Journal of Convex Analysis
Volume 25 (2018), No. 3, 1045–1058

Dentable Point and Ball-Covering Property in Banach Spaces

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Received: April 29, 2015
Revised manuscript received: March 14, 2017
Accepted: August 2, 2017

We prove that if every bounded subset of X^* is w^*-separable, X is compactly locally uniformly convex, X is 2-strictly convex and X is nonsquare, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of dentable points of $B(X)$ such that $S(X) \subset \cup_{n=1}^{\infty} B(x_n, r_n)$, where $r_n < 1$ for all $n \in N$. Moreover, we also prove that if A is a bounded closed convex subset of X, then $x \in A$ is a strongly exposed point of A if and only if x is a dentable point of A and x is a w^*-exposed point of $\overline{A}^{w^*}$.

Keywords: Compactly locally uniformly convex, ball-covering property, dentable point, nonsquare space, 2-strictly convex space.

2010 Mathematics Subject Classification: 46B20

1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and unit ball of X, respectively. By X^* denotes the dual space of X. $B(x, r)$ denotes the open ball centered at x and of radius $r > 0$. $C(\overline{C^w}, \overline{C^{w^*}})$ denotes closed hull of C(weak closed hull, closed hull weak* closed hull). $\mathrm{dist}(x, C)$ denote the distance of x and C. Let N, R and R^+ denote the sets of natural number, reals and nonnegative reals, respectively.

The study of geometric and topological properties of unit balls of Banach spaces plays a central role in the geometry of Banach spaces. Almost all properties of Banach spaces, such as convexity, smoothness, reflexivity and the Radon-Nikodym property, can be viewed as properties of the unit ball. We should mention here that there are many topics studying behavior of ball collections.

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ISSN 0944-6532 / $ 2.50

For example, the Mazur intersection property, the packing sphere problem of unit balls, the measure of non-compactness with respect to topological degree, and the ball topology have also brought great attention of many mathematicians.

Starting with a different viewpoint, a notion of ball-covering property is introduced by Cheng in [2].

Definition 1.1. A Banach space is said to have the *ball-covering property* if its unit sphere can be contained in the union of countably many balls off the origin.

In [2], Cheng proved that if X is smooth and with $\dim X = n < \infty$, in particular, $X = R^n$, then the sphere can be covered by $n+1$ balls. In [3], Cheng proved that if X is locally uniformly convex and $B(X^*)$ is w^*-separable, then X has the ball-covering property. In [4], Cheng and Liu proved that by constructing equivalent norms on l^∞, there exists a Banach space $(l^\infty, \|\cdot\|^0)$ such that $(l^\infty, \|\cdot\|^0)$ has not the ball-covering property. In [7], Cheng proved that for every $\varepsilon > 0$, every Banach space with a w^*-separable dual has an $1+\varepsilon$-equivalent norm with the ball-covering property. In [11], Shang and Cui proved that if a separable space X has the Radon-Nikodym property, then X^* has the ball-covering property. In [12], Shang and Cui proved that if X is separable, X is locally 2-uniformly convex and X is uniformly nonsquare, then there exists a sequence $\{x_n\}_{n=1}^\infty$ of strongly extreme points such that $\cup_{n=1}^\infty B(x_n, r_n)$ is a ball-covering of X.

Let us recall some geometrical notions and lemmas which will be used in the further part of the paper.

Definition 1.2. A Banach space X is said to be *nonsquare* if for any $x, y \in S(X)$, we have $\min\{\|x+y\|, \|x-y\|\} < 2$.

Definition 1.3. Given $A \subset X$, a point $x \in A$ is said to be a *dentable point* of A if for any $\varepsilon > 0$, we have $x \notin \overline{co}(A \backslash B(x, \varepsilon))$.

Definition 1.4. Given $G \subset X^*$, a point $x^* \in G$ is said to be a w^**-exposed point* of G if there exists $x \in S(X)$ such that $x^*(x) > y^*(x)$ whenever $y^* \in G \setminus \{x^*\}$.

Definition 1.5. Given $A \subset X$, a point $x \in A$ is said to be an *exposed point* of A if there exists $x^* \in S(X^*)$ such that $x^*(x) > x^*(y)$ whenever $y \in A \setminus \{x\}$.

Definition 1.6. Given $C \subset X$, a point $x \in C$ is said to be a *strongly exposed point* of C if for any $\{x_n\}_{n=1}^\infty \subset C$, there exists $x^* \in S(X^*)$ such that $x_n \to x$ whenever $x^*(x_n) \to \sup\{x^*(x) : x \in C\}$.

Definition 1.7. Given $A \subset X$, a point $x \in A$ is said to be a *pc-point* of A if the identity mapping $(A, w) \to (A, \|\cdot\|)$ is continuous at x.

Definition 1.8. A point $x \in S(X)$ is called a *smooth point* if it has an unique supporting functional f_x. If every $x \in S(X)$ is a smooth point, then X is called *smooth.*

Definition 1.9. A point $x \in A$ is said to be an *extreme point* of A if $2x = y+z$ and $y, z \in A$ imply $y = z$. The set of all extreme points of A is denoted by $ExtA$. If $ExtB(X) = S(X)$, then X is said to be a strictly convex space.

Definition 1.10. A Banach space X is said to be *compactly locally uniformly convex* if for any $x \in S(X)$ and $\{x_n\}_{n=1}^{\infty} \subset S(X)$, if $\|x_n + x\| \to 2$, then $\{x_n\}_{n=1}^{\infty}$ has a Cauchy subsequence.

Definition 1.11. A Banach space X is said to be *k-strictly convex* if for any $k+1$ elements $x_1, x_2, \cdots, x_{k+1} \in S(X)$, if $\|x_1 + x_2 + \cdots + x_{k+1}\| = k+1$, then $x_1, x_2, \cdots, x_{k+1}$ are linearly dependent.

I. Singer defined the k-strictly convex spaces in [15]. It is well known that X is strictly convex if and only if X is 1-strictly convex. Moreover, it is well known that if $x \in A$ is a strongly exposed point of A, then x is a dentable point of A.

Lemma 1.12. (see [9]) *Suppose that A is a bounded closed convex subset of X. Then $x \in A$ is a dentable point of A if and only if*
(1) *$x \in A$ is an extreme point of A;*
(2) *$x \in A$ is a pc-point of A.*

In this paper, we prove that if every bounded subset of X^* is w^*-separable, X is compactly locally uniformly convex, X is 2-strictly convex and X is nonsquare, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of dentable points of $B(X)$ such that $S(X) \subset \cup_{n=1}^{\infty} B(x_n, r_n)$, where $r_n < 1$ for all $n \in N$. Moreover, we also prove that if A is a bounded closed convex set, then $x \in A$ is a strongly exposed point of A if and only if x is a dentable point of A and x is a w^*-exposed point of $\overline{A}^{w^*}$. The topic of this paper is related to the topic of [11]–[13].

2. Main results

Theorem 2.1. *Suppose that*

(1) *Every bounded subset of X^* is w^*-separable;*

(2) *X is compactly locally uniformly convex and 2-strictly convex;*

(3) *X is nonsquare.*

Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of dentable points of $B(X)$ such that $S(X) \subset \cup_{n=1}^{\infty} B(x_n, r_n)$, where $r_n < 1$ for all $n \in N$.

Proof. (a) We will prove that if $x \in J(x^*)$, then $J(x^*) \cap \overline{co}(B(X) \backslash U_{J(x^*)}) = \emptyset$ for any open set $U_{J(x^*)} \supset J(x^*)$, where

$$J(x^*) = \{x \in X : x^*(x) = \|x^*\| = \|x\| = 1\}.$$

We claim that there exists $k(x^*) > 0$ such that

$$k(x^*) \leq 1 - \sup\left\{x^*(x) : x \in B(X)\backslash U_{J(x^*)}\right\} \tag{1}$$

for any open set $U_{J(x^*)} \supset J(x^*)$. Otherwise there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset B(X)\backslash U_{J(x^*)}$ such that $x^*(x_n) \to 1$ as $n \to \infty$. Pick $x \in J(x^*)$. Then

$$\begin{aligned} 2 &\geq \limsup_{n\to\infty} \|x_n + x\| \geq \liminf_{n\to\infty} \|x_n + x\| \geq \liminf_{n\to\infty} (x^*, x_n + x) \\ &= \lim_{n\to\infty} x^*(x_n) + \lim_{n\to\infty} x^*(x) = 2. \end{aligned}$$

This implies that $\|x_n + x\| \to 2$ as $n \to \infty$. Since X is compactly locally uniformly convex, we obtain that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to y$ as $k \to \infty$. Hence $y \in S(X)$ and $x^*(y + x) = 2$. It is easy to see that $y \in J(x^*)$. This implies that $x_{n_k} \in U_{J(x^*)}$ for enough large k, a contradiction. Therefore, by formula (1), we obtain that

$$\begin{aligned} x^*(x) - k(x^*) &\geq \sup\left\{x^*(x) : x \in B(X)\backslash U_{J(x^*)}\right\} \\ &= \sup\left\{x^*(x) : x \in co\left(B(X)\backslash U_{J(x^*)}\right)\right\} \\ &= \sup\left\{x^*(x) : x \in \overline{co}\left(B(X)\backslash U_{J(x^*)}\right)\right\} \end{aligned}$$

for any $x \in J(x^*)$. This implies that $x \notin \overline{co}(B(X)\backslash U_{J(x^*)})$ for any open set $U_{J(x^*)} \supset J(x^*)$. Hence $J(x^*) \cap \overline{co}(B(X)\backslash U_{J(x^*)}) = \emptyset$ for any open set $U_{J(x^*)} \supset J(x^*)$.

(b) We will prove that if $x^* \in S(X^*)$ is norm attainable on $S(X)$, then the set $J(x^*)$ is a line segment or a singleton. In fact, for any $z_1 \in J(x^*)$, $z_2 \in J(x^*)$ and $z_3 \in J(x^*)$, we have

$$3 \geq \|z_1 + z_2 + z_3\| \geq x^*(z_1 + z_2 + z_3) = 3 = \|z_1\| + \|z_2\| + \|z_3\|.$$

Since X is 2-strictly convex, we may assume that $z_3 = t_1 z_1 + t_2 z_2$. Thus

$$1 = x^*(z_3) = x^*(t_1 z_1 + t_2 z_2) = t_1 x^*(z_1) + t_2 x^*(z_2) = t_1 + t_2.$$

Since $J(x^*)$ is compact, there exist $x_1 \in J(x^*)$ and $x_2 \in J(x^*)$ such that

$$\sup\{\|x - y\| : x \in J(x^*), y \in J(x^*)\} = \|x_1 - x_2\|.$$

Let $x_1 = x_2$. Then $J(x^*)$ is a singleton. Let $x_1 \neq x_2$. Then, for any $y \in J(x^*)$, if $x_1 = ty + (1-t)x_2$, then $t \neq 0$. Otherwise, we obtain that $x_1 = x_2$. Hence, for any $y \in J(x^*)$, we obtain that $y = \alpha x_1 + (1-\alpha)x_2$. Suppose that $\alpha < 0$. Then $x_2 = y/(1-\alpha) + (-\alpha x_1)/(1-\alpha)$. Hence

$$\|x_1 - x_2\| = \left\|x_1 - \frac{1}{1-\alpha} y - \frac{-\alpha}{1-\alpha} x_1\right\| = \frac{-\alpha}{1-\alpha}\|x_1 - y\| < \|x_1 - y\|,$$

a contradiction. Similarly, we have $1-\alpha \geq 0$. This implies that $J(x^*) = [x_1, x_2]$. We claim that x_1 is an extreme point of $B(X)$. In fact, suppose that there exist $y_1 \in B(X)$ and $y_2 \in B(X)$ such that $2x_1 = y_1 + y_2$. Hence

$$2 = x^*(2x_1) = x^*(y_1 + y_2) = x^*(y_1) + x^*(y_2). \tag{2}$$

This implies that $x^*(y_1) = x^*(y_2) = 1$. Then $y_1 \in J(x^*)$ and $y_2 \in J(x^*)$. Noticing that $J(x^*) = [x_1, x_2]$, we obtain that $x_1 = y_1 = y_2$. This implies that x_1 is an extreme point of $B(X)$.

(c) We next will prove that x_1 is a pc-point of $B(X)$. Let $\{x_\alpha\}_{\alpha\in\Delta} \subset B(X)$ be a net and $x_\alpha \xrightarrow{w} x_1$. Suppose that $\{x_\alpha\}_{\alpha\in\Delta}$ dose not converge to x_1. Then there exists $\varepsilon > 0$ such that for any $\alpha \in \Delta$, there exists $\beta > \alpha$, for which $\|x_\beta - x_1\| \geq \varepsilon$. Hence we define a subnet $\{x_\beta\}_{\beta\in\Delta_1}$ of $\{x_\alpha\}_{\alpha\in\Delta}$. By $x_\alpha \xrightarrow{w} x_1$, we obtain that for any weak neighbourhood V of x_1, there exists $\alpha_0 \in \Delta$ such that $x_\alpha \in V$ whenever $\alpha > \alpha_0$. Therefore, by the definition of $\{x_\beta\}_{\beta\in\Delta_1}$, there exists $\beta_0 > \alpha_0$ such that $x_\beta \in V$ whenever $\beta > \beta_0 > \alpha_0$. This implies that $x_\beta \xrightarrow{w} x_1$. Put

$$\varepsilon_{x_1} = \sup\left\{\varepsilon > 0 : \begin{array}{l} \text{there exists a subnet } \{x_\beta\} \text{ of } \{x_\alpha\}_{\alpha\in\Delta} \\ \text{such that } \|x_\beta - x_1\| \geq \varepsilon, \quad x_\beta \xrightarrow{w} x_1 \end{array}\right\}.$$

Then $\varepsilon_{x_1} > 0$. Moreover, it is easy to see that the net $\{x_\alpha\}_{\alpha\in\Delta}$ does not converge to z, where $z \in (x_1, x_2] = \{\lambda x_1 + (1-\lambda)x_2 : \lambda \in [0,1)\}$. Put

$$\varepsilon_z = \sup\left\{\varepsilon > 0 : \begin{array}{l} \text{there exists a subnet } \{x_\beta\} \text{ of } \{x_\alpha\}_{\alpha\in\Delta} \\ \text{such that } \|x_\beta - z\| \geq \varepsilon, \quad x_\beta \xrightarrow{w} x_1 \end{array}\right\},$$

where $z \in (x_1, x_2]$. Then it is easy to see that $\varepsilon_z > 0$. We claim that

$$\eta = \inf\{\varepsilon_x > 0 : x \in [x_1, x_2]\} > 0. \tag{3}$$

Otherwise, there exists a sequence $\{z_n\}_{n=1}^\infty \subset [x_1, x_2]$ such that $\varepsilon_{z_n} \to 0$ as $n \to \infty$. Since the set $[x_1, x_2]$ is compact, we may assume without loss of generality that $z_n \to z_0 \in [x_1, x_2]$ as $n \to \infty$. Therefore, by $\varepsilon_{z_n} \to 0$ and $z_n \to z_0$, we obtain that there exists a natural number $n(0)$ such that

$$\left\|z_{n(0)} - z_0\right\| < \frac{1}{4}\varepsilon_{z_0} \qquad \text{and} \qquad \varepsilon_{z_{n(0)}} < \frac{1}{4}\varepsilon_{z_0} \tag{4}$$

From the previous proof, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}_{\alpha\in\Delta}$ such that $\|x_\beta - z_0\| \geq (3/4)\varepsilon_{z_0}$ and $x_\beta \xrightarrow{w} x_1$. Therefore, by formula (4), we obtain that

$$\left\|x_\beta - z_{n(0)}\right\| \geq \|x_\beta - z_0\| - \left\|z_0 - z_{n(0)}\right\| \geq \frac{3}{4}\varepsilon_{z_0} - \frac{1}{4}\varepsilon_{z_0} = \frac{1}{2}\varepsilon_{z_0}.$$

This implies that $\varepsilon_{z_n(0)} \geq (1/2)\varepsilon_{z_0}$, which contradict formula (4). Hence there exists a finite set $\{y_i\}_{i=1}^k$ such that

$$\{y_i\}_{i=1}^k \subset [x_1, x_2] \subset \bigcup_{i=1}^{k} B\big(y_i, \frac{1}{2}\eta\big).$$

From the previous proof there exists a subnet $\{x_{\beta_1}\}$ of $\{x_\alpha\}$ such that $\|x_{\beta_1} - y_1\| \geq \eta$ and $x_{\beta_1} \xrightarrow{w} x_1$. From the previous proof, there exists a subnet $\{x_{\beta_2}\}$ of $\{x_{\beta_1}\}$ such that $\|x_{\beta_2} - y_1\| \geq \eta$, $\|x_{\beta_2} - y_2\| \geq \eta$ and $x_{\beta_2} \xrightarrow{w} x_1$. Hence there exists a subnet $\{x_{\beta_k}\}$ of $\{x_\alpha\}$ such that

$$\|x_{\beta_k} - y_1\| \geq \eta,\ \|x_{\beta_k} - y_2\| \geq \eta\ ,...,\ \|x_{\beta_k} - y_n\| \geq \eta \ \text{ and } \ x_{\beta_k} \xrightarrow{w} x_1. \tag{5}$$

Let

$$U_{J(x^*)} = \bigcup_{i=1}^{k} B\big(y_i, \frac{1}{2}\eta\big) \supset [x_1, x_2] = J(x^*).$$

Then, by formula (5), we obtain that $U_{J(x^*)} \cap \{x_{\beta_k}\} = \emptyset$. Therefore, by $\{x_{\beta_k}\} \subset B(X)$ and $U_{J(x^*)} \cap \{x_{\beta_k}\} = \emptyset$, we have $\{x_{\beta_k}\} \subset B(X) \backslash U_{J(x^*)}$. Moreover, by the proof of (a), we obtain that

$$J(x^*) \cap \overline{co}(B(X) \backslash U_{J(x^*)}) = \emptyset.$$

Therefore, by the separation theorem, there exist $y^* \in S(X^*)$ and $r > 0$ such that

$$\inf\{y^*(x) : x \in J(x^*)\} - r \geq \sup\left\{y^*(x) : x \in \overline{co}\left(B(X) \backslash U_{J(x^*)}\right)\right\}.$$

Since $\{x_{\beta_k}\} \subset B(X) \backslash U_{J(x^*)}$, we obtain that $y^*(x_1) - r \geq \sup\{y^*(x) : x \in \{x_{\beta_k}\}\}$, which contradict $x_{\beta_k} \xrightarrow{w} x_1$. This implies that $x_\alpha \to x_1$. Hence we obtain that x_1 is a pc-point of $B(X)$. Since x_1 is an extreme point of $B(X)$, by Lemma 1.12, we obtain that x_1 is a dentable point of $B(X)$. Similarly, we obtain that x_2 is a dentable point of $B(X)$.

(d) We first define a set

$$A(X^*) = \{x^* \in S(X^*) : x^* \text{ is norm attainable on } S(X)\}.$$

Let $\mathrm{diam} J(x^*) = \sup\{\|x - y\| : x \in J(x^*),\ y \in J(x^*)\}$. Since X is a nonsquare space, we obtain that

$$\min\{\|x + y\|, \|x - y\|\} < 2.$$

for any $x \in S(X)$ and $y \in S(X)$. Moreover, if $x \in J(x^*)$ and $y \in J(x^*)$, then

$$2 \geq \|x + y\| \geq x^*(x + y) = x^*(x) + x^*(y) = 2.$$

This implies that $\|x - y\| = \min\{\|x - y\|, \|x + y\|\} < 2$ for all $x \in J(x^*)$ and $y \in J(x^*)$. Since $J(x^*)$ is a line segment or a singleton, we have $\mathrm{diam} J(x^*) < 2$

for all $x^* \in J(X^*)$. Moreover, there exists a dense subset $\{r(i)\}_{i=1}^{\infty}$ of $[1/8, 2)$. It is easy to see that $16r(i) - [2 - r(i)] < 16r(i)$ for all $i \in N$. Put

$$H_i^* = \left\{ x^* \in A(X^*) : r(i) - \frac{1}{16}[2 - r(i)] \le \operatorname{diam} J(x^*) \le r(i) \right\},$$

$$H^* = \left\{ x^* \in A(X^*) : \operatorname{diam} J(x^*) \le \frac{1}{8} \right\},$$

$$H_i = \{x \in S(X) : \text{ there exists } x^* \in H_i^* \text{ such that } x^*(x) = 1\}$$

and

$$H = \{x \in S(X) : \text{ there exists } x^* \in H^* \text{ such that } x^*(x) = 1\}.$$

Since every bounded set of X^* is w^*-separable, we obtain that H_i^* and H^* are w^*-separable. Hence there exist two sequences $\{x^*(n)\}_{n=1}^{\infty} \subset H^*$ and $\{x^*(i,n)\}_{n=1}^{\infty} \subset H_i^*$ such that $\{x^*(i,n)\}_{n=1}^{\infty}$ is a w^*-dense subset of H_i^* and $\{x^*(n)\}_{n=1}^{\infty}$ is a w^*-dense subset of H^*. Let $J(x^*(i,n)) = [x(1,i,n), x(2,i,n)]$ and $J(x^*(n)) = [x(1,n), x(2,n)]$. We define open balls

$$B\left(x(1,i,n),\ 1 - \frac{2 - r(i)}{16}\right),\ B\left(x(2,i,n),\ 1 - \frac{2 - r(i)}{16}\right)$$

and

$$B\left(x(1,n), \frac{1}{2}\right),\ B\left(x(2,n), \frac{1}{2}\right)$$

for all $i, n \in N$. From the proof of (c), we obtain that $x(1,i,n)$, $x(2,i,n)$, $x(1,n)$ and $x(2,n)$ are dentable points of $B(X)$ for all $i, n \in N$. We next will prove that

$$S(X) \subset \bigcup_{j=1}^{2} \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ B\left(x(j,i,n), 1 - \frac{2 - r(i)}{16}\right) \cup B\left(x(j,n), \frac{1}{2}\right) \right\}.$$

It is easy to see that $S(X) \subset (\cup_{i=1}^{\infty} H_i^*) \cup H$. For clarity, we will divide the proof into two cases.

Case 1. Let $x \in \cup_{i=1}^{\infty} H_i$. Then there exists a natural number $i_0 \in N$ such that $x \in H_{i_0}$. Hence there exists $y_0^* \in J(x)$ such that $y_0^* \in H_{i_0}^*$, where

$$J(x) = \{x^* \in S(X^*) : x^*(x) = \|x\| = 1\}.$$

Since $\{x^*(i_0, n)\}_{n=1}^{\infty}$ is a w^*-dense subset of $H_{i_0}^*$, there exists a net $\{x_\alpha^*\}_{\alpha \in \Delta} \subset \{x^*(i_0,n)\}_{n=1}^{\infty}$ such that $x_\alpha^* \xrightarrow{w^*} y_0^*$. Then there exists a subsequence $\{x^*(i_0, n_k)\}_{k=1}^{\infty}$ of $\{x^*(i_0,n)\}_{n=1}^{\infty}$ such that $x^*(i_0, n_k) \xrightarrow{w^*} y_0^*$ as $k \to \infty$. Otherwise, there exists a weak* neighborhood U of y_0^* such that $\{x^*(i_0,n)\}_{n=1}^{\infty} \cap U = \emptyset$. This implies that $\{x_\alpha^*\}_{\alpha\in\Delta} \cap U = \emptyset$, which contradict $x_\alpha^* \xrightarrow{w^*} y_0^*$. Hence, for every $k \in N$, we define a weak* neighborhood

$$V_k = \left\{ z^* \in X^* : |z^*(x) - y_0^*(x)| < \frac{1}{2k} \right\}$$

of y_0^*. Since $x^*(i_0, n_k) \xrightarrow{w^*} y_0^*$ as $k \to \infty$, we may assume without loss of generality that $x^*(i_0, n_k) \in V_k$. Then

$$(x^*(i_0, n_k), x) > y_0^*(x) - \frac{1}{2k} > 1 - \frac{1}{k}.$$

This implies that

$$2 \geq \|x(1, i_0, n_k) + x\| \geq (x^*(i_0, n_k), x(1, i_0, n_k) + x) > 2 - \frac{1}{k}$$

and

$$2 \geq \|x(2, i_0, n_k) + x\| \geq (x^*(i_0, n_k), x(2, i_0, n_k) + x) > 2 - \frac{1}{k}.$$

Hence $\|x(1, i_0, n_k) + x\| \to 2$ and $\|x(2, i_0, n_k) + x\| \to 2$ as $k \to \infty$. Since X is compactly locally uniformly convex, we may assume without loss of generality that

$$\lim_{k\to\infty} x(1, i_0, n_k) = x(1, i_0) \qquad \text{and} \qquad \lim_{k\to\infty} x(2, i_0, n_k) = x(2, i_0). \tag{6}$$

Since $(x^*(i_0, n_k), x(1, i_0, n_k)) = 1$ and $\lim\limits_{k\to\infty} x(1, i_0, n_k) = x(1, i_0)$, we obtain that

$$\begin{aligned} 0 &\leq |(x^*(i_0, n_k), x(1, i_0)) - 1| \\ &= |(x^*(i_0, n_k), x(1, i_0)) - (x^*(i_0, n_k), x(1, i_0, n_k))| \\ &\leq \|x^*(i_0, n_k)\| \cdot \|x(1, i_0, n_k) - x(1, i_0)\| \to 0 \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

Since $x^*(i_0, n_k) \xrightarrow{w^*} y_0^*$ and $(x^*(i_0, n_k), x(1, i_0)) \to 1$ as $k \to \infty$, we obtain that $(y_0^*, x(1, i_0)) = 1$. Similarly, we have $(y_0^*, x(2, i_0)) = 1$. Noticing that $y_0^* \in J(x)$, we have $x(1, i_0) \in J(y_0^*)$, $x(2, i_0) \in J(y_0^*)$ and $x \in J(y_0^*)$. Moreover, by formula (6), there exists $k_0 \in N$ such that

$$\|x(1, i_0, n_{k_0}) - x(1, i_0)\| < \frac{2 - r(i_0)}{64^2}, \quad \|x(2, i_0, n_{k_0}) - x(2, i_0)\| < \frac{2 - r(i_0)}{64^2}. \tag{7}$$

and

$$\big|\|x(1, i_0) - x(2, i_0)\| - \|x(1, i_0, n_{k_0}) - x(2, i_0, n_{k_0})\|\big| < \frac{2 - r(i_0)}{64^2} \tag{8}$$

Since $J(x_0^*(i_0, n_{k_0})) = [x(1, i_0, n_{k_0}), x(2, i_0, n_{k_0})] \subset H_{i_0}^*$, we obtain that

$$\|x(1, i_0, n_{k_0}) - x(2, i_0, n_{k_0})\| \geq r(i_0) - \frac{1}{16}[2 - r(i_0)].$$

Therefore, by formula (8), we obtain that

$$\begin{aligned}\|x(1, i_0) - x(2, i_0)\| &> \|x(1, i_0, n_{k_0}) - x(2, i_0, n_{k_0})\| - \frac{2 - r(i_0)}{64^2} \\ &\geq r(i_0) - \frac{2 - r(i_0)}{16} - \frac{2 - r(i_0)}{64^2} \qquad (9)\end{aligned}$$

Moreover, by $y_0^* \in H_{i_0}^*$, we have $\operatorname{diam} J(y_0^*) \leq r(i_0)$. Therefore, by $x(1, i_0) \in J(y_0^*)$ and $x(2, i_0) \in J(y_0^*)$, we obtain that

$$\|x(1, i_0) - x(2, i_0)\| \leq \operatorname{diam} J(y_0^*) \leq r(i_0). \qquad (10)$$

Suppose that $x \in [x(1, i_0), x(2, i_0)]$. Then, by formulas (7) and (10), we obtain that

$$\begin{aligned}&\|x - x(1, i_0, n_{k_0})\| + \|x - x(2, i_0, n_{k_0})\| \\ &\leq \|x - x(1, i_0)\| + \|x(1, i_0) - x(1, i_0, n_{k_0})\| + \|x - x(2, i_0)\| \\ &\qquad + \|x(2, i_0) - x(2, i_0, n_{k_0})\| \\ &= \|x(1, i_0) - x(2, i_0)\| + \|x(1, i_0) - x(1, i_0, n_{k_0})\| + \|x(2, i_0) - x(2, i_0, n_{k_0})\| \\ &< r(i_0) + \frac{2 - r(i_0)}{64^2} + \frac{2 - r(i_0)}{64^2} < \left(1 - \frac{2 - r(i_0)}{16}\right) + \left(1 - \frac{2 - r(i_0)}{16}\right).\end{aligned}$$

This implies that

$$\|x - x(1, i_0, n_{k_0})\| < 1 - \frac{2 - r(i_0)}{16} \quad \text{or} \quad \|x - x(2, i_0, n_{k_0})\| < 1 - \frac{2 - r(i_0)}{16}.$$

Hence

$$x \in B\left(x(1, i_0, n_{k_0}), 1 - \frac{2 - r(i_0)}{16}\right) \cup B\left(x(2, i_0, n_{k_0}), 1 - \frac{2 - r(i_0)}{16}\right).$$

Suppose that $x \notin [x(1, i_0), x(2, i_0)]$. Then, by formulas (9) and (10), we obtain that

$$\|x(1, i_0) - x(2, i_0)\| \in \left(r(i_0) - \frac{2 - r(i_0)}{16} - \frac{2 - r(i_0)}{64^2},\ r(i_0)\right). \qquad (11)$$

Let $J(y_0^*) = [y_1, y_2]$. Then $x(1, i_0) \in [y_1, y_2]$ and $x(2, i_0) \in [y_1, y_2]$. Noticing that $y_0^* \in H_{i_0}^*$, we obtain that

$$\|y_1 - y_2\| \in \left(r(i_0) - \frac{2 - r(i_0)}{16},\ r(i_0)\right). \qquad (12)$$

Since $x \notin [x(1,i_0), x(2,i_0)]$, $[x(1,i_0), x(2,i_0)] \subset [y_1, y_2]$ and $x \in [y_1, y_2]$, we obtain that

$$x(1,i_0) \in [x, x(2,i_0)] \subset [y_1, y_2] \quad \text{or} \quad x(2,i_0) \in [x(1,i_0), x] \subset [y_1, y_2]. \tag{13}$$

We we may assume that $x(1,i_0) \in [x, x(2,i_0)] \subset [y_1, y_2]$. Then $\|x - x(1,i_0)\| < [2 - r(i_0)]/8$. Suppose that $\|x - x(1,i_0)\| \geq [2 - r(i_0)]/8$. Then, by formulas (11)–(13), we obtain that

$$\begin{aligned} \|y_1 - y_2\| &\geq \|x - x(1,i_0)\| + \|x(1,i_0) - x(2,i_0)\| \\ &\geq \frac{2 - r(i_0)}{8} + r(i_0) - \frac{2 - r(i_0)}{16} - \frac{2 - r(i_0)}{64^2} \\ &= r(i_0) + \frac{2 - r(i_0)}{16} - \frac{2 - r(i_0)}{64^2} > r(i_0), \end{aligned}$$

which contradict formula (12). Therefore, by formula (7), we obtain that

$$\begin{aligned} \|x - x(1,i_0,n_{k_0})\| &\leq \|x - x(1,i_0)\| + \|x(1,i_0) - x(1,i_0,n_{k_0})\| \\ &< \frac{2 - r(i_0)}{8} + \frac{2 - r(i_0)}{64^2} < 1 - \frac{2 - r(i_0)}{16}. \end{aligned}$$

This implies that

$$x \in B\left(x(1,i_0,n_{k_0}), 1 - \frac{2 - r(i_0)}{16}\right) \cup B\left(x(2,i_0,n_{k_0}), 1 - \frac{2 - r(i_0)}{16}\right).$$

Case 2. Let $x \in H$. Analogous to the proof of Case 1, we obtain that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} x(1,n_k) = x(1) \qquad \text{and} \qquad \lim_{k\to\infty} x(2,n_k) = x(2).$$

Then there exists $k_0 \in N$ such that $\|x(1,n_{k_0}) - x(1)\| < 1/16$. Analogous to the proof of Case 1, we obtain that there exists $z^* \in H^*$ such that $x(1) \in J(z^*)$, $x(2) \in J(z^*)$ and $x \in J(z^*)$. Since $z^* \in H^*$, we obtain that $\operatorname{diam} J(z^*) \leq 1/8$. Hence

$$\|x - x(1,n_{k_0})\| \leq \|x - x(1)\| + \|x(1) - x(1,n_{k_0})\| \leq \frac{1}{8} + \frac{1}{16} < \frac{1}{2}.$$

This implies that $x \in B(x(1,n_{k_0}), 1/2)$. Combing Case 1 with Case 2, we obtain that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of dentable points of $B(X)$ such that $S(X) \subset \cup_{n=1}^{\infty} B(x_n, r_n)$, where $r_n < 1$ for all $n \in N$. □

Theorem 2.2. *Suppose that A is a bounded closed convex subset of X. Then $x \in A$ is a strongly exposed point of A if and only if*

(1) *$x \in A$ is a dentable point of A;*

(2) *$x \in A$ is a w^*-exposed point of $\overline{A}^{w^*}$.*

Proof. Necessity. Let $x \in A$ be a strongly exposed point of A. Then $x \in A$ is a dentable point of A. Moreover, since A is a closed convex subset of X and $x \in A$ is a strongly exposed point of A, there exists $x^* \in S(X^*)$ such that $z_n \to x$ whenever $\{z_n\}_{n=1}^{\infty} \subset A$ and $x^*(z_n) \to \sup\{x^*(y) : y \in A\}$ as $n \to \infty$. Suppose that there exists $x^{**} \in \overline{A^{w^*}} \setminus \{x\}$ such that $x^*(x^{**}) = x^*(x)$. Since $x^{**} \neq x$, there exist a weak* neighborhood V of x^{**} and a weak* neighborhood U of x such that $U \cap V = \emptyset$. Hence, for any $n \in N$, we define a weak* neighborhood

$$V_n = V \cap \left\{ z^{**} \in X^{**} : |x^*(z^{**}) - x^*(x^{**})| < \frac{1}{n} \right\}$$

of x^{**}. Since $x^{**} \in \overline{A^{w^*}}$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \in V_n$ for all $n \in N$. This implies that $U \cap V_n = \emptyset$. Moreover, it is easy to see that

$$\lim_{n\to\infty} x^*(x_n) = x^*(x^{**}) = x^*(x) = \sup\{x^*(y) : y \in A\}.$$

Since $x \in A$ is a strongly exposed point of A, we obtain that $x_n \to x$ as $n \to \infty$. Then there exists a natural number $n_0 \in N$ such that $x_{n_0} \in U$, a contradiction. Hence $x^*(x) > x^*(y^{**})$ whenever $y^{**} \in \overline{A^{w^*}} \setminus \{x\}$. This implies that $x \in A$ is a w^*-exposed point of $\overline{A^{w^*}}$.

Sufficiency. Since $x \in A$ is a w^*-exposed point of $\overline{A^{w^*}}$, there exists $x^* \in S(X^*)$ such that $x^*(x) > x^*(y^{**})$ whenever $y^{**} \in \overline{A^{w^*}}$. Let $x^*(x_n) \to x^*(x) = \sup\{x^*(y) : y \in A\}$ as $n \to \infty$. Suppose that x_n dose not converge to x. Then we may assume without loss of generality that there exists $\varepsilon > 0$ such that $\|x_n - x\| \geq \varepsilon$. Since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. there exists $x^{**} \in \overline{A^{w^*}}$ such that x^{**} is a weak* accumulation point of $\{x_n\}_{n=1}^{\infty}$. Since $x^*(x_n) \to x^*(x) = \sup\{x^*(y) : y \in A\}$ as $n \to \infty$, we obtain that $x^*(x^{**}) = x^*(x)$ and there exists a net $\{x_\alpha\}_{\alpha\in\Delta} \subset \{x_n\}_{n=1}^{\infty}$ such that $x_\alpha \xrightarrow{w^*} x^{**}$. Since x is a w^*-exposed point of $\overline{A^{w^*}}$, we obtain that $x^{**} = x$. Hence $x_\alpha \xrightarrow{w} x$. Since x is a dentable point of A, we obtain that $x \notin \overline{co}(A \backslash B(x, \varepsilon))$. Therefore, by the separation theorem, there exist $y^* \in S(X^*)$ and $r > 0$ such that

$$y^*(x) - r \geq \sup\{y^*(y) : y \in \overline{co}(A \backslash B(x, \varepsilon))\}. \tag{14}$$

Since $\{x_\alpha\}_{\alpha\in\Delta} \subset \{x_n\}_{n=1}^{\infty} \subset \overline{co}(A \backslash B(x, \varepsilon))$, by formula (14), we have $y^*(x) - r \geq y^*(x_\alpha)$, a contradiction. Hence $x \in A$ is a strongly exposed point of A. □

Definition 2.3. (see [9]) A Banach space X is said to be *nearly very convex* if for any $x^* \in S(X^*)$, if $x^{**}(x^*) = \|x^{**}\| = 1$, then $x^{**} \in S(X)$.

Theorem 2.4. *Suppose that X is a nearly very convex space. Then $x \in S(X)$ is a strongly exposed point of $B(X)$ if and only if*

(1) *$x \in S(X)$ is a dentable point of $B(X)$;*

(2) *$x \in S(X)$ is a exposed point of $B(X)$.*

Proof. Necessity. Let x be a strongly exposed point of $B(X)$. Then, by Theorem 2.2, we obtain that $x \in S(X)$ is a dentable point of $B(X)$ and $x \in S(X)$ is a w^*-exposed point of $\overline{B(X)}^{w^*} = B(X^{**})$. This implies that x is a exposed point of $B(X)$.

Sufficiency. Let $x \in S(X)$ be a dentable point of $B(X)$ and be a exposed point of $B(X)$. Then there exists $x^* \in S(X^*)$ such that $x^*(x) > x^*(y)$ whenever $y \in B(X) \setminus \{x\}$. Since X is nearly very convex, we obtain that if $x^{**}(x^*) = \|x^{**}\| = 1$, then $x^{**} \in S(X)$. This implies that $x^*(x) > x^*(y^{**})$ whenever $y^{**} \in B(X^{**}) \setminus \{x\}$. Hence x is a w^*-exposed point of $\overline{B(X)}^{w^*} = B(X^{**})$. Therefore, by Theorem 2.2, we obtain that x is a strongly exposed point of $B(X)$. □

Corollary 2.5. *Suppose that X is reflexive. Then $x \in S(X)$ is a strongly exposed point of $B(X)$ if and only if*

(1) *$x \in S(X)$ is a dentable point of $B(X)$;*

(2) *$x \in S(X)$ is a exposed point of $B(X)$.*

3. Some examples

Suppose that every bounded subset of X^* is w^*-separable, X is compactly locally uniformly convex and X is 2-strictly convex. Then the conclusion of Theorem 2.1 is not necessarily true.

Example 3.1. Let $(R^2, \|\cdot\|_\infty) = \{(x, y) : x, y \in R, \|(x, y)\|_\infty = \max(|x|, |y|)\}$. It is easy to see that $(R^2, \|\cdot\|_\infty)$ is compactly locally uniformly convex and is 2-strictly convex. Moreover, it is easy to see that $\{(1, 1), (1, -1), (-1, -1), (-1, 1)\}$ is the set of dentable points of $S(R^2, \|\cdot\|_\infty)$. Suppose that there exist $0 < r_1 < 1$, $0 < r_2 < 1$, $0 < r_3 < 1$ and $0 < r_4 < 1$ such that

$$S((R^2, \|\cdot\|_\infty)) \subset B((1, 1), r_1) \cup B((1, -1), r_2) \cup B((-1, -1), r_3) \cup B((-1, 1), r_4).$$

It is easy to see that

$$(0, 1) \notin B((1, 1), r_1) \cup B((1, -1), r_2) \cup B((-1, -1), r_3) \cup B((-1, 1), r_4),$$

a contradiction.

Suppose that every bound subset of X^* is w^*-separable, X is compactly locally uniformly convex, X is 2-strictly convex and X is nonsquare. Then by Theorem 2.1 and Theorem 2.2, it is very natural to ask whether for any $J(x^*) = [x_1, x_2]$, x_1 and x_2 are strongly exposed points of $B(X)$. Unfortunately, it is not true.

Example 3.2. Let $A = \{(x, y) \in R^2 : (x - 1/2)^2 + y^2 \leq 1\}$, $B = \{(x, y) \in R^2 : (x + 1/2)^2 + y^2 \leq 1\}$ and $C = co(A \cup B)$. Define

$$(R^2, \|\cdot\|_0) = \left\{(x, y) \in R^2 : \|(x, y)\|_0 = \inf \left\{\lambda \in R : \frac{1}{\lambda}(x, y) \in C\right\}\right\}.$$

Then, by the definition of smooth space, it is easy to see that $(R^2, \|\cdot\|_0)$ is smooth. Since $(R^2, \|\cdot\|_0)$ is a 2-dimensional space, we obtain that $(R^2, \|\cdot\|_0)$ is compactly locally uniformly convex and 2-strictly convex. Moreover, it is easy to see that $(R^2, \|\cdot\|_0)$ is nonsquare. Since $[(-1/2, 1), (1/2, 1)] \subset S((R^2, \|\cdot\|_0))$, we obtain that $(R^2, \|\cdot\|_0)$ is not strictly convex. Hence there exists $z^* \in S\left((R^2, \|\cdot\|_0)^*\right)$ such that $A(z^*) = [(x_1, y_1), (x_2, y_2)]$ and $(x_1, y_1) \neq (x_2, y_2)$. Therefore, by Theorem 2.1, we obtain that (x_1, y_1) is a dentable point of $B((R^2, \|\cdot\|_0))$. Suppose that (x_1, y_1) is a strongly exposed point of $B((R^2, \|\cdot\|_0))$. Then we obtain that (x_1, y_1) is a w^*-exposed point of $\overline{B((R^2, \|\cdot\|_0)^*)}^{w^*}$ by Theorem 2.2. Since $(R^2, \|\cdot\|_0)$ is reflexive, we obtain that (x_1, y_1) is an exposed point of $B((R^2, \|\cdot\|_0))$. Hence there exists $h^* \in S\left((R^2, \|\cdot\|_0)^*\right)$ such that

$$1 = (h^*, (x_1, y_1)) > (h^*, (x, y)) \quad \text{whenever} \quad (x, y) \in B\left((R^2, \|\cdot\|_0)\right) \setminus \{(x_1, y_1)\}.$$

Since $(R^2, \|\cdot\|_0)$ is smooth, we have $h^* = z^*$. Then $(h^*, (x_2, y_2)) = (z^*, (x_2, y_2)) = 1$, a contradiction. This implies that (x_1, y_1) is not a strongly exposed point of $B((R^2, \|\cdot\|_0))$.

Corollary 3.3. *There exist a finite dimensional Banach space X and $x \in S(X)$ such that x is a dentable point of $B(X)$ and x is not a strongly exposed point of $B(X)$.*

Acknowledgement. This research was supported by the China Natural Science Fund under grant numbers 11401084 and 11561053.

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