Journal of Convex Analysis
Volume 25 (2018), No. 4, 1139–1158

Some Geometric and Proximinality Properties in Banach Spaces

P. Shunmugaraj
Dept. of Mathematics and Statistics, Indian Inst. of Technology, Kanpur 201816, India
psraj@iitk.ac.in

V. Thota
Dept. of Mathematics and Statistics, Indian Inst. of Technology, Kanpur 201816, India
vamst@iitk.ac.in

Received: February 14, 2017
Revised manuscript received: July 07, 2017
Accepted: August 03, 2017

We present some geometric and approximation theoretic characterizations of locally uniform convex, compactly locally uniformly convex, nearly uniformly convex, compactly uniformly convex and uniformly convex Banach spaces. The approximation theoretic characterizations of compactly locally uniformly convex spaces presented in this paper provide some answers to a question posed by F. Deutsch and J. Lambert in their paper "On continuity of metric projections", J. Approx. Theory 29 (1980) 116–131.

Keywords: Compactly locally uniformly convex, nearly uniformly convex, compactly uniformly convex, approximative compactness, metric projection.

2000 Mathematics Subject Classification: 41A65, 46B20

1. Introduction

Let X be a real Banach space and X^* its dual. The *closed unit ball* and *unit sphere* of X are denoted by B_X and S_X respectively. For $x^* \in S_{X^*}$ and $0 \leq \delta < 1$, let $S(X, x^*, \delta) = \{x \in B_X : x^*(x) \geq 1 - \delta\}$ and $S(X^{**}, x^*, \delta) = \{x^{**} \in B_{X^{**}} : x^{**}(x^*) \geq 1 - \delta\}$. The sets $S(X, x^*, \delta)$ and $S(X^{**}, x^*, \delta)$ are called *slices* of B_X and $B_{X^{**}}$ respectively defined by x^* and δ.

For a non-empty subset M of X and $x \in X$, let $P_M(x) = \{y \in M : \|x - y\| = d(x, M)\}$, the set of best approximations of x in M. Here, $d(x, M)$ denotes the distance from x to M. If $P_M(x) \neq \emptyset$ for a given $x \in X$, we say that M is *proximinal* at x and if $P_M(x)$ is a singleton then we say that M is *Chebyshev* at x. A sequence (x_n) in M is called a *minimizing sequence* with respect to $x \in X$, if $\|x - x_n\| \to d(x, M)$. We say that M is *approximatively compact* at $x \in X$, if every minimizing sequence in M with respect to x has a convergent subsequence converging to an element of M. The set M is said to be *approximatively compact*

ISSN 0944-6532 / \$ 2.50

if it is approximatively compact at all points of X. Proximinal and Chebyshev subsets are defined similarly. If M is proximinal then the set-valued mapping $x \mapsto P_M(x)$ is called the *metric projection* onto M. All subsets we consider in this paper are non-empty.

We say that the space X has the *Kadec-Klee property* if for any sequence on S_X, the weak and strong convergence coincide. The space X is called *strongly convex* [16] if it is reflexive, strictly convex and it has the Kadec-Klee property.

Consider the following well known characterizations of uniform convexity and strong convexity in terms of a geometric property and a proximinality property respectively.

Theorem 1.1. *The following three statements are equivalent.*

(1) *X is uniformly convex.*
(2) *For every $\epsilon > 0$, there exists $\delta > 0$ such that diam$(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$.*
(3) *For every $\epsilon > 0$, there exists $\delta > 0$ such that diam$(C) < \epsilon$ for any convex subset C of B_X such that $d(0, C) \geq 1 - \delta$.*

Theorem 1.2. *The following two statements are equivalent.*

(1) *X is strongly convex.*
(2) *Every closed convex subset of X is approximatively compact and Chebyshev.*

The implication (2) $\Rightarrow$ (3) of Theorem 1.1 follows from the separation theorem whereas (3) $\Rightarrow$ (2) is obvious and (1) $\Leftrightarrow$ (2) of Theorem 1.1 is well known (see e.g. [24]). The implications (1) $\Leftrightarrow$ (3) of Theorem 1.1 are also proved in [1, 10]. Theorem 1.2 is due to Fan and Glicksberg [10] (see e.g. [16]).

Characterizations which are similar to the preceding results are also obtained for some other geometric properties in literature. For instance, it is shown in [18, 23] that nearly uniform convexity (i.e., reflexivity with uniform Kadec-Klee property) has characterizations analogous to Theorem 1.1 (see Theorem 4.1 of this paper). A characterization analogous to Theorem 1.2 is obtained for mid point locally uniformly convex space (in short, MLUC) in [17, Theorem B.8] which states that X is MLUC if and only if B_X is approximatively compact and Chebyshev. We refer to [13, 21, 22] for similar characterizations.

It is natural to ask whether characterizations analogous to Theorem 1.1 or/and Theorem 1.2 exist for well known geometric properties such as local uniform convexity, compactly uniform convexity, compactly local uniform convexity, nearly uniform convexity and uniform convexity. Such a question is not new and has already been raised in literature (see e.g. [6, page 119]). In this paper, we show that such characterizations exist and we present them with some of their consequences. The definitions of the geometric properties mentioned above are presented in Section 2.

We refer to the recent papers [3, 5, 8, 13, 21, 22, 27] and also [6, 10, 17, 19, 26] for some results, which relate some geometric and proximinality properties in Banach spaces.

The paper is organized as follows. In Section 2, we recall several notions which are needed for the rest of the sections. In Section 3, we present some characterizations of compactly local uniform convexity and local uniform convexity which are analogous to Theorems 1.1 and 1.2. The proximinality results presented in this section provide some answers to a question posed by Deutsch and Lambert in [6, page 119]. Approximation theoretic properties of nearly uniform convexity are discussed in Section 4. Section 5 deals with some geometric and proximinality properties of compactly uniform convexity. Section 6 is devoted to uniform convexity.

2. Preliminaries

In this section, we first recall the notion of measure of non-compactness and present some elementary known results involving convergence of sequence of subsets which are needed. We then recall the definitions of the classes of Banach spaces which are used in this paper.

Let A be a subset of X. The *complement*, *closure* and *diameter* of A are denoted by A^c, $\overline{A}$ and $diam(A)$ respectively. For $\epsilon > 0$, let $B_\epsilon(A) = \{x \in X : d(x, A) < \epsilon\}$. If A is bounded, let

$$\alpha(A) = inf\,\{\delta > 0 : A \subseteq B_\delta(F) \text{ for some finite set } F \subset X\}.$$

The number $\alpha(A)$ is called the *Hausdorff index of measure of non-compactness of A*. It is known [2] that $\alpha(A) = \alpha(\overline{A})$.

Definition 2.1. Let (C_n) be a sequence of subsets of X and C_0 a subset of X. We say that

(1) (C_n) converges to C_0 in the upper Hausdorff sense, denoted by $C_n \xrightarrow{H^+} C_0$, if for every $\epsilon > 0$, $C_n \subseteq C_0 + \epsilon B_X$ eventually;

(2) (C_n) converges to C_0 in the upper Vietoris sense, denoted by $C_n \xrightarrow{V^+} C_0$, if $C_n \subseteq V$ eventually whenever $C_0 \subseteq V$ for some open subset V of X.

We need the following result which follows from Proposition 2.2 and Corollary 2.3 of [4].

Proposition 2.2. *Let (C_n) be a sequence of closed bounded subsets of X satisfying the condition that $C_{n+1} \subseteq C_n$ for all n and let $C = \cap_{n=1}^{\infty} C_n$. Consider the following statements.*

(1) $\alpha(C_n) \to 0$.

(2) *Every sequence (x_n) with $x_n \in C_n, n \in \mathbb{N}$, has a convergent subsequence.*

(3) *C is non-empty compact and $C_n \xrightarrow{V^+} C$.*

(4) *C is non-empty compact and $C_n \xrightarrow{H^+} C$.*
(5) *$diam(C_n) \to 0$.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). *If C is a singleton, then* (4) $\Leftrightarrow$ (5).

For $x \in S_X$ and $0 \leq \delta \leq 1$, let

$$C[x, \delta] = \{y \in S_X : \left\| \frac{x+y}{2} \right\| \geq 1 - \delta\}.$$

Definition 2.3. A Banach space X is said to be

(1) *locally uniformly convex* (LUC) if $x_n \to x$ whenever $x, x_n \in S_X$ and $\|x_n + x\| \to 2$;
(2) *compactly locally uniformly convex* ($CLUC$), if (x_n) has a convergent subsequence whenever $x, x_n \in S_X$ and $\|x_n + x\| \to 2$;
(3) *nearly uniformly convex* (NUC) if it is reflexive and uniformly Kadec-Klee (i.e, for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that $\|x\| < \delta$ whenever (x_n) in B_X, $x_n \to x$ weakly and $\inf\{\|x_n - x_m\| : n \neq m\} > \epsilon$);
(4) *compactly uniformly convex* (CUC) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(C[x, \delta]) < \epsilon$ for all $x \in S_X$;
(5) *uniformly convex* (UC) if $\|x_n - y_n\| \to 0$ whenever $x_n, y_n \in S_X$ and $\|x_n + y_n\| \to 2$.

Remark 2.4. (1) It is clear from the definitions that $UC \Rightarrow LUC \Rightarrow CLUC$. If X is $CLUC$ and strictly convex then it is LUC.

(2) It follows from [7, 18, 23] that $UC \Rightarrow CUC \Rightarrow NUC$ and from the definition of NUC that NUC spaces are reflexive.

(3) It is observed in [7] that $CUC \Rightarrow CLUC$.

(4) It is also known that none of the implications mentioned above can be reversed. For instance, we refer to

(a) [20, Example 4.1] for $CLUC \not\Rightarrow LUC$;
(b) [20, Example 4.2] for $CLUC \not\Rightarrow NUC$;
(c) [20, Example 4.2] for $CLUC \not\Rightarrow CUC$;
(d) [16, Example 5.3.6] for $LUC \not\Rightarrow UC$;
(e) [21, Example 2.8] for $CUC \not\Rightarrow LUC$;
(f) [21, Example 2.9] for $NUC \not\Rightarrow CLUC$.

3. *LUC* and *CLUC* spaces

In this section, we present some geometric and proximinality properties of CLUC and LUC. LUC spaces are well known [16] and CLUC spaces are discussed in [6, 7, 19, 20, 21, 26].

3.1. Geometric properties of $CLUC$ and LUC

We first present a characterization of $CLUC$ in terms of slices of X and X^{**} which is analogous to (1) $\Leftrightarrow$ (2) of Theorem 1.1.

The following notations are needed.

For $x \in S_X$, let $J(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$ and for $0 \le \delta < 1$, let

$$A_\delta(X^{**}, x) = \cup\{S(X^{**}, x^*, \delta) : x^* \in S_{X^*} \text{ and } x \in S(X^{**}, x^*, \delta)\}.$$

Note that $A_0(X^{**}, x) = \cup\{S(X^{**}, x^*, 0) : x^* \in J(x)\}$. Let

$$A_\delta(X, x) = \cup\{S(X, x^*, \delta) : x^* \in S_{X^*} \text{ and } x \in S(X, x^*, \delta)\}.$$

and $A_0(X, x) = \cup\{S(X, x^*, 0) : x^* \in J(x)\}$. It is clear that $A_\delta(X, x) = A_\delta(X^{**}, x) \cap X$. Observe that for $0 \le \delta \le \beta$ and $x \in S_X$, $A_\delta(X^{**}, x) \subseteq A_\beta(X^{**}, x)$ and $A_\delta(X, x) \subseteq A_\beta(X, x)$. We will denote $A_\delta(X, x)$ simply by $A_\delta(x)$.

For $x \in S_X$ and $0 \le \delta \le 1$, let

$$D[x, \delta] = \{y \in B_X : \left\|\frac{x+y}{2}\right\| \ge 1 - \delta\}.$$

Observe that $C[x, \delta] = D[x, \delta] \cap S_X$.

We say that X is CLUC at $x \in S_X$ if (x_n) has a convergent subsequence whenever (x_n) in S_X and $\|\frac{x_n+x}{2}\| \to 1$. LUC at $x \in S_X$ is defined similarly.

It follows from the definitions and Proposition 2.2 that

$$X \text{ is } CLUC \text{ at } x \iff \alpha(D[x, \frac{1}{n}]) \to 0 \iff \alpha(C[x, \frac{1}{n}]) \to 0.$$

Similarly, X is LUC at $x \iff diam(C[x, \frac{1}{n}]) \to 0 \iff diam(D[x, \frac{1}{n}]) \to 0$.

Lemma 3.1. *Let $x \in S_X$ and $0 \le \delta < 1$. Then $D[x, \frac{\delta}{2}] \subseteq A_\delta(x) \subseteq D[x, \delta]$ and $\overline{A_\delta(x)} \subseteq D[x, \delta]$.*

Proof. Suppose $y \in D[x, \frac{\delta}{2}]$. Find $x^* \in S_{X^*}$ such that $x^*(\frac{x+y}{2}) = \|\frac{x+y}{2}\|$. Since $y \in D[x, \frac{\delta}{2}]$,

$$x^*\left(\frac{x+y}{2}\right) \ge 1 - \frac{\delta}{2}.$$

This implies that $x^*(x) \ge 1 - \delta$ and $x^*(y) \ge 1 - \delta$. Hence $y \in A_\delta(x)$.

Let $y \in A_\delta(x)$. Then, there exists $x^* \in S_{X^*}$ such that $x^*(x) \ge 1 - \delta$ and $x^*(y) \ge 1 - \delta$. Therefore

$$\left\|\frac{x+y}{2}\right\| \ge x^*\left(\frac{x+y}{2}\right) \ge 1 - \delta.$$

Hence $y \in D[x, \delta]$. Observe: $D[x, \delta]$ is a closed set and $\overline{A_\delta(x)} \subseteq D[x, \delta]$. $\square$

Observe from the preceding result that for any $x \in S_X$, $A_0(x) = D[x,0] = C[x,0]$. It is easy to see that $D[x,0]$ is a non-empty closed subset of S_X for every $x \in S_X$.

Lemma 3.2. *If $A_{\frac{1}{n}}(x) \xrightarrow{V^+} A_0(x)$ for some $x \in S_X$, then $A_0(x)$ is compact and $\overline{A_{\frac{1}{n}}(x)} \xrightarrow{V^+} A_0(x)$.*

Proof. Suppose $A_{\frac{1}{n}}(x) \xrightarrow{V^+} A_0(x)$. Let $x \in S_X$ and (x_n) be a sequence in $A_0(x)$. Then there exists $x_n^* \in S_{X^*}$ such that $x_n^*(x_n) = 1$ and $x_n^*(x) = 1$ for all n. For each n, define $y_n = (1 - \frac{1}{n})x_n$. Note that $y_n \in A_{\frac{1}{n}}(x) \backslash A_0(x)$ and $\|x_n - y_n\| = \frac{1}{n}$ for all n. If (y_n) does not have a convergent subsequence, then $A_0(x) \subseteq \{y_n : n \in \mathbb{N}\}^c$ which contradicts the fact that $A_{\frac{1}{n}}(x) \xrightarrow{V^+} A_0(x)$. Therefore (y_n) has a convergent subsequence and hence (x_n) has a convergent subsequence. This shows that $A_0(x)$ is compact. It follows from Lemma 3.1 that $\overline{A_{\frac{1}{n}}(x)} \xrightarrow{V^+} A_0(x)$. $\square$

For a subset A of X, let $w^* - cl(A)$ denote the w^*-closure of A in X^{**}.

Theorem 3.3. *Let $x \in S_X$. Then the following statements are equivalent.*

(1) *X is CLUC at x.*

(2) $\alpha(A_{1/n}(x)) \to 0$.

(3) $A_{\frac{1}{n}}(x) \xrightarrow{V^+} A_0(x)$.

(4) $A_{\frac{1}{n}}(X^{**}, x) \xrightarrow{V^+} A_0(X^{**}, x)$.

(5) $\alpha(A_{1/n}(X^{**}, x)) \to 0$.

(6) *X^{**} is CLUC at x.*

Proof. (1) $\Leftrightarrow$ (2): These implications follow from Lemma 3.1.

(2) $\Leftrightarrow$ (3): These implications follow from Lemma 3.2 and Proposition 2.2.

(3) $\Rightarrow$ (4): Suppose (3) holds true and $\epsilon > 0$. Then by (3) there exists $\delta > 0$ such that $A_\delta(x) \subseteq A_0(x) + \epsilon B_X$ and by Lemma 3.2, $A_0(x)$ is compact. Suppose $x^* \in S_{X^*}$ and $x \in S(X^{**}, x^*, \delta)$. Observe that $x \in S(X, x^*, \delta)$. Since $A_0(x) + \epsilon B_{X^{**}}$ is w^*-compact, by Lemma 2.1 of [11],

$$S(X^{**}, x^*, \delta) \subseteq w^* - cl(S(X, x^*, \delta)) \subseteq w^* - cl(A_\delta(x)) \subseteq A_0(x) + \epsilon B_{X^{**}}.$$

Since the previous inclusion is true for all $x^* \in S_{X^*}$ satisfying $x \in S(X^{**}, x^*, \delta)$,

$$A_\delta(X^{**}, x) \subseteq A_0(x) + \epsilon B_{X^{**}} \subseteq A_0(X^{**}, x) + \epsilon B_{X^{**}}.$$

From the above inclusion, note that $A_0(X^{**}, x) \subseteq A_0(x) + \epsilon B_{X^{**}}$. This shows that $\alpha(A_0(X^{**}, x)) \leq 2\epsilon$. Hence $A_0(X^{**}, x)$ is compact. Observe that

$$\overline{A_\delta(X^{**}, x)} \subseteq A_0(X^{**}, x) + \epsilon B_{X^{**}}.$$

Therefore (4) follows from Proposition 2.2.

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6): These implications follow from (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(5) $\Rightarrow$ (2): This follows from the fact that $A_{\frac{1}{n}}(x) \subseteq A_{\frac{1}{n}}(X^{**}, x)$ for every n. $\square$

We now present some consequences of the preceding result.

The following known result [21, Proposition 2.5] can also be obtained as an immediate consequence of Theorem 3.3 and Proposition 2.2.

Corollary 3.4. *If X is CLUC at $x \in S_X$, then $\alpha(S(X, x^*, \frac{1}{n})) \to 0$ for all $x^* \in J(x)$.*

The characterization of LUC in terms of slices is addressed in the next result.

Corollary 3.5. *Let $x \in S_X$. Then the following statements are equivalent.*

(1) *X is LUC at x.*
(2) *$diam(A_{1/n}(x)) \to 0$.*
(3) *$diam(A_{1/n}(X^{**}, x)) \to 0$.*
(4) *X^{**} is LUC at x.*

Proof. (1) $\Leftrightarrow$ (2): Observe that X is strictly convex if and only if $A_0(x) = \{x\}$ for every $x \in S_X$. The implications follow from Theorem 3.3 and Proposition 2.2.

(2) $\Rightarrow$ (3): Suppose (2) holds true. The proof of (3) $\Rightarrow$ (4) of Theorem 3.3 implies that for $\epsilon > 0$, $A_0(X^{**}, x) \subseteq A_0(x) + \epsilon B_{X^{**}}$. Since $diam(A_0(x)) = 0$, $A_0(X^{**}, x) = \{x\}$. The rest follows from Theorem 3.3 and Proposition 2.2.

(3) $\Rightarrow$ (2): This is obvious.

(3) $\Leftrightarrow$ (4): These implications follow from (1) $\Leftrightarrow$ (2). $\square$

We now present some geometric characterizations of CLUC which is analogous to (1) $\Leftrightarrow$ (3) of Theorem 1.1 as a consequence of Theorem 3.3. We need the following definition.

For $x, y \in X$ and $x \neq y$, let $[x, y] = \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$ and $(x, y) = \{\lambda x + (1-\lambda)y : \lambda \in (0, 1)\}$.

Definition 3.6. A subset C of X is said to be *convex at x* if $[x, y] \subseteq C$ whenever $y \in C$.

Theorem 3.7. *Let $x \in S_X$. The following statements are equivalent.*

(1) *X is CLUC at x.*
(2) *If (C_n) is a sequence of subsets of B_X which are convex at x and assume $d(0, C_n) \to 1$, then $C_n \xrightarrow{V^+} A_0(x)$.*
(3) *If (C_n) is a sequence of convex subsets of B_X such that $x \in C_n$ for all n and $d(0, C_n) \to 1$ then $C_n \xrightarrow{V^+} A_0(x)$.*

(4) *If (C_n) is a sequence of convex subsets of B_X such that $x \in C_n$ for all n and $d(0, C_n) \to 1$ then every sequence (x_n) satisfying $x_n \in C_n$ for all n has a convergent subsequence.*

Proof. (1) ⇒ (2): Let V be an open subset of X such that $A_0(x) \subseteq V$. By (1) and Theorem 3.3, there exists $\delta > 0$ such that $A_\delta(x) \subseteq V$. Let C be a subset of B_X such that C is convex at x and $d(0, C) \geq 1 - \frac{\delta}{2}$. Suppose $y \in C$. Then $[x, y] \subseteq C$. Since $[x, y] \subseteq (B[0, 1 - \delta])^c$, by separating $[x, y]$ and $B[0, 1 - \delta]$, we find some $x^* \in S_{X^*}$ such that $x, y \in S(X, x^*, \delta)$. This shows that $C \subseteq A_\delta(x)$ and hence $C \subseteq V$ which proves (2).

(2) ⇒ (3): This is obvious.

(3) ⇒ (4): We first show that $A_0(x)$ is compact. Let (x_n) be in $A_0(x)$. Then, for each n, there exists $x_n^* \in S_{X^*}$ such that $x_n^*(x_n) = 1$ and $x_n^*(x) = 1$. Let $C_n = S(X, x_n^*, \frac{1}{n})$ for each n. It is easy to verify that $d(0, C_n) \to 1$. Hence, by (3), $C_n \xrightarrow{V+} A_0(x)$. For each n, consider $y_n = (1 - \frac{1}{n})x_n$ in $C_n \backslash A_0(x)$. Following the argument involved in the proof of Lemma 3.2, we show that $A_0(x)$ is compact.

Suppose (C_n) is a sequence of convex subsets of B_X such that $x \in C_n$ for all n and $d(0, C_n) \to 1$. By (3), $C_n \xrightarrow{V+} A_0(x)$. Let (x_n) be such that $x_n \in C_n$ for all n. If a subsequence of (x_n) is in $A_0(x)$, then by compactness of $A_0(x)$, (x_n) has a convergent subsequence. Suppose $x_n \in C_n \backslash A_0(x)$ for every n. If (x_n) does not have a convergent subsequence, then $A_0(x) \subseteq \{x_n : n \in \mathbb{N}\}^c$ which contradicts the fact that $C_n \xrightarrow{V+} A_0(x)$. Hence (x_n) has a convergent subsequence.

(4) ⇒ (1): Let $x_n \in D[x, \frac{1}{2n}]$ for all n. Then, by Lemma 3.1, $x_n \in A_{\frac{1}{n}}(x)$ for each n. Therefore, for each n, there exists $x_n^* \in S_{X^*}$ such that $x_n, x \in S(X, x_n^*, \frac{1}{n})$. For each n, let $C_n = S(X, x_n^*, \frac{1}{n})$. By (4), (x_n) has a convergent subsequence. Therefore, by Proposition 2.2, $\alpha(D[x, \frac{1}{2n}]) \to 0$ which implies that X is $CLUC$ at x. □

The following corollary which is a consequence of Corollary 3.5 and Theorem 3.7 provides some variants of the statement of Theorem 2.7 of [1].

Corollary 3.8. *Let $x \in S_X$. The following statements are equivalent.*

(1) *X is LUC at x.*

(2) *If (C_n) is a sequence of subsets of B_X which are convex at x and $d(0, C_n) \to 1$ then $diam(C_n) \to 0$.*

(3) *If (C_n) is a sequence of convex subsets of B_X such that $d(0, C_n) \to 1$ and $x \in C_n$ for every n, then $diam(C_n) \to 0$.*

(4) *For every $\epsilon > 0$ there exists $\delta > 0$ such that for any convex subset C of B_X satisfying the conditions $d(0, C) \geq 1 - \delta$ and $x \in C$, $diam(C) < \epsilon$.*

(5) *If (C_n) is a sequence of convex subsets of B_X such that $d(0, C_n) \to 1$ and $x \in C_n$ for every n, then every sequence (x_n) satisfying $x_n \in C_n$ for all n converges to x.*

Proof. It is easy to verify that each of the preceding five statements implies that $A_0(x) = \{x\}$. Hence the corollary follows from Theorem 3.7. □

We remark that (1) ⇔ (4) of the preceding corollary is observed in [10].

For $x \in X$ and $r > 0$, we denote $\{y \in X : \|x - y\| \leq r\}$ by $B_X[x, r]$ and $\{y \in X : \|x - y\| < r\}$ by $B_X(x, r)$.

We now see another consequence of Theorem 3.3. For $x \in S_X$, $0 < r < 1$ and $\delta > 0$, consider the lenses

$$N_r^{**}(x, \delta) = B_{X^{**}} \backslash B_{X^{**}}[-rx, 1 + r - \delta]$$

in X^{**} and $N_r(x, \delta) = N_r^{**}(x, \delta) \cap X$ in X.

Lemma 3.9. *For $x \in S_X$, $0 < r < 1$ and $0 < \delta < r$, $N_r(x, \delta) \subseteq A_{\frac{\delta}{r}}(x)$, $A_\delta(x) \subseteq N_r(x, 2\delta)$, $N_r^{**}(x, \delta) \subseteq A_{\frac{\delta}{r}}(X^{**}, x)$ and $A_\delta(X^{**}, x) \subseteq N_r^{**}(x, 2\delta)$.*

Proof. Let $y^{**} \in N_r^{**}(x, \delta)$. Find $\widehat{y} \in S_{X^{***}}$ such that $\widehat{y}(y^{**} + rx) = \|y^{**} + rx\|$. Since

$$1 + r - \delta < \|y^{**}\| + r\widehat{y}(x) \leq 1 + r\widehat{y}(x),$$

$1 - \frac{\delta}{r} < \widehat{y}(x)$. Further, since $\widehat{y}(y^{**}) + r\widehat{y}(x) > 1 + r - \delta$, $\widehat{y}(y^{**}) > 1 - \delta > 1 - \frac{\delta}{r}$. Therefore $y^{**} \in A_{\frac{\delta}{r}}(X^{**}, x)$.

Let $y^{**} \in A_\delta(X^{**}, x)$. Then there exists $\widehat{x} \subset S_{X^*} \subseteq S_{X^{***}}$ such that we have $x \in S(X^{**}, \widehat{x}, \delta)$ and $y^{**} \in S(X^{**}, \widehat{x}, \delta)$. Hence

$$\|y^{**} + rx\| \geq \widehat{x}(y^{**} + rx) \geq 1 + r - (1 + r)\delta > 1 + r - 2\delta$$

which implies that $y^{**} \in N_r^{**}(x, 2\delta)$.

The other two inclusions now follow easily. □

In the following result $CLUC$ is characterized in terms of lenses in X and X^{**}. The equivalences (1) ⇔ (3) of the next result are known [15].

Proposition 3.10. *Let $x \in S_X$. Then the following statements are equivalent.*

(1) *X is $CLUC$ at x.*

(2) *For every $0 < r < 1$, $\alpha(N_r^{**}(x, \frac{1}{n})) \to 0$.*

(3) *For every $0 < r < 1$, $\alpha(N_r(x, \frac{1}{n})) \to 0$.*

(4) *For every $0 < r < 1$, $\alpha(N_r(x, \frac{1}{n}) \cap S_X) \to 0$.*

(5) *$\alpha(N_r(x, \frac{1}{n}) \cap S_X) \to 0$ for some $r \in (0, 1)$.*

Proof. (1) ⇒ (2): This follows from Theorem 3.3 and Lemma 3.9.

(2) ⇒ (3) ⇒ (4) ⇒ (5): These implications are immediate.

(5) ⇒ (1): Let $x, y_n \in S_X$ and $\|\frac{x+y_n}{2}\| \to 1$. Suppose $\alpha(N_r(x, \frac{1}{n}) \cap S_X) \to 0$ for some $r \in (0, 1)$. We claim that for every $\delta > 0$, $y_n \in N_r(x, \delta) \cap S_X$ eventually.

Suppose this is not true. Then, for some $\delta > 0$, there exists a subsequence (y_{n_k}) of (y_n) such that y_{n_k} does not belong to $N_r(x,\delta) \cap S_X$ for every n_k. Observe that $\|y_{n_k} + rx\| \leq 1 + r - \delta$ for all n_k. Hence, for every n_k,

$$\left\|\frac{x + y_{n_k}}{2}\right\| = \left\|\frac{(1-r)x}{2} + \frac{rx + y_{n_k}}{2}\right\| \leq \frac{(1-r)}{2} + \frac{1+r-\delta}{2} = 1 - \frac{\delta}{2}$$

which is a contradiction. Therefore, for every $\delta > 0$, $y_n \in N_r(x,\delta) \cap S_X$ eventually. Since $\alpha(N_r(x, \frac{1}{n}) \cap S_X) \to 0$, by Proposition 2.2, (y_n) has a convergent subsequence. □

From Proposition 3.10 we can obtain a characterization of *LUC* which can be stated by replacing the measure of non-compactness α into *diam* in the statements of the preceding result.

3.2. Proximinality properties of *LUC* and *CLUC*

In this subsection we present some approximation theoretic characterizations of *LUC* and *CLUC* which provide some answers to a question posed in [6, page 119].

For a non-empty subset M of X, $x \in X$ and $\delta \geq 0$, let

$$P_M(x,\delta) = \{y \in M : \|x - y\| \leq d(x,M) + \delta\}.$$

Note that $P_M(x,0) = P_M(x)$.

We now present a characterization of *CLUC* which is analogous to Theorem 1.2. The following lemma which is used in the next theorem is also needed for Section 5.

Lemma 3.11. *Let* $0 < \delta < 1$.

(1) *For* $0 < r < 1$, $P_{S_X}(rx, r\delta) \subseteq C[x,\delta]$ *for all* $x \in S_X$.

(2) *For* $y \in S_X$ *and* $x \in C[y,\delta]$, $\frac{x+y}{\|x+y\|} \in P_{S_X}(\frac{y}{2}, \delta)$.

Proof. (1) Let $x \in S_X$ and $y \in P_{S_X}(rx, r\delta)$. Then $y \in S_X$ and

$$\|rx - y\| \leq (1-r) + r\delta.$$

Find $y^* \in S_{X^*}$ such that $y^*(y) = 1$. Then

$$1 = y^*(y - rx + rx) \leq \|y - rx\| + ry^*(x) \leq 1 - r + r\delta + ry^*(x)$$

which implies that $y^*(x) \geq 1 - \delta$. Therefore

$$\left\|\frac{x+y}{2}\right\| \geq y^*\left(\frac{x+y}{2}\right) \geq \frac{1}{2}(1-\delta) + \frac{1}{2} = 1 - \frac{\delta}{2} \geq 1 - \delta$$

which shows that $y \in C[x,\delta]$.

(2) Let $x \in C[y,\delta]$. Then $\|\frac{x+y}{2}\| \geq 1-\delta$ and $d(\frac{x+y}{2}, S_X) \leq \delta$. Let $z = \frac{x+y}{\|x+y\|}$. Then $d(\frac{x+y}{2}, S_X) = \|\frac{x+y}{2} - z\|$ and

$$\left\|\frac{y}{2} - z\right\| \leq \left\|\frac{x+y}{2} - z\right\| + \left\|\frac{x}{2}\right\| \leq \delta + \frac{1}{2}.$$

Therefore $z \in P_{S_X}(\frac{y}{2}, \delta)$ which proves (2). $\square$

Theorem 3.12. *The following statements are equivalent.*

(1) *X is CLUC.*

(2) *S_X is approximatively compact at every $x \in X\backslash\{0\}$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in S_X$ and $r \geq 1$. Let us verify that S_X is approximatively compact at rx. Let $x_n \in S_X$ for all n and $\|rx - x_n\| \to r-1$. Let $x^* \in S_{X^*}$ be such that $x^*(x) = 1$. Since

$$r - 1 \leq x^*(rx - x_n) \leq \|rx - x_n\|,$$

$x^*(x_n) \to 1$. Observe that $1 \geq \|\frac{x_n+x}{2}\| \geq x^*(\frac{x_n+x}{2})$ and this implies that $\|\frac{x_n+x}{2}\| \to 1$. Since X is CLUC, (x_n) has a convergent subsequence.

Let $0 < r < 1$ and (x_n) be a minimizing sequence in S_X with respect to rx. Then, for all n, $x_n \in P_{S_X}(rx, r\delta_n)$ for some (δ_n) such that $\delta_n > 0$ for all n and $\delta_n \to 0$. By Lemma 3.11, $x_n \in C[x, \delta_n]$ and hence (x_n) has a convergent subsequence.

(2) $\Rightarrow$ (1): Let $x_n, y \in S_X$ be such that $\|\frac{x_n+y}{2}\| \to 1$. It is easy to verify that $x_n \in C[y, \delta_n]$ for some (δ_n) such that $\delta_n > 0$ for all n and $\delta_n \to 0$. By Lemma 3.11, $\frac{x_n+y}{\|x_n+y\|} \in P_{S_X}(\frac{y}{2}, \delta_n)$ for all n. Since S_X is approximatively compact at $\frac{y}{2}$, $(\frac{x_n+y}{\|x_n+y\|})$ has a convergent subsequence and hence (x_n) has a convergent subsequence. $\square$

As a consequence of the preceding result, we present a characterization of LUC. We need the following elementary observation.

Lemma 3.13. *The following statements are equivalent.*

(1) *X is strictly convex.*

(2) *For every $x \in S_X$, $P_{S_X}(rx) = \{x\}$ for all $r > 0$.*

(3) *For every $x \in S_X$, $P_{S_X}(rx) = \{x\}$ for all $0 < r < 1$.*

Proof. To prove (1) $\Rightarrow$ (2), it is sufficient to show that $P_{S_X}(x)$ is a singleton for all $x \in B_X\backslash\{0\}$. Let $x \in B_X\backslash(\{0\} \cup S_X)$. Suppose $y, z \in P_{S_X}(x)$. Then $\|y - x\| = 1 - \|x\| = \|y\| - \|x\|$ and hence $\|y\| = \|y - x\| + \|x\|$. By strict convexity, there exists $\alpha > 0$ such that $y - x = \alpha x$. Similarly, there exists $\beta > 0$ such that $z - x = \beta x$. Since $y = (1+\alpha)x$, $z = (1+\beta)x$ and $\|y\| = \|z\|$, $\alpha = \beta$. To prove (3) $\Rightarrow$ (1), let $[y,z] \subset S_X$ for some $y, z \in S_X$ and $y \neq z$. Let $x = \frac{y+z}{4}$. Then $\|x\| = \frac{1}{2}$, $d(x, S_X) = \frac{1}{2}$ and $\frac{y+z}{2}, \frac{3y+z}{4} \in P_{S_X}(x)$ which is a contradiction. $\square$

Theorem 3.14. *Consider the following statements.*

(1) *X is LUC.*

(2) *For every $x \in S_X$ and $r > 0$, S_X is approximatively compact and Chebyshev at rx.*

(3) *For every $x \in S_X$ and $r \geq 1$, S_X is approximatively compact and Chebyshev at rx.*

(4) *X is MLUC.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

Proof. The implications (1) $\Leftrightarrow$ (2) follow from Theorem 3.12 and Lemma 3.13. It is clear that (2) $\Rightarrow$ (3). The proofs of the implications (3) $\Leftrightarrow$ (4) follow from the proof of Theorem 5.3.28 of [16]. □

We now present some consequences of Theorem 3.12 and Theorem 3.14. We need the following definitions.

Definition 3.15. (1) (see [6]) Let M be a proximinal subset of X. The metric projection P_M is said to be *outer radially lower semicontinuous* (in short, o.r.l.s.c.) at x_0 if for every $y_0 \in P_M(x_0)$ and an open set V satisfying $V \cap P_M(x_0) \neq \emptyset$ there exists a neighborhood U of x_0 such that $V \cap P_M(x) \neq \emptyset$ for all x in $U \cap \{x_0 + \lambda(x_0 - y_0) : \lambda \geq 0\}$.

(2) Let M be a proximinal subset of X. An element $z \in X$ is called a *DL-point* of M if $z \in (x, y)$ for some $x \in X$ and $y \in P_M(x)$.

The implication (1) $\Rightarrow$ (2) in each of the next two results are proved in [6, Theorem 2.6 and Theorem 2.2]. We discuss the reverse implications as consequences of Theorems 3.12 and 3.14.

Theorem 3.16. *The following two statements are equivalent.*

(1) *X is CLUC.*

(2) *Every proximinal subset M of X is approximatively compact at all DL-points of M.*

Proof. It is sufficient to show (2) $\Rightarrow$ (1). Let $M = S_X$. It is clear that M is proximinal and all the points of $M^c \backslash \{0\}$ are DL-points of M. Therefore, by (2), S_X is approximatively compact at all $x \in X \backslash \{0\}$. By Theorem 3.12, X is *CLUC.* □

Theorem 3.17. *Consider the following statements.*

(1) *X is CLUC.*

(2) *For any proximinal subset M of X and $x \in X$, if P_M is o.r.l.s.c. at x and $P_M(x)$ is compact then M is approximatively compact at x.*

Then (1) $\Rightarrow$ (2). *If X is strictly convex then* (2) $\Rightarrow$ (1).

Proof. Let X be strictly convex, $M = S_X$ and $x \in S_X$. By Lemma 3.13, $P_{S_X}(rx) = \{x\}$ for all $r > 0$. Hence P_M is o.r.l.s.c. at all points of $X\backslash\{0\}$. Hence by (2), S_X is approximatively compact at all points of $X\backslash\{0\}$. By Theorem 3.14, X is LUC. □

4. Nearly uniformly convex spaces

In this section we present some proximinality properties of NUC spaces. We refer to [14, 18, 21, 23] for some examples and properties of NUC spaces.

The following result which is analogous to Theorem 1.1 is proved in [18, 23].

Theorem 4.1. *The following three statements are equivalent.*

(1) *X is nearly uniformly convex.*

(2) *For every $\epsilon > 0$, there exists $\delta > 0$ such that $\alpha(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$.*

(3) *For every $\epsilon > 0$, there exists $\delta > 0$ such that $\alpha(C) < \epsilon$ for any convex subset C of B_X such that $d(0, C) \geq 1 - \delta$.*

Proximinality properties of *NUC* are derived in the next result.

For a subset A of X and $0 \leq \gamma \leq \beta$, let $D_\gamma^\beta(A) = \{x \in X : \gamma \leq d(x, A) \leq \beta\}$ and $D_1(A) = \{x \in X : d(x, A) = 1\}$.

Theorem 4.2. *The following statements are equivalent.*

(1) *X is NUC.*

(2) *For every $\epsilon > 0$ and $0 < \gamma \leq \beta$, there exists $\delta > 0$ such that $\alpha(P_A(x, \delta)) < \epsilon$ for any closed convex subset A of X and $x \in D_\gamma^\beta(A)$.*

(3) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(P_M(x, \delta)) < \epsilon$ for any hyperplane $M = \{y \in X : x^*(y) = 0\}$, $x^* \in S_{X^*}$ and $x \in D_1(M)$.*

(4) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(P_H(0, \delta)) < \epsilon$ for any $H = \{y \in X : x^*(y) = 1\}$, $x^* \in S_{X^*}$.*

Proof. (1) $\Rightarrow$ (2): Let $\epsilon > 0$. Since X in NUC, by Theorem 4.1, there exists $\delta > 0$ such that $\alpha(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$. We assume that $\delta < 1$. Let $0 < \gamma \leq \beta$ for some γ and β and let A be a closed convex subset of X. We show that $\alpha(P_A(x, \gamma\delta)) \leq 2\beta\epsilon$ for all $x \in D_\gamma^\beta(A)$. Suppose $x \in D_\gamma^\beta(A)$ and $d(x, A) = d$. Let $C = A - x$ and observe that $d(0, C) = d$. Find $x^* \in S_{X^*}$ such that $x^*(y) \geq d$ for all $y \in C$. We now show that $\alpha(P_C(0, d\delta)) \leq 2\beta\epsilon$. Choose $y \in P_C(0, d\delta)$ and let $\overline{y} = \frac{y}{d(1+\delta)}$. Then $\overline{y} \in S(X, x^*, \delta)$. This shows that $P_C(0, d\delta) \subseteq d(1 + \delta)S(X, x^*, \delta)$. Therefore

$$\alpha(P_C(0, d\delta)) \leq d(1 + \delta)\epsilon < 2d\epsilon \leq 2\beta\epsilon.$$

Since $P_A(x, d\delta) - x = P_C(0, d\delta)$ we get $\alpha(P_A(x, d\delta)) \leq 2\beta\epsilon$, which implies that $\alpha(P_A(x, \gamma\delta)) \leq 2\beta\epsilon$.

(2) ⇒ (3) : This is immediate.

(3) ⇒ (4): Let $x^* \in S_{X^*}$, $H = \{x \in X : x^*(x) = 1\}$, $M = \{x \in X : x^*(x) = 0\}$ and $x_0 \in H$. Observe that $d(0, H) = d(x_0, M) = 1$ and $P_H(0, \delta) = x_0 - P_M(x_0, \delta)$ for any $\delta \geq 0$. The rest follows from (3).

(4) ⇒ (1): Let $\epsilon > 0$. Then by (4), there exists $\delta > 0$ such that $\delta < \epsilon$ and $\alpha(P_H(0, \delta)) < \epsilon$ for any $H = \{y \in X : x^*(y) = 1\}$ defined by some $x^* \in S_{X^*}$. Let $x_0^* \in S_{X^*}$ and $H_0 = \{x \in X : x_0^*(x) = 1\}$. Suppose $x_0 \in S(X, x_0^*, \frac{\delta}{2})$. Since $d(x_0, H_0) < \delta$, find $\overline{x} \in H_0$ such that $\|x_0 - \overline{x}\| < \delta$. Note that

$$\|\overline{x}\| \leq \|x_0\| + \|x_0 - \overline{x}\| \leq 1 + \delta.$$

Therefore $\overline{x} \in P_{H_0}(0, \delta)$. Since $\|x_0 - \overline{x}\| < \epsilon$, $\alpha(S(X, x^*, \frac{\delta}{2})) < 2\epsilon$. By Theorem 4.1, X is *NUC*. □

We now discuss a continuity property of the metric projection in *NUC* spaces.

Definition 4.3. [12] Let $F\colon X \to 2^X$ be a set valued mapping and $C \subseteq X$. We say that F is *uniformly α-upper semi-continuous* on $C \subseteq X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(F(B_X(x, \delta) \cap C)) < \epsilon$ for every $x \in C$.

The argument of the proof of the next result will also be used in Section 5.

Corollary 4.4. *Let X be NUC and A a closed convex subset of X. The metric projection P_A is uniformly α-upper semi-continuous on $D_\gamma^\beta(A)$ for any γ and β satisfying* $0 < \gamma \leq \beta$.

Proof. Let A be a closed convex subset of X, $0 < \gamma \leq \beta$ and $\epsilon > 0$. By Theorem 4.2, there exists $\delta > 0$ such that $\alpha(P_A(z, \delta)) < \epsilon$ for all $z \in D_\gamma^\beta(A)$. Let $x_0 \in D_\gamma^\beta(A)$ and $x \in B_X(x_0, \frac{\delta}{2})$ and $y \in P_A(x)$. Then

$$\|x_0 - y\| \leq \|x_0 - x\| + \|x - y\| \leq \frac{\delta}{2} + d(x, A) \leq \frac{\delta}{2} + \frac{\delta}{2} + d(x_0, A).$$

Therefore $y \in P_A(x_0, \delta)$ and hence $P_A(x) \subseteq P_A(x_0, \delta)$. Since x is an arbitrary element in $B_X(x_0, \frac{\delta}{2})$, $\cup\{P_A(x) : x \in B_X(x_0, \frac{\delta}{2})\} \subseteq P_A(x_0, \delta)$. By the assumption, $\alpha(P_A(B_X(x_0, \frac{\delta}{2}))) < \epsilon$. □

A proximinality property of the closed unit ball of any NUC space is observed in the following result. We will also use the next result in Section 5.

Proposition 4.5. *Consider the following statements.*

(1) *X is NUC.*

(2) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(P_{B_X}(x, \delta)) < \epsilon$ for every $x \in X$.*

Then (1) ⇒ (2).

Proof. Let $\epsilon > 0$. Since X is NUC, by Theorem 4.1, there exists $\delta > 0$ such that $\alpha(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$. Let $z \in X \backslash B_X$ and $x \in P_{B_X}(z, \delta)$. Observe that $d(z, B_X) = \|z\| - 1$. Find $x^* \in S_{X^*}$ such that $x^*(z) = \|z\|$. Then

$$x^*(z - x) \leq \|z - x\| \leq (\|z\| - 1) + \delta.$$

This implies that $x \in S(X, x^*, \delta)$ and hence $\alpha(P_{B_X}(z, \delta)) < \epsilon$. □

5. Compactly uniformly convex spaces

We present some geometric and proximinality properties of *CUC* spaces. *CUC* spaces are discussed in [7, 15, 21].

The proof of the following result follows from the proof of Theorem 3.3.

Theorem 5.1. *The following statements are equivalent.*

(1) *X is CUC.*

(2) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(A_\delta(x)) \leq \epsilon$ for all $x \in S_X$.*

(3) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(A_\delta(X^{**}, x)) \leq \epsilon$ for all $x \in S_X$.*

The following well known result is immediate from Theorem 4.1 and Theorem 5.1.

Corollary 5.2. *Every CUC space is NUC.*

Proof. Let $x^* \in S_{X^*}$ and $\delta > 0$. Choose $x \in S(X, x^*, \delta) \cap S_X$. Then $S(X, x^*, \delta) \subseteq A_\delta(x)$. Apply (2) of Theorem 5.1 to get (2) of Theorem 4.1. □

We now derive a result which is analogous to Theorem 1.1 and Theorem 4.1.

Theorem 5.3. *The following statements are equivalent.*

(1) *X is CUC.*

(2) *For $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in S_X$, $\alpha(\cup\{C : C$ is a convex subset of B_X, $x \in C$ and $d(0, C) \geq 1 - \delta\}) < \epsilon$.*

Proof. (1) ⇒ (2): Let $\epsilon > 0$. By Theorem 5.1, there exists $\delta > 0$ such that $\alpha(A_\delta(x)) \leq \epsilon$ for all $x \in S_X$. Let $x \in S_X$. Suppose C is a convex subset of B_X, $x \in C$ and $d(0, C) \geq 1 - \frac{\delta}{2}$. By following the proof of (1) ⇒ (2) of Theorem 3.7, we show that $C \subseteq A_\delta(x)$ which proves the implication.

(2) ⇒ (1): Observe that for a given $\delta > 0$ and $x \in S_X$,

$$A_\delta(x) \subseteq \cup\{C : \ C \text{ is a convex subset of } B_X, x \in C \text{ and } d(0, C) \geq 1 - \delta\}.$$

Hence (2) ⇒ (1) follows from Theorem 5.1. □

Remark 5.4. We remark that Corollary 5.2 can also be obtained from Theorem 5.3 as follows. We show that (3) of Theorem 4.1 can be obtained from (2) of Theorem 5.3. Suppose (2) of Theorem 5.3 holds true. Let ϵ and δ be as in

(2) of Theorem 5.3. Suppose C is a convex subset of B_X and $d(0, C) \geq 1 - \delta$. By separating C and $B[0, 1-\delta]$ by some $x^* \in S_{X^*}$, find $x \in S_X \cap S(X, x^*, \delta)$ and let C_0 be the convex hull of $\{x\} \cup C$. Hence C_0 is a convex subset of B_X containing x and $d(0, C_0) \geq 1 - \delta$. Therefore, by the assumption $\alpha(C) < \epsilon$.

In the following result we present a proximinality property of *CUC* spaces.

Theorem 5.5. *The following statements are equivalent.*

(1) *X is CUC.*

(2) *For every $r > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\alpha(P_{S_X}(rx, \delta)) < \epsilon$ for all $x \in S_X$.*

Proof. (1) $\Rightarrow$ (2): Let $r > 1$. In this case, the implication follows from Corollary 5.2 and Proposition 4.5. Suppose $0 < r < 1$. In this case, the implication follows from Lemma 3.11 and the definition of *CUC*.

(2) $\Rightarrow$ (1): Let $\epsilon > 0$ and $0 < r < 1$. Then there exists $\delta > 0$ such that $\alpha(P_{S_X}(rx, \delta)) < \epsilon$ for all $x \in S_X$. Assume that $\delta < \epsilon$ and let $x \in S_X$. Define

$$M_r(x, \delta) = B_X[rx, 1 - r + \delta] \backslash B_X(0, 1).$$

Let $y \in M_r(x, \delta)$. Then $\|y\| \geq 1$ and $\|rx - y\| \leq 1 - r + \delta$. Let $z = [rx, y] \cap S_X$. Since $\|rx - z\| \geq \|rx - x\|$, $\|y - z\| \leq \delta$. Observe that

$$z \in M_r(x, \delta) \cap S_X \subseteq P_{S_X}(rx, \delta).$$

This shows that $\alpha(M_r(x, \delta)) \leq \alpha(P_{S_X}(rx, \delta)) + \delta$. By assumption we know that $\alpha(M_r(x, \delta)) \leq 2\epsilon$. It follows from Theorem 2 of [15] and Proposition 2.5 of [7] that X is *CUC*. $\square$

6. Uniformly convex space

We present some geometric and proximinality properties of uniform convexity in this section. In the next two characterizations of uniform convexity, we include some known equivalences for the sake of completeness.

Proposition 6.1. *The following statements are equivalent.*

(1) *X is uniformly convex.*

(2) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $diam(S(X^{**}, x^*, \delta)) \leq \epsilon$ for all $x^* \in S_{X^*}$.*

(3) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $diam(S(X, x^*, \delta)) \leq \epsilon$ for all $x^* \in S_{X^*}$.*

(4) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $diam(A_\delta(x)) \leq \epsilon$ for all $x \in S_X$.*

(5) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $diam(A_\delta(X^{**}, x)) \leq \epsilon$ for all $x \in S_X$.*

(6) *For every* $\epsilon > 0$ *and every* r *with* $0 < r < 1$ *there exists* $\delta > 0$ *such that* $diam(N_r(x,\delta)) \leq \epsilon$ *for all* $x \in S_X$.

(7) *For every* $\epsilon > 0$ *and every* r *with* $0 < r < 1$ *there exists* $\delta > 0$ *such that* $diam(N_r^{**}(x,\delta)) \leq \epsilon$ *for all* $x \in S_X$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): This is known (see e.g. Proposition 6.6 of [4]).

(3) $\Rightarrow$ (4): Let $\epsilon > 0$. It follows from (3) that there exists $\delta > 0$ such that $diam(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$. Let $x \in S_X$ and $y \in A_\delta(x)$. Since $y \in A_\delta(x)$, there exists $x^* \in S_{X^*}$ such that $y \in S(X, x^*, \delta)$ and $x \in S(X, x^*, \delta)$. Hence $y \subseteq B(x, 2\epsilon)$ and therefore $A_\delta(x) \subseteq B(x, 2\epsilon)$.

(4) $\Rightarrow$ (3): This is obvious.

(2) $\Leftrightarrow$ (5): The proofs are similar to the proofs of (3) $\Leftrightarrow$ (4).

(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7): The proofs follow from Lemma 3.9. □

We remark that the implication (1) $\Leftrightarrow$ (6) of Proposition 6.1 is known (see e.g. [15]).

Recall that for a non-empty closed subset A of X and $\gamma \geq 0$ the notation $D_0^\gamma(A) = \{x \in X : d(x, A) \leq \gamma\}$ is used.

Definition 6.2. Let A be a closed subset of X and B be a subset of X. We say that A is *uniformly strongly Chebyshev* on B if for $\epsilon > 0$ there exists $\delta > 0$ such that $diam(P_A(x, \delta)) < \epsilon$ for all $x \in B$.

We present some proximinality properties of uniform convexity.

Theorem 6.3. *The following statements are equivalent.*

(1) *X is uniformly convex.*

(2) *For a given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $diam(P_H(0, \delta)) < \epsilon$ *for any* $H = \{y \in X : x^*(y) = 1\}$ *defined by some* $x^* \in S_{X^*}$.

(3) *For a given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $diam(P_M(x, \delta)) < \epsilon$ *for any hyperplane* $M = \{y \in X : x^*(y) = 0\}$, $x^* \in S_{X^*}$ *and* $x \in D_1(M)$.

(4) *For a given* $\epsilon > 0$ *and* $\gamma \geq 0$, *there exists* $\delta > 0$ *such that* $diam(P_A(x, \delta)) < \epsilon$ *for any closed convex subset* A *of* X *and* $x \in D_0^\gamma(A)$.

(5) *Every closed convex subset* A *of* X *is uniformly strongly Chebyshev on* $D_0^\gamma(A)$ *for any* $\gamma \geq 0$.

(6) *Every closed convex subset is uniformly strongly Chebyshev on any bounded subset of* X.

(7) *The unit ball is uniformly strongly Chebyshev on* $2S_X$.

(8) *The unit ball is uniformly strongly Chebyshev on* X.

(9) *The unit sphere is uniformly strongly Chebyshev on* $\frac{1}{2}S_X$.

(10) *For a given $\varepsilon > 0$ there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$ holds for every closed subspace M of X and $x \in D_1(M)$.*

(11) *For a given $\varepsilon > 0$ there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$ holds for every closed hyperplane M of X and $x \in D_1(M)$.*

Proof. (1) $\Rightarrow$ (4): Let $\epsilon > 0$ and $\gamma > 0$. Since X is uniformly convex, by Theorem 6.1, there exists $\delta > 0$ such that $diam(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$. Let A be any closed convex subset of X. By following the proof of (1) $\Rightarrow$ (2) of Theorem 4.2, we show that $diam(P_A(x, \gamma\delta)) \leq 2\gamma\epsilon$ for all $x \in D^\gamma_\gamma(A)$. We now show that $diam(P_A(z, \gamma\delta)) < 2\gamma\epsilon$ for all $z \in D^\gamma_0(A)$. Let $y \in D^\gamma_0(A) \backslash (D^\gamma_\gamma(A) \cup A)$, $\lambda = \frac{\gamma}{d(y,A)}$ and $z = \lambda y + (1 - \lambda)P_A(y)$. Then it is easy to verify that $z \in D^\gamma_\gamma(A)$ and $P_A(y, \gamma\delta) \subseteq P_A(z, \gamma\delta)$. This shows that $diam(P_A(y, \gamma\delta)) < \gamma\epsilon$ for all $y \in D^\gamma_0(A)$.

(4) $\Rightarrow$ (3): This is immediate.

(3) $\Rightarrow$ (2): The proof is similar to the proof of (3) $\Rightarrow$ (4) of Theorem 4.2.

(2) $\Rightarrow$ (1): The proof is similar to the proof of (4) $\Rightarrow$ (1) of Theorem 4.2.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7): These are immediate.

(7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1): These implications are proved in [25].

(1) $\Leftrightarrow$ (10) $\Leftrightarrow$ (11): These implications are proved in [9]. □

Corollary 6.4. *If X is uniformly convex and A is a closed convex subset of X then the metric projection P_A is uniformly continuous on $\{x \in X : d(x, A) \leq \gamma\}$ for every $\gamma \geq 0$. In particular, P_A is uniformly continuous on bounded subsets of X.*

Proof. The proof is similar to the proof of Corollary 4.4. □

Acknowledgments. We thank the referee for suggestions which led to an improvement of this paper.

References

[1] A. Aizpuru, F. J. Garcia-Pacheco: Some questions about rotundity and renormings in Banach spaces, J. Aust. Math. Soc. 79 (2005) 131–140.

[2] J. Banaś, K. Goebel: Measures of Non-Compactness in Banach Spaces, Lecture Notes Pure Appl. Math. 60, Marcel Dekker, New York (1980).

[3] P. Bandyopadhyay, Y. Li, B.-L. Lin, D. Narayana: Proximinality in Banach spaces, J. Math. Anal. Appl. 341 (2008) 309–317.

[4] A. K. Chakrabarty, P. Shunmugaraj, C. Zălinescu: Continuity properties for the subdifferential and ε-subdifferential of a convex function and its conjugate, J. Convex Analysis 14 (2007) 479–514.

[5] S. Cobzas: Geometric properties of Banach spaces and the existence of nearest and farthest points, Abstr. Appl. Anal. 3 (2005) 259–285.

[6] F. Deutsch, J. Lambert: On continuity of metric projections, J. Approx. Theory 29 (1980) 116–131.

[7] S. J. Dilworth, D. Kutzarova, N. L. Randrianarivony, J. P. Revalski, N. V. Zhivkov: Compactly uniformly convex spaces and property (β) of Rolewich, J. Math. Anal. Appl. 402 (2013) 297–307.

[8] S. Dutta, P. Shunmugaraj: Strong proximinality of closed convex sets, J. Approx. Theory 163 (2011) 547–553.

[9] S. Dutta, P. Shunmugaraj, V. Thota: Uniform strong proximinality and continuity of metric projection, J. Convex Analysis 24(4) (2017) 1263–1279.

[10] K. Fan, I. Glicksberg: Some geometric properties of the spheres in a normed linear space, Duke Math. J. 25 (1958) 553–568.

[11] J. R. Giles, D. A. Gregory, B. Sims: Geometrical implications of upper semicontinuity of the duality mapping on a Banach space, Pacific J. Math. 79 (1978) 99–109.

[12] J. R. Giles, W. B. Moors: A continuity property related to Kuratowski's index of non-compactness, its relevance to the drop property and its implications for differentiabilty theory, J. Math. Anal. Appl. 178 (1993) 247–268.

[13] A. J. Guirao, V. Montesinos: A note in approximative compactness and continuity of metric projection in Banach spaces, J. Convex Analysis 18 (2011) 397–401.

[14] R. Huff: Banach spaces which are nearly uniformly convex, Rocky Mountain J. Math. 10 (1980) 743–749.

[15] D. N. Kutzarova, P. L. Papini: On a characterization of property (β) and LUR, Boll. Un. Mat. Ital. (A)(7) 6 (1992) 209–214.

[16] R. E. Megginson: An Introduction to Banach Space Theory, Springer, New York (1998).

[17] R. E. Megginson: The semi-Kadec-Klee condition and nearest point properties of sets in normed linear spaces, Ph.D. Thesis, University of Illinois at Urbana-Champaign (1984).

[18] V. Montesinos, J. R. Torregrosa: Some properties which generalize uniform rotundity of Banach spaces, Arch. Math. 59 (1992) 457–467.

[19] B. B. Panda, O. P. Kapoor: Approximative compactness and continuity of metric projections, Bull. Austral. Math. Soc. 11 (1974) 47–55.

[20] B. B. Panda, O. P. Kapoor: A generalization of local uniform convexity of the norm, J. Math. Anal. Appl. 52 (1975) 300–308.

[21] J. P. Revalski, N. V. Zhivkov: Best approximation problems in compactly uniformly rotund spaces, J. Convex Analysis 19 (2012) 1153–1166.

[22] J. P. Revalski, N. V. Zhivkov: Small sets in best approximation theory, J. Global Optim. 50 (2011) 77–91.

[23] S. Rolewicz: On Δ-uniform convexity and drop property, Studia Math. 87 (1987) 181–191.

[24] P. Shunmugaraj: Convergence of slices, geometric aspects in Banach spaces and proximinality, Nonlinear Analysis 61–107, part of Trends in Mathematics, Springer, New Delhi (2014).

[25] P. Shunmugaraj, V. Thota: Uniform convexity, strong convexity and property UC, J. Math. Anal. Appl. 446 (2017) 1769–1775.

[26] L. P. Vlasov: Approximative properties of sets in normed linear spaces, Russian Math. Surveys 28 (1973) 1–66.

[27] Z. Zhang, C. Liu, Y. Zhou: Some examples concerning proximinality in Banach spaces, J. Approx. Theory 200 (2015) 136–143.