h-convex* if,

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for all $t \in [0,1]$ and $x, y \in D$,

$$f\big(tx + (1-t)y\big) \le h(t)f(x) + h(1-t)f(y).$$

If the inequality above holds with equality, then f is termed to be *h-affine*.

This generalization is due to Varošanec [19] and was studied intensively, among many others, by Házy [9]. The concept of h-convexity subsumes several earlier attempts of extensions. Clearly, the choice $h = \mathrm{id}$ reduces to the classical setting. Similarly, the cases of Breckner-convexity [6], (with $h(t) = t^s$, where $t \in]0,1[$ and $s \in \mathbb{R}$), Godunova–Levin-convexity [8], (with $h(t) = 1/t$ where $t \in]0,1[$), and P-convexity [18], (with $h \equiv 1$) are also included.

Affine and convex separation problems play a central role in the theory of convexity. A result of Baron, Matkowski, and Nikodem [1] describes those pairs of functions that can be separated by a convex one. A correspondence for the existence of an affine separator is due to Nikodem and Wąsowicz [15]. The main motivation of the forthcoming investigations is also a separation result which has recently been obtained by Olbryś [16]. This result characterizes pairs of functions possessing an h-convex separator. In the particular setting $h = \mathrm{id}$, its statement reduces to the dimension-free version of the Baron–Matkowski–Nikodem Theorem. Let us recall here the statement of the Olbryś Theorem explicitly:

Theorem 1.2. *Let D be a convex subset of a real linear space X and let $h\colon [0,1] \to \mathbb{R}$ be multiplicative, furthermore let $f, g\colon D \to \mathbb{R}$ be given. There exists an h-convex function $\phi\colon D \to \mathbb{R}$ fulfilling $f \le \phi \le g$ if and only if, for all $x_1, \dots, x_n \in D$ and $\lambda_1, \dots, \lambda_n \in [0,1]$ with $\lambda_1 + \cdots + \lambda_n = 1$ the next inequality is satisfied:*

$$f\Big(\sum_{k=1}^{n} \lambda_k x_k\Big) \le \sum_{k=1}^{n} h(\lambda_k) g(x_k).$$

Remarkable, that in the standard and finite dimensional setting, the length of the involved convex combinations is controlled uniformly. This phenomenon is a consequence of the Carathéodory Theorem [7]. Concerning the result of Olbryś, the next problems are posed evidently.

- *What can be stated about h-affine separation?*
- *Can the length of the convex combinations be reduced if X is finite dimensional?*
- *Can the assumption on multiplicativity be relaxed?*

In this note, we are going to give complete answers to these questions. The first two problems are closely related to the structure of h-affine functions. According to one of our main results, there are two possibilities: either h-affine functions are rather simple, or h-convexity is just the classical one. In the first case, h-affine separation problems are (almost) trivial, while the second case coincides to the Nikodem–Wąsowicz Theorem. For similar reasons, the Carathéodory Theorem has no correspondence if the set of h-affine functions is poor. Therefore, the convex combinations must have arbitrary length in the Olbryś Theorem. Finally, our other main result gives positive answer to the third question.

The lack of h-affine functions has serious consequences. If $h \neq \mathrm{id}$, then h-convex functions cannot be reconstructed via their supports anymore, and the duality between convexity of functions and convexity of epigraphs disappears. These facts suggest that, at least in view of Convex Geometry or Convex Analysis, the notion of h-convexity may have only particular importance.

2. Notions, notations, basic properties

Throughout in this note, $\mathbb{N}$ denotes the set of positive integers, while $\mathbb{R}_+$ stands for the set of nonnegative reals; that is, $\mathbb{R}_+ = [0, +\infty[$. We shall need the next concept, as well. Assume that D is a set in a real linear space X. A point $p \in D$ is an *algebraic interior point*, if, for all $v \in X$, there exists $\varepsilon > 0$ such that $p + tv \in D$ whenever $t \in]0, \varepsilon[$. The set D is called *algebraic open*, if its points are algebraic interior points. Our first three propositions formulate some easy but important properties of h-convex and h-affine functions. Their proofs are omitted.

Proposition 2.1. *Let D be a convex subset of a real vector space X and $h\colon [0,1] \to \mathbb{R}$.*

(i) If $h(t) + h(1-t) > 1$ for some $t \in [0,1]$, then any h-convex function on D is nonnegative.

(ii) If $h(t) + h(1-t) < 1$ for some $t \in [0,1]$, then any h-convex function on D is nonpositive.

In particular, if (i) and (ii) are fulfilled simultaneously, then the only h-convex function is the constant zero function.

Proposition 2.2. *Let D be a convex subset of a real vector space X and $h\colon [0,1] \to \mathbb{R}$. Then, the next assertions are equivalent:*

(i) there exists a nonzero constant h-affine function;

(ii) each constant function is h-affine;

(iii) $h(t) + h(1-t) = 1$ holds for all $t \in [0,1]$.

Proposition 2.3. *The set of h-convex functions, acting on a convex subset D of a real vector space X, forms a cone. This cone is closed under taking maximum. Furthermore, if $\mathrm{id} \leq h$, then any convex function is h-convex.*

Let us note, that the cone of h-convex functions, in general, is not invariant under shifting by scalars. Some authors do not assume the nonnegativity of h in the definition. As it has been indicated, Proposition 2.1 and Proposition 2.2

are still valid without this assumption. However, the cone of h-convex functions in Proposition 2.3 is not necessarily closed under taking maximum if we drop nonnegativity. The next nontrivial proposition remains also true without any restriction on the range of h.

Proposition 2.4. *Assume that D is an algebraic open convex subset of a real vector space X, furthermore $f\colon D \to \mathbb{R}$ and $h\colon [0,1] \to \mathbb{R}$ satisfy the functional equation*

$$f\big(tx + (1-t)y\big) = h(t)f(x) + h(1-t)f(y).$$

If h has a continuity point and f is nonconstant, then $h = \mathrm{id}$.

Proof. Assume that D is an open interval, and f is a solution to the functional equation above. Substituting $x = y$ and then subtracting it from the original one, the term $h(1-t)f(y)$ cancels, and the next equation is obtained:

$$f\big(tx + (1-t)y\big) - f(y) = h(t)\big(f(x) - f(y)\big), \quad x, y \in D; \quad t \in [0,1]. \tag{1}$$

The main point of this equation is that the variables x, y and t appear *separately* on the right hand-side. This phenomenon allows to develop the regularity properties of the involved functions f and h step by step, still the standard methods of differential equations can be applied.

Claim 1: Continuity of h at a single point of $[0,1]$ implies continuity of f everywhere.

Fix $u_0 \in D$ arbitrarily. Assume first that $t_0 \in]0,1[$ is a continuity point of h. Since D is open, there exist elements $x < y$ of D such that $u_0 = t_0 x + (1-t_0)y$. Take a sequence (u_n) from D such that $u_n \to u_0$. Then, for large enough indices, $u_n \in]x, y[$ holds, providing the convex representation $u_n = t_n x + (1-t_n)y$ with some $t_n \in [0,1]$. Therefore, $|u_n - u_0| = |t_n - t_0||y - x|$, yielding the property $t_n \to t_0$. Applying now (1), one can arrive at the identity

$$\begin{aligned} f(u_n) - f(u_0) &= f\big(t_n x + (1-t_n)y\big) - f\big(t_0 x + (1-t_0)y\big) \\ &= \Big(f\big(t_n x + (1-t_n)y\big) - f(y)\Big) - \Big(f\big(t_0 x + (1-t_0)y\big) - f(y)\Big) \\ &= \big(h(t_n) - h(t_0)\big)\big(f(x) - f(y)\big). \end{aligned}$$

The latter term tends to zero as n tends to infinity, since t_0 is a continuity point of h and $t_n \to t_0$ also holds. This results $f(u_n) \to f(u_0)$, the continuity property of f at u_0. Now assume that $t_0 = 1$. Then choose $x = u_0$ and $y \in I$ such that $x < y$. Then a similar argument to the previous one guarantees that f is continuous from the right at u_0. The left continuity of f at u_0 can be obtained analogously, choosing now $y < x$. The case $t_0 = 0$ can be investigated in the same way, and the proof of the first claim is completed.

Claim 2: Continuity of f implies differentiability of h everywhere.

According to the continuity, f has an antiderivative F. Let $x_0 \in D$ be fixed, and denote F that particular antiderivative, which vanishes at x_0. Substituting

$x = u$ into (1) and then integrating the sides between x_0 and any element x of D, the next formula follows:

$$\int_{x_0}^{x} f\big(tu + (1-t)y\big)du - (x - x_0)f(y) = h(t)\big(F(x) - (x - x_0)f(y)\big). \tag{2}$$

If $F(x) - (x - x_0)f(y) = 0$ for all $x \in D$, then, after differentiating, $f(x) = f(y)$ is obtained. In this case, f is a constant function. Assume now that $F(x) - (x - x_0)f(y) \neq 0$ for some $x \in D$. Then substituting $v = tu + (1-t)y$ and applying change of variables, one can arrive at

$$\begin{aligned}\int_{x_0}^{x} f\big(tu + (1-t)y\big)du &= \frac{1}{t}\int_{tx_0+(1-t)y}^{tx+(1-t)y} f(v)dv \\ &= \frac{1}{t}\Big(F\big(tx + (1-t)y\big) - F\big(tx_0 + (1-t)y\big)\Big).\end{aligned}$$

The last expression, due to the continuity of f again, is differentiable on $]0,1]$. Therefore, the left hand-side of (2) is differentiable with respect to t on $]0,1]$. Moreover, a similar reasoning gives that it is differentiable on the entire $[0,1]$. Hence the left hand-side of (2) is differentiable with respect to t, as well. Then, the assumption $F(x) - (x - x_0)f(y) \neq 0$ guarantees that h is differentiable on its domain.

Claim 3: Differentiability of h implies differentiability of f everywhere.

This statement can be proved via the same methods that have been already applied in Claim 1. Details are omitted.

Having the differentiability property of f, differentiate (1) with respect to the variable x and then substitute $y = x$. This process yields the equation

$$tf'(x) = h(t)f'(x); \qquad x \in D, \quad t \in [0,1].$$

If $f'(x) = 0$ for all $x \in D$, then f must be a constant function. If this is not the case, there exists some $x \in D$ such that $f'(x) \neq 0$. After canceling this term, $h(t) = t$ follows for all $t \in [0,1]$, which was to be proved.

Now assume that D is an algebraically open convex subset of a real linear space X. Fix $x_0 \in D$, and let $x \in D \setminus \{x_0\}$ be arbitrary. Then the line determined by x and x_0 meets D in an open interval, so the previous part completes the proof. □

This proposition will be reformulated into a more adequate form in the next section. However, in its present form, Proposition 2.4 has immediate consequences for the classes of Breckner-, Godunova–Levin-, and P-convex functions. In each cases, the sets of the corresponding affine functions are constant. More precisely, they consist of only the zero function.

The next proposition can be considered as a generalization of the Cantor Intersection Theorem. In fact, its statement is a direct consequence of the Helly Theorem [10]. To make the paper self-contained, we present here an independent and direct proof. Finally, a separation theorem via constant functions is presented. Its role is going to be enlightened later.

Proposition 2.5. *If $\mathcal{H}$ is a family of compact intervals such that each two members of $\mathcal{H}$ intersect, then $\bigcap \mathcal{H}$ is nonempty, as well.*

Proof. The first ingredient of the proof is known as the Riesz Lemma on compactness: *If $\mathcal{H}$ is a family of closed sets in a topological space such that any finite subcollection of $\mathcal{H}$ intersects, and $\mathcal{H}$ contains a compact member, then $\bigcap \mathcal{H} \neq \emptyset$.* Indeed, assume to the contrary, that $\bigcap \mathcal{H} = \emptyset$. Then the complements of the members in $\mathcal{H}$ cover the entire space. In particular, these complements provide an open covering for the compact member K. Then, due to the compactness, a finite family of the complements also covers K. However, in this case, the corresponding original members and K form such a disjoint family which consists of finite members. This is a contradiction.

In view of Riesz' Lemma, only the finite intersection property should be verified in the second step. Consider a finite subsystem $\{[a_k, b_k] \in \mathcal{H} \mid k = 1, \dots, n\}$, where the intervals are labeled in increasing order of their left endpoints: $a_1 \leq \cdots \leq a_n$. Since any two members are intersect, therefore $a_n \in [a_k, b_k]$ holds for all $k \in \{1, \dots, n\}$, yielding the desired property. □

Proposition 2.6. *If $f, g\colon D \to \mathbb{R}$ are given on a nonempty set D, then the necessary and sufficient condition for the existence of $c \in \mathbb{R}$ fulfilling $f \leq c \leq g$ is that $f(x) \leq g(y)$ holds for all $x, y \in D$.*

Proof. If there exists some $c \in \mathbb{R}$ fulfilling $f \leq c \leq g$, then $f(x) \leq g(y)$ holds evidently for all elements $x, y \in D$. For the converse implication, consider the family

$$\mathcal{H} := \{[f(x), g(x)] \mid x \in D\}.$$

Clearly, $\mathcal{H}$ consists of nonempty compact intervals. Moreover, any two members of $\mathcal{H}$ intersect. Indeed, suppose to the contrary that $[f(x_1), g(x_1)]$ and $[f(x_2), g(x_2)]$ are disjoint. Then, there are two possibilities: either $g(x_1) < f(x_2)$ or $g(x_2) < f(x_1)$. However, both one contradict to the assumption $f(x) \leq g(y)$. Therefore, Proposition 2.5 finishes the proof. □

It is worth mentioning that Proposition 2.6 follows from other results if D is a real interval. For details and further references, consult the papers [4] and [5]. Besides some cited ones, these papers clarify the role of the Helly Theorem in separation problems.

3. Affine and convex separation by h-convex functions

The first main result states that nonconstant h-affine functions appear only in the classical convexity case. The key tool of the proof is the next fact: If a Jensen-affine function (in other words, a midpoint affine function) is bounded on an interval, then it is affine in the usual sense. This fact is an immediate consequence of the celebrated Bernstein–Doetch Theorem [3].

Theorem 3.1. *If D is a convex subset of a real linear space X, then either each h-affine function on D is a constant function, or $h = \mathrm{id}$.*

Proof. Assume that D is a real interval, and f is a nonconstant h-affine function. Then, for simplicity, there exist elements $x < y$ of D such that $f(x) < f(y)$. Consider now the identities

$$f\big(tx + (1-t)y\big) - f(y) = \big(f(x) - f(y)\big)h(t),$$
$$f\big(tx + (1-t)y\big) - f(x) = \big(f(y) - f(x)\big)h(1-t).$$

Since h is nonegative, the right-hand side of the first equation is nonpositive, while the right-hand side of the second one is nonnegative. This proves that f is bounded on the interval $[x, y]$. On the other hand, by subtracting these identities, the nonzero term $f(x) - f(y)$ can be canceled, and

$$h(t) + h(1-t) = 1, \quad t \in [0, 1]$$

follows. In particular, $h(1/2) = 1/2$ holds, resulting in the Jensen-affinity of f. However, Jensen-affinity and boundedness together give that $f(u) = \alpha u + \beta$ for all $u \in [x, y]$. Moreover, $\alpha \neq 0$ since f is supposed to be nonconstant. Substituting this form of f into the equation

$$f\big(tx + (1-t)y\big) - f(y) = \big(f(x) - f(y)\big)h(t),$$

the left hand-side turns out to be $\alpha t(x - y)$, while the right-hand side is equal to $\alpha h(t)(x - y)$. Thus $h(t) = t$ for all $t \in [0, 1]$, as it has been stated.

Assume now that D is a convex subset of a real linear space X having at least two elements. Fix $x_0 \in D$, and let $x \in D \setminus \{x_0\}$ be arbitrary. Then the line determined by x and x_0 meets D in an interval, so the previous part completes the proof. □

Using Theorem 3.1 and Proposition 2.2, there are two possibilities for h-affine separation in the case $h \neq \mathrm{id}$. The first one is when $h(t)+h(1-t) \neq 1$ holds with some $t \in [0, 1]$. Then, the only h-affine function is the constant zero function. Hence two given functions possess an h-affine separator if and only if one of them is nonegative, while the other one is nonpositive. The second case is when $h(t) + h(1-t) = 1$ holds for all $t \in [0, 1]$. Then, the set of h affine functions coincides with the set of constant functions. In this case, Proposition 2.6 gives the complete answer for the existence of h-affine separator between two given functions.

Similarly, Theorem 3.1 suggests that the length of the involved convex combinations cannot be uniformly controlled in the result of Olbryś. Indeed, for fixed $n \in \mathbb{N}$, define

$$S_n := \Big\{(\lambda_1, \dots, \lambda_n) \in [0,1]^n \mid \sum_{k=1}^{n} \lambda_k = 1\Big\}, \quad H(\lambda_1, \dots, \lambda_n) := \sum_{k=1}^{n} h(\lambda_k),$$

and $\beta := \sup_{S_n} H$. Assume now that $1 < \beta < +\infty$. For example, this occurs in the case of P-convexity. Then Proposition 2.1 guarantees that the range of h-convex functions is nonnegative. In particular, there is no h-convex separator between the functions $f \equiv -\beta$ and $g \equiv -1$, where f and g are defined in a convex subset of a real linear space. On the other hand, the inequality of the Olbryś Theorem is fulfilled if the length of the convex combination is at most n. Indeed, if $m \in \mathbb{N}$ and $m \le n$, then

$$f\Big(\sum_{k=1}^{m} \lambda_k x_k\Big) = -\beta \le -\sum_{k=1}^{m} h(\lambda_k) = \sum_{k=1}^{m} h(\lambda_k) g(x_k).$$

There is another significant result of Olbryś [17] which worth to compare with Theorem 3.1. *If D is an algebraic open, convex subset of a real linear space X, and $h\colon [0,1] \to \mathbb{R}$ satisfies $h(t) + h(1-t) = 1$ for all $t \in [0,1]$, then either each h-convex function is a constant function, or* $h = \mathrm{id}$. Observe that nonnegativity on h is not assumed here.

In the particular case when D is an open and connected subset of a real or complex normed space X, Theorem 3.1 can also be reduced from a result of Maksa and Páles [14]. In their paper, the functional equation

$$f(tx + sy) = pf(x) + qf(y)$$

has been studied, which is an obvious generalizaton of the defining equation of h-affine functions. The authors express the solution its terms of additive functions. In our opinion, these result of Olbryś [17] and of Maksa and Páles [14] belong definitly to the deepest results of the topic.

In order to give the answer to the last motivating question, the notion of h-convexity has to be refined:

Definition 3.2. Let D be a convex subset of a real vector space X and let $h\colon [0,1] \to \mathbb{R}_+$ be given. For a fixed $n \in \mathbb{N}$, a function $f\colon D \to \mathbb{R}$ is said to be *h_n-convex* if, for all $\lambda_1, \dots, \lambda_n \in [0,1]$ fulfilling $\lambda_1 + \dots + \lambda_n = 1$ and for all $x_1, \dots, x_n \in D$, the next inequality holds:

$$f\Big(\sum_{k=1}^{n} \lambda_k x_k\Big) \le \sum_{k=1}^{n} h(\lambda_k) f(x_k).$$

We say that f is *h_∞-convex*, if it is h_n-convex for all $n \in \mathbb{N}$.

In this framework, h-convexity coincides to h_2-convexity. If $m \le n$, then h_n-convexity implies h_m-convexity. Contrary to the classical setting, h_2-convexity does not provide h_∞-convexity in general. However, as Olbryś has proved [16], this implication is still valid whenever h is multiplicative. Now the second main result reads as follows.

Theorem 3.3. *Let D be a convex subset of a real vector space X and assume that $h\colon [0,1] \to \mathbb{R}_+$ fulfills $h(1) = 1$. If, for all $n; m_1, \dots, m_n \in \mathbb{N}$, the functions $f, g\colon D \to \mathbb{R}$ satisfy the inequalities*

$$f\Big(\sum_{k=1}^{n} \lambda_k x_k\Big) \le \sum_{k=1}^{n} \prod_{j=1}^{m_k} h(\lambda_{kj}) g(x_k)$$

whenever $x_1, \dots, x_n \in D$ and $\lambda_{kj} \in [0,1]$ with $\lambda_k = \lambda_{k1} \cdot \ldots \cdot \lambda_{km_k}$ and $\lambda_1 + \dots + \lambda_n = 1$, then there exists a h_∞-convex separator $\phi\colon D \to \mathbb{R}$ fulfilling $f \le \phi \le g$.

Proof. For convenience, denote the set of convex coefficients by S and the set of product representations of $\alpha \in [0,1]$ by $P(\alpha)$, respectively; that is,

$$S := \bigcup_{n \in \mathbb{N}} \{\lambda \in [0,1]^n \mid \lambda = (\lambda_k)_{k=1}^n,\ \lambda_1 + \dots + \lambda_n = 1\},$$

$$P(\alpha) := \bigcup_{n \in \mathbb{N}} \{\lambda \in [0,1]^n \mid \lambda = (\lambda_k)_{k=1}^n,\ \lambda_1 \cdot \ldots \cdot \lambda_n = \alpha\}.$$

Define

$$\phi(x) = \inf\Big\{\sum_{k=1}^{n} \prod_{j=1}^{m_k} h(\lambda_{kj}) g(x_k) \,\Big|\, x = \sum_{k=1}^{n} \lambda_k x_k,\ (\lambda_k)_{k=1}^n \in S,\ (\lambda_{kj})_{j=1}^{m_k} \in P(\lambda_k)\Big\}.$$

The assumption of the theorem guarantees that $\phi\colon D \to \mathbb{R}$ is a function, indeed. It is also clear that $f \le \phi$ holds. The property $h(1) = 1$ implies the inequality $\phi \le g$. The only task is to verify the h_∞-convexity of ϕ. To do this, fix $\varepsilon > 0$ and let $x = \lambda_1 x_1 + \dots + \lambda_n x_n$ be a convex representation. Since $\phi(x_k) + \varepsilon$ is not an infimum, there exist coefficients $\lambda_{ljk} \in [0,1]$ and vectors $x_{lk} \in D$

$$\phi(x_k) + \varepsilon > \sum_{l=1}^{n(k)} \prod_{j=1}^{m(l,k)} h(\lambda_{ljk}) g(x_{lk}), \tag{3}$$

where

$$x_k = \sum_{l=1}^{n(k)} \lambda_{lk} x_{lk}, \quad 1 = \sum_{l=1}^{n(k)} \lambda_{lk}, \quad \lambda_{lk} = \prod_{j=1}^{m(l,k)} \lambda_{ljk}.$$

For $k \in \{1, \dots, n\}$ and $l \in \{1, \dots, n(k)\}$, consider the coefficients

$$\mu_{lk} := \lambda_k \lambda_{lk} = \prod_{j=0}^{m(l,k)} \lambda_{ljk}, \quad \lambda_{l0k} := \lambda_k$$

Then, $(\lambda_k, \lambda_{ljk})_{j=1}^{m(l,k)} \in P(\mu_{lk})$. Moreover,

$$\sum_{k=1}^{n}\sum_{l=1}^{n(k)} \mu_{lk}x_{lk} = \sum_{k=1}^{n}\lambda_k\sum_{l=1}^{n(k)}\lambda_{lk}x_{lk} = \sum_{k=1}^{n}\lambda_k x_k = x,$$

and

$$\sum_{k=1}^{n}\sum_{l=1}^{n(k)} \mu_{lk} = \sum_{k=1}^{n}\lambda_k\sum_{l=1}^{n(k)}\lambda_{lk} = \sum_{k=1}^{n}\lambda_k = 1.$$

This proves that the value

$$y = \sum_{k=1}^{n}\sum_{l=1}^{n(k)}\Big(h(\lambda_k)\cdot\prod_{j=1}^{m(l,k)} h(\lambda_{ljk})g(x_{lk})\Big) = \sum_{k=1}^{n}\sum_{l=1}^{n(k)}\prod_{j=0}^{m(l,k)} h(\lambda_{ljk})g(x_{lk})$$

belongs to the defining set of $\phi(x)$, and hence $\phi(x) \le y$ remains true. Therefore, multiplying the sides of (3) by the nonnegative value $h(\lambda_k)$ and then summing the obtained inequalities,

$$\sum_{k=1}^{n} h(\lambda_k)\phi(x_k) + \varepsilon\sum_{k=1}^{n} h(\lambda_k) > \sum_{k=1}^{n}\sum_{l=1}^{n(k)}\prod_{j=0}^{m(l,k)} h(\lambda_{ljk})g(x_{lk}) = y \ge \phi(x).$$

Passing to the limit for $\varepsilon \to 0$, the desired convexity property is obtained. □

Observe that Theorem 3.3 is a generalization of the Olbryś Theorem and hence the result of Baron, Matkowski, and Nikodem [1]. Indeed, if h is multiplicative, then h is nonnegative, fulfills the condition $h(1) = 1$, further h-convexity and h_∞-convexity are equivalent as it has already been mentioned (see [16] for details). Therefore the inequality of Theorem 3.3 reduces to that of Olbryś. On the other hand, Theorem 3.3 is not only sufficient, but also necessary if h is multiplicative. This part is a straightforward calculation.

Note also that the inequality in Theorem 3.3 suggests a possible definition of ε-h-convexity. A similar result to stability property in [16] also remains true in this situation. The details are omitted.

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