Journal of Convex Analysis
Volume 25 (2018), No. 4, 1075–1104

Quasi-Newton Methods for Solving Nonsmooth Equations: Generalized Dennis-Moré Theorem and Broyden's Update

Samir Adly
XLIM UMR-CNRS 7252, Université de Limoges, 87060 Limoges, France
samir.adly@unilim.fr

Huynh Van Ngai*
Department of Mathematics, University of Quynhon,
170 An Duong Vuong, Qui Nhon, Vietnam
ngaivn@yahoo.com

Received: April 03, 2017
Accepted: June 21, 2017

We study the quasi-Newton method by using set-valued approximations for solving generalized equations without smoothness assumptions. The set-valued approximations appear naturally when dealing with nonsmooth problems, or even in smooth cases, data in almost concrete applications are not exact. We present a generalization of the classical Dennis-Moré theorem, which gives a characterization of the q-superlinear convergence of the quasi-Newton iterates. The local linear and superlinear convergences of the method, especially, a modification of the Broyden update method are investigated. We present an example showing that the classical Broyden update method is no longer linearly convergent when the function involved in the nonlinear equation is not smooth. A modified version of the Broyden update is proposed and its convergence is proved. These results are new, and can be considered as both an improvement and an extension of some results appeared recently in the literature on this subject.

Keywords: Quasi-Newton methods, Broyden's update, Dennis-Moré theorem, metric regularity, generalized equations.

2010 Mathematics Subject Classification: 47N10, 49J40, 49J53, 65K10, 65K15, 65J15, 90C30.

1. Introduction and preliminaries

We consider the following generalized equation

$$\text{Find} \quad x \in X \quad \text{such that} \quad 0 \in f(x) + F(x), \tag{1}$$

where $f\colon X \to Y$ is a mapping and $F\colon X \rightrightarrows Y$ is a multifunction between Banach spaces X, Y. The model of generalized equations (1) (this terminology

*The research of this author is supported by NAFOSTED, under Grant: 101.01-2016.27.

ISSN 0944-6532 / $ 2.50

is due to Robinson) covers a large number of problems in applied mathematics such as nonlinear equations, variational inequalities, complementarity problems, variational inclusions, optimality conditions in constrained optimization, etc. When F is the zero mapping, inclusion (1) reduces to equation $f(x) = 0$. In the literature of applied mathematics, engineering and sciences, it is well-known that the Newton's method and as well as the quasi-Newton method is one of the most famous and effective method for solving numerically nonlinear equations $f(x) = 0$, for a given function $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ (see e.g. [10] and [18]).

The study of generalized equations was initiated by Robinson (1980). Later, Josephy [13] investigated the local convergence of the Newton and quasi-Newton methods for the variational inequality of the form $0 \in f(x) + N_C(x)$, where C is a closed convex subset of $\mathbb{R}^m$ and N_C stands for the forward-normal cone of C in the sense of convex analysis. The extension of Newton's method to generalized equations were considered by many authors, for example, Dontchev [7], Dontchev and Rockafellar [9] and the references therein.

When f is a smooth function, Newton's method starts from some point x_0 near a solution $x^* \in X$ of inclusion (1) and generates a sequence (x_k) defined by solving the following auxiliary problem

$$0 \in f(x_k) + Df(x_k)(x_{k+1} - x_k) + F(x_{k+1}), \tag{2}$$

where, Df denotes the Fréchet derivative of f. Under the assumptions that f is of classe C^1 and F has closed graph with $(f+F)^{-1}$ being Aubin continuous at $(x^*, 0)$, Dontchev [7] has established the local q-quadratic convergence of Newton's method.

Besides the Newton method, the quasi-Newton methods for solving nonlinear equations and optimization problems have attracted the study of many researchers (see, e.g. [5], [8], [14], [15], [16], [21], [22] and references therein). The quasi-Newton methods, consist of replacing the derivative $Df(x_k)$ (or its kind of generalizations such as, e.g., subdifferentials) in the auxiliary problems (2) by a continuous linear operator B_k which is constructed in a specific way. One of central questions related to the quasi-Newton methods is how to update the sequence (B_k) to ensure the (at least linear) convergence of the generated sequence? A classical result due to Dennis-Moré ([6], Theorem 2.2) gave a characterization of the q-superlinear convergence of the quasi-Newton method for solving smooth equations $f(x) = 0$. More precisely, it establishes for $F \equiv 0$ a characterization of the q-superlinear convergence of the quasi-Newton iterates

$$\lim_{k\to+\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0,$$

under the smoothness of the function f and the nonsingularity of its Jacobian $\nabla f(x^*)$ at the solution x^*. This result was generalized recently to the quasi-Newton methods for solving generalized equations in [8] under the assumption that that the mapping $x \mapsto f(x^*)+\nabla f(x^*)(x-x^*)+F(x)$ is strongly subregular at $(x^*, 0)$.

In this paper, we consider the generalized quasi-Newton method by using set-valued approximations for solving generalized equation (1): Given a starting point $x_0 \in X$, and to generate a sequence $(x_k)_{k\in\mathbb{N}}$ iteratively by taking x_{k+1} to be a solution to the auxiliary inclusion

$$0 \in f(x_k) + \mathcal{B}_k(x_{k+1} - x_k) + F(x_{k+1}) \quad \text{for } k \in \mathbb{N}, \tag{3}$$

where $\mathcal{B}_k : X \rightrightarrows Y$ is a multifunction from X to Y. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a given sequence of positive reals. An inexact version of (3) is given by

$$0 \in f(x_k) + \mathcal{B}_k(x_{k+1} - x_k) + F(x_{k+1}) + \gamma_k \mathbb{B}_Y \quad \text{for } k \in \mathbb{N}. \tag{4}$$

The set-valued approximations used in (3) appears naturally when one deals with nonsmooth equations (see e.g. [1] and [23]) or even in smooth cases, data in almost real-world problems are not exact. An important special case, when each $\mathcal{B}_k$ is generated by a family of continuous linear operators from X to Y, is given by semismooth (quasi-)Newton methods. In this work, we are interested in the local convergence around the reference point. The aim of the paper is twofold. Firstly, we show that the classical Dennis-Moré theorem can be extended to the generalized quasi-Newton method (3) using set-valued approximations. This set-valued version generalizes some results established recently in the papers [5] and [8]. Secondly, we investigate the local linear/superlinear convergence of the quasi-Newton method when the sequence $(\mathcal{B}_k)$ is updated in a suitable way. We give an example showing that the classical Broyden update method is no longer linearly convergent when the function defining the equation is not smooth. A modified version of the Broyden update is proposed and its convergence is proved.

Below we recall some notations and definitions which are needed in the sequel. Let X be a Banach space endowed with norm $\|\cdot\|$, denote by $[x, y]$ the closed line segment with endpoints $x, y \in X$ and by $\mathbb{B}(x, r)$, $\mathbb{B}_X$ the open ball centered at x with radius $r > 0$, the unit ball in X, respectively. The convex hull of $C \subseteq X$ is denoted by $\operatorname{co} C$. Given $\alpha \in \mathbb{R}$, we set $\alpha C = \{\alpha x : \ x \in C\}$. The *Minkowski sum* of sets A and B in X is the set $A+B := \{a+b : \ a \in A,\ b \in B\}$, and the *excess of A beyond B* is defined by

$$e(A, B) = \sup_{x\in A} d(x, B) = \inf\{\tau > 0 : \ A \subset B + \tau\mathbb{B}_X\},$$

with the convention that $e(\emptyset, B) := 0$ when $B \neq \emptyset$, and that $e(\emptyset, \emptyset) := +\infty$. The *Pompeiu-Hausdorff distance between A and B* is then defined by

$$h(A, B) = \max\{e(A, B), e(B, A)\} = \inf\{\tau > 0 : \ A \subset B+\tau\mathbb{B}_X,\ B \subset A+\tau\mathbb{B}_X\}.$$

For two Banach spaces X, Y, as usual, denote $\mathcal{L}(X, Y)$, the space of continuous linear operators between the Banach spaces X and Y. For a set-valued map

$F\colon X \rightrightarrows Y$, the *graph* of F, denoted by $\operatorname{gph} F$, is defined by

$$(x,y) \in \operatorname{gph} F \text{ iff } y \in F(x),\ x \in X,\ y \in Y.$$

In the sequel, we need some regular properties of multifunctions. Recall (see e.g. [17], [19]) that a mapping $F\colon X \rightrightarrows Y$ is called *metrically regular at* $(\bar{x},\bar{y}) \in \operatorname{gph} F$ *with a constant* $\kappa > 0$ *on a neighborhood* $U \times V$ *of* $(\bar{x},\bar{y})$ *in* $X \times Y$ if

$$d\left(x, F^{-1}(y)\right) \le \kappa d(y, F(x)) \quad \text{whenever} \quad (x,y) \in U \times V. \tag{5}$$

The mapping F is *metrically regular at* $(\bar{x},\bar{y}) \in \operatorname{gph} F$, if there is a constant $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x},\bar{y})$ in Y such that F is metrically regular at $(\bar{x},\bar{y})$ with the constant κ on the neighborhood $U \times V$. If F is metrically regular at $(\bar{x},\bar{y})$ and F^{-1} has a localization at $(\bar{y},\bar{x})$ which is single valued, then F is called *strongly metrically regular* at $(\bar{x},\bar{y})$. In case of a single valued mapping $f\colon X \to Y$ we simply speak about (strong) metric regularity at $\bar{x}$. Let us recall also a weaker property than the strong metric regularity, called the strong metric subregularity: A multifunction $F : X \rightrightarrows Y$ is said to be *strongly metric subregular* at $(\bar{x},\bar{y})$ with $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there are a constant $\kappa > 0$ and a neighborhood U of $\bar{x}$ such that

$$\|x - \bar{x}\| \le \kappa d(\bar{y}, F(x)) \quad \text{for all } x \in U. \tag{6}$$

We also need the following uniform properties of the metric regularity and the strong metric subregularity when considering a collection $\mathcal{F}$ of multifunctions $F : X \rightrightarrows Y$ and a point $(\bar{x},\bar{y}) \in X \times Y$ such that $(\bar{x},\bar{y}) \in \operatorname{gph} F$ for all $F \in \mathcal{F}$. We say that $\mathcal{F}$ is *uniformly* metrically regular (strongly metrically subregular, respectively) at $(\bar{x},\bar{y})$ if there exists a constant $\kappa > 0$ and a neighborhood $U \times V$ of $(\bar{x},\bar{y})$ in $X \times Y$ (a neighborhood U of $\bar{x}$ in X, respectively) such that (5) ((6), respectively) holds for all $F \in \mathcal{F}$.

Finally, recall that a sequence (x_k) of elements in a Banach space X which converges to $\bar{x} \in X$ is said to be

- at least q-linearly convergent if $\displaystyle\limsup_{k\to\infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} < 1$;
- q-superlinearly convergent if $\displaystyle\lim_{k\to\infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} = 0$;
- q-quadratically convergent if $\displaystyle\limsup_{k\to\infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|^2} < +\infty$.

2. Generalized Dennis-Moré theorem

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence generated by (4). For a given point $\bar{x} \in X$ (which maybe is not yet a solution of the inclusion (1)), set

$$e_k := x_k - \bar{x}, \quad d_k := x_{k+1} - x_k.$$

Firstly, we state the following theorem which is a set-valued version of the classical Dennis-Moré theorem for smooth nonlinear equations.

Theorem 2.1. *Let* (γ_k) *be a sequence of positives. Let* $\mathcal{B}_k : X \rightrightarrows Y$, $k \in \mathbb{N}$ *be a sequence of multifunctions from* X *to* Y*. Suppose that the sequence* $(x_k)_{k\in N_0}$ *generated by* (4) *converges to* $\bar{x}$*. Let* $(\mathcal{A}_k)_{k\in\mathbb{N}}$ *be a sequence of multifunctions* $\mathcal{A}_k \colon X \rightrightarrows Y$ *such that*

$$\lim_{k\to\infty} \frac{\sup\limits_{w_k\in\mathcal{A}_k d_k} \|f(x_{k+1}) - f(x_k) - w_k\|}{\|d_k\|} = 0. \tag{7}$$

Assume further that $\lim_{k\to\infty} \gamma_k/\|e_k\| = 0$*. Then one has*

(i) *If* $x_k \to \bar{x}$ *q-superlinearly, then*

$$\lim_{k\to\infty} \frac{e(\mathcal{A}_k d_k, f(x_{k+1}) + \mathcal{B}_k d_k + F(x_{k+1}))}{\|d_k\|} = 0. \tag{8}$$

In addition, if $f + F$ *is sub-Lipschitz around* $\bar{x}$*, in the sense that there exist* $L, \delta > 0$ *such that* $f(x) + F(x) \subseteq f(\bar{x}) + F(\bar{x}) + L\|x - \bar{x}\|\mathbb{B}_Y$*, for all* $x \in B(\bar{x}, \delta)$*, then*

$$\lim_{k\to\infty} \frac{e(\mathcal{A}_k d_k, f(\bar{x}) + \mathcal{B}_k d_k + F(\bar{x}))}{\|d_k\|} = 0.$$

(ii) *Conversely, suppose that*

$$\lim_{k\to\infty} \frac{e(\mathcal{B}_k d_k, \mathcal{A}_k d_k)}{\|d_k\|} = 0. \tag{9}$$

If the gph $(f + F)$ *is closed, then* $\bar{x}$ *is a solution of the generalized equation* (1)*. In addition, if* $f + F$ *is strongly metrically regular at* $(\bar{x}, 0)$*, then* $x_k \to \bar{x}$ *q-superlinearly.*

Proof. For (i), suppose that (x_k) converges to $\bar{x}$ q-superlinearly, then for each $\varepsilon > 0$, there is k_0 such that

$$\gamma_k \le \varepsilon\|e_k\|; \quad \|e_{k+1}\| \le \varepsilon\|d_k\| < \delta \quad \text{for all } k \ge k_0, \tag{10}$$

from which follows that $\|e_k\| \le \|e_{k+1}\| + \|d_k\| \le (1+\varepsilon)\|d_k\| \;\; \forall k \ge k_0$. By (7), for given $\varepsilon > 0$, when k is sufficiently large, say $k \ge k_0$, one has

$$\|f(x_{k+1}) - f(x_k) - w_k\| \le \varepsilon\|d_k\| \quad \text{for all } \; k \ge k_0, \quad w_k \in \mathcal{A}_k d_k. \tag{11}$$

Since $0 \in f(x_k) + \mathcal{B}_k d_k + F(x_{k+1}) + \gamma_k\mathbb{B}_Y,\;\; k = 0, 1, ...,$ then for all $k \in \mathbb{N}$,

$$f(x_{k+1}) - f(x_k) \in f(x_{k+1}) + \mathcal{B}_k d_k + F(x_{k+1}) + \gamma_k\mathbb{B}_Y.$$

Therefore, for all $w_k \in \mathcal{A}_k d_k$, one has

$$d(w_k, f(x_{k+1}) + \mathcal{B}_k d_k + F(x_{k+1})) \leq \|f(x_{k+1}) - f(x_k) - w_k\| + \gamma_k. \tag{12}$$

From (10), (11), one has

$$d(w_k, f(x_{k+1}) + \mathcal{B}_k d_k + F(x_{k+1})) \leq (2+\varepsilon)\varepsilon\|d_k\| \ \ \forall k \geq k_0, \ \ \forall w_k \in \mathcal{A}_k d_k. \tag{13}$$

which shows (8). Assume now $f + F$ is sub-Lipschitz around $\bar{x}$, i.e. there are $L > 0$, and $\delta > 0$ such that

$$f(x) + F(x) \subseteq f(\bar{x}) + F(\bar{x}) + L\|x - \bar{x}\|\mathbb{B}_Y, \ \text{ for all } x \in B(\bar{x}, \delta).$$

Without loss of generality, we can assume $\varepsilon\|d_k\| < \delta$, for all $k \geq k_0$. Then, by (13), for all $w_k \in \mathcal{A}_k d_k$, one has

$$\begin{aligned} d(w_k, f(\bar{x}) + \mathcal{B}_k d_k + F(\bar{x})) &\leq d(w_k, f(x_{k+1}) + \mathcal{B}_k d_k + F(x_{k+1})) + L\|e_{k+1}\| \\ &\leq (L + 1 + \varepsilon)\varepsilon\|d_k\|, \end{aligned}$$

and (8) has already been proved.

Let us now prove (ii). Assume that (9) is fullfilled. For each index k, there exists $v_k \in \mathcal{B}_k d_k$ such that

$$0 \in f(x_k) + v_k + F(x_{k+1}) + \gamma_k \mathbb{B}_Y,$$

equivalently,

$$f(x_{k+1}) - f(x_k) - v_k \in f(x_{k+1}) + F(x_{k+1}) + \gamma_k \mathbb{B}_Y. \tag{14}$$

By (7) and (9), we have

$$\lim_{k\to\infty} [f(x_{k+1}) - f(x_k) - v_k] = 0,$$

This together with (14) and the closedness of the graph of $f + F$ imply $0 \in f(\bar{x}) + F(\bar{x})$. Assume now that $f + F$ is strongly metrically subregular at $(\bar{x}, 0)$, say, there are $\tau, \delta > 0$ such that

$$\|x - \bar{x}\| \leq \tau d(0, f(x) + F(x)), \text{ for all } \ x \in \mathbb{B}(x_0, \delta).$$

Let k_0 be an index such that $x_k \in \mathbb{B}(\bar{x}, \delta)$. Then

$$\|x_{k+1} - \bar{x}\| \leq \tau d(0, f(x_{k+1}) + F(x_{k+1})), \ \text{ for } \ k \geq k_0. \tag{15}$$

By (9), and the fact that $\gamma_k/\|e_k\| \to 0$ as $k \to \infty$, for each $\varepsilon \in (0, 1/(5\tau))$, we can find, say k_0 such that

$$\gamma_k \leq \varepsilon\|e_k\|, \quad e(\mathcal{B}_k d_k, \mathcal{A}_k d_k) \leq \varepsilon\|d_k\| \ \ \forall k \geq k_0.$$

Therefore, for v_k as in (14), there exists $w_k \in \mathcal{A}_k d_k$ such that

$$\|v_k - w_k\| \le e(\mathcal{B}_k d_k, \mathcal{A}_k d_k) + \varepsilon\|d_k\| \le 2\varepsilon\|d_k\| \text{ for } k \ge k_0.$$

From this equality as well as (11), (14, (15), one has the following estimations, for $k \ge k_0$,

$$\begin{aligned}\|x_{k+1} - \bar{x}\| &\le \tau\|f(x_{k+1}) - f(x_k) - v_k\| + \gamma_k \\ &\le \tau\left(\|f(x_{k+1}) - f(x_k) - w_k\| + \|v_k - w_k\| + \varepsilon\|e_k\|\right) \\ &\le \tau(\varepsilon\|d_k\| + 2\varepsilon\|d_k\| + (1+\varepsilon)\varepsilon\|d_k\|) = \tau(4+\varepsilon)\varepsilon\|d_k\|.\end{aligned}$$

As $\|d_k\| \le \|e_{k+1}\| + \|e_k\|$, then

$$\frac{\|e_{k+1}\|}{\|e_k\|} \le \frac{\tau(4+\varepsilon)\varepsilon}{1-\tau(4+\varepsilon)\varepsilon}.$$

Therefore, $\lim\limits_{k\to\infty} \dfrac{\|e_{k+1}\|}{\|e_k\|} = 0$, and the proof of Theorem 2.1 is completed. □

Remark 2.2. When f is Fréchet differentiable in a neighborhood of $\bar{x}$, and the derivative mapping is continuous at $\bar{x}$, then $(\mathcal{A}_k)$ with $\mathcal{A}_k := Df(x_k)$ verifies relation (7). Therefore, Theorem 2.1 covers Theorem 3 in [8]. Still, the essential difference between the above theorem and Theorem 3.2 in [5] is that the conditions used in Theorem 2.1 involve general set-valued approximations $\mathcal{A}_k \colon X \rightrightarrows Y$, without added assumption on $\mathcal{A}_k$. While Theorem 3.2 in [5] uses linear approximations $A_k \in \mathcal{L}(X, Y)$ along with a multifunction $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ which satisfies a certain condition of a strict pre-derivative of f at $\bar{x}$ (see its definition below). We note that relation (8) is different from the one used in (3.5) Theorem 3.2 in [5] in which $f(\bar{x})$ is used instead of $f(x_{k+1})$. Our condition (8) is more realistic than (3.5) Theorem 3.2 in [5], which involves the unknown solution $\bar{x}$. Hence, Theorem 2.1 can be seen as both an improvement and an extension of Theorem 3.2 in [5]. □

Next, we consider an extension of the Dennis-Moré to generalized equations involving *semismooth functions*. We need to recall some notions and backgrounds from nonsmooth analysis. For a locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$, thanks to Rademacher's Theorem, the set of points $x \in \mathbb{R}^n$ at which the derivative $f'(x)$ does not exist, has Lebesgue measure zero. The *B-subdifferential* of g at $u \in \mathbb{R}^n$, denoted by $\partial_B g(x)$, is the set defined by

$$\partial_B g(x) = \left\{ M \in \mathbb{R}^{m\times n} : M = \lim_{k\to+\infty} f'(x_k) \text{ for some } (x_k)_{k\in\mathbb{N}} \text{ converging to } x \right\}.$$

The set $\partial f(x) := \operatorname{co} \partial_B f(x)$ is referred to the *Clarke generalized Jacobian* of f at x. Recall that a mapping $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is called *semismooth* (see [10, Section

7.4]) at $x \in \mathbb{R}^n$ if it is locally Lipschitz continuous around x, directionally differentiable near x, and the following condition is satisfied

$$\lim_{0 \neq v \to 0} \frac{\sup\limits_{M \in \partial g(x+v)} \|f(x+v) - f(x) - Mv\|}{\|v\|} = 0. \tag{16}$$

For a given map $f\colon X \to Y$, recall from [12] that, a positively homogeneous set-valued mapping $G\colon X \rightrightarrows Y$ (i.e., gph G is a cone in $X \times Y$) is called the *strict pre-derivative of f at $x_0 \in X$* if for each $c > 0$, there exists $\delta > 0$ such that

$$f(x_1) \in f(x_2) + G(x_1 - x_2) + c\|x_1 - x_2\|\mathbb{B}_Y \text{ whenever } x_1, x_2 \in \mathbb{B}(x_0, \delta). \tag{17}$$

An important special case, a strict pre-derivative is generated by a family of continuous linear operators, that is, there is a subset $\mathcal{S}$ of $\mathcal{L}(X,Y)$ such that $G(x) = \{S(x) : S \in \mathcal{S}\}$ for each $x \in X$. In this case, one sets $G := \mathcal{S}$. It is well known that when X, Y are finite dimensional spaces, $\partial_B f(x)$, $\partial f(x)$ are strict pre-derivatives of f at x. Note that both the two mappings $\partial_B f$ and ∂f are upper semicontinuous in the sense of set-valued maps.

Denote by $\mathcal{F}(X,Y)$ the set of multifunctions from X to Y. We introduce a generalization of the semismooth notion, associated to a mapping $\mathcal{H}\colon X \rightrightarrows \mathcal{F}(X,Y)$.

Definition 2.3. Let $f : X \to Y$ be a mapping between two Banach spaces, and let $\mathcal{H} : X \rightrightarrows \mathcal{F}(X,Y)$ be given. The mapping f is said to be $\mathcal{H}$-semismooth at $\bar{x} \in X$ if

$$\lim_{\substack{x \to \bar{x} \\ x \notin \bar{x}}} \frac{\sup\limits_{w \in \mathcal{H}(x)(\bar{x}-x)} \|f(\bar{x}) - f(x) - w\|}{\|x - \bar{x}\|} = 0. \tag{18}$$

We make use of the following assumption for a set-valued mapping $\mathcal{H}\colon X \rightrightarrows \mathcal{F}(X,Y)$ under consideration at a reference point $\bar{x} \in X$.

Assumption (A1).

(A1.1) For each $x \in X$, $\mathcal{H}(x)$ is positively homogeneous and

$$\mathcal{H}(x)(u_1 + u_2) \subseteq \mathcal{H}(x)(u_1) + \mathcal{H}(x)(u_2), \quad \forall u_1, u_2 \in X.$$

(A1.2) The mapping $\mathcal{H}$ is upper semicontinuous at $\bar{x} \in X$ in the sense that for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mathcal{H}(x)u \subseteq \mathcal{H}(\bar{x})u + \varepsilon\mathbb{B}_Y, \quad \forall x \in \mathbb{B}(\bar{x}, \delta), \ \forall u \in \mathbb{B}_X.$$

Obviously, if $\mathcal{H}(x) \subseteq \mathcal{L}(X,Y)$, for all $x \in X$, then (A1.1) is automatically satisfied. In this case, the following assumption of upper semicontinuity which is stronger than (A1.2) is made use of:

(A1.2') For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mathcal{H}(x) \subseteq \mathcal{H}(\bar{x}) + \varepsilon \mathbb{B}_{\mathcal{L}(X,Y)}, \quad \forall x \in \mathbb{B}(\bar{x}, \delta).$$

In particular, if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is semismooth at $\bar{x}$ then both ∂f and $\partial_B f$ satisfy Assumption (A1.2').

A general version of the Dennis-Moré theorem for generalized equations associated to $\mathcal{H}$-semismooth functions f is stated in the following theorem.

Theorem 2.4. *Let (γ_k) be a sequence of positive reals and let $\bar{x} \in X$ be given. Let $f\colon X \to Y$ be $\mathcal{H}$-semismooth with respect to $\mathcal{H}\colon X \rightrightarrows \mathcal{F}(X,Y)$ for which,* (A1) *is satisfied and $\mathcal{H}(\bar{x})(\mathbb{B}_X)$ is bounded. Let $\mathcal{B}_k\colon X \rightrightarrows Y$, $k \in \mathbb{N}$ be a sequence of multifunctions from X to Y. Suppose that the sequence $(x_k)_{k\in N_0}$ generated by* (3) *converges to $\bar{x}$ and $\lim_{k\to\infty} \gamma_k/\|e_k\| = 0$.*

(i) *If $x_k \to \bar{x}$ q-superlinearly then one has*

$$\lim_{k\to\infty} \frac{e(\mathcal{H}(x_k)d_k, f(\bar{x}) + \mathcal{B}_k d_k + F(x_{k+1}))}{\|d_k\|} = 0. \tag{19}$$

(ii) *Suppose that*

$$\lim_{k\to\infty} \frac{e(\mathcal{B}_k d_k, \mathcal{H}(x_k)d_k)}{\|d_k\|} = 0. \tag{20}$$

Then $\bar{x}$ is a solution of (1) *provided F has closed graph. Moreover, if the multifunction $f(\bar{x}) + \mathcal{H}(\bar{x})(\cdot - \bar{x}) + F(\cdot)$ is strongly metrically subregular at $(\bar{x}, 0)$, then $x_k \to \bar{x}$ q-superlinearly.*

Proof. (i). For $\varepsilon > 0$ be given, let an index $k_0 \in \mathbb{N}$ such that

$$\gamma_k \le \varepsilon\|e_k\|; \quad \|e_{k+1}\| \le \varepsilon\|d_k\| < \delta \quad \text{for all } k \ge k_0. \tag{21}$$

Since f is $\mathcal{H}$-semismooth, assume that

$$\|f(x_k) - f(\bar{x}) + w\| \le \varepsilon\|x_k - \bar{x}\| \quad \text{for all } \; k \ge k_0, \; w \in \mathcal{H}(x_k)(-e_k). \tag{22}$$

Since $\mathcal{H}(\bar{x})(\mathbb{B}_X)$ is bounded, and by virtue of (A1.2), $\mathcal{H}(x_k)(\mathbb{B}_X)$ is bounded when k is sufficiently large. So assume that for some $L > 0$,

$$\|w\| \le L \quad \text{for all} \quad w \in \mathcal{H}(x_k)(\mathbb{B}_X) \quad \text{with } k \ge k_0.$$

Let $k \ge k_0$ and $w_k \in \mathcal{H}(x_k)d_k$ be given. According to (A1.1), there is $w_{1,k} \in \mathcal{H}(x_k)e_{k+1}$ and $w_{2,k} \in \mathcal{H}(x_k)(-e_k)$ such that $w_k = w_{1,k} + w_{2,k}$. By

$$0 \in f(x_k) + \mathcal{B}_k(x_{k+1} - x_k) + F(x_{k+1}) + \gamma_k \mathbb{B}_Y,$$

one has

$$\begin{aligned} d(w_k, f(\bar{x}) + \mathcal{B}_k d_k + F(x_{k+1})) &\le \|f(x_k) - f(\bar{x}) + w_{2,k}\| + \|w_{1,k}\| + \gamma_k \\ &\le \varepsilon\|e_k\| + L\|e_{k+1}\| + \varepsilon\|e_k\| \\ &\le (L + 2 + 2\varepsilon)\varepsilon\|d_k\|, \end{aligned}$$

which implies (19).

(ii). By virtue of (A1.2), for $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\mathcal{H}(x)u \subseteq \mathcal{H}(\bar{x})u + \varepsilon\mathbb{B}_Y \ \ \forall x \in \mathbb{B}(\bar{x}, \delta_0), \ \forall u \in \mathbb{B}_X. \tag{23}$$

For each index $k \geq k_0$, let $v_k \in \mathcal{B}_k(x_{k+1} - x_k) = \mathcal{B}_k d_k$ be such that

$$0 \in f(x_k) + v_k + F(x_{k+1}) + \gamma_k\mathbb{B}_Y. \tag{24}$$

Let an index k_0 such that $x_k \in \mathbb{B}(\bar{x}, \delta_0)$ and $\gamma_k \leq \varepsilon\|e_k\|$ for all $k \geq k_0$. By relation(20), we can find $w_k \in \mathcal{H}(x_k)d_k$ such that

$$\|v_k - w_k\| \leq \varepsilon\|d_k\|, \ \text{ for all } k \geq k_0.$$

Then, by (A1.1), there are $w_{1,k} \in \mathcal{H}(x_k)e_{k+1}$ and $w_{2,k} \in \mathcal{H}(x_k)(-e_k)$ such that $w_k = w_{1,k} + w_{2,k}$. Suppose gph F is closed. Relation (24) yields

$$-f(x_k) + f(\bar{x}) - w_{2,k} \in f(\bar{x}) + (v_k - w_k) + w_{1,k} + F(x_{k+1}) + \gamma_k\mathbb{B}_Y. \tag{25}$$

Since $\mathcal{H}(\bar{x})$ is bounded, relation (23) yields $\lim_{k\to\infty} w_{1,k} = 0$. By this and since gph F is closed, and

$$\lim_{k\to\infty}[-f(x_k) + f(\bar{x}) - w_{2,k}] = \lim_{k\to\infty}(v_k - w_k) = \lim_{k\to\infty}\gamma_k = 0,$$

relation (25) implies $0 \in f(\bar{x}) + F(\bar{x})$.

Assume now that the multifunction $f(\bar{x}) + \mathcal{H}(\bar{x})(\cdot - \bar{x}) + F(\cdot)$ is strongly metrically subregular at $(\bar{x}, 0)$, that is, there are $\kappa, \delta > 0$ such that

$$\|x - \bar{x}\| \leq \kappa d(0, f(\bar{x}) + \mathcal{H}(\bar{x})(x - \bar{x}) + F(x)) \quad \text{for all } \ x \in \mathbb{B}(\bar{x}, \delta). \tag{26}$$

Let $\varepsilon \in (0, 1/2\kappa)$ be given, and pick $\delta_0 \in (0, \delta)$ such that (23) holds. Then by (26) along with (23), one has

$$\begin{aligned}\|x_{k+1} - \bar{x}\| &\leq \kappa d(0, f(\bar{x}) + \mathcal{H}(\bar{x})e_{k+1} + F(x_{k+1}))\\ &\leq \kappa d(0, f(\bar{x}) + \mathcal{H}(x_k)e_{k+1} + F(x_{k+1})) + \kappa\varepsilon\|e_{k+1}\| \ \text{ for all } k \geq k_0.\end{aligned}$$

Thus, $\|x_{k+1} - \bar{x}\| \leq 2\kappa d(0, f(\bar{x}) + w_{1,k} + F(x_{k+1}))$ for $k \geq k_0$.

As $f(\bar{x}) - f(x_k) - v_k + w_{1,k} \in f(\bar{x}) + w_{1,k} + F(x_{k+1}) + \gamma_k\mathbb{B}_Y$, one has

$$\begin{aligned}\|e_{k+1}\| &\leq 2\kappa(\|f(\bar{x}) - f(x_k) - w_{2,k}\| + \|v_k - w_k\|) + \kappa\gamma_k\\ &\leq 2\kappa\varepsilon(\|e_k\| + \|d_k\|) + \kappa\varepsilon\|e_k\|\\ &\leq 2\kappa\varepsilon(\|e_k\| + \|e_k\| + \|e_{k+1}\|) + \kappa\varepsilon\|e_k\|.\end{aligned}$$

Hence, $\dfrac{\|e_{k+1}\|}{\|e_k\|} \leq \dfrac{5\kappa\varepsilon}{1 - 2\kappa\varepsilon}$, which shows the superlinear convergence of (x_k).

The proof is thereby complete. □

Remark 2.5. Consider the special case in which $\mathcal{H}(x)$ is generated by a subset of $\mathcal{L}(X,Y)$.

(1) When $\mathcal{H}(\bar{x})$ is a compact set, then the strong metric subregularity of the multifunction

$$\Phi_{\mathcal{H}(\bar{x})} := f(\bar{x}) + \mathcal{H}(\bar{x})(\cdot - \bar{x}) + F(\cdot)$$

is equivalent to the strong metric subregularity of all multifunctions

$$\Phi_H := f(\bar{x}) + H(\cdot - \bar{x}) + F(\cdot),$$

with $H \in \mathcal{H}(\bar{x})$, at $(\bar{x}, 0)$. Indeed, assume that for each $\bar{H} \in \mathcal{H}(\bar{x})$, there are $\tau_{\bar{H}}, \delta_{\bar{H}} > 0$ such that

$$\|x - \bar{x}\| \le \tau_{\bar{H}} d(0, f(\bar{x}) + \bar{H}(x - \bar{x}) + F(x))$$

for all $x \in \mathbb{B}(\bar{x}, \delta_{\bar{H}})$. Then, by setting $\varepsilon_{\bar{H}} := 1/(2\tau_{\bar{H}})$, one has, for all $H \in \mathbb{B}(\bar{H}, \varepsilon_{\bar{H}})$ and for all $x \in \mathbb{B}(\bar{x}, \delta_{\bar{H}})$

$$\begin{aligned}\|x - \bar{x}\| &\le \tau_{\bar{H}} d(0, f(\bar{x}) + H(x - \bar{x}) + F(x)) + \|H - \bar{H}\|\|x - \bar{x}\| \\ &\le \tau_{\bar{H}} d(0, f(\bar{x}) + H(x - \bar{x}) + F(x)) + \varepsilon_{\bar{H}}\|x - \bar{x}\|.\end{aligned}$$

Hence, for all $x \in \mathbb{B}(\bar{x}, \delta_{\bar{H}}),\ H \in \mathbb{B}(\bar{H}, \varepsilon_{\bar{H}})$, one has

$$\|x - \bar{x}\| \le 2\tau_{\bar{H}} d(0, f(\bar{x}) + H(x - \bar{x}) + F(x)) + \|H - \bar{H}\|\|x - \bar{x}\|$$

Because of the compactness of $\mathcal{H}(\bar{x})$, there exists a finite collection of balls $\mathbb{B}(H_i, \varepsilon_{H_i})$, $i = 1, ..., n$, with $H_i \in \mathcal{H}(\bar{x})$ such that

$$\mathcal{H}(\bar{x}) \subseteq \bigcup_{i=1}^{n} \mathbb{B}(H_i, \varepsilon_{H_i}).$$

Let $\kappa := 2\min\{\tau_{H_i} : i = 1, ..., n\}$, $\quad \delta := \min\{\delta_{H_i} : i = 1, ..., n\}$. For any $H \in \mathcal{H}(\bar{x})$, then $H \in \mathbb{B}(H_i, \varepsilon_{H_i})$ for some $i \in \{1, ..., n\}$, and therefore one has

$$\|x - \bar{x}\| \le \kappa d(0, f(\bar{x}) + H(x - \bar{x}) + F(x)) \quad \text{for all } \ x \in \mathbb{B}(\bar{x}, \delta),$$

which implies immediately the strong metric subregularity of the multifunction $f(\bar{x}) + \mathcal{H}(\bar{x})(\cdot - \bar{x}) + F(\cdot)$ at $(\bar{x}, 0)$.

(2) In [5], an assumption to guarantee the strong metric regularilty of the multifunction $\Phi_{\mathcal{H}(\bar{x})}$ is given. That assumption which is weaker than the compactness of $\mathcal{H}(\bar{x})$, concerns the relationship between the so-called mesure of noncompactness of $\mathcal{H}(\bar{x})$ and modulus of the strong subregularity of $\Phi_{\bar{H}}$, $\bar{H} \in \mathcal{H}(\bar{x})$. $\square$

3. Convergence analysis of generalized semismooth quasi-Newton methods

Consider the quasi-Newton method (3) along with a sequence of sets of continuous linear operators $\mathcal{B}_k \subseteq \mathcal{L}(X, Y)$. In what follow we assume that $f : X \to Y$ is a continuous mapping and $F\colon X \rightrightarrows Y$ is a closed multifunction. We consider in this section and the next one the convergence of the quasi-Newton method when the sequence $(\mathcal{B}_k)$ is updated in a suitable way. When f is a continuously differentiable function, some updated ways ensuring the convergence (at least q-linear) are available in the literature (see, e.g, [2] and references given therein). However, there are few results related to this questions when f is not continuously differentiable, but rather semismooth, even for nonsmooth equations without set-valued part. We establish in this section a sufficient condition ensuring the local (at least q-linear) convergence of the quasi-Newton iteration (3). In the next section, we consider an extension of the Broyden update method.

We make use of the following result established in [1] on the stability of the metric regularity when the multifunction under consideration is perturbed by set-valued mappings. In fact, in the proof of the following convergence result, we need only the stability established by Dontchev-Rockafellar ([9]) under single-valued perturbation mappings. However, since the parameters in these results are different, we shall make use of the following result given in [1].

Theorem 3.1. [Theorem 3.4, [1]] *Given Banach spaces X and Y, let $\Phi\colon X \rightrightarrows Y$ be a set-valued mapping with closed graph and let $(\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$. Suppose that Φ is metrically regular at $(\bar{x}, \bar{y})$ with a constant $\kappa > 0$ on a neighborhood $\mathbb{B}(\bar{x}, a) \times \mathbb{B}(\bar{y}, b)$ of $(\bar{x}, \bar{y})$ for some $a > 0$ and $b > 0$. Let $\delta > 0$, $L \in (0, \kappa^{-1})$, and set $\tau = \kappa/(1 - \kappa L)$. Let $\beta > 0$ be such that*

$$\beta\tau < \min\{a, \delta/2\}, \quad \beta(\tau + \kappa) < \delta, \quad \beta(1 + c\tau) < \min\{b, \delta\}, \tag{27}$$

with $c := \max\{1, 1/\kappa\}$. Then for any set-valued mapping $G\colon X \rightrightarrows Y$ with closed graph and a point $\bar{z} \in G(\bar{x})$ such that

$$e\big(G(x_1) \cap \mathbb{B}(\bar{z}, \delta), G(x_2)\big) \le L\|x_1 - x_2\| \quad \text{whenever} \quad x_1, x_2 \in \mathbb{B}(\bar{x}, \delta), \tag{28}$$

one has

$$d\big(\bar{x}, (\Phi + G)^{-1}(y)\big) \le \tau d(y, \Phi(\bar{x}) + \bar{z}) \quad \text{for all} \quad y \in \mathbb{B}(\bar{y} + \bar{z}, \beta). \tag{29}$$

Theorem 3.2. *Let $\bar{x} \in X$ be a solution to the generalized equation* (1). *Let $\mathcal{H}\colon X \rightrightarrows \mathcal{L}(X, Y)$ be such that the assumptions* (A1.2') *is fullfilled. Suppose that f is $\mathcal{H}$-semismooth at $\bar{x}$ and that the collection of multifunctions*

$$\left\{\Phi_{\bar{H}}(x) := f(\bar{x}) + \bar{H}(x - \bar{x}) + F(x),\ x \in X,\ \bar{H} \in \mathcal{H}(\bar{x})\right\}$$

is uniformly metrically regular at $(\bar{x}, 0)$. Let $c, s \in [0, 1]$, $\gamma > 0$ be given. Consider the quasi-Newton (3) *with a starting point $x_0 \in X$. Then there exist* *a neighborhood U of $\bar{x}$ and $r_0 > 0$ such that for any starting point $x_0 \in U$, and*

$$e(\mathcal{B}_0, \mathcal{H}(x_0)) < r_0,$$

there is a sequence $(x_k)_{k\in\mathbb{N}}$ generated by (3) *along with the sequence $(\mathcal{B}_k) \subseteq \mathcal{L}(X,Y)$ updated for $k = 0, 1, ...$ in the following way*

$$e(\mathcal{B}_{k+1}, \mathcal{H}(x_{k+1})) \le se(\mathcal{B}_k, \mathcal{H}(x_k)) + c(\|x_k - \bar{x}\|^\gamma + \|x_{k+1} - \bar{x}\|^\gamma), \tag{30}$$

converging linearly to $\bar{x}$. Moreover, if $s < 1$, then there is a sequence $(x_k)_{k\in\mathbb{N}}$ generated by (3) *converging superlinearly to $\bar{x}$.*

Proof. By the assumption of the uniform metric regularity, there are $\kappa > 0$, $a > 0$ and $b > 0$ such that for each $\bar{H} \in \mathcal{H}(\bar{x})$ the mapping $\Phi_{\bar{H}}$ is metrically regular at $(\bar{x}, 0)$ with the constant κ on $\mathbb{B}(\bar{x}, a) \times \mathbb{B}(0, b)$. Let $\beta > 0$ such that (27) is satisfied. Then, fix $L, \varepsilon, r_0, r > 0$ and $\mu \in (0, 1)$ such that

$$\frac{2\kappa\left(\varepsilon + r_0 + \frac{3cr^\gamma}{1-\mu^\gamma}\right)}{1 - \kappa\left(r_0 + \frac{3cr^\gamma}{1-\mu^\gamma} + L\right)} \le \mu; \quad \left(\varepsilon + r_0 + \frac{3cr^\gamma}{1-\mu^\gamma}\right) r < \beta. \tag{31}$$

By (A1.2') as well as the $\mathcal{H}$-semismoothness of f, there is $\delta > 0$ such that

$$\|f(x) - f(\bar{x}) - A(x - \bar{x})\| \le \varepsilon\|x - \bar{x}\| \text{ for all } x \in \mathbb{B}(\bar{x}, \delta),\ A \in \mathcal{H}(x), \tag{32}$$

$$\mathcal{H}(x) \subset \mathcal{H}(\bar{x}) + L\mathbb{B}_{\mathcal{L}(X,Y)} \text{ whenever } x \in \mathbb{B}(\bar{x}, \delta) \subset \operatorname{dom}\mathcal{H}. \tag{33}$$

Set $U = \mathbb{B}(\bar{x}, r)$. Let x_0 be an arbitrary point in U. Let $B_0 \in \mathcal{B}_0$ be given. Then, there is $A_0 \in \mathcal{H}_0$ such that $\|B_0 - A_0\| < r_0$. By (33), we can find $\bar{H} \in \mathcal{H}(\bar{x})$ such that

$$\|B_0 - \bar{H}\| \le \|B_0 - A_0\| + \|A_0 - \bar{H}\| < r_0 + L.$$

Applying Theorem 3.1 with $\Phi := \Phi_{\bar{H}}$, $G(x) := f(x_0) + B_0(x - x_0) - f(\bar{x}) - \bar{H}(x - \bar{x})$, $\bar{z} := G(\bar{x}) = f(x_0) + B_0(\bar{x} - x_0) - f(\bar{x})$, one has

$$\begin{aligned} d(\bar{x}, (\Phi_{\bar{H}} + G)^{-1}(y)) &\le \frac{\kappa}{1 - \kappa(r_0 + L)} d(y, (\Phi_{\bar{H}} + G)(\bar{x})) \\ &= \frac{\kappa}{1 - \kappa(r_0 + L)} d(y, \bar{z} + f(\bar{x}) + F(\bar{x})) \\ &\le \frac{\kappa}{1 - \kappa(r_0 + L)} \|y - \bar{z}\| \ \ \forall y \in \mathbb{B}(\bar{z}, \beta). \end{aligned} \tag{34}$$

Since

$$\begin{aligned} \|\bar{z}\| &= \|f(x_0) + B_0(\bar{x} - x_0) - f(\bar{x})\| \\ &\le \|f(x_0) + A_0(\bar{x} - x_0) - f(\bar{x})\| + \|B_0 - A_0\|\|\bar{x} - x_0\| \\ &\le (\varepsilon + r_0)\|x_0 - \bar{x}\| < \beta \end{aligned}$$

there is $x_1 \in (\Phi_{\bar{H}} + G)^{-1}(0))$, which means that $0 \in f(x_0) + B_0(x_1 - x_0) + F(x_1)$, such that

$$\|x_1 - \bar{x}\| \leq \frac{2\kappa}{1 - \kappa(r_0 + L)}\|\bar{z}\| \leq \frac{2\kappa(\varepsilon + r_0)}{1 - \kappa(r_0 + L)}\|x_0 - \bar{x}\| \leq \mu\|x_0 - \bar{x}\|.$$

Assume that for an index k, we have founded $x_1, ..., x_k \in U$, verifying

$$0 \in f(x_i) + \mathcal{B}_i(x_{i+1} - x_i) + F(x_{i+1}), \quad i = 0, ..., k-1, \tag{35}$$

and

$$\|x_{i+1} - \bar{x}\| \leq \mu\|x_i - \bar{x}\|, \quad i = 0, ..., k-1. \tag{36}$$

By (30),

$$\begin{aligned} e(\mathcal{B}_k, \mathcal{H}(x_k)) &\leq s^k e(\mathcal{B}_0, \mathcal{H}(x_0)) + c\sum_{i=1}^{k} s^{k-i}(\|x_i - \bar{x}\|^\gamma + \|x_{i-1} - \bar{x}\|^\gamma) \\ &\leq s^k r_0 + 2c\sum_{i=0}^{k}\|x_i - \bar{x}\|^\gamma \\ &\leq r_0 + 2c\|x_0 - \bar{x}\|^\gamma \sum_{k=0}^{\infty}\mu^{\gamma i} \\ &\leq r_0 + \frac{2cr^\gamma}{1 - \mu^\gamma}. \end{aligned} \tag{37}$$

Consequently, for any $B_k \in \mathcal{B}_k$, there exist $A_k \in \mathcal{H}(x_k)$ and $\bar{H}_k \in \mathcal{H}(\bar{x})$ such that

$$\begin{cases} \|B_k - A_k\| < r_0 + \frac{3cr^\gamma}{1-\mu^\gamma} \\ \|B_k - \bar{H}_k\| \leq e(\mathcal{B}_k, \mathcal{H}_k) + e(\mathcal{H}(x_k), \mathcal{H}(\bar{x})) < r_0 + \frac{3cr^\gamma}{1-\mu^\gamma} + L. \end{cases}$$

As

$$\begin{aligned} \|f(x_k) + B_k(\bar{x} - x_k) - f(\bar{x})\| &\leq \\ &\leq \|f(x_k) + A_k(\bar{x} - x_k) - f(\bar{x})\| + \|B_k - A_k\|\|\bar{x} - x_k\| \\ &\leq (\varepsilon + r_0 + \frac{3cr^\gamma}{1 - \mu^\gamma})\|x_k - \bar{x}\| < \beta, \end{aligned}$$

by applying again Theorem 3.1 with $\Phi := \Phi_{\bar{H}_k} = f(\bar{x}) + \bar{H}_k(x - \bar{x}) + F(x)$,

$$G(x) = G_k(x) := f(x_k) + B_k(x - x_k) - f(\bar{x}) - \bar{H}_k(x - \bar{x}),$$

$$\bar{z} = \bar{z}_k := G_k(\bar{x}) = f(x_k) + B_k(\bar{x} - x_k) - f(\bar{x}),$$

one obtains

$$d(\bar{x},(\Phi_{\bar{H}}+G_k)^{-1}(0)) \le \frac{\kappa}{1-\kappa(r_0+\frac{3cr^\gamma}{1-\mu^\gamma}+L)} d(0,(\Phi_{\bar{H}_k}+G_k)(\bar{x})) = \frac{\kappa}{1-\kappa(r_0+\frac{3cr^\gamma}{1-\mu^\gamma}+L)}\|\bar{z}_k\|. \tag{38}$$

Therefore, there is $x_{k+1} \in (\Phi_{\bar{H}_k}+G_k)^{-1}(0)$, that is,

$$0 \in f(x_k)+B_k(x_{k+1}-x_k)+F(x_{k+1}),$$

such that

$$\begin{aligned}\|x_{k+1}-\bar{x}\| &\le \frac{2\kappa}{1-\kappa(r_0+\frac{3cr^\gamma}{1-\mu^\gamma}+L)}\|\bar{z}_k\| \\ &\le \frac{2\kappa(\varepsilon+r_0+\frac{3cr^\gamma}{1-\mu^\gamma})}{1-\kappa(r_0+\frac{3cr^\gamma}{1-\mu^\gamma}+L)}\|x_k-\bar{x}\| \qquad (39)\\ &\le \mu\|x_k-\bar{x}\|.\end{aligned}$$

So by induction, we constructed a sequence (x_k) generated by (3) satisfying

$$\|x_{k+1}-\bar{x}\| \le \mu\|x_k-\bar{x}\|, \quad \text{for all } k \in \mathbb{N}.$$

That is, (x_k) converges linearly to $\bar{x}$.

Suppose now $s<1$. By (37), one has

$$\begin{aligned}e(\mathcal{B}_k,\mathcal{H}(x_k)) &\le s^k e(\mathcal{B}_0,\mathcal{H}(x_0)) + c\sum_{i=1}^{k} s^{k-i}(\|x_i-\bar{x}\|^\gamma+\|x_{i-1}-\bar{x}\|^\gamma) \\ &\le s^k e(\mathcal{B}_0,\mathcal{H}(x_0)) + 2c\|x_0-\bar{x}\|^\gamma \frac{s^k}{\mu^\gamma}\sum_{i=0}^{k}\left(\frac{\mu^\gamma}{s}\right)^i \\ &= s^k e(\mathcal{B}_0,\mathcal{H}(x_0)) + 2c\|x_0-\bar{x}\|^\gamma \frac{s^{k+1}-\mu^{\gamma(k+1)}}{\mu^\gamma(s-\mu^\gamma)},\end{aligned}$$

which implies $\lim_{k\to\infty} e(\mathcal{B}_k,\mathcal{H}(x_k))=0$. Therefore, for any $\varepsilon>0$, there is an index k_0 such that for any $B_k \in \mathcal{B}_k$, we can find $A_k \in \mathcal{H}(x_k)$ such that

$$\begin{aligned}\|f(x_k)+B_k(\bar{x}-x_k)-f(\bar{x})\| &\le \|f(x_k)+A_k(\bar{x}-x_k)-f(\bar{x})\| + \\ &+\|B_k-A_k\|\|\bar{x}-x_k\| \le \varepsilon\|x_k-\bar{x}\| \quad \forall\, k \ge k_0.\end{aligned}$$

Combining this relation with (39) one obtains

$$\|x_{k+1}-\bar{x}\| \le \frac{2\kappa}{1-\kappa(r_0+\frac{3cr^\gamma}{1-\mu^\gamma}+L)}\varepsilon\|x_k-\bar{x}\|.$$

As ε is arbitrary, this shows that (x_k) converges superlinearly to $\bar{x}$. $\square$

Remark 3.3. As was shown in Theorem 4.3, [1], if $\mathcal{H}(\bar{x})$ is a compact set, then the uniform metric regularity of

$$\left\{\Phi_{\bar{H}}(x) := f(\bar{x}) + \bar{H}(x-\bar{x}) + F(x),\ x \in X,\ \bar{H} \in \mathcal{H}(\bar{x})\right\}$$

is equivalent to the metric regularity of all multifunctions $\Phi_{\bar{H}}(x)$, $\bar{H} \in \mathcal{H}(\bar{x})$ at $(\bar{x}, 0)$. When f is Fréchet differentiable at $\bar{x}$, then by taking $\mathcal{H}(x) = \{Df(\bar{x})\}$, which satisfied obviously (A1), the preceding theorem implies Theorem 3.1, [2].

4. Convergence of the Broyden update method

Let X be a Hilbert space endowed with scalar product $\langle \cdot, \cdot \rangle$ and let Y be a Banach space. As before, let $f : X \to Y$, $F : X \rightrightarrows Y$ be a continuous mapping and a closed multitifunction between spaces X, Y. Consider the quasi-Newton iteration for solving the inclusion $0 \in f(x) + F(x)$:

$$0 \in f(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1}), \quad k \in \mathbb{N}. \tag{40}$$

Where (B_k) is a (updated) sequence of continuous linear operators. The well-known Broyden update for (B_k) is defined by [3]

$$B_{k+1} = B_k + \frac{(f(x_{k+1}) - f(x_k) - B_k d_k)\langle d_k, \cdot \rangle}{\|d_k\|^2},$$

here, $d_k = x_{k+1} - x_k$. In the finite dimensional setting, the superlinear convergence of the Broyden method for solving the smooth nonlinear equations has been established by Broyden-Dennis-Moré in [4]. The convergence of this method for smooth equations in the infinite dimensional settings was established by several authors, e.g., by Kelly-Sachs [14], Sachs [21]. An interesting discussion on the history and the effectiveness of the Broyden updating was given in [11]. Recently, the Broyden method for solving generalized equations was considered in [2]. In that paper, the authors have established some results on the (linear and superlinear) convergence of the method when X, Y are Hilbert spaces and the function f is assumed to be Fréchet differentiable around the reference point and its derivative is Lipschitz continuous with respect to the Hilbert-Schmidt norm.

In this final section, we consider a modified Broyden update method for solving the generalized equations (1) when f is a semismooth function associated to a multifunction $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ which, in addition the Assumption (A1), verifies the following assumption.

Assumption (A2). There are a neighborhood U of $\bar{x}$, $c > 0$ and $\gamma \in (0, 1]$ such that

(A2.1) for each $x \in U \setminus \{\bar{x}\}$, there exists $s_x > 0$ such that

$$\mathcal{H}(x) \subseteq \mathcal{H}(x') + c\|x - \bar{x}\|^{\gamma}\mathbb{B}_{\mathcal{L}(X,Y)}, \text{ for all } x' \in \mathbb{B}(x, s_x);$$

(A2.2) $\|f(x) - f(\bar{x}) - H(x - \bar{x})\| \leq c\|x - \bar{x}\|^{1+\gamma}$ for all $x \in U$, $H \in \mathcal{H}(x)$.

For each $x \in U \setminus \{\bar{x}\}$, set

$$r_x := \sup\{s_x : \; s_x \text{ verifies (A2.1)}\}. \tag{41}$$

The modified Broyden update is stated as follows.

(MBA) Let (θ_k) be a sequence of reals with $\theta_k \in (0, 1)$, converging to 0. Giving a starting point $x_0 \in X$; $B_0 \in \mathcal{L}(X, Y)$ with

$$d(B_0, \mathcal{H}(x_0)) \leq c\|x_0 - \bar{x}\|^{\gamma}, \tag{42}$$

at the kth iteration, let p_k be a solution of the auxiliary inclusion

$$0 \in f(x_k) + B_k p_k + F(x_k + p_k). \tag{43}$$

If $\|p_k\| < r_k := r_{x_k}$, then $x_{k+1} = x_k + p_k$, otherwise,

$$x_{k+1} = x_k + \frac{r_k(1 - \theta_k)}{\|p_k\|} p_k,$$

$$B_{k+1} = B_k + \frac{(f(x_{k+1}) - f(x_k) - B_k d_k)\langle d_k, \cdot\rangle}{\|d_k\|^2}, \quad k = 0, 1, ..., \tag{44}$$

where, $d_k := x_{k+1} - x_k$.

Remark 4.1. Note that if f is differentiable at $\bar{x}$ and that

$$\|f(x) - f(\bar{x}) - Df(\bar{x})(x - \bar{x})\| \leq c\|x - \bar{x}\|^{1+\gamma} \quad \text{for all } x \in U,$$

then we can take $\mathcal{H}(x) = Df(\bar{x})$ for all $x \in X$, Obviously, for this $\mathcal{H}$, (A1), (A2) are fulfilled, and for this case, one has $r_x = +\infty$ for all $x \in X$. It is worth to note that in the modified Broyden update, one can replace r_k by its lower bound. In general, the quantity r_x is unknown, however, in some important particular cases, for example, when f is the *Fischer-Burmeister gap function* for complementarity problems (see e.g. [10] p. 629), it is possible to compute explicitly a lower bound of the quantity r_x.

An examination of the proof of Theorem 2.4, allows us to have the following sufficient condition of Dennis-Moré type for the superlinear convergence of (MBA).

Lemma 4.2. *Let $\bar{x} \in X$ be a solution of the generalized equation (1). Let $f : X \to Y$ be $\mathcal{H}$-semismooth for $\mathcal{H}\colon X \rightrightarrows \mathcal{L}(X, Y)$. Suppose that the sequence $(x_k)_{k \in N_0}$ generated by* (MBA) *converges to $\bar{x}$. Suppose that Assumption* (A1.2') *is satisfied, the multifunction*

$$\Phi_{\mathcal{H}(\bar{x})}(x) := f(\bar{x}) + \mathcal{H}(\bar{x})(x - \bar{x}) + F(x), \quad x \in X$$

is strongly metric subregular at $\bar{x}$ *for* 0 *and that*

$$\liminf_{x\to\bar{x}} \frac{r_x}{\|x-\bar{x}\|} \geq 1, \tag{45}$$

if

$$\lim_{k\to\infty} \frac{d(B_k d_k, \mathcal{H}(x_k)d_k)}{\|d_k\|} = 0, \tag{46}$$

where $d_k = x_{k+1} - x_k$*, then* $x_k \to \bar{x}$ *q-superlinearly.*

Proof. Let $\kappa, \delta > 0$ such that

$$\|x-\bar{x}\| \leq \kappa d(0, f(\bar{x}) + H(x-\bar{x}) + F(x)) \text{ for all } H \in \mathcal{H}(\bar{x}),\ x \in \mathbb{B}(\bar{x},\delta). \tag{47}$$

Let $\varepsilon \in (0, 1/6\kappa)$ be given. By (A1.2'), there is $\delta_0 \in (0,\delta)$ such that

$$\mathcal{H}(x) \subseteq \mathcal{H}(\bar{x}) + \varepsilon \mathbb{B}_{\mathcal{L}(X,Y)} \ \ \forall x \in \mathbb{B}(\bar{x},\delta_0). \tag{48}$$

We consider an index k_0 such that $x_k \in \mathbb{B}(\bar{x},\delta_0)$ for all $k \geq k_0$. By relation(46), we can find $A_k \in \mathcal{H}(x_k)$ such that $\|B_k d_k - A_k d_k\| \leq \varepsilon \|d_k\|$, for all $k \geq k_0$.

Consequently, $\|B_k p_k - A_k p_k\| \leq \varepsilon \|p_k\|$, for all $k \geq k_0$.

For $k \geq k_0$, pick $\bar{A}_k \in \mathcal{H}(\bar{x})$ such that $\|A_k - \bar{A}_k\| \leq \varepsilon$. Then by (47), one has, for all $k \geq k_0$,

$$\begin{aligned}\|x_k + p_k - \bar{x}\| &\leq \kappa d(0, f(\bar{x}) + \bar{A}_k(x_k + p_k - \bar{x}) + F(x_k + p_k)) \\ &\leq \kappa d(0, f(\bar{x}) + A_k(x_k + p_k - \bar{x}) + F(x_k + p_k)) + \kappa\varepsilon\|x_k + p_k - \bar{x}\|.\end{aligned}$$

Thus,

$$\|x_k + p_k - \bar{x}\| \leq 2\kappa d(0, f(\bar{x}) + A_k(x_k + p_k - \bar{x}) + F(x_k + p_k)) \ \text{ for } \ k \geq k_0.$$

As

$$f(\bar{x}) - f(x_k) - v_k + A_k(x_k + p_k - \bar{x}) \in f(\bar{x}) + A_k(x_k + p_k - \bar{x}) + F(x_k + p_k),$$

one has

$$\begin{aligned}\|x_k + p_k - \bar{x}\| &\leq 2\kappa(\|f(\bar{x}) - f(x_k) + A_k(x_k - \bar{x})\| + \|B_k p_k - A_k p_k\|) \\ &\leq 2\kappa\varepsilon(\|x_k - \bar{x}\| + \|p_k\|) \\ &\leq 2\kappa\varepsilon(\|x_k - \bar{x}\| + \|x_k - \bar{x}\| + \|x_k + p_k - \bar{x}\|).\end{aligned}$$

Hence,

$$\|x_k + p_k - \bar{x}\| \leq \frac{4\kappa\varepsilon}{1 - 2\kappa\varepsilon}\|x_k - \bar{x}\|. \tag{49}$$

Consequently,

$$\left(1-\frac{4\kappa\varepsilon}{1-2\kappa\varepsilon}\right)\|x_k-\bar{x}\|\le\|p_k\|\le\left(1+\frac{4\kappa\varepsilon}{1-2\kappa\varepsilon}\right)\|x_k-\bar{x}\|.$$

If $\liminf_{x\to\bar{x}}\frac{r_x}{\|x-\bar{x}\|}>1$, then $\|p_k\|<r_k$ when k is sufficiently large, and therefore $x_{k+1}=x_k+p_k$. Therefore, the conclusion follows from (49). Otherwise,

$$\liminf_{x\to\bar{x}}\frac{r_x}{\|x-\bar{x}\|}=1,$$

and one has

$$\begin{aligned}\|x_{k+1}-\bar{x}\|&\le\|x_k+p_k-\bar{x}\|+|1-r_{x_k}(1-\theta_k)/\|p_k\|\|\|p_k\|\\&\le\left(\frac{4\kappa\varepsilon}{1-2\kappa\varepsilon}+2|1-r_{x_k}(1-\theta_k)/\|p_k\||\right)\|x_k-\bar{x}\|,\end{aligned}$$

which implies the desired assertion. $\square$

Theorem 4.3. *Let X be a Hilbert space and let Y be a Banach space and let $\bar{x}\in X$ be a solution of the generalized equation* (1). *Let $\bar{x}\in int(\operatorname{dom}\mathcal{H})$ and the assumptions* (A1.2'), (A2) *be fulfilled. Suppose further that $\mathcal{H}(\bar{x})$ is a closed set in $\mathcal{L}(X,Y)$ and the collection*

$$\left\{\Phi_{\bar{H}}(x):=f(\bar{x})+\bar{H}(x-\bar{x})+F(x),\ x\in X,\ \bar{H}\in\mathcal{H}(\bar{x})\right\}$$

is uniformly metrically regular at $(\bar{x},0)$ and

$$\liminf_{x\to\bar{x}}\frac{r_x}{\|x-\bar{x}\|}>0. \tag{50}$$

Then there exists a neighborhood U of $\bar{x}$ such that for any starting point $x_0\in U$ and B_0 verifying (42), *there is a sequence $(x_k)_{k\in\mathbb{N}}$ generated by* (MBA) *along with the sequence (B_k) updated in the way* (44), *converging linearly to $\bar{x}$.*

Moreover, if (45) *is satisfied and $\Phi^{-1}_{\mathcal{H}(\bar{x})}(0)=\{\bar{x}\}$, where*

$$\Phi_{\mathcal{H}(\bar{x})}(x)=f(\bar{x})+\mathcal{H}(\bar{x})(x-\bar{x})+F(x),\ x\in X,$$

and for every $\bar{H}\in\mathcal{H}(\bar{x})$, $B_0-\bar{H}$ is a compact operator, then the sequence $(x_k)_{k\in\mathbb{N}}$ converges superlinearly to $\bar{x}$.

Proof. By the uniform metric regularity, there are $\kappa>0$, $a>0$ and $b>0$ such that for each $\bar{H}\in\mathcal{H}(\bar{x})$ the mapping $\Phi_{\bar{H}}$ is metrically regular at $(\bar{x},0)$ with the constant κ on $\mathbb{B}(\bar{x},a)\times\mathbb{B}(0,b)$. Let $\beta>0$ such that (27) in Theorem 3.1 is satisfied. By (50), assume that, for some $\theta\in(0,1)$, $\delta>0$,

$$r_x>\theta\|x-\bar{x}\|\quad\text{for all }\ x\in\mathbb{B}(\bar{x},\delta). \tag{51}$$

Then, fix $L, \varepsilon, r \in (0, \delta)$ and $\mu \in (0, 1)$ such that

$$\frac{6\kappa(1+\varepsilon)c(1+2(1-\eta)^{-1})r^{\gamma}}{(1-\eta^{\gamma})\left[1-\kappa\left(\frac{2c(1+2(1-\eta)^{-1})r^{\gamma}}{1-\eta^{\gamma}}+L\right)\right]} \le \mu, \quad \frac{3c(1+2(1-\eta)^{-1})r^{1+\gamma}}{1-\eta^{\gamma}} < \beta,$$
$$\eta := \max\{1+2\mu-\theta(1-\theta_k) : k = 0, 1, ...\} < 1. \tag{52}$$

By (A1.2'), there is, say, $r > 0$ such that

$$\mathcal{H}(x) \subset \mathcal{H}(\bar{x}) + L\mathbb{B}_{\mathcal{L}(X,Y)} \quad \text{for all} \quad x \in \mathbb{B}(\bar{x}, r). \tag{53}$$

Assume also that (A2) holds for $U := \mathbb{B}(\bar{x}, r)$. Let x_0 be an arbitrary point in U. Let $B_0 \in \mathcal{L}(X, Y)$ such that

$$d(B_0, \mathcal{H}(x_0)) \le c\|x_0 - \bar{x}\|^{\gamma}.$$

By (53), we can find $\bar{H} \in \mathcal{H}(\bar{x})$ such that

$$\|B_0 - \bar{H}\| \le L + c\|x_0 - \bar{x}\|^{\gamma} < L + cr^{\gamma}. \tag{54}$$

Applying Theorem 3.1 with $\Phi := \Phi_{\bar{H}}$

$$G(x) := f(x_0) + B_0(x - x_0) - f(\bar{x}) - \bar{H}(x - \bar{x}), \text{ and}$$
$$\bar{z} := G(\bar{x}) = f(x_0) + B_0(\bar{x} - x_0) - f(\bar{x}),$$

one has

$$\begin{aligned} d(\bar{x}, (\Phi_{\bar{H}} + G)^{-1}(y)) &\le \frac{\kappa}{1-\kappa(L+cr^{\gamma})} d(y, (\Phi_{\bar{H}} + G)(\bar{x})) \\ &= \frac{\kappa}{1-\kappa(L+cr^{\gamma})} d(y, \bar{z} + f(\bar{x}) + F(\bar{x})) \\ &\le \frac{\kappa}{1-\kappa(L+cr^{\gamma})} \|y - \bar{z}\| \ \ \forall y \in \mathbb{B}(\bar{z}, \beta). \end{aligned} \tag{55}$$

In view of (A2.2) and (54),

$$\|\bar{z}\| = \|f(x_0) + B_0(\bar{x} - x_0) - f(\bar{x})\| \le 2c\|x_0 - \bar{x}\|^{1+\gamma} < 2cr^{1+\gamma} < \beta.$$

Thus, there is $p_0 \in X$ with $x_0 + p_0 \in (\Phi_{\bar{H}} + G)^{-1}(0))$, i.e.,

$$0 \in f(x_0) + B_0(p_0) + F(x_0 + p_0),$$

such that

$$\|x_0 + p_0 - \bar{x}\| \le \frac{2\kappa(1+\varepsilon)}{1-\kappa(L+cr^{\gamma})}\|\bar{z}\| \le \frac{2\kappa(1+\varepsilon)c}{1-\kappa(L+cr^{\gamma})}\|x_0 - \bar{x}\|^{1+\gamma} \le \mu\|x_0 - \bar{x}\|.$$

Thus

$$(1-\mu)\|x_0 - \bar{x}\| \le \|p_0\| \le (1+\mu)\|x_0 - \bar{x}\|.$$

By the definition of (x_k), if $p_0 < r_0$, then

$$\|x_1 - \bar{x}\| = \|x_0 + p_0 - \bar{x}\| \le \mu \|x_0 - \bar{x}\|,$$

otherwise,

$$\begin{aligned} \|x_1 - \bar{x}\| &\le \|x_0 + p_0 - \bar{x}\| + (1 - r_0(1-\theta_0)/\|p_0\|)\|p_0\| \\ &\le (\mu + (1 - \theta(1-\theta_0)(1+\mu)^{-1})(1+\mu)\|x_0 - \bar{x}\| \\ &= (1 + 2\mu - \theta(1-\theta_0))\|x_0 - \bar{x}\|. \end{aligned} \tag{56}$$

Therefore, $\|x_1 - \bar{x}\| \le \eta \|x_0 - \bar{x}\|$, and

$$\|d_0\| \ge \|x_0 - \bar{x}\| - \|x_1 - \bar{x}\| \ge (1-\eta)\|x_0 - \bar{x}\| \ge (1-\eta)\eta^{-1}\|x_1 - \bar{x}\|.$$

According to (A2.1), for any $A_0 \in \mathcal{H}(x_0)$, there is $A_1 \in \mathcal{H}(x_1)$ such that

$$\|A_0 - A_1\| \le c\|x_0 - \bar{x}\|^\gamma.$$

Observe that

$$\left\| u - \frac{\langle x, u\rangle x}{\|x\|^2} \right\|^2 = \|u\|^2 - \frac{\langle x, u\rangle^2}{\|x\|^2} \le 1 \quad \forall x \in X, \ \forall u \in \mathbb{B}_X.$$

Therefore,

$$\left\| (B_0 - A_0) - \frac{(B_0 - A_0)d_0\langle d_0, \cdot\rangle}{\|d_0\|^2} \right\| \le \|B_0 - A_0\|, \ k \in \mathbb{N}.$$

In consequence of relation (44) and (A2.2), by making of use of the latter inequalities, we obtain the following estimation

$$\begin{aligned} d(B_1, \mathcal{H}(x_1)) &\le \|B_1 - A_1\| \\ &\le \left\| B_0 - A_0 - \frac{(B_0 - A_0)d_0\langle d_0, \cdot\rangle}{\|d_0\|^2} \right\| + \|A_0 - A_1\| + \frac{\|f(x_1) - f(x_0) - A_0 d_0\|}{\|d_0\|} \\ &\le \|B_0 - A_0\| + c\|x_0 - \bar{x}\|^\gamma + \frac{\|f(x_1) - f(\bar{x}) - A_1(x_1 - \bar{x})\|}{\|d_0\|} + \\ &\qquad + \frac{\|f(x_0) - f(\bar{x}) - A_0(x_0 - \bar{x})\|}{\|d_0\|} + \frac{\|A_1 - A_0\|\|x_1 - \bar{x}\|}{\|d_0\|} \\ &\le \|B_0 - A_0\| + c\|x_0 - \bar{x}\|^\gamma + c\eta(1-\eta)^{-1}\|x_1 - \bar{x}\|^\gamma + \\ &\qquad + c(1-\eta)^{-1}\|x_0 - \bar{x}\|^\gamma + c\eta(1-\eta)^{-1}\|x_0 - \bar{x}\|^\gamma \\ &\le \|B_0 - A_0\| + M\left(\|x_1 - \bar{x}\|^\gamma + \|x_0 - \bar{x}\|^\gamma\right), \end{aligned} \tag{57}$$

where $M := c(1 + (1-\eta)^{-1} + \eta(1-\eta)^{-1})$. Thus,

$$d(B_1, \mathcal{H}(x_1)) \le d(B_0, \mathcal{H}(x_0)) + M\left(\|x_1 - \bar{x}\|^\gamma + \|x_0 - \bar{x}\|^\gamma\right).$$

Assume by induction that for an index k, we have founded $x_1, ..., x_k \in U$, verifying

$$0 \in f(x_i) + B_i(x_{i+1} - x_i) + F(x_{i+1}), \quad i = 0, ..., k-1, \tag{58}$$

$$\|x_{i+1} - \bar{x}\| \le \eta \|x_i - \bar{x}\|, \quad i = 0, ..., k-1 \tag{59}$$

and, for $i = 0, ..., k-1$,

$$d(B_{i+1}, \mathcal{H}(x_{i+1})) \le d(B_i, \mathcal{H}(x_i)) + M\left(\|x_i - \bar{x}\|^\gamma + \|x_{i+1} - \bar{x}\|^\gamma\right). \tag{60}$$

Therefore,

$$\begin{aligned} d(B_k, \mathcal{H}(x_k)) &\le M \sum_{i=1}^{k} (\|x_i - \bar{x}\|^\gamma + \|x_{i-1} - \bar{x}\|^\gamma) \le 2M \sum_{i=0}^{k} \|x_i - \bar{x}\|^\gamma \\ &< 2M \|x_0 - \bar{x}\|^\gamma \sum_{k=0}^{\infty} (\eta^\gamma)^k \le \frac{2c(1 + 2(1-\eta)^{-1}) r^\gamma}{1 - \eta^\gamma}. \end{aligned} \tag{61}$$

Consequently, we can find $A_k \in \mathcal{H}(x_k)$ and $\bar{H}_k \in \mathcal{H}(\bar{x})$ such that

$$\begin{cases} \|B_k - A_k\| < \frac{2c(1+2(1-\eta)^{-1}) r^\gamma}{1-\eta^\gamma}; \\ \|B_k - \bar{H}_k\| \le d(B_k, \mathcal{H}_k) + e(\mathcal{H}(x_k), \mathcal{H}(\bar{x})) < \frac{2c(1+2(1-\eta)^{-1}) r^\gamma}{1-\eta^\gamma} + L. \end{cases}$$

By (A2.2),

$$\begin{aligned} &\|f(x_k) + B_k(\bar{x} - x_k) - f(\bar{x})\| \le \\ &\qquad \le \|f(x_k) + A_k(\bar{x} - x_k) - f(\bar{x})\| + \|B_k - A_k\| \|\bar{x} - x_k\| \\ &\qquad \le \frac{3c(1 + 2(1-\eta)^{-1}) r^\gamma}{1 - \eta^\gamma} \|x_k - \bar{x}\| < \beta, \end{aligned}$$

By applying again Theorem 3.1 with $\Phi := \Phi_{\bar{H}_k} = f(\bar{x}) + \bar{H}_k(x - \bar{x}) + F(x)$:

$$\begin{cases} G(x) = G_k(x) := f(x_k) + B_k(x - x_k) - f(\bar{x}) - \bar{H}_k(x - \bar{x}); \\ \bar{z} = \bar{z}_k := G_k(\bar{x}) = f(x_k) + B_k(\bar{x} - x_k) - f(\bar{x}), \end{cases}$$

one obtains

$$\begin{aligned} d(\bar{x}, (\Phi_{\bar{H}_k} + G_k)^{-1}(0)) &\le \frac{\kappa}{1 - \kappa(\frac{2c(1+2(1-\eta)^{-1}) r^\gamma}{1-\eta^\gamma} + L)} d(0, (\Phi_{\bar{H}_k} + G_k)(\bar{x})) \\ &= \frac{\kappa}{1 - \kappa\left(\frac{2c(1+2(1-\eta)^{-1}) r^\gamma}{1-\eta^\gamma} + L\right)} \|\bar{z}_k\|. \end{aligned} \tag{62}$$

Therefore, there is $p_k \in X$ with $x_k + p_k \in (\Phi_{\bar{H}_k} + G_k)^{-1}(0)$, that is,

$$0 \in f(x_k) + B_k(p_k) + F(x_k + p_k),$$

such that

$$\|x_k + p_k - \bar{x}\| \leq \frac{2\kappa(1+\varepsilon)}{1-\kappa(\frac{2c(1+2(1-\eta)^{-1})r^\gamma}{1-\eta^\gamma}+L)}\|\bar{z}_k\|$$

$$\leq \frac{6\kappa(1+\varepsilon)c(1+2(1-\eta)^{-1})r^\gamma}{(1-\eta^\gamma)\left[1-\kappa\left(\frac{2c(1+2(1-\eta)^{-1})r^\gamma}{1-\eta^\gamma}+L\right)\right]}\|x_k-\bar{x}\| \leq \mu\|x_k-\bar{x}\|,$$

Hence, by the definition of x_{k+1}, using the same argument as in (56), one obtains $\|x_{k+1}-\bar{x}\| \leq \eta\|x_k-\bar{x}\|$. Then, $\|d_k\| \geq (1-\eta)\|x_k-\bar{x}\|$.

By virtue of (A2.1), for any $A_k \in \mathcal{H}(x_k)$, there is $A_{k+1} \in \mathcal{H}(x_{k+1})$ such that

$$\|A_{k+1}-A_k\| \leq c\|x_{k+1}-x_k\|^\gamma \leq c(\|x_{k+1}-\bar{x}\|^\gamma + \|x_k-\bar{x}\|^\gamma).$$

As

$$\left\|(B_k-A_k) - \frac{(B_k-A_k)d_k\langle d_k,\cdot\rangle}{\|d_k\|^2}\right\| \leq \|B_k-A_k\|,\ k\in\mathbb{N},$$

similar to estimation (57), we obtain

$$\begin{aligned}
&d(B_{k+1},\mathcal{H}(x_{k+1})) \leq \|B_{k+1}-A_{k+1}\| \leq \\
&\quad \leq \left\|(B_k-A_k) - \tfrac{(B_k-A_k)d_k\langle d_k,\cdot\rangle}{\|d_k\|^2}\right\| + \|A_k-A_{k+1}\| + \tfrac{\|f(x_{k+1})-f(x_k)-A_kd_k\|}{\|d_k\|} \\
&\quad \leq \|B_k-A_k\| + \|A_k-A_{k+1}\| + \tfrac{\|f(x_{k+1})-f(\bar{x})-A_{k+1}(x_{k+1}-\bar{x})\|}{\|d_k\|} + \\
&\qquad\qquad + \tfrac{\|f(x_k)-f(\bar{x})-A_k(x_k-\bar{x})\|}{\|d_k\|} + \tfrac{\|A_{k+1}-A_k\|\|x_{k+1}-\bar{x}\|}{\|d_k\|} \qquad (63) \\
&\quad \leq \|B_k-A_k\| + c(\|x_k-\bar{x}\|^\gamma + \|x_{k+1}-\bar{x}\|^\gamma) + c\eta(1-\eta)^{-1}\|x_{k+1}-\bar{x}\|^\gamma \\
&\qquad\qquad + c(1-\eta)^{-1}\|x_k-\bar{x}\|^\gamma + c\eta(1-\eta)^{-1}(\|x_k-\bar{x}\|^\gamma + \|x_{k+1}-\bar{x}\|^\gamma) \\
&\quad \leq \|B_k-A_k\| + M\left(\|x_{k+1}-\bar{x}\|^\gamma + \|x_k-\bar{x}\|^\gamma\right).
\end{aligned}$$

As $A_k \in \mathcal{H}(x_k)$ is arbitrary, we obtain

$$d(B_{k+1},\mathcal{H}(x_{k+1})) \leq d(B_k,\mathcal{H}(x_k)) + M\left(\|x_k-\bar{x}\|^\gamma + \|x_{k+1}-\bar{x}\|^\gamma\right). \qquad (64)$$

By induction, we constructed a sequence (x_k) generated by (3) satisfying

$$\|x_{k+1}-\bar{x}\| \leq \eta\|x_k-\bar{x}\|, \quad \text{for all } k\in\mathbb{N}. \qquad (65)$$

That is, (x_k) converges linearly to $\bar{x}$.

For the second part, suppose now (45) is satisfied, $B_0-\bar{H}$ is a compact operator, and for all $\bar{H} \in \mathcal{H}(\bar{x})$, $\Phi_{\bar{H}}^{-1}(0) = \{\bar{x}\}$. The later property along with the fact that $\{\Phi_{\bar{H}} : \bar{H} \in H(\bar{x})\}$ is uniformly metrically regular at $(\bar{x},0)$ imply immediately the strong metric subregularity of the multifunction $\Phi_{\mathcal{H}(\bar{x})}$ at $(\bar{x},0)$.

According to Lemma 4.2, it suffices to show that there exists a sequence (A_k) with $A_k \in \mathcal{H}(x_k)$ such that

$$\lim_{k\to\infty} \frac{\|(B_k - A_k)d_k\|}{\|d_k\|} = 0. \tag{66}$$

Pick $A_0 \in \mathcal{H}(x_0)$ with $\|A_0 - B_0\| \le c\|x_0 - \bar{x}\|^\gamma$. By the definition of the sequence (x_k) as well as assumption (A2.1), at the step k, when A_k is defined, we can find $A_{k+1} \in \mathcal{H}(x_{k+1})$ such that

$$\|A_{k+1} - A_k\| \le c\|x_k - \bar{x}\|^\gamma.$$

Since

$$\sum_{k=0}^{\infty} \|x_k - \bar{x}\|^\gamma \le \frac{\|x_0 - \bar{x}\|^\gamma}{1-\eta^\gamma} < \infty,$$

then (A_k) is a Cauchy sequence in $\mathcal{L}(X,Y)$, therefore, by the upper semicontinuity of $\mathcal{H}$ at $\bar{x}$, as well as the closedness of $\mathcal{H}(\bar{x})$, which converges to some $A \in \mathcal{H}(\bar{x})$, and one has, for all $k \in \mathbb{N}$,

$$\|A_k - A\| \le \sum_{i=0}^{\infty} \|A_{k+i} - A_{k+i+1}\| \le c\sum_{i=0}^{\infty} \|x_{k+i} - \bar{x}\|^\gamma \le \frac{c}{1-\eta^\gamma}\|x_k - \bar{x}\|^\gamma. \tag{67}$$

Hence,

$$\frac{\|f(x_{k+1})-f(x_k)-Ad_k\|}{\|d_k\|} \le \frac{\|f(x_{k+1})-f(\bar{x})-A_{k+1}(x_{k+1}-\bar{x})\|}{\|d_k\|} + \frac{\|f(x_k)-f(\bar{x})-A_k(x_k-\bar{x})\|}{\|d_k\|} +$$

$$+\frac{\|A_{k+1}-A\|\|x_{k+1}-\bar{x}\|}{\|d_k\|} + \frac{\|A_k-A\|\|x_k-\bar{x}\|}{\|d_k\|} \tag{68}$$

$$\le c(\|x_{k+1} - \bar{x}\|^\gamma + \|x_k - \bar{x}\|^\gamma + \eta(1-\eta)^{-1}(1-\eta^\gamma)^{-1}\|x_{k+1} - \bar{x}\|^\gamma$$
$$+(1-\eta)^{-1}(1-\eta^\gamma)^{-1}\|x_k - \bar{x}\|^\gamma) \le \rho\|x_k - \bar{x}\|^\gamma,$$

where,

$$\rho := c\left(1 + \eta^\gamma + \eta^{1+\gamma}(1-\eta)^{-1}(1-\eta^\gamma)^{-1} + (1-\eta)^{-1}(1-\eta^\gamma)^{-1}\right).$$

By virtue of the estimation(63), one has

$$\|B_{k+1} - A_{k+1}\| \le \|B_k - A_k\| + M\left(\|x_{k+1} - \bar{x}\|^\gamma + \|x_k - \bar{x}\|^\gamma\right) \quad k \in \mathbb{N}.$$

This implies that the sequence $(B_k - A)$ is bounded.
Relation (66) follows from the following two claims whose proofs are similar to Theorem 6.3 in [22] and Lemma 2.4 in [14], respectively.

Claim 1. *For all $y^* \in Y$, one has* $\displaystyle\lim_{k\to\infty} \frac{\langle y^*, (B_k - A)d_k\rangle}{\|d_k\|} = 0.$

Proof of Claim 1. Set $y_k = f(x_{k+1}) - f(x_k)$, for any $u \in X$, with $\|u\| = 1$, one has

$$\begin{aligned}\langle y^*, (B_{k+1} - A)u\rangle &= \langle y^*, (B_k - A)u\rangle + \tfrac{\langle d_k, u\rangle}{\|d_k\|^2}\langle y^*, y_k - B_k d_k\rangle \\ &= \left\langle y^*, (B_k - A)\left(u - \tfrac{\langle d_k, u\rangle}{\|d_k\|^2} d_k\right)\right\rangle + \tfrac{\langle d_k, u\rangle}{\|d_k\|^2}\langle y^*, y_k - A d_k\rangle.\end{aligned}$$

By setting $a_k := (B_k - A)^* y^*,\ k \in \mathbb{N}$, where $(B_k - A)^*$ denotes the adjoint operator of $B_k - A$, one has

$$\sup_{\|u\|=1}\left\langle y^*, (B_k - A)\left(u - \frac{\langle d_k, u\rangle}{\|d_k\|^2} d_k\right)\right\rangle = \sup_{\|u\|=1}\left\langle a_k, u - \frac{\langle d_k, u\rangle}{\|d_k\|^2} d_k\right\rangle =$$

$$= \sup_{\|u\|=1}\left\langle a_k - \frac{\langle a_k, d_k\rangle}{\|d_k\|^2} d_k, u\right\rangle = \left\| a_k - \frac{\langle a_k, d_k\rangle}{\|d_k\|^2} d_k\right\| = \left(\|a_k\|^2 - \frac{\langle a_k, d_k\rangle^2}{\|d_k\|^2}\right)^{1/2}.$$

By virtue of (68), one also has

$$\sup_{\|u\|=1}\frac{\langle d_k, u\rangle}{\|d_k\|^2}\langle y^*, y_k - A d_k\rangle \le \rho\|y^*\|\|x_k - \bar{x}\|^\gamma.$$

From the latter relations, we obtain

$$\begin{aligned}\|a_{k+1}\| &\le \left(\|a_k\|^2 - \frac{\langle a_k, d_k\rangle^2}{\|d_k\|^2}\right)^{1/2} + \rho\|y^*\|\|x_k - \bar{x}\|^\gamma \\ &\le \|a_k\| - \frac{\langle a_k, d_k\rangle^2}{2\|d_k\|^2\|a_k\|} + \rho\|y^*\|\|x_k - \bar{x}\|^\gamma.\end{aligned}$$

As (a_k) is bounded and $\sum_{k=0}^\infty \|x_k - \bar{x}\|^\gamma$ is finite, this relation implies that

$$\sum_{k=0}^\infty \frac{\langle a_k, d_k\rangle^2}{2\|d_k\|^2\|a_k\|} < \infty.$$

Hence, $\lim\limits_{k\to\infty} \dfrac{\langle a_k, d_k\rangle^2}{\|d_k\|^2\|a_k\|} = 0.$

By the boundedness of (a_k), one has $\lim\limits_{k\to\infty} \dfrac{\langle a_k, d_k\rangle}{\|d_k\|} = 0$, and the claim is showed.

Claim 2. $\displaystyle\bigcup_{k=0}^\infty (B_k - A)\mathbb{B}_X$ *is relatively compact in* Y.

Proof of Claim 2. Set $v_k = \dfrac{f(x_{k+1}) - f(x_k) - A d_k}{\|d_k\|},\quad k \in \mathbb{N}$. From the relations (68) and (65), one has

$$\|v_k\| \le \rho\|x_k - \bar{x}\|^\gamma \le \rho\eta^{\gamma k}\|x_0 - \bar{x}\|^\gamma, \quad \forall k \in N_0.$$

It follows that $\sum_{k=0}^{\infty} \|v_k\| < +\infty$. By the definition of Broyden's update, we have

$$B_{k+1} - A = (B_k - A) - (B_k - A)\frac{d_k\langle d_k, \cdot\rangle}{\|d_k\|^2} + \frac{v_k\langle d_k, \cdot\rangle}{\|d_k\|}.$$

Since

$$\left\| u - \frac{\langle x, u\rangle x}{\|x\|^2} \right\|^2 = \|u\|^2 - \frac{\langle x, u\rangle^2}{\|x\|^2} \leq 1 \quad \forall x \in X,\ \forall u \in \mathbb{B}_X,$$

one has $(B_{k+1} - A)\mathbb{B}_X \subseteq (B_k - A)\mathbb{B}_X + [-1, 1]v_k$ for all $k \in \mathbb{N}$. Therefore,

$$\bigcup_{k=0}^{\infty}(B_k - A)\mathbb{B}_X \subseteq (B_0 - A)\mathbb{B}_X + \sum_{k=0}^{\infty}[-1, 1]v_k.$$

Since $(B_0 - A)$ is relatively compact, it suffices to show that the set $C := \sum_{k=0}^{\infty}[-1, 1]v_k$ is compact. Indeed, let (u_n) be a sequence of elements in C with

$$u_n = \sum_{k=0}^{\infty} \alpha_{k,n} v_k, \quad n \in \mathbb{N},$$

where, $\alpha_{k,n} \in [-1, 1]$ for all $k, n \in N_0$. For each $k \in \mathbb{N}$, let us pick a subsequence $(\alpha_{k,n})_{n\in I_k}$ (I_k is an infinite subset of $\mathbb{N}$) of $(\alpha_{k,n})_{n\in\mathbb{N}}$ which converges to some $\alpha_k \in [-1, 1]$. Of course, we can take the index sets I_k such that $I_{k+1} \subseteq I_k$ for all $k \in \mathbb{N}$. Denote by n_k the k-th element in I_k, $k \in \mathbb{N}$, we show that the subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of (u_n) converges to $u = \sum_{k=0}^{\infty} \alpha_k v_k \in C$.

Let $\varepsilon > 0$ be given arbitrarily. Since $\sum_{k=0}^{\infty} \|v_k\| < +\infty$, we can find an index $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0+1} \|v_k\| < \varepsilon.$$

Since $(\alpha_{k,n})_{n\in I_k} \to \alpha_k$, for each $k \in \mathbb{N}$, there exists an index $k_1 \geq k_0$ such that

$$|\alpha_{k,n} - \alpha_k| < \varepsilon \quad \forall k = 0, ..., k_0,\ \forall n \in I_k,\ n \geq k_1.$$

Hence, for $k \geq k_1$, one has

$$\begin{aligned}\|u_{n_k} - u\| &\leq \sum_{j=0}^{k_0} |\alpha_{j,n_k} - \alpha_j|\|v_j\| + \sum_{j=k_0+1}^{\infty} |\alpha_{j,n_k}|\|v_j\| + \sum_{j=k_0+1}^{\infty} |\alpha_j|\|v_j\| \\ &\leq \varepsilon \sum_{j=0}^{\infty} \|v_j\| + 2\varepsilon.\end{aligned}$$

This shows that $(u_{n_k})_{k\in\mathbb{N}}$ converges to u.

Let us return to the proof of the theorem, by Claim 1, the sequence $((B_k - A)(d_k/\|d_k\|))$ converges weakly to 0. On the other hand, every subsequence of this sequence has a strongly convergent subsequence. Thus, the sequence $((B_k - A)(d_k/\|d_k\|))$ converges strongly to 0. Finally, as the sequence (A_k) converges strongly to A, (66) holds and the proof is completed. □

Remark 4.4. In [2], the authors have established the linear/superlinear convergence of the quasi-Newton method with the Broyden update for generalized equations in Hilbert spaces when the function f is assumed to be Fréchet differentiable on a neighborhood of $\bar{x}$ such that Df is Lipschitz continuous around $\bar{x}$ with respect to the Hilbert-Schmidt norm and $B_0 - Df(\bar{x})$ is a Hilbert-Schmidt operator. As mentioned in the preceeding remark, if f is Fréchet differentiable at $\bar{x}$ such that

$$\|f(x) - f(\bar{x}) - Df(\bar{x})(x - \bar{x})\| \leq c\|x - \bar{x}\|^{1+\gamma} \quad \text{for all } x \in U,$$

for some $c > 0$, $\gamma \in (0, 1]$, a neighborhood U of $\bar{x}$, then assumptions (A1), (A2) are satisfied for $\mathcal{H}(x) = \{Df(\bar{x})\}$ for all $x \in X$, and moreover, in (41), $r_x = +\infty$. Since a Hilbert-Schmidt operator is also a compact operator, Theorem 4.3 covers with weaker assumptions the result established in ([2], Theorem 4.9).

The following example gives a semismooth equation for which, the modified Broyden method is q-superlinearly convergent, although the standard Broyen method is not even q-linearly convergent.

Example 4.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x - x^2 & \text{if } \quad x \geq 0, \\ -x + x^2 & \text{if } \quad x < 0. \end{cases}$$

We have,

$$\mathcal{H}(x) := \partial_B f(x) = \begin{cases} \{1 - 2x\} & \text{if } \quad x \geq 0, \\ \{1, -1\} & \text{if } \quad x = 0, \\ \{-1 + 2x\} & \text{if } \quad x < 0. \end{cases}$$

Consider the equation $f(x) = 0$ with a solution $\bar{x} = 0$. Obviously, f is H-semismooth at 0; the collection of functions

$$\{\Phi_{\bar{H}} : \ \bar{H} \in \mathcal{H}(0)\} = \{x, -x\},$$

is uniformly strongly metrically regular at $(0, 0)$, and the assumptions (A1), (A2) are fulfilled. Moreover, by checking directly, assumption (A2) is verified with $c = 2$, $\gamma = 1$ and $r_x = |x|$ for all $x \in \mathbb{R} \setminus \{0\}$. Therefore, the condition (45) is satisfied, and the modified Broyden update method (MBA) applied to the equation $f(x) = 0$ is locally q-superlinearly convergent to the solution 0. Despite of this, all iterative sequences generated by the standard Broyden update (SBA, shortly) do not q-linearly convergent to 0. Indeed, assume to the contrary that there is a iterative sequence (x_k) with $x_0 \neq 0$, generated by (SBA) converging q-linearly to 0, that is, there are $\eta \in (0,1)$ and $k_0 \in \mathbb{N}$ such that

$$\|x_{k+1}\| \leq \eta \|x_k\| \quad \text{for all } k \geq k_0.$$

Obviously, $x_k \neq 0$ for all $k \in \mathbb{N}$. For $k \geq k_0$, consider the case $x_k > 0$ (the case $x_k < 0$ is similar). One has

$$\begin{aligned} B_{k+1} &= B_k + \frac{(f(x_{k+1}) - f(x_k) - B_k(x_{k+1} - x_k))(x_{k+1} - x_k)}{|x_{k+1} - x_k|^2} \\ &= \frac{f(x_k) - f(x_{k+1})}{x_k - x_{k+1}}. \end{aligned}$$

If $x_{k+1} < 0$, then

$$\begin{aligned} B_{k+1} &= \frac{x_k - x_k^2 + x_{k+1} - x_{k+1}^2}{x_k - x_{k+1}} \geq \frac{x_k - x_k^2 - \eta x_k - \eta^2 x_k^2}{(1+\eta)x_k} \\ &= \frac{1 - \eta - (1+\eta^2)x_k}{1+\eta}. \end{aligned}$$

From

$$f(x_{k+1}) + B_{k+1}(x_{k+2} - x_{k+1}) = 0, \tag{69}$$

we derive

$$\frac{x_{k+2}}{x_{k+1}} = \frac{1 + B_{k+1}}{B_{k+1}} - \frac{x_{k+1}}{B_{k+1}},$$

which shows that $|x_{k+2}| > |x_{k+1}|$ when k is sufficiently large, a contradiction.

If $x_{k+1} > 0$, one has

$$B_{k+1} = \frac{x_k - x_k^2 - x_{k+1} + x_{k+1}^2}{x_k - x_{k+1}} = 1 - x_k - x_{k+1}.$$

Therefore, by (69),

$$x_{k+1} - x_{k+1}^2 + (1 - x_k - x_{k+1})(x_{k+2} - x_{k+1}) = 0,$$

which implies that $x_{k+2} = \frac{-x_k x_{k+1}}{1 - x_k - x_{k+1}} < 0$, when k is sufficiently large.
By repeating the preceeding arguments for x_{k+2} instead of x_{k+1}, we derive a contradiction that $|x_{k+3}| > |x_{k+2}|$.

5. Concluding remarks

In this contribution, we derive a general version of the Dennis-Moré theorem for generalized equations associated to semismooth functions. This result allows us to obtain a new superlinear convergence results for generalized quasi-Newton type methods, especially for the Broyden update method. The results established in the paper furnish a methodology for considering quasi-Newton type methods for solving generalized equations by using tools from modern variational analysis. As pointed by one of the reviewers, it would be interesting to study, from a computational point of view, all the numerical methods proposed in this paper. This is out of the scope of this manuscript and will be the subject of a futur research project.

Acknowledgement. The authors would like to thank the reviewers for their careful reading and for their insightful comments, which helped us to improve the initial version of our manuscript.

References

[1] S. Adly, R. Cibulka, H. V. Ngai: Newton's method for solving inclusions using set-valued approximations, SIAM J. Optim. 25 (2015) 159–184.

[2] F. J. Aragón Artacho, A. Belyakov, A. L. Dontchev, M. López: Local convergence of quasi-Newton methods under metric regularity, Comp. Optim. Appl. 58(1) (2014) 225–247.

[3] C. G. Broyden: A class of methods for solving nonlinear simultaneous equations, Math. Comp. 19 (1965) 577–593.

[4] C. G. Broyden, J. E. Dennis, J. J. Moré: On the local and superlinear convergence of quasi-Newton methods, IMA J. 12 (1973) 223–246.

[5] R. Cibulka, A. L. Dontchev, M. H. Geoffroy: Inexact Newton methods and Dennis-Moré theorems for nonsmooth generalized equations, SIAM J. Control and Optim. 53(2) (2015) 1003–1019.

[6] J. E. Dennis, J. J. Moré: A characterization of superlinear convergence and its application to quasi-Newton methods, Math. Comput. 28 (1974) 549–560.

[7] A. L. Dontchev: Local analysis of Newton-type method based on linearization, in: Letters in Applied Mathematics 32, Amer. Math. Soc., Providence (1996) 295–306.

[8] A. L. Dontchev: Generalizations of the Dennis-More theorem, SIAM J. Optim. 22 (2012) 821–830.

[9] A. L. Dontchev, R. T. Rockafellar: Implicit Functions and Solution Mappings. A View from Variational Analysis, Springer Monographs in Mathematics, Springer, New York (2014).

[10] F. Facchinei, J.-S. Pang: Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. II, Springer Series in Operations Research, Springer, New York (2003).

[11] A. Griewank: Broyden updating, the good and the bad! Optimization stories, Doc. Math. (2012) 301–315.

[12] A. D. Ioffe: Nonsmooth analysis: differential calculus of nondifferentiable mappings, Trans. Amer. Math. Soc. 266 (1981) 1–56.

[13] N. H. Josephy: Newton's method for generalized equations, Technical Summary Report no. 1965, Mathematics Research Center, University of Wisconsin, Madison (1979).

[14] C. T. Kelly, E. Sachs: A new proof of superlinear convergence for Broyden's method in Hilbert space, SIAM J. Optim. 1(1) (1991) 146–150.

[15] D. Klatte, B. Kummer: Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications, Kluwer Academic Publishers, Dordrecht (2002).

[16] A. S. Lewis, M. L. Overton: Nonsmooth optimization via quasi-Newton methods, Math. Program., Ser. A 141 (2013) 135–163.

[17] H. V. Ngai, M. Théra: Error bounds in metric spaces and application to the perturbation stability of metric regularity, SIAM J. Optim. 19 (2008) 1–20.

[18] M. Ortega, W. C. Rheinboldt: Iterative Solution of Nonlinear Equations in Several Variables, Classics in Applied Mathematics 30, SIAM, Philadelphia (2000).

[19] J.-P. Penot: Calculus without derivatives, Graduate Texts in Mathematics 266, Springer, New York (2013).

[20] S. M. Robinson: Strongly regular generalized equations, Math. Oper. Res. 5 (1980) 43–62.

[21] E. W. Sachs: Broyden's method in Hilbert spaces, Math. Program. 35 (1986) 71–82.

[22] E. Sachs: Convergence rates of quasi-Newton algorithms for some nonsmooth optimization problems, SIAM J. Control Optim. 23(3) (1985) 401–418.

[23] H. Xu: Set-valued approximations and Newton's methods, Math. Program., Ser. A 84 (1999) 401–420.