Journal of Convex Analysis
Volume 25 (2018), No. 4, 1059–1074

A New Class of Sets Regularity

Chadi Nour*
Department of Computer Science and Mathematics, Lebanese American University, Byblos Campus, P.O. Box 36, Byblos, Lebanon
cnour@lau.edu.lb

Jean Takche
Department of Computer Science and Mathematics, Lebanese American University, Byblos Campus, P.O. Box 36, Byblos, Lebanon
jtakchi@lau.edu.lb

Received: February 02, 2017
Revised manuscript received: July 12, 2017
Accepted: July 14, 2017

Let $A \subset \mathbb{R}^n$ be a closed set and let $S \subset \mathbb{R}^n$ be a set containing A. In this paper, we study a new regularity class for A, called S-convexity, introduced in [10] where an inner approximation of a closed set by sets satisfying the interior sphere condition is given. We prove that this new class covers several known regularity properties including the proximal smoothness, the exterior sphere condition and the union of closed balls. As an application of such results, we provide a new sufficient condition for the equivalence between proximal smoothness and the exterior sphere condition studied in [11].

Keywords: S-convexity, proximal smoothness, exterior sphere condition, union of closed balls property, proximal analysis, nonsmooth analysis.

2010 Mathematics Subject Classification: 49J52, 52A20, 93B27.

1. Introduction

Let A be a nonempty and closed subset of $\mathbb{R}^n$. For $a \in A$ a boundary point and $\zeta \in \mathbb{R}^n$ a unit vector, we recall that ζ is said to be *proximal normal* to A at a if there exists $r > 0$ such that

$$B\left(a + r\frac{\zeta}{\|\zeta\|}; r\right) \cap A = \emptyset,$$

where $B(z;\rho)$ denotes the open ball of radius ρ centered at z. In this case, we say that ζ is *realized by an r-sphere*. The set of all proximal normals to A at a is a convex cone, and we denote it by $N_A^P(a)$.

For $r > 0$, the set A is said to be *r-proximally smooth* if for any boundary point a of A and *for all* $0 \neq \zeta \in N_A^P(a)$, ζ is realized by an r-sphere. The set A is simply

*Corresponding author.

ISSN 0944-6532 / $ 2.50

said to be *uniformly proximally smooth* if A is r-proximally smooth for some positive r. The uniform proximal smoothness was first introduced by Federer [9] under the name *positive reach*. In Clarke, Stern and Wolenski [6] (see also Canino [1] and Shapiro [20]), the proximal smoothness was studied in depth in a Hilbert space setting, and equivalence were made with uniquenesses of the metric projection mapping and smoothness properties of the Euclidean distance function on an open tube around A. Other properties related to proximal smoothness are *φ-convexity*; see Colombo and Marigonda [7] and Colombo, Marigonda and Wolenski [8], and *prox-regularity*; see Poliquin and Rockafellar [16], Poliquin, Rockafellar and Thibault [17] and Rockafellar and Wets [19].

For $r > 0$, we say that A satisfies the *exterior r-sphere condition* if for any boundary point a of A, *there exits* $0 \neq \zeta \in N_A^P(a)$ such that ζ is realized by an r-sphere. Paralleling the terminology above, we simply say that A satisfies the *uniform exterior sphere condition* if it satisfies the exterior r-sphere condition for some $r > 0$. Clearly, when A equals to the closure of its interior, then the uniform exterior sphere condition coincides with the uniform *interior* sphere condition of the complement of the interior of A. The latter condition is a well known one in control theory, and is important in deriving regularity properties of the minimal time function; see e.g. Cannarsa and Frankowska [2], Cannarsa and Sinestrari [3, 4] and Sinestrari [21].

The relationship between proximal smoothness and the exterior sphere condition was studied, apparently for the first time, in Nour, Stern and Takche [11]. The nonequivalence between these two properties, unless *epi-Lipschitzness* (or *wedgedness*) is assumed, was verified via counterexamples, thus clarifying a semantic ambiguity in the control theory literature.

Another important regularity property related to proximal smoothness and the exterior sphere condition is the *union of uniform closed balls property*. Indeed, the equivalence between the interior sphere condition and the union of uniform closed balls property was stated as a conjecture in Nour, Stern and Takche [11, 12] and proved two years later by the same authors in [14].

The goal of this paper is to study the new regularity property, called *S-convexity* for $S \supset A$, introduced in [10] where an inner approximation of a closed set by sets satisfying the interior sphere condition is given. We prove that this new regularity class contains (for a suitable and remarkable choice for S) several known regularity properties including the proximal smoothness, the exterior sphere condition and the union of closed balls. As an application, we provide a new sufficient condition for the equivalence between proximal smoothness and the exterior sphere condition.

The paper is organized as follows. In the next section we present some notations, definitions and preliminaries from nonsmooth analysis. Section 3 is devoted to the study of the S-convexity and its relation to some other known regularity properties.

2. Preliminaries

We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, B and $\overline{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\overline{B}(x;\rho) := x + \rho\overline{B}$. The segment joining two points x and y in $\mathbb{R}^n$ is denoted by $[x,y]$, that is,

$$[x,y] := \{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}.$$

For a set $A \subset \mathbb{R}^n$, A^c, $\operatorname{int} A$, ∂A, and $\operatorname{cl} A$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary, and the closure of A, respectively. We also denote by A' the complement of the interior of A, that is, $A' = (\operatorname{int} A)^c$. The distance from a point x to a set A is denoted by $d_A(x)$, and $\operatorname{proj}_A(x)$ denotes the set of closest points in A to x, that is, the set of points $a \in A$ satisfying $\|a - x\| = d_A(x)$. For $x \in \mathbb{R}^n$ a point and $\zeta \in \mathbb{R}^n$ a unit vector, the directional distance from x to a closed set A in the direction ζ, denoted by $d_A(x,\zeta)$, is defined by

$$d_A(x,\zeta) := \min\{t \geq 0 : x + t\zeta \in A\}.$$

Note that here the minimum of an empty set is taken to be ∞.

Now we provide certain definitions from proximal analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [5]; see also [15] and [19]. Let A be a nonempty and closed subset of $\mathbb{R}^n$. For $x \in A$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to A at x* provided that there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle\zeta, s - x\rangle \leq \sigma\|a - x\|^2 \quad \forall a \in A. \tag{1}$$

The relation (1) is commonly referred to as the *proximal normal inequality*. No nonzero ζ satisfying (1) exists if $x \in \operatorname{int} A$, but this may also occur for $x \in \partial A$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0,0)$. For such a point, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to A at x is a convex cone, and we denote it by $N_A^P(x)$. Now let $x \in \partial A$, $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ be such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap A = \emptyset. \tag{2}$$

Then ζ is a proximal normal to A at x and we say that ζ is realized by an r-sphere. Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = 1/2r$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, a - x \right\rangle \leq \frac{1}{2r}\|a - x\|^2 \quad \forall a \in A. \tag{3}$$

For $r > 0$, the set of all boundary points of A that have at least one proximal normal vector realized by an r-sphere will be denoted by $\partial_r A$, and the set of all points $x \in A'$ that belongs to a closed r-ball in A' will be denoted by $A'(r)$. Finally, for $x \in \partial A$, $r > 0$ and $0 \neq \zeta \in N_A^P(x)$, we define

$$r(x,\zeta) := \max\{r' > 0 : r' \leq r \text{ and } \zeta \text{ is realized by an } r'\text{-sphere}\}. \tag{4}$$

We proceed to introduce two well-known regularity properties for a nonempty and closed subset A of $\mathbb{R}^n$: the uniform interior sphere condition and the uniform proximal smoothness. For more information about these two properties, we invite the reader to see the papers [1, 6, 9, 11, 13].

Definition 2.1. For $r > 0$, a nonempty and closed set $A \subset \mathbb{R}^n$ is said to satisfy an *exterior r-sphere condition* if and only if for all $x \in \partial A$ there exits $0 \neq \zeta \in N_A^P(x)$ such that ζ is realized by an r-sphere, that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, a - x \right\rangle \leq \frac{1}{2r}\|a - x\|^2 \quad \forall a \in A.$$

We simply say that A satisfies the *uniform exterior sphere condition* if it satisfies the exterior r-sphere condition for some $r > 0$. □

Remark 2.2. Using the equivalence between (2) and (3), we get that the exterior r-sphere condition is equivalent to the existence, for each boundary point x, of $y_x \in A^c$ such that $B\left(y_x; r\right) \subset A^c$ and $\|y_x - x\| = r$.

Definition 2.3. For $r > 0$, a nonempty and closed set $A \subset \mathbb{R}^n$ is said to be *r-proximally smooth* if and only if for all $x \in \partial A$ and $0 \neq \zeta \in N_A^P(x)$, ζ is realized by an r-sphere, that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, a - x \right\rangle \leq \frac{1}{2r}\|a - x\|^2 \quad \forall a \in A.$$

We simply say that A is uniformly proximally smooth if A is r-proximally smooth for some positive r. □

The relationship between proximal smoothness and the exterior sphere condition was studied, apparently for the first time, in Nour, Stern and Takche [11]. The authors proved, by several counterexamples see [11, Section 2], that these two properties are not equivalent unless epi-Lipschitzness (or wedgedness) is assumed on the set A, see [11, Corollary 3.12].

On the other hand, one can easily verify that if A' is the union of closed r-balls (in this case we say that A' is the union of uniform closed balls) then A satisfies the exterior r-sphere condition. The reverse implication is not necessarily true (for the same radius r) as it is shown in [11, Example 4.7]. Nour, Stern and Takche proved in [14] that the reverse implication holds if we replace r by $r/2$, that is, if A satisfies the exterior r-sphere condition then A' is union of closed balls with same radius $r/2$. A strong version of this result (in which $r/2$ is replaced by $nr/2\sqrt{n^2-1}$) remains an open question, see [14, Conjecture 1.3].

3. S-convexity

Let $S \subset \mathbb{R}^n$ be a nonempty and closed set. In [10], Nour, Saoud and Takche gave sufficient conditions for the existence of an inner approximation for S by sets satisfying the interior sphere condition. One of these sufficient conditions was the existence of a closed set $A \subset S$ satisfying a new geometric property called S-convexity. The following is the definition.

Definition 3.1. Let A be a closed and nonempty subset of $\mathbb{R}^n$ and let S be a set containing A. We say that A is *S-convex* if and only if for all $s \in S \cap A^c$ and $a \neq a' \in \partial A$ satisfying

$$\|s-a\| \leq d_{\partial S}\left(a, \frac{s-a}{\|s-a\|}\right) \quad \text{and} \quad \|s-a'\| \leq d_{\partial S}\left(a', \frac{s-a'}{\|s-a'\|}\right), \tag{5}$$

we have $s-a \notin N_A^P(a)$ or $s-a' \notin N_A^P(a')$.

Remark 3.2. The two conditions of (5) are equivalent to $[a,s] \subset S$ and $[a',s] \subset S$, respectively. Hence if for $a \in \partial A$, $\zeta \in N_A^P(a)$ and $t \geq 0$, the segment $[a, a+t\zeta]$ is called *normal segment* to A at a, then the S-convexity of A means that no two normal segments to A (at two distinct points) contained in S, intersect in S. See Figure 3.1.

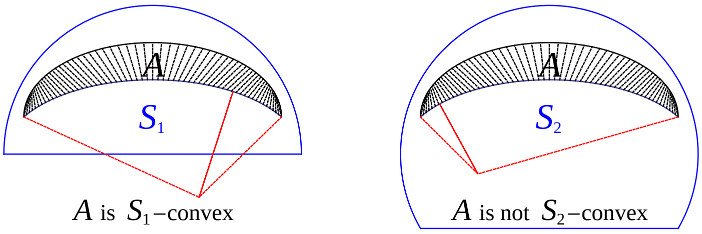


Figure 3.1: S-Convexity

Let us illustrate Definition 3.1 with some basic properties and examples. We begin by the following proposition in which we give some straightforward properties.

Proposition 3.3. *Let A and S be two nonempty sets in $\mathbb{R}^n$ such that A is closed and $A \subset S$. Then we have the following:*

(i) *A is convex if and only if A is $\mathbb{R}^n$-convex.*
(ii) *If A is S-convex and $A \subset S_1 \subset S$ then A is S_1-convex.*
(iii) *If A is S-convex and $s \in S$ such that $[s, \mathrm{proj}_A(s)] \subset S$, then s has a unique projection on A.*

In the following example, we will prove that the converse of (iii) (of the preceding proposition) is not necessarily true. More precisely, we will provide a closed set $A \subset \mathbb{R}^2$ and a set $S \supset A$ such that any point in S has a unique projection on A but A fails to be S-convex.

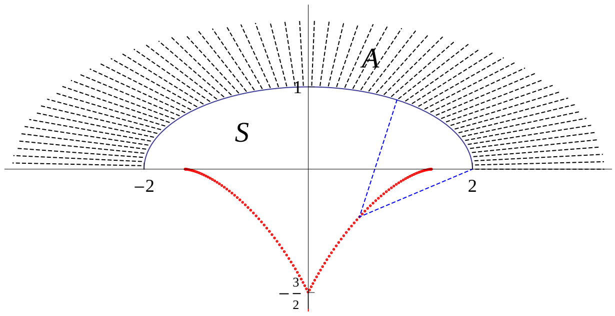


Figure 3.2: Example 3.4

Example 3.4. Let $S := \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ and let A be the set of points inside S but outside the ellipse $\frac{x^2}{4} + y^2 = 1$, that is,

$$A := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 \geq 1 \text{ and } y \geq 0 \right\}.$$

One can easily prove that each point in S has a unique projection on A. In fact, all the points in $\mathbb{R}^2$, except the points of the red curve (see Figure 3.2), have a unique projection. But A fails to be S-convex since the origin $(0,0)$ belongs to the two normal segments $[(-2,0),(0,0)]$ and $[(2,0),(0,0)]$. $\square$

We proceed to introduce three special sets that will play an important role in our study. For $A \subset \mathbb{R}^n$ a nonempty and closed set and $r > 0$ we denote by A_r the set of all point $x \in \mathbb{R}^n$ with distance to A less than r, that is,

$$A_r =: \{x \in \mathbb{R}^n : d_A(x) < r\}.$$

Using the definition of $r(a,\zeta)$, see (4), one can easily verify that

$$A_r = A \cup \bigcup_{\substack{a\in\partial A,\, \zeta\in N_A^P(a)\\ r(a,\zeta)=r}} [a, a + r(a,\zeta)\zeta[\cup \bigcup_{\substack{a\in\partial A,\, \zeta\in N_A^P(a)\\ r(a,\zeta)<r}} [a, a + r(a,\zeta)\zeta].$$

We also denote by $\tilde{A}_r$ and $\bar{A}_r$ the following two sets:

- $$\tilde{A}_r := A \cup \bigcup_{\substack{a\in\partial A,\, \zeta\in N_A^P(a)\\ [a,a+r(a,\zeta)\zeta]\subset A'(r)}} [a, a + r(a,\zeta)\zeta[\cup \bigcup_{\substack{a\in\partial A,\, \zeta\in N_A^P(a)\\ [a,a+r(a,\zeta)\zeta]\not\subset A'(r)}} [a, a + r(a,\zeta)\zeta],$$
- $$\bar{A}_r := A \cup \bigcup_{a\in\partial_r A,\, \zeta\in N_A^P(a)} [a, a + r(a,\zeta)\zeta[\cup \bigcup_{a\in\partial A\cap(\partial_r A)^c,\, \zeta\in N_A^P(a)} [a, a + r(a,\zeta)\zeta].$$

Proposition 3.5. *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set and let $r > 0$. Then $\bar{A}_r \subset \tilde{A}_r \subset A_r$.*

Proof. The inclusion $\bar{A}_r \subset \tilde{A}_r$ follows directly from the following facts:

- $$A \cup \bigcup_{a \in \partial_r A,\, \zeta \in N_A^P(a)} [a, a + r(a,\zeta)\zeta[\;\cup \bigcup_{a \in \partial A \cap (\partial_r A)^c,\, \zeta \in N_A^P(a)} [a, a + r(a,\zeta)\zeta[\subset \tilde{A}_r.$$
- If $x = a_0 + r(a_0,\zeta_0)\zeta_0$, where $a_0 \in \partial A \cap (\partial_r A)^c$ and $\zeta_0 \in N_A^P(a_0)$, then $a_0 \notin A'(r)$ and hence $a_0 \in \partial A$, $\zeta_0 \in N_A^P(a_0)$ and $[a_0, a_0 + r(a_0,\zeta_0)\zeta_0] \not\subset A'(r)$. Therefore
$$x \in \bigcup_{\substack{a \in \partial A,\, \zeta \in N_A^P(a) \\ [a, a+r(a,\zeta)\zeta] \not\subset A'(r)}} [a, a + r(a,\zeta)\zeta] \subset \tilde{A}_r.$$

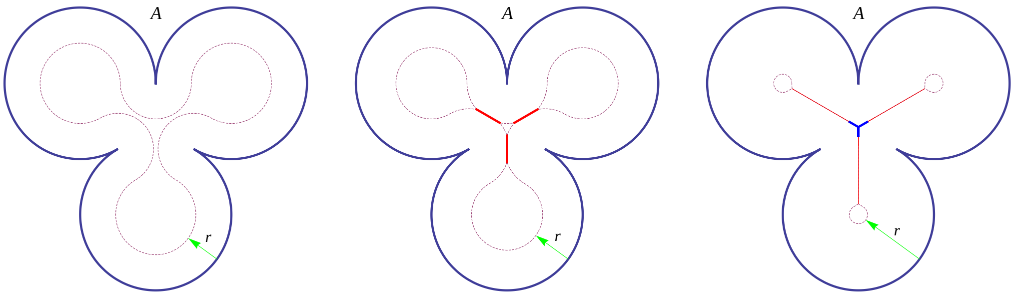
For the inclusion $\tilde{A}_r \subset A_r$, it is sufficient to remark the following:

- For all $x \in [a, a + r(a,\zeta)\zeta[$, where $a \in \partial A$ and $\zeta \in N_A^P(a)$, we have
$$d(x,A) \le d(x,a) < r(a,\zeta) \le r,$$
and then
$$A \cup \bigcup_{\substack{a \in \partial A,\, \zeta \in N_A^P(a) \\ [a, a+r(a,\zeta)\zeta] \subset A'(r)}} [a, a + r(a,\zeta)\zeta[\;\cup \bigcup_{\substack{a \in \partial A,\, \zeta \in N_A^P(a) \\ [a, a+r(a,\zeta)\zeta] \not\subset A'(r)}} [a, a + r(a,\zeta)\zeta[\subset A_r.$$
- If $a \in \partial A$ and $\zeta \in N_A^P(a)$ then
$$[a, a + r(a,\zeta)\zeta] \not\subset A'(r) \Longrightarrow r(a,\zeta) < r.$$
Hence for $x := a_0 + r(a_0,\zeta_0)\zeta_0$, where $a_0 \in \partial A$, $\zeta_0 \in N_A^P(a_0)$ and
$$[a_0, a_0 + r(a_0,\zeta_0)\zeta_0] \not\subset A'(r),$$
we get that
$$d(x,A) \le d(x,a_0) = r(a_0,\zeta_0) < r.$$
This gives that $x \in A_r$.

The details are left to the reader. □

Example 3.6. In Figure 3.3 and for the same set A (the region outside the three blue circles), we give three examples (by increasing the radius r) for possible relations between the sets $\bar{A}_r$, $\tilde{A}_r$ and A_r. More precisely:

- In Figure 3.3a, the three sets coincide.
- In Figure 3.3b, the sets $\bar{A}_r$ and $\tilde{A}_r$ coincides but are proper subsets of A_r (the three red segments are in A_r but not in $\tilde{A}_r$).



(a) $\bar{A}_r = \tilde{A}_r = A_r$ (b) $\bar{A}_r = \tilde{A}_r \subsetneq A_r$ (c) $\bar{A}_r \subsetneq \tilde{A}_r \subsetneq A_r$

Figure 3.3: Example 3.6

- In Figure 3.3c, $\bar{A}_r$ is a proper subset of $\tilde{A}_r$ (the three red segments are in $\tilde{A}_r$ but not in $\bar{A}_r$) and $\tilde{A}_r$ is a proper subset of A_r (the three blue segments are in A_r but not in $\tilde{A}_r$).

The main result of this paper is the following theorem in which we prove that the three properties, uniform proximal smoothness, uniform exterior sphere condition and the union of uniform closed balls, belong to the class of S-convexity.

Theorem 3.7. *Let A and S be two nonempty sets in $\mathbb{R}^n$ such that A is closed and $A \subset S$. For $r > 0$, the following equivalences hold:*

(i) A is r-proximally smooth $\Longleftrightarrow$ A is A_r-convex.

(ii) A satisfies the exterior r-sphere condition $\Longleftrightarrow$ A is $\bar{A}_r$-convex.

(iii) A' is the union of closed r-balls $\Longleftrightarrow$ A is $\tilde{A}_r$-convex.

Proof. (i) "$\Longleftarrow$": If A is A_r-convex then by (iii) of Proposition 3.3, each point in A_r has a unique projection on A. This gives, using [6, Theorem 4.11], that A is r-proximally smooth.

(i) "$\Longrightarrow$": Assume that A is not A_r-convex. Then, by Remark 3.2, there exist $x \in A_r \cap A^c$ and $a \neq a' \in \partial A$ such that

- $x - a \in N_A^P(a)$ and $x - a' \in N_A^P(a')$,
- $[a, x] \subset A_r$ and $[a', x] \subset A_r$.

We claim that $\|x - a\| \leq r$. Indeed, if not then we have

$$a + r\frac{x-a}{\|x-a\|} \in [a, x] \subset A_r,$$

and then

$$d_A\left(a + r\frac{x-a}{\|x-a\|}\right) < r. \tag{6}$$

But the proximal normal vector $\dfrac{x-a}{\|x-a\|}$ is realized by an r-sphere, then

$$B\left(a+r\frac{x-a}{\|x-a\|};r\right)\subset A^c,$$

which gives that

$$d_A\left(a+r\frac{x-a}{\|x-a\|}\right)=r.$$

This contradicts (6). Therefore, $\|x-a\|\leq r$. Now, using again that the proximal normal vector $\dfrac{x-a}{\|x-a\|}$ is realized by an r-sphere, we have that

$$B(x;\|x-a\|)\subset B\left(a+r\frac{x-a}{\|x-a\|};r\right)\subset A^c.$$

Hence $d_A(x)=\|x-a\|$ and then $a\in \operatorname{proj}_A(x)$. Similarly, we have $a'\in \operatorname{proj}_A(x)$. But this contradicts the r-proximal smoothness of A since x has now two distinct projections on A, the points a and a' (see [6, Theorem 4.11]). The proof of (i) is terminated.

(ii) "$\Longrightarrow$": Since A satisfies the exterior r-sphere condition we have that $\partial_r A=\partial A$. Then

$$\bar{A}_r:=A\cup\bigcup_{a\in\partial A,\,\zeta\in N_A^P(a)}[a,a+r(a,\zeta)\zeta[. \tag{7}$$

Assume that A is not $\bar{A}_r$-convex. Then there exist $x\in\bar{A}_r\cap A^c$ and $a\neq a'\in\partial A$ such that

- $x-a\in N_A^P(a)$ and $x-a'\in N_A^P(a')$,
- $[a,x]\subset\bar{A}_r$ and $[a',x]\subset\bar{A}_r$.

Since $x\in\bar{A}_r\cap A^c$ and using (7), we get the existence of $a_0\in\partial A$ and $\zeta_0\in N_A^P(a_0)$ such that $x\in[a_0,a_0+r(a_0,\zeta_0)\zeta_0[$. But ζ_0 is realized by an $r(a_0,\zeta_0)$-sphere, hence we have

$$\overline{B}(x;\|x-a_0\|)\subset\{a_0\}\cup B(a_0+r(a_0,\zeta_0)\zeta_0;r(a_0,\zeta_0))\subset\{a_0\}\cup A^c. \tag{8}$$

Clearly we can assume that $a\neq a_0$. We denote by $\zeta:=\dfrac{x-a}{\|x-a\|}$.

Case 1: $\|x-a\|\leq r(a,\zeta)$.
Then $B(x;\|x-a\|)\subset B(a+r(a,\zeta)\zeta;r(a,\zeta))\subset A^c$, which gives that $d_A(x)=\|x-a\|$. Hence $\|x-a_0\|\geq\|x-a\|$. Then using (8) we get that

$$a\in\overline{B}(x;\|x-a_0\|)\subset\{a_0\}\cup A^c.$$

This gives the desired contradiction.

Case 2: $\|x-a\| > r(a,\zeta)$.

Then the point $x' := a + r(a,\zeta)\zeta \in [a,x] \subset \bar{A}_r$. But $x' \notin A \cup [a, a+r(a,\zeta)\zeta[$, then there exists $a_1 \in \partial A$ and $\zeta_1 \in N_A^P(a_1)$ such that $x' \in [a_1, a_1 + r(a_1,\zeta_1)\zeta_1[$, and $a_1 \neq a$. Since ζ_1 is realized by an $r(a_1,\zeta_1)$-sphere, we have

$$\overline{B}(x'; \|x'-a_1\|) \subset \{a_1\} \cup B(a_1 + r(a_1,\zeta_1)\zeta_1; r(a_1,\zeta_1)) \subset \{a_1\} \cup A^c.$$

This gives since $a \notin \{a_1\} \cup A^c$ that $\|x'-a\| > \|x'-a_1\|$. Hence

$$a_1 \in B(x'; \|x'-a\|) = B(a + r(a,\zeta)\zeta; r(a,\zeta)) \subset A^c.$$

The desired contradiction is reached.

(ii) "$\Longleftarrow$": Assume that A does not satisfy the exterior r-sphere condition. Then $\partial_r A \neq \partial A$. Let $a \in (\partial_r A)^c \cap \partial A$. Since the set of points $x \in \partial A$ for which $N_A^P(x) \neq \{0\}$ is dense in ∂A (see [6, Corollary 1.6.2]) and $\partial_r A$ is closed, we can assume that $N_A^P(a) \neq \{0\}$. Let $0 \neq \zeta \in N_A^P(a)$. Since $a \notin \partial_r A$, we have by the definition of $\bar{A}_r$ that

$$[a,x] \subset \bar{A}_r \text{ where } x := a + r(a,\zeta)\zeta. \tag{9}$$

The fact the ζ is realized by an $r(a,\zeta)$-sphere gives that $B(x; r(a,\zeta)) \subset A^c$.

Case 1: There exists $a' \neq a$ such that $a' \in \overline{B}(x; r(a,\zeta)) \cap \partial A$.

Let $\zeta' := \dfrac{x-a'}{\|x-a'\|}$. Clearly $\zeta' \in N_{A'}^P(a')$ and ζ' is realized by an $r(a,\zeta)$-sphere. Then we have $r(a',\zeta') \geq r(a,\zeta)$ and hence

$$[a', x[\subset [a', a' + r(a',\zeta')\zeta'[\subset \bar{A}_r.$$

Therefore

- $x - a \in N_A^P(a)$ and $x - a' \in N_A^P(a')$,
- $[a,x] \subset \bar{A}_r$ and $[a',x] \subset \bar{A}_r$.

This contradicts the $\bar{A}_r$-convexity of A.

Case 2: $\overline{B}(x; r(a,\zeta)) \cap \partial A = \{a\}$ (see Figure 3.4).

Since $a \notin \partial_r A$ we have that $r(a,\zeta) < r$. Let $r' \in]r(a,\zeta), r[$. Using the definition of $r(a,\zeta)$ we get

$$B(x'; r') \cap A \neq \emptyset \quad \text{where} \quad x' := a + r'\zeta.$$

Let $y' \in B(x'; r') \cap A$. Clearly $y' \in \operatorname{int} A$. Since $x' \notin A$ (we can assume this by taking r' close to $r(a,\zeta)$), we get that the line joining y' to x' intersects ∂A at a point $b' \neq y'$. We have

$$\|x'-b'\| < \|x'-y'\| < r' = \|x'-a\|.$$

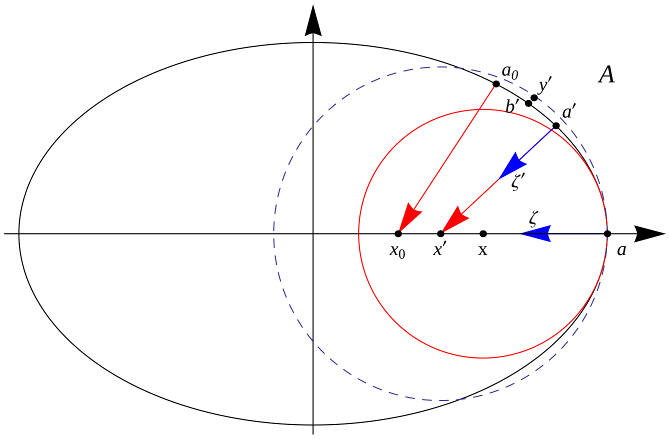


Figure 3.4: $\overline{B}(x; r(a,\zeta)) \cap \partial A = \{a\}$

Now let $a' \in \mathrm{proj}_A(x')$ and $\zeta' := \dfrac{x'-a'}{\|x'-a'\|}$. Clearly we have

- ζ' is a proximal normal vector to A at a' realized by a $\|x'-a'\|$-sphere,
- $\|x'-a'\| \le \|x'-b'\| < r' < r$,
- $a \neq a'$.

Case 2.1: $\exists\, r_0 \in]r(a,\zeta), r[$ such that $\forall\, r' \in]r(a,\zeta), r_0]$ we have $a' \notin \partial_r A$. Then, using the definition of $\bar{A}_r$, we get

$$[a', a' + r(a',\zeta')\zeta'] \subset \bar{A}_r \text{ for all } r' \in\,]r(a,\zeta), r_0].$$

But $r(a',\zeta') \ge \|x'-a'\|$, then

$$x' \in [a', a' + r(a',\zeta')\zeta'] \subset \bar{A}_r \text{ for all } r' \in\,]r(a,\zeta), r_0]. \tag{10}$$

Hence for $x_0 := a + r_0\zeta$, we have

$$\begin{aligned}[a, x_0] &= [a,x] \cup]x, x_0] = [a,x] \cup]a + r(a,\zeta)\zeta, a + r_0\zeta] \\ &= [a,x] \cup \bigcup_{r' \in\,]r(a,\zeta), r_0]} \{a + r'\zeta\} = [a,x] \cup \bigcup_{r' \in\,]r(a,\zeta), r_0]} \{x'\} \subset \bar{A}_r,\end{aligned}$$

where the last inclusion follows from (9) and (10). On the other hand, the inclusion (10) also gives that $[a_0, x_0] \subset \bar{A}_r$, where a_0 denotes the point a' that corresponds to the choice x_0 for x'. Therefore

- $x_0 - a \in N_A^P(a)$ and $x_0 - a_0 \in N_A^P(a_0)$,
- $[a, x_0] \subset \bar{A}_r$ and $[a_0, x_0] \subset \bar{A}_r$.

This contradicts the $\bar{A}_r$-convexity of A.

Case 2.2: $\forall\, r_0 \in]r(a,\zeta), r[\ \ \exists\, r' \in]r(a,\zeta), r_0]$ such that $a' \in \partial_r A$.

Then there exists a sequence $r'_n \in\,]r(a,\zeta), r[$ such that r'_n converges to $r(a,\zeta)$ and for all n, a'_n, the a' that corresponds to r'_n, belongs to $\partial_r A$. Hence we have

- $x'_n = a + r'_n\zeta$ converges to x,
- $a'_n \in \mathrm{proj}_A(x'_n)$ converges to a point $a'_0 \in \partial_r A \cap \mathrm{proj}_A(x)$,
- $\|x - a'_0\| \leq r(a,\zeta)$ (since $\|x'_n - a'_n\| < r'_n$).

But $\overline{B}(x; r(a,\zeta)) \cap A = \{a\}$ then $a'_0 = a$. This gives that $a \in \partial_r A$, contradiction. The proof of (ii) is terminated.

(iii) "$\Longrightarrow$": Since A' is the union of closed r-balls, we get that A satisfies the exterior r-sphere condition and then $\partial_r A = \partial A$. Moreover, $A'(r) = A'$. Therefore,

$$\bar{A}_r = \tilde{A}_r = A \cup \bigcup_{a \in \partial A,\, \zeta \in N_A^P(a)} [a, a + r(a,\zeta)\zeta[.$$

By (ii), we have A is $\bar{A}_r$-convex and then $\tilde{A}_r$-convex.

(iii) "$\Longleftarrow$": Assume that A' is not the union of closed r-balls. Then $A'(r) \subsetneq A'$. Let $y_0 \in A' \cap (A'(r))^c$. Since A is $\tilde{A}_r$-convex and $\bar{A}_r \subset \tilde{A}_r$ (by Proposition 3.5), we have that A is $\bar{A}_r$-convex. Then by (ii), A satisfies the exterior r-sphere condition. This gives that $\partial_r A = \partial A$ and hence $y_0 \notin \partial A$. Now let $a \in \mathrm{proj}_A(y_0)$ and let $\zeta := \dfrac{y_0 - a}{\|y_0 - a\|} \in N_A^P(a)$. Since $y_0 \notin A'(r)$ we have $r(a,\zeta) < r$. Hence

$$[a, y] \subset \tilde{A}_r \text{ where } y := a + r(a,\zeta)\zeta. \tag{11}$$

Claim 1: There exists $r_0 > 0$ such that $\overline{B}(y_0; r_0) \subset A'$ and $\overline{B}(y_0; r_0) \cap A'(r) = \emptyset$.

Indeed, if not then, using the fact that $\overline{B}(y_0; \|y_0 - a\|) \subset A'$, there exists a sequence y_n satisfying

$$y_n \in \overline{B}\left(y_0; \frac{1}{n}\right) \cap A'(r). \tag{12}$$

Since $y_n \in A'(r)$, there exists a sequence z_n such that

$$y_n \in \overline{B}(z_n; r) \subset A'. \tag{13}$$

Clearly the sequence z_n is bounded and then it has a convergent subsequence (we do not relabel). Let z_0 be the limit of z_n. By (12) we have that y_n converges to y_0 and then (13) yields

$$y_0 \in \overline{B}(z_0; r) \subset A'.$$

This yields $y_0 \in A'(r)$ – a contradiction. Hence, Claim 1 holds.

Claim 2: $\mathrm{proj}_A(y) = \{a\}$.

Indeed, if not, there exists $a' \in \mathrm{proj}_A(y_0)$ such that $a' \neq a$. Clearly we have

- $y - a \in N_A^P(a)$ and $y - a' \in N_A^P(a')$,
- $[a, y] \subset \tilde{A}_r$ (by (11)) and $[a', y] = [a', y[\cup \{y\} \subset \tilde{A}_r$ (by definition of $\tilde{A}_r$).

This contradicts the $\tilde{A}_r$-convexity of A and hence, Claim 2 holds.

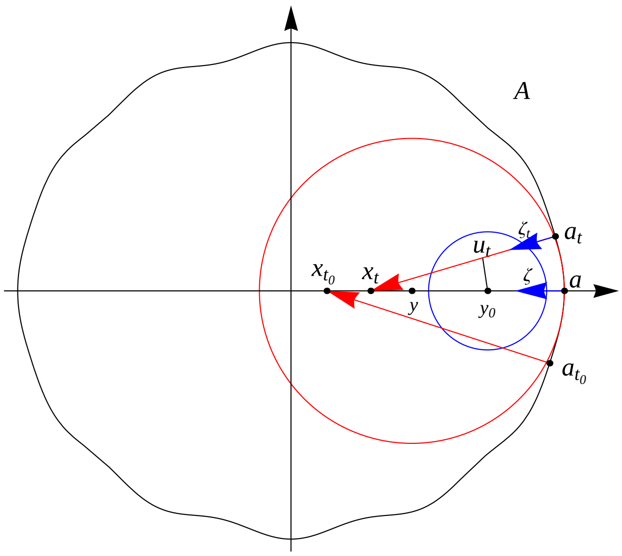


Figure 3.5: $\mathrm{proj}_A(y) = \{a\}$

We proceed by defining for each $t \in \,]0, r - r(a,\zeta)]$ the point

$$x_t := a + (t + r(a,\zeta))\zeta = y + t\zeta,$$

and a_t to be a point in $\mathrm{proj}_A(x_t)$, see Figure 3.5. By the definition of $r(a,\zeta)$, we get

$$a_t \neq a \quad \text{for all} \quad t \in \,]0, r - r(a,\zeta)],$$

and

$$\|x_t - a_t\| < \|x_t - a\| = t + r(a,\zeta) \leq r.$$

This gives, for $\zeta_t := \dfrac{x_t - a_t}{\|x_t - a_t\|}$, that $r(a_t,\zeta_t) \geq \|x_t - a_t\|$ and then

$$[a_t, x_t[\, \subset [a_t, a_t + r(a_t,\zeta_t)\zeta_t[\, \subset \tilde{A}_r.$$

When $t \to 0$, we have $x_t \to y$ and then by Claim 2, $a_t \to a$. Hence there exists $t_0 \in \,]0, r - r(a,\zeta)]$ such that

$$\|a_t - a\| \leq r_0 \quad \text{for all} \quad t \leq t_0.$$

Now let $u_t := \varepsilon_t a_t + (1 - \varepsilon_t)x_t$ where

$$\varepsilon_t := \frac{\|x_t - y_0\|}{\|x_t - a\|} = \frac{\|x_t - y_0\|}{\|x_t - y_0\| + \|y_0 - a\|} < 1.$$

Clearly we have $u_t \in [a_t, x_t]$. Moreover,

$$\begin{aligned}
\|u_t - y_0\| &= \|\varepsilon_t a_t + (1-\varepsilon_t)x_t - y_0\| = \|\varepsilon_t(a_t - x_t) + x_t - y_0\| \\
&= \|\varepsilon_t(a_t - x_t) + \varepsilon_t(x_t - a)\| = \|\varepsilon_t(a_t - a)\| \\
&\leq \varepsilon_t r_0 < r_0 \quad \text{for all} \quad t \leq t_0.
\end{aligned}$$

Hence using Claim 1 we get that for all $t \leq t_0$

$$u_t \in \overline{B}(y_0; r_0) \implies u_t \notin A'(r) \implies [a_t, x_t] \subset \tilde{A}_r. \tag{14}$$

On the other hand,

$$[a, x_{t_0}] = [a, y] \cup]y, x_{t_0}] = [a, y] \cup \bigcup_{0<t\leq t_0} \{x_t\} \subset \tilde{A}_r\,,$$

where the last inclusion follows from (11) and (14). Therefore

- $x_{t_0} - a_{t_0} \in N_A^P(a_{t_0})$ and $x_{t_0} - a \in N_A^P(a)$,
- $[a_{t_0}, x_{t_0}] \subset \tilde{A}_r$ (by (14)) and $[a, x_{t_0}] \subset \tilde{A}_r$.

This contradicts the $\tilde{A}_r$-convexity of A, and the proof of *(iii)* is complete. □

In [11], Nour, Stern and Takche studied the equivalence between the uniform proximal smoothness and the uniform exterior sphere condition. They proved, via counterexamples see [11, Section 2], that these two properties are not necessarily equivalent. Moreover, they provided the following result, see [11, Corollary 3.12], in which a sufficient geometric condition for the equivalence is given. We recall that a closed set $A \subset \mathbb{R}^n$ is said to be *wedged* (or *epi-Lipschitz*) if for any boundary point a of A, the set A can be viewed, after application of an orthogonal linear transformation, as the epigraph of a Lipschitz continuous function. This geometric definition was introduced by Rockafellar in [18], and is also characterizable in terms of the nonemptiness of the topological interior of the *Clarke tangent cone*, which is in turn equivalent to the pointedness of the *Clarke normal cone*; see [5, 19].

Theorem 3.8. ([11], Corollary 3.12) *Let $A \subset \mathbb{R}^n$ be a nonempty and wedged set with compact boundary. Assume that A satisfies an exterior r-sphere condition. Then A is uniformly proximally smooth.*

In the following corollary, we provide a new sufficient condition for the equivalence between the proximal smoothness and the exterior sphere condition.

Corollary 3.9. *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set satisfying the exterior r-sphere condition. Assume the existence of $r' \leq r$ such that $\bar{A}_{r'} = A_{r'}$. Then A is uniformly proximally smooth.*

Proof. Since A satisfies the exterior r-sphere condition, we have by Theorem 3.7 that A is $\bar{A}_r$-convex and then $\bar{A}_{r'}$-convex. But $\bar{A}_{r'} = A_{r'}$, then A is $A_{r'}$-convex which gives, by Theorem 3.7, that A is r'-proximally smooth. □

Remark 3.10. In this remark, we will prove that our new result, Corollary 3.9, is more general than the previous one, Theorem 3.8. More precisely, we will prove the following:

(i) If A is wedged with compact boundary and satisfies the exterior r-sphere condition then there exists $r' \leq r$ such that $\bar{A}_{r'} = A_{r'}$.

(ii) If A satisfies the exterior r-sphere condition and there exists $r' \leq r$ such that $\bar{A}_{r'} = A_{r'}$ then A is not necessarily wedged.

For (i), assume that A is wedged with compact boundary and satisfies the exterior r-sphere condition. Then by Theorem 3.8, there exists r' (can be taken to be less than r) such that A is r'-proximally smooth. Therefore, $\partial A = \partial_{r'} A$ and $r(a, \zeta) = r'$ for all $a \in \partial A$ and $\zeta \in N_A^P(a)$. This gives, using the definition of A_r and $\bar{A}_{r'}$, that

$$\bar{A}_{r'} = A_{r'} = A \cup \bigcup_{a \in \partial A,\, \zeta \in N_A^P(a)} [a, a + r(a, \zeta)\zeta[.$$

For (ii), let A be the curved triangle of Figure 3.6 (the two curves are part of two tangent circles with same radius r_0). Clearly A satisfies the r_0-exterior sphere condition and $\bar{A}_{r_0} = A_{r_0}$. But A fails to be wedged due to the cusp points. □

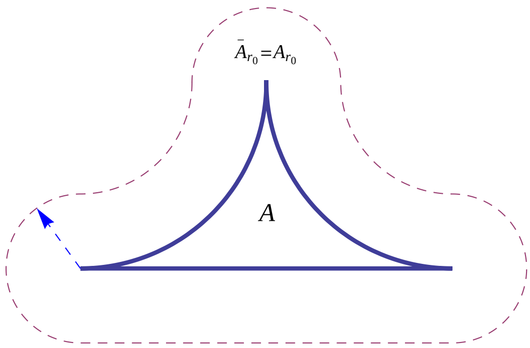


Figure 3.6: Curved triangle

References

[1] A. Canino: On p-convex sets and geodesics, J. Differential Equations 75(1) (1988) 118–157.

[2] P. Cannarsa, H. Frankowska: Interior sphere property of attainable sets and time optimal control problems, ESAIM: Control Optim. Calc. Var. 12 (2006) 350–370.

[3] P. Cannarsa, C. Sinestrari: Convexity properties of the minimum time function, Calc. Var. 3 (1995) 273–298.

[4] P. Cannarsa, C. Sinestrari: Semiconcave Functions, Hamiton-Jacobi Equations and Optimal Control, Birkhäuser, Boston (2004).

[5] F. H. Clarke, Yu. Ledyaev, R. J. Stern, P. R. Wolenski: Nonsmooth Analysis and Control Theory, Graduate Texts in Mathematics 178, Springer, New York (1998).

[6] F. H. Clarke, R. J. Stern, P. R. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2(1-2) (1995) 117–144.

[7] G. Colombo, A. Marigonda: Differentiability properties for a class of non-convex functions, Calc. Var. 25 (2005) 1–31.

[8] G. Colombo, A. Marigonda, P. R. Wolenski: Some new regularity properties for the minimal time function, SIAM J. Control Optim. 44(6) (2006) 2285–2299.

[9] H. Federer: Curvature measures, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[10] C. Nour, H. Saoud, J. Takche: Regularization via sets satisfying the interior sphere condition, J. Convex Analysis 25(1) (2018) 1–19.

[11] C. Nour, R. J. Stern, J. Takche: Proximal smoothness and the exterior sphere condition, J. Convex Analysis 16(2) (2009) 501–514.

[12] C. Nour, R. J. Stern, J. Takche: The union of uniform closed balls conjecture, Control and Cybernetics 38(4B) (2009) 1525–1534.

[13] C. Nour, R. J. Stern, J. Takche: The θ-exterior sphere condition, φ-convexity and local semiconcavity, Nonlinear Analysis 73(2) (2010) 573–589.

[14] C. Nour, R. J. Stern, J. Takche: Validity of the union of uniform closed balls conjecture, J. Convex Analysis 18(2) (2011) 589–600.

[15] J.-P. Penot: Calculus without Derivatives, Graduate Texts in Mathematics 266, Springer, New York (2013).

[16] R. A. Poliquin, R. T. Rockafellar: Prox-regular functions in variational analysis, Trans. Amer. Math. Soc. 348 (1996) 1805–1838.

[17] R. A. Poliquin, R. T. Rockafellar, L. Thibault: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[18] R. T. Rockafellar: Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$, Nonlinear Anal. TMA 3 (1979) 145–154.

[19] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Grundlehren der Mathematischen Wissenschaften 317, Springer, Berlin (1998).

[20] A. S. Shapiro: Existence and differentiability of metric projections in Hilbert spaces, SIAM J. Optim. 4 (1994) 231–259.

[21] C. Sinestrari: Semiconcavity of the value function for exit time problems with nonsmooth target, Commun. Pure Appl. Anal. 3(4) (2004) 757–774.