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Quasi-Nonexpansive Mappings Involving Pseudomonotone Bifunctions on Convex Sets

Tran Viet Anh
Department of Scientific Fundamentals,
Posts and Telecommunications Institute of Technology, Hanoi, Vietnam
tranvietanh@outlook.com

Le Dung Muu
TIMAS, Thang Long University; and: Institute of Mathematics, VAST, Hanoi
ldmuu@math.ac.vn

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We present new quasi-nonexpansive mappings involving pseudomonotone bifunctions defined on convex sets in real Hilbert space. We investigate some properties concerning their fixed point sets. Some applications to equilibrium problems are discussed.

Keywords: Convexity, pseudomonotonicity, quasi-nonexpansive mapping, fixed point, equilibria.

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1. Introduction and motivation

Let $\mathcal{H}$ be a real Hilbert space with inner product and its reduced norm $\|\cdot\|$, and C be a nonempty closed convex set in $\mathcal{H}$. A mapping $T\colon C \to C$ is said to be *nonexpansive* on C, if $\|T(x) - T(y)\| \leq \|x - y\|$ for every $x, y \in C$. It is said to be *quasi-nonexpansive* [1] on C if $\|T(x) - p\| \leq \|x - p\|$ for every $x \in C$ and every fixed point p of T whenever T admits a fixed point. It is clear that every nonexpansive mapping (operator) having a fixed point is quasi-nonexpansive, but the converse is not true. Quasi-nonexpansive mappings have been studied by some authors and some solution methods for finding a fixed point of quasi-nonexpansive mappings have been developed in the literature (see e.g. the monograph [4] and the papers [1, 6, 8, 13, 16]).

The approach based on the fixed point of nonexpansive and generalized nonexpansive mappings are commonly used in various fields of mathematics. In [5] Combettes et al. have introduced a nonexpansive mapping P_f, called the

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proximal operator, by taking for each $x \in \mathcal{H}$

$$P_f(x) := \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ \forall y \in C\}$$

where $r > 0$ and $f : C \times C \to \mathbb{R}$. Under the assumptions that f is monotone, lower semicontinuous, convex on C with respect to the second argument and hemicontinuous on C with respect to the first argument, they have proved, among others, that P_f is single valued, nonexpansive on C, and that the fixed point set of P_f coincides the solution set of the equilibrium problem

EP(C,f) find $x^* \in C : f(x^*, y) \geq 0 \ \ \forall y \in C.$

It is worth mentioning that Problem *EP(C,f)*, although it has a simple formulation, contains a lot number of problems such as optimization, variational inequality, the Kakutani fixed point, Nash equilibrium models as special cases (see e.g. [2, 3, 14]). The proximal operator P_f becomes a powerful tool for the monotone variational inequality, equilibrium problem and related topics. However, for pseudomonotone bifunctions, this operator, in general, is not nonexpansive, even its values may not be convex. Motivated by this fact, in this paper, based upon the extragradient method introduced by Korpelevich [11] to the saddle point problem, we define mappings for pseudomonotone bifunctions with and without a Lipschitz-type condition. We show that these mappings are quasi-nonexpansive, demi-closed and that their fixed point sets coincide with the solution set of the equilibrium problem defined by the bifunction. This result thus allows that the existing results for quasi-nonxpansive mappings can be applied to pseudomonotone equilibrium problems and their special cases such as variational inequalities, the Nash equilibira, optimization and other related problems.

The remaining part of the paper is organized as follows. The next section contains some preliminaries. In the third section we define a quasi-nonexpansive mapping for a bifunction satisfying a Lipschitz-type condition, and investigate some properties concerning its fixed point set. The last section is devoted to the case when the bifunction is not required to be Lipschiz-type.

2. Preliminaries

In what follows, for a real Hilbert space $\mathcal{H}$, by $x_n \to x$ and $x_n \rightharpoonup x$, we denote the *strong and weak convergence of a sequence* $\{x_n\}$ to x in $\mathcal{H}$, respectively. For a mapping $T : C \to C$, by $\mathrm{Fix}(T)$, we denote the *set of fixed points* of T, i.e., $\mathrm{Fix}(T) = \{x \in C : T(x) = x\}$. The mapping $T : C \to C$ is called *demi-closed at zero* if, for every sequence $\{x_n\}$ contained in C, one has

$$x_n \rightharpoonup x \ \text{ and } \ \|T(x_n) - x_n\| \to 0 \implies x \in \mathrm{Fix}(T).$$

The following well-known definitions are widely used (see e.g. [5, 12]).

Definition 2.1. A bifunction $f\colon \mathcal{H}\times\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ is said to be

(a) *β-strongly monotone* on C if there exists $\beta>0$ such that

$$f(x,y)+f(y,x)\le -\beta\|x-y\|^2 \quad \forall x,y\in C;$$

(b) *monotone* on C if

$$f(x,y)+f(y,x)\le 0 \quad \forall x,y\in C;$$

(c) *pseudomonotone* on C

$$f(x,y)\ge 0 \Longrightarrow f(y,x)\le 0 \quad \forall x,y\in C;$$

(d) *Lipschitz-type* on C with constants $c_1>0$ and $c_2>0$, if

$$f(x,y)+f(y,z)\ge f(x,z)-c_1\|x-y\|^2-c_2\|y-z\|^2 \quad \forall x,y,z\in C.$$

The concepts of monotonicity and Lipschitz-type for bifunctions are generalizations of the following definitions for operators:

Definition 2.2. A mapping $F\colon C\to\mathcal{H}$ is said to be

(a) *β-strongly monotone* on C if there exists $\beta>0$ such that

$$\langle F(x)-F(y),x-y\rangle\ge \beta\|x-y\|^2 \quad \forall x,y\in C;$$

(b) *monotone* on C if

$$\langle F(x)-F(y),x-y\rangle\ge 0 \quad \forall x,y\in C;$$

d) *pseudomonotone* on C if

$$\langle F(x),y-x\rangle\ge 0 \implies \langle F(y),y-x\rangle\ge 0 \quad \forall x,y\in C;$$

(e) *L-Lipschitz continuous* on C if there exists $L>0$ such that

$$\|F(x)-F(y)\|\le L\|x-y\| \quad \forall x,y\in C.$$

It is easy to see that the bifunction $f(x,y):=\langle F(x),y-x\rangle$ is β-strongly monotone (resp. monotone, pseudomonotone) on C if and only if F is β- strongly monotone (resp. monotone, pseudomonotone) on C.

It has been proved [17] that if F is L-Lipschitz continuous on C, then $f(x,y):=\langle F(x),y-x\rangle$ is Lipschitz-type on C with constants $c_1=c_2=L/2$. Note that if $f(x,y)=\varphi(y)-\varphi(x)$, then it is Lipschitz-type for any function φ.

3. A quasi-nonexpansive mapping and its properties

In what follows we suppose that $f\colon \mathcal{H}\times\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ and that $C\times C\subseteq \mathrm{dom}f := \{(x,y) : f(x,y) < +\infty\}$. Let $\mathrm{Sol}(C,f)$ denote the solution set of the equilibrium problem *EP(C,f)*. Let us define the mapping T_f from C to itself by taking, for every $x\in C$,

$$T_f(x) := \mathrm{argmin}\Big\{\lambda f(s(x),y)+\frac{1}{2}\|y-x\|^2 : y\in C\Big\},$$

where $\lambda > 0$ fixed and

$$s(x) := \mathrm{argmin}\Big\{\lambda f(x,y)+\frac{1}{2}\|y-x\|^2 : y\in C\Big\}.$$

Proposition 3.1. *Suppose that*

(A_0) *Either int$C\neq\emptyset$ or, for each $x\in C$, the function $f(x,.)$ is continuous at a point of C;*

(A_1) *$f(x,\cdot)$ is lower semicontinuous, convex on $\mathcal{H}$, subdifferentiable on C and $f(x,x)=0$ for all $x\in C$;*

(A_2) *f is pseudomonotone on C;*

(A_3) *f is Lipschitz-type on C with constants $c_1>0$ and $c_2>0$.*

Then the mapping T_f is well defined and, for every $x^*\in\mathrm{Sol}(C,f)$, one has

$$\|T_f(x)-x^*\|^2\le\|x-x^*\|^2-(1-2\lambda c_1)\|x-s(x)\|^2-(1-2\lambda c_2)\|s(x)-T_f(x)\|^2.$$

Proof. First, note that since $f(x,\cdot)$ is lower semicontinuous, convex on the closed convex set C, both problems defining $T_f(x)$ and $s(x)$ are uniquely solvable. It is obvious that

$$\begin{aligned} s(x) &= \underset{y\in C}{\mathrm{argmin}}\Big\{\lambda f(x,y)+\frac{1}{2}\|y-x\|^2\Big\}\\ &= \underset{y\in\mathcal{H}}{\mathrm{argmin}}\Big\{\hat f(x,y) := \lambda f(x,y)+\frac{1}{2}\|y-x\|^2+\delta_C(y)\Big\}, \end{aligned}$$

where δ_C stands for the indicator function of C. Then $s(x)$ solves the latter unconstrained optimization problem if and only if $0\in\partial_2\hat f(x,s(x))$, where $\partial_2\hat f(x,s(x))$ denotes the subdifferential of the convex function $\hat f(x,.)$ at $s(x)$. Since $C\subseteq\mathrm{dom}f(x,.)$, using Assumption ($A_0$), we can apply the Moreau-Rockafellar theorem [7, p. 47], to obtain

$$0\in\partial_2\Big(\lambda f(x,y)+\frac{1}{2}\|y-x\|^2\Big)(s(x))+N_C(s(x)),$$

where $N_C(s(x))$ is the (outward) normal cone of C at $s(x)\in C$. Thus, there exist $w\in\partial_2 f(x,s(x))$ and $\overline{w}\in N_C(s(x))$ such that

$$0=\lambda w+s(x)-x+\overline{w}.$$

Since $\overline{w} \in N_C(s(x))$, we have $\langle \overline{w}, y - s(x)\rangle \leq 0$ for all $y \in C$. Note that $\overline{w} = x - \lambda w - s(x)$ we can write

$$\lambda\langle w, y - s(x)\rangle \geq \langle x - s(x), y - s(x)\rangle \ \ \forall y \in C. \tag{1}$$

Since $w \in \partial_2 f(x, s(x))$, we have

$$f(x, y) - f(x, s(x)) \geq \langle w, y - s(x)\rangle \ \ \forall y \in C,$$

which together with (1) implies

$$\lambda[f(x, y) - f(x, s(x))] \geq \langle x - s(x), y - s(x)\rangle \ \ \forall y \in C. \tag{2}$$

Substituting $y = T_f(x) \in C$ in (2) we obtain

$$\lambda[f(x, T_f(x)) - f(x, s(x))] \geq \langle x - s(x), T_f(x) - s(x)\rangle. \tag{3}$$

Similarly, from

$$T_f(x) = \operatorname{argmin}\Big\{\lambda f(s(x), y) + \frac{1}{2}\|y - x\|^2 : y \in C\Big\},$$

we can show that

$$\lambda[f(s(x), y) - f(s(x), T_f(x))] \geq \langle x - T_f(x), y - T_f(x)\rangle \ \ \forall y \in C. \tag{4}$$

Since $x^* \in \mathrm{Sol}(C, f)$ and f is pseudomonotone on C, we have $f(s(x), x^*) \leq 0$. Then substituting $y = x^* \in C$ into (4), we obtain

$$\begin{aligned}\langle T_f(x) - x, x^* - T_f(x)\rangle &\geq \lambda[f(s(x), T_f(x)) - f(s(x), x^*)] \\ &\geq \lambda f(s(x), T_f(x)).\end{aligned} \tag{5}$$

Now applying the Lipschitz-type condition of f on C with constants $c_1 > 0$ and $c_2 > 0$ and (3), we get

$$\begin{aligned}&\lambda f(s(x), T_f(x)) \geq \\ &\quad \geq \lambda[f(x, T_f(x)) - f(x, s(x)) - c_1\|x - s(x)\|^2 - c_2\|s(x) - T_f(x)\|^2] \\ &\quad \geq \langle x - s(x), T_f(x) - s(x)\rangle - \lambda c_1\|x - s(x)\|^2 - \lambda c_2\|s(x) - T_f(x)\|^2,\end{aligned}$$

which together with (5) implies

$$\begin{aligned}&\langle T_f(x) - x, x^* - T_f(x)\rangle \geq \\ &\quad \geq \langle x - s(x), T_f(x) - s(x)\rangle - \lambda c_1\|x - s(x)\|^2 - \lambda c_2\|s(x) - T_f(x)\|^2.\end{aligned} \tag{6}$$

Taking into account (6) and

$$\langle T_f(x) - x, x^* - T_f(x)\rangle = \frac{1}{2}[\|x^* - x\|^2 - \|T_f(x) - x\|^2 - \|x^* - T_f(x)\|^2],$$

$$\langle x - s(x), T_f(x) - s(x)\rangle = \frac{1}{2}[\|x - s(x)\|^2 + \|T_f(x) - s(x)\|^2 - \|T_f(x) - x\|^2],$$

we deduce that

$$\|T_f(x) - x^*\|^2 \leq \|x - x^*\|^2 - (1 - 2\lambda c_1)\|x - s(x)\|^2 - (1 - 2\lambda c_2)\|s(x) - T_f(x)\|^2,$$

which completes the proof. □

Remark 3.1 Assumptions (A_1) is commonly used for equilibrium problems, see e.g., the survey paper [2]. In the variational inequality case, where $f(x, y) := \langle F(x), y - x\rangle$, this assumption is automatically satisfied. The Lipschitz-type assumption (A3) was first used in [12] and then in some papers, see e.g., [14]. Note that if $f(x, y) := \langle F(x), y - x\rangle$, then f is Lipschitz-type on C if and only if so the operator F is.

If $\mathcal{H}$ is finite dimensional, Assumption (A_0) can be replaced by

$$\mathrm{ri}C \cap \mathrm{ri}(\mathrm{dom}f(x, .)) \neq \emptyset,$$

see e.g. [18, Theorem 23.8]. □

Proposition 3.2. *Under the assumptions* $(A_0) - (A_3)$ *of Proposition* 3.1*, for* $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$*, the fixed point set of* T_f *coincides with the solution set of the equilibrium problem*

EP(C,f) *find* $x^* \in C : f(x^*, y) \geq 0 \quad \forall y \in C,$

provided the solution set $\mathrm{Sol}(C, f)$ *of EP(C,f) is nonempty.*

Proof. Suppose that $u^* \in \mathrm{Sol}(C, f)$ then $f(u^*, y) \geq 0$ for all $y \in C$. From $f(x, x) = 0$ for every $x \in C$, it follows that

$$s(u^*) = \mathrm{argmin}\left\{\lambda f(u^*, y) + \frac{1}{2}\|y - u^*\|^2 : y \in C\right\} = u^*,$$

$$\begin{aligned} T_f(u^*) &= \mathrm{argmin}\left\{\lambda f(s(u^*), y) + \frac{1}{2}\|y - u^*\|^2 : y \in C\right\} \\ &= \mathrm{argmin}\left\{\lambda f(u^*, y) + \frac{1}{2}\|y - u^*\|^2 : y \in C\right\} = u^*, \end{aligned}$$

which implies $u^* \in \mathrm{Fix}(T_f)$. Hence $\mathrm{Sol}(C, f) \subseteq \mathrm{Fix}(T_f)$.

Conversely, suppose that $u^* \in \mathrm{Fix}(T_f)$. Then $T_f(u^*) = u^*$. From Proposition 3.1, for $x^* \in \mathrm{Sol}(C, f)$, we have

$$\|u^* - x^*\|^2 \leq \|u^* - x^*\|^2 - (1 - 2\lambda c_1)\|u^* - s(u^*)\|^2 - (1 - 2\lambda c_2)\|s(u^*) - u^*\|^2.$$

Since $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, it follows from the last inequality that $s(u^*) = u^*$. Thus, from (2) with $x = u^*$, we have $f(u^*, y) \geq 0$ for all $y \in C$, i.e., $u^* \in \mathrm{Sol}(C, f)$. Hence $\mathrm{Fix}(T_f) \subseteq \mathrm{Sol}(C, f)$. Combining with $\mathrm{Sol}(C, f) \subseteq \mathrm{Fix}(T_f)$, we obtain $\mathrm{Fix}(T_f) = \mathrm{Sol}(C, f)$. □

Proposition 3.3. *Suppose the bifunction f satisfies the conditions $(A_1)-(A_3)$ of Proposition* 3.1 *and, additionally, the following conditions:*

(A_4) *f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.*

(A_5) *The solution set* $\mathrm{Sol}(C, f)$ *of the equilibirum problem $EP(C,\, f)$ is nonempty.*

Then the mapping T_f is quasi-nonexpansive on C with $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and demi-closed at zero.

Proof. By Proposition 3.2, we have $\mathrm{Fix}(T_f) = \mathrm{Sol}(C, f)$, which together with Proposition 3.1 and the fact that $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ imply

$$\|T_f(x) - x^*\| \leq \|x - x^*\| \ \ \forall x \in C, \forall x^* \in \mathrm{Fix}(T_f).$$

Thus the mapping T_f is quasi-nonexpansive on C.

To prove demiclosedness we suppose that sequence $\{x_n\} \subset C$, $x_n \rightharpoonup x$ and $\|T_f(x_n) - x_n\| \to 0$. We will show that $x \in \mathrm{Fix}(T_f)$. First, we see that since $\{x_n\} \subset C$, $x_n \rightharpoonup x$ and C is closed and convex, it is also weakly closed, and thus $x \in C$. Since $x_n \rightharpoonup x$, the sequence $\{x_n\}$ is bounded. Since T_f is quasi-nonexpansiveness, $\{T_f(x_n)\}$ is bounded. Since $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, from Proposition 3.1, with $x^* \in \mathrm{Sol}(C, f)$, we have

$$\begin{aligned}(1 - 2\lambda c_1)\|x_n - s(x_n)\|^2 &\leq \|x_n - x^*\|^2 - \|T_f(x_n) - x^*\|^2 \\ &\leq (\|x_n - x^*\| + \|T_f(x_n) - x^*\|)\|x_n - T_f(x_n)\|.\end{aligned}$$

Using the above inequality, the boundedness of $\{x_n\}$, $\{T_f(x_n)\}$ and $\|T_f(x_n) - x_n\| \to 0$, we have

$$\lim_{n\to\infty} \|x_n - s(x_n)\| = 0. \tag{7}$$

From (2), it follows that

$$\lambda[f(x_n, y) - f(x_n, s(x_n))] \geq \langle x_n - s(x_n), y - s(x_n)\rangle \ \ \forall y \in C. \tag{8}$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - s(x_n)\| = 0$, the sequence $\{s(x_n)\}$ is bounded too. Applying the Cauchy-Schwarz inequality and using the boundedness of $\{s(x_n)\}$ and (7), we get $\lim_{n\to\infty} \langle x_n - s(x_n), y - s(x_n)\rangle = 0$. From $x_n \rightharpoonup x$ and (7), we have $s(x_n) \rightharpoonup x$. Now, using (8), the jointly weak continuity of f and the weak convergence of the two sequences $\{x_n\}$, $\{s(x_n)\}$ to x and $\lim_{n\to\infty} \langle x_n - s(x_n), y - s(x_n)\rangle = 0$, we obtain $f(x, y) \geq 0$ $\forall y \in C$, i.e. $x \in \mathrm{Sol}(C, f)$. So $x \in \mathrm{Fix}(T_f)$. Thus T_f is demi-closed at zero. □

Conditions for which $\mathrm{Sol}(C,f) \neq \emptyset$, thereby $\mathrm{Fix}(T_f) \neq \emptyset$, can be found, for example, in [2, 3].

An important special case of T_f occurs in variational inequality problems. In this case $f(x,y) := \langle F(x), y-x\rangle$, where F is a mapping from C to $\mathcal{H}$. The mapping T_f then takes the form

$$T_f(x) = P_C(x - \lambda F(P_C(x - \lambda F(x)))),$$

where P_C stands for the metric projection mapping onto C. It is well known [17] that if F is pseudomonotone and Lipschitz continuous on C, then the bifunction $f(x,y) := \langle F(x), y-x)\rangle$ is Lipschitz-type on C. In this case Problem $EP(C,\ f)$ is reduced to the variational inequality problem

$$\textit{VIP(C, F)} \qquad\qquad \text{find} \quad x^* \in C : \langle F(x^*), y - x^*)\rangle \geq 0 \quad \forall y \in C.$$

4. Avoiding the Lipschitz-type condition

So far we have supposed that f is Lipschitz-type which may not always be satisfied. In this section, without the Lipschitz-type condition for the bifunction f, we define a quasi-nonexpansive mapping as follows:
For a fixed number $\eta \in (0,1)$, define the mapping $L^f_\eta \colon C \to C$ by taking, for each $x \in C$,

$$L^f_\eta(x) := \begin{cases} x \text{ if } x = y, \\ P_C(x - \sigma g), \text{ otherwise,} \end{cases} \tag{9}$$

where

$$y := \operatorname{argmin}\left\{ f(x,t) + \frac{1}{2}\|t-x\|^2 : t \in C\right\},$$

$g \in \partial_2 f(z,x)$, $\sigma := \dfrac{f(z,x)}{\|g\|^2}$, with $z := (1-\eta^m)x + \eta^m y$, and m being the smallest nonnegative integer number k satisfying

$$f((1-\eta^k)x + \eta^k y, x) - f((1-\eta^k)x + \eta^k y, y) \geq \frac{\|x-y\|^2}{2}.$$

We need the following lemma for proving the main results of this section.

Lemma 4.1. ([19] Proposition 4.3) *Let $\Delta \subseteq \mathcal{H}$ be an open convex set containing C and $f\colon \Delta\times\Delta \to \mathbb{R}$ be a bifunction satisfying conditions (A_1) on C and (A_4) on $\Delta\times\Delta$. Let $\overline{x}, \overline{z} \in \Delta$ and $\{x_n\}, \{z_n\}$ be two sequences in Δ converging weakly to $\overline{x}, \overline{z}$, respectively. Then, for any $\varepsilon > 0$, there exist $\gamma > 0$ and $n_\varepsilon \in \mathbb{N}$ such that*

$$\partial_2 f(z_n, x_n) \subset \partial_2 f(\overline{z}, \overline{x}) + \frac{\varepsilon}{\gamma}B,$$

for every $n \geq n_\varepsilon$, where B denotes the closed unit ball in $\mathcal{H}$.

Proposition 4.2. *Suppose that $f\colon \Delta\times\Delta\to\mathbb{R}$ be a bifunction satisfying conditions (A_1), (A_2) on C, (A_4) on $\Delta\times\Delta$ and (A_5), where $\Delta\subseteq\mathcal{H}$ is an open convex set containing C. Then the mapping L^f_η in (9) is well defined and the followings hold true:*

(i) *If $y\neq x$ and $x^*\in \mathrm{Sol}(C,f)$, then* $\|L^f_\eta(x)-x^*\|<\|x-x^*\|$;

(ii) *For all $x\in C$ and $x^*\in\mathrm{Sol}(C,f)$ one has* $\|L^f_\eta(x)-x^*\|\leq\|x-x^*\|$;

(iii) $\mathrm{Fix}(L^f_\eta)=\mathrm{Sol}(C,f)$;

(iv) *L^f_η is demi-closed at zero, that is, for every sequence $\{x_n\}$ contained in C, one has*

$$x_n\rightharpoonup\overline{x}\quad\textit{and}\quad \|L^f_\eta(x_n)-x_n\|\to 0\implies \overline{x}\in\mathrm{Fix}(L^f_\eta).$$

Proof. (i): We first show that the number m is finite. Suppose by contradiction that, for every nonnegative integer m and $z_m=(1-\eta^m)x+\eta^m y$ one has

$$f(z_m,x)-f(z_m,y)<\frac{\|x-y\|^2}{2}.$$

Taking the limit as $m\to\infty$, from $z_m\to x$ as $m\to\infty$, it follows that

$$-f(x,y)\leq\frac{\|x-y\|^2}{2}. \tag{10}$$

Since $y=\operatorname{argmin}\left\{f(x,t)+\frac{1}{2}\|t-x\|^2 : t\in C\right\}$, from (2), we have

$$f(x,t)-f(x,y)\geq\langle x-y,t-y\rangle\ \ \forall t\in C. \tag{11}$$

Choose $t=x\in C$, we get $-f(x,y)\geq\|x-y\|^2$.

Combining with (10) yields $\|x-y\|^2\leq\dfrac{\|x-y\|^2}{2}$, which is in contradiction to the fact that $x\neq y$.

Since f is convex with respect to the second argument, we have

$$(1-\eta^m)f(z,x)+\eta^m f(z,y)\geq f(z,(1-\eta^m)x+\eta^m y)=f(z,z)=0,$$

which implies

$$f(z,x)\geq\eta^m(f(z,x)-f(z,y))\geq\frac{\eta^m\|x-y\|^2}{2}>0. \tag{12}$$

If $0\in\partial_2 f(z,x)$ then $f(z,t)-f(z,x)\geq 0$ for all $t\in C$. Choose $t=z\in C$ to get $f(z,x)\leq 0$, which contradicts (12).

Thus $0\notin\partial_2 f(z,x)$. Then, it follows from $g\in\partial_2 f(z,x)$ and $f(z,x)>0$ that

$$g\neq 0,\quad \sigma=\frac{f(z,x)}{\|g\|^2}>0. \tag{13}$$

Let $x^* \in \mathrm{Sol}(C, f)$ and $z \in C$, then $f(x^*, z) \geq 0$. Using the pseudomonotonicity of f, we obtain

$$f(z, x^*) \leq 0. \tag{14}$$

It follows from $g \in \partial_2 f(z, x)$ that $f(z,t) - f(z,x) \geq \langle g, t - x\rangle$ for all $t \in C$. Replacing $t = x^* \in C$ in the above inequality, from (14), we can write

$$\langle g, x - x^*\rangle \geq f(z, x) - f(z, x^*) \geq f(z, x). \tag{15}$$

Since the projection operator P_C is nonexpansive, by (15) and (13), we have

$$\begin{aligned}\|L^f_\eta(x) - x^*\|^2 &= \|P_C(x - \sigma g) - P_C(x^*)\|^2 \leq \|x - \sigma g - x^*\|^2 \qquad (16)\\ &= \|x - x^*\|^2 + \sigma^2\|g\|^2 - 2\sigma\langle g, x - x^*\rangle \leq \|x - x^*\|^2 + \sigma^2\|g\|^2 - 2\sigma f(z, x)\\ &= \|x - x^*\|^2 + \sigma^2\|g\|^2 - 2\sigma^2\|g\|^2 = \|x - x^*\|^2 - (\sigma\|g\|)^2 < \|x - x^*\|^2.\end{aligned}$$

Thus

$$\|L^f_\eta(x) - x^*\| < \|x - x^*\|.$$

(ii): The assertion follows from (i) and $L^f_\eta(x) = x$ if $y = x$.

(iii): If $x \in \mathrm{Sol}(C, f)$ then $f(x, t) \geq 0$ for all $t \in C$. Since $f(x, x) = 0$, it follows that

$$y = \mathrm{argmin}\left\{f(x,t) + \frac{1}{2}\|t - x\|^2 : t \in C\right\} = x.$$

From the definition of L^f_η, we have $L^f_\eta(x) = x$, i.e., $x \in \mathrm{Fix}(L^f_\eta)$. Hence $\mathrm{Sol}(C, f) \subseteq \mathrm{Fix}(L^f_\eta)$.

Conversely, suppose that $x \in \mathrm{Fix}(L^f_\eta)$, then $L^f_\eta(x) = x$. If $y \neq x$ then from (i), we have $\|L^f_\eta(x) - x^*\| < \|x - x^*\|$, where $x^* \in \mathrm{Sol}(C, f)$. This contradicts to the fact that $L^f_\eta(x) = x$. Thus $y = x$. From (11), we find $f(x,t) \geq 0$ for all $t \in C$, i.e., $x \in \mathrm{Sol}(C, f)$. Hence $\mathrm{Fix}(L^f_\eta) \subseteq \mathrm{Sol}(C, f)$. Combined with $\mathrm{Sol}(C, f) \subseteq \mathrm{Fix}(L^f_\eta)$, we obtain $\mathrm{Fix}(L^f_\eta) = \mathrm{Sol}(C, f)$.

(iv): Suppose that sequence $\{x_n\} \subset C$, $x_n \rightharpoonup \overline{x}$ and $\|L^f_\eta(x_n) - x_n\| \to 0$. We will prove that $\overline{x} \in \mathrm{Fix}(L^f_\eta)$. First, we observe that from $\{x_n\} \subset C$, $x_n \rightharpoonup \overline{x}$ and C is closed and convex, it follows $\overline{x} \in C$. Since $x_n \rightharpoonup x$, the sequence $\{x_n\}$ is bounded. Thus, from (ii), the sequence $\{L^f_\eta(x_n)\}$ is bounded too.

Case 1: Suppose that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $L^f_\eta(x_{n_i}) = x_{n_i}$ for all i. In this case $x_{n_i} \in \mathrm{Fix}(L^f_\eta) = \mathrm{Sol}(C, f)$. Thus

$$f(x_{n_i}, y) \geq 0 \quad \forall y \in C. \tag{17}$$

From (17) and the weak convergence of the sequence $\{x_{n_i}\}$ to $\overline{x}$, we get $f(\overline{x}, y) \geq 0$ for all $y \in C$, i.e. $\overline{x} \in \mathrm{Sol}(C, f) = \mathrm{Fix}(L^f_\eta)$.

Case 2: Suppose that there exists a subsequence of $\{x_n\}$, which for simplicity we denote also by $\{x_n\}$, such that $L_\eta^f(x_n) \neq x_n$ for all n. Let $\{m_n\}$ be the sequence of the smallest non-negative integers such that, for all n

$$f((1-\eta^{m_n})x_n + \eta^{m_n}y_n, x) - f((1-\eta^{m_n})x_n + \eta^{m_n}y_n, y_n) \geq \frac{\|x_n - y_n\|^2}{2}.$$

where

$$y_n = \operatorname{argmin}\Big\{f(x_n, t) + \frac{1}{2}\|t - x_n\|^2 : t \in C\Big\},$$

and $z_n = (1-\eta_n)x_n + \eta_n y_n,\ \ \eta_n = \eta^{m_n}$. From (16) we get

$$\|L_\eta^f(x_n) - x^*\|^2 \leq \|x_n - x^*\|^2 - (\sigma_n\|g_n\|)^2,$$

where $g_n \in \partial_2 f(z_n, x_n)$ and $\sigma_n = \dfrac{f(z_n, x_n)}{\|g_n\|^2}$. Thus

$$\begin{aligned}(\sigma_n\|g_n\|)^2 &\leq \|x_n - x^*\|^2 - \|L_\eta^f(x_n) - x^*\|^2 \\ &\leq (\|x_n - x^*\| + \|L_\eta^f(x_n) - x^*\|)\|x_n - L_\eta^f(x_n)\|.\end{aligned}$$

This inequality together with the boundedness of two sequences $\{x_n\}$, $\{L_\eta^f(x_n)\}$ and the fact $\|L_\eta^f(x_n) - x_n\| \to 0$ imply

$$\lim_{n\to\infty} \sigma_n\|g_n\| = 0. \tag{18}$$

Now, we prove that $\{y_n\}$ is bounded. From

$$y_n = \operatorname{argmin}\Big\{f(x_n, t) + \frac{1}{2}\|t - x_n\|^2 : t \in C\Big\},$$

it follows that

$$f(x_n, t) + \frac{1}{2}\|t - x_n\|^2 \geq f(x_n, y_n) + \frac{1}{2}\|y_n - x_n\|^2 \quad \forall t \in C.$$

With $t = x_n$ the above inequality becomes

$$f(x_n, y_n) + \frac{1}{2}\|y_n - x_n\|^2 \leq 0.$$

For every $\zeta_n \in \partial_2 f(x_n, x_n)$, since $y_n \in C$, it holds that $f(x_n, y_n) \geq \langle \zeta_n, y_n - x_n\rangle$. Thus $\langle \zeta_n, y_n - x_n\rangle + \dfrac{1}{2}\|y_n - x_n\|^2 \leq 0$, which implies

$$-\|\zeta_n\|.\|y_n - x_n\| + \frac{1}{2}\|y_n - x_n\|^2 \leq 0.$$

Hence
$$\|y_n - x_n\| \le 2\|\zeta_n\|. \tag{19}$$

Since $x_n \rightharpoonup \overline{x}$, in virue of Lemma 4.1, the sequence $\{\zeta_n\}$ is bounded. Combining this fact with the boundedness of $\{x_n\}$ and (19), we see that the sequence $\{y_n\}$ is also bounded. The sequence $\{z_n\}$, being a convex combination of the sequences $\{x_n\}$ and $\{y_n\}$ is also bounded, and therefore there exists a subsequence of $\{z_n\}$, that for simplicity we also denote by $\{z_n\}$, converging weakly to $\overline{z} \in C$. Since $g_n \in \partial_2 f(z_n, x_n)$, from Lemma 4.1 we deduce that $\{g_n\}$ is bounded. So from (18) and $f(z_n, x_n) = \sigma_n\|g_n\|^2 = (\sigma_n\|g_n\|).\|g_n\|$, we obtain $f(z_n, x_n) \to 0$ as $n \to \infty$. From (12), we have

$$f(z_n, x_n) \ge \frac{\eta_n\|x_n - y_n\|^2}{2}. \tag{20}$$

Letting $\lim\limits_{n\to\infty} f(z_n, x_n) = 0$, (20), we obtain

$$\lim_{n\to\infty} \eta_n\|x_n - y_n\|^2 = 0. \tag{21}$$

From (11), it follows that

$$f(x_n, t) - f(x_n, y_n) \ge \langle x_n - y_n, t - y_n\rangle \quad \forall t \in C. \tag{22}$$

Now we distinguish two cases:

Case 2.1: $\limsup\limits_{n\to\infty} \eta_n > 0$. In this case, there exist $\overline{\eta}$ and a subsequence of $\{\eta_n\}$, denoted by $\{\eta_{n_i}\}$ such that $\eta_{n_i} \to \overline{\eta}$. Then, by (21), we obtain that $\lim\limits_{i\to\infty} \|x_{n_i} - y_{n_i}\| = 0$. Since $x_{n_i} \rightharpoonup \overline{x}$ we have $y_{n_i} \rightharpoonup \overline{x}$. It follows from (22) that

$$f(x_{n_i}, t) - f(x_{n_i}, y_{n_i}) \ge \langle x_{n_i} - y_{n_i}, t - y_{n_i}\rangle \quad \forall t \in C. \tag{23}$$

Applying the Cauchy-Schwarz inequality, we get

$$|\langle x_{n_i} - y_{n_i}, t - y_{n_i}\rangle| \le \|x_{n_i} - y_{n_i}\|.\|t - y_{n_i}\|. \tag{24}$$

Since $\|x_{n_i} - y_{n_i}\| \to 0$ and the sequence $\{y_{n_i}\}$ is bounded, from (24), it ensures that $\lim\limits_{i\to\infty}\langle x_{n_i} - y_{n_i}, t - y_{n_i}\rangle = 0$. So, using (23) and the weak convergence of the two sequences $\{x_{n_i}\}$, $\{y_{n_i}\}$ to $\overline{x}$, we get

$$f(\overline{x}, t) \ge 0 \quad \forall t \in C,$$

i.e. $\overline{x} \in \mathrm{Sol}(C, f) = \mathrm{Fix}(L^f_\eta)$.

Case 2.2: $\lim\limits_{n\to\infty} \eta_n = 0$. From the boundedness of $\{y_n\}$, without loss of generality, we may assume that $y_n \rightharpoonup \overline{y}$ as $n \to \infty$. Since $\eta_n = \eta^{m_n} \to 0$, it follows that $m_n > 1$ for n sufficiently large and consequently, that

$$f(\overline{z}_n, x_n) - f(\overline{z}_n, y_n) < \frac{\|x_n - y_n\|^2}{2}, \tag{25}$$

where $\overline{z}_n = (1-\eta^{m_n-1})x_n + \eta^{m_n-1}y_n$. Choose $t = x_n$ in (22) to obtain

$$-f(x_n, y_n) \geq \|x_n - y_n\|^2. \tag{26}$$

From (25) and (26) we can write

$$f(\overline{z}_n, x_n) - f(\overline{z}_n, y_n) < -\frac{1}{2} f(x_n, y_n). \tag{27}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $\eta_n \to 0$, we have

$$\|\overline{z}_n - x_n\| = \frac{\eta_n}{\eta}\|x_n - y_n\| \to 0.$$

From $x_n \rightharpoonup \overline{x}$ and $\|\overline{z}_n - x_n\| \to 0$, it follows that $\overline{z}_n \rightharpoonup \overline{x}$. Since $y_n \rightharpoonup \overline{y}$, from (27), we have

$$-f(\overline{x}, \overline{y}) \leq -\frac{1}{2} f(\overline{x}, \overline{y}).$$

Thus

$$f(\overline{x}, \overline{y}) \geq 0. \tag{28}$$

From (26) follows

$$f(x_n, y_n) \leq 0. \tag{29}$$

Since $x_n \rightharpoonup \overline{x}$, $y_n \rightharpoonup \overline{y}$, from (29), we have $f(\overline{x}, \overline{y}) \leq 0$. In combination with (28) we obtain $f(\overline{x}, \overline{y}) = 0$. Note that

$$\lim_{n\to\infty} f(x_n, y_n) = f(\overline{x}, \overline{y}) = 0.$$

Thus from (26) we can write $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

Since $x_n \rightharpoonup \overline{x}$ we have $y_n \rightharpoonup \overline{x}$. From $\|x_n - y_n\| \to 0$ and the boudedness of the sequence $\{y_n\}$, it follows that $\lim\limits_{k\to\infty} \langle x_n - y_n, t - y_n\rangle = 0$. So, by using (22) and the weak convergence of the two sequences $\{x_n\}$, $\{y_n\}$ to $\overline{x}$, we obtain

$$f(\overline{x}, t) \geq 0 \quad \forall t \in C,$$

i.e. $\overline{x} \in \mathrm{Sol}(C, f) = \mathrm{Fix}(L^f_\eta)$. □

Remark 3.2 (1) A contraction fixed point approach for strongly monotone equilibrium problem (EP) has been studied in [15] by defining the following mapping

$$s(x) := \mathrm{argmin}\{\rho f(x, y) + \|y - x\|^2 : \ y \in C\},$$

where $\rho > 0$.

It has been proved [15] that if f is strongly monotone, Lipschitz-type on C, then one can find $\rho > 0$, depending on the strongly monotone modulus and

Lipschitz-type constant, such that s is contractive on C and its fixed point is the unique solution of equilibrium problem *EP(C, f)*;

Another fixed point formulation for equilibrium problems has been used in [5] by defining a mapping as

$$P_f(x) := \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ \ \forall y \in C\}.$$

It has been shown that if f is a monotone bifunction satisfying certain assumptions, then P_f is single-valued, nonexapansive on C and its fixed point set coincides with the solution set of Problem *EP(C, f)*. However for pseudomonotone bifunctions the mapping P_f, in general, is multi-valued, even its values may not be convex.

(2) The extragradient method introduced by Korpelevich [11] for the saddle point problem has been extended and modified for solving the equilibrium problem *EP(C, f)* and its special cases such as variational inequality, see e.g. [9, 10, 17, 19, 20] and the references therein. The results obtained in the above propositions give a quasi-nonexpansive fixed point approach to the extragradient method for pseudomonotone equilibrium problems with and without Lipschitz-type condition. The existing results for quasi-nonexpansive mappings in real Hilbert spaces thus can be applied to pseudomonotone equilibrium as well as variational inequality problems. □

5. Conclusion

We have introduced quasi-nonexpansive mappings for a pseudomonotone monotone equilibrium problem defined by a convex set C and a bifunction f in real Hilbert spaces and investigated some properties concerning their fixed point sets related to the solution set of the problem.

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