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Tight SDP Relaxations for a Class of Robust SOS-convex Polynomial Programs without the Slater Condition

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We examine a robust SOS-convex polynomial program under a general class of bounded uncertainty, called intersection ellipsoidal uncertainty, which is described by the intersection of a family of ellipsoids and covers many commonly used uncertainty classes of robust optimization. The class of SOS-convex polynomials is a numerically tractable subclass of convex polynomials and, in particular, contains convex quadratic functions and convex separable polynomials. We present a conical linear program with a coupled sum-of-squares polynomial and linear matrix inequality constraints as its relaxation problem and show that the relaxation problem can equivalently be reformulated as a semidefinite linear program (SDP). Under a mild well-posedness condition, we establish that the so-called SDP relaxation is *tight* in the sense that the optimal values of the robust SOS-convex polynomial program and its relaxation problem are equal. We also show, under a general regularity condition, that the SDP relaxation is *exact* in the sense that the relaxation problem not only shares the same optimal value with the robust SOS-convex polynomial program but also attains its optimum, extending the corresponding known results in the robust optimization literature to the general class of intersection ellipsoidal uncertainty.

Keywords: Robust optimization, semi-infinite convex program, convex polynomial, semidefinite linear program, SDP relaxation.

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1. Introduction

In this paper we study a class of convex polynomial optimization problems for which the constraint data are not specified exactly and they are only known to belong to given uncertainty sets, yet the constraints must be satisfied for all possible values of the data from the *uncertainty* sets. The resulting optimization problems are known as the *robust* convex optimization problems [3, 6]. The robust convex optimization problem is a *semi-infinite* [7, 8, 10, 12, 13, 14, 24, 28, 29] convex optimization problem and it cannot, in general, efficiently be solved numerically. A great deal of attention has been focused on developing semidefinite linear programming (SDP) relaxation schemes for solving classes of robust convex polynomial problems [20, 27, 35, 36]. These relaxation schemes generally rely on the structure of the considered problem as well as on the geometry of the uncertainty sets [5].

Recent research has shown that a robust SOS-convex polynomial program with *polytopic* or *restricted ellipsoidal* uncertainty exhibits an *exact* SDP relaxation in the sense that the optimal values of the robust SOS-convex polynomial program and its SDP relaxation problem are equal and the relaxation problem attains its optimum whenever the Slater constraint qualification is satisfied [20, 21, 25]. The SOS-convexity [1, 11] is a restricted, but numerically tractable, notion of convexity for polynomials. Moreover, the scheme applies only to a limited class of uncertainty sets and the Slater constraint qualification may fail for robust SOS-convex polynomial programs with broad classes of uncertainty sets (see e.g., Example 2.6 and Example 3.4 below).

The purpose of this paper is to examine the robust SOS-convex polynomial program under a general class of bounded intersection ellipsoidal uncertainty sets, which is described by the intersection of a family of ellipsoids (see [4] for a more general concept). These uncertainty sets are closed convex sets and possess a broad spectrum of infinite convex sets including polytopes, balls and ellipsoids that commonly appear in robust optimization [3, 6]. For example, the restricted ellipsoidal uncertainty sets considered in [20] are particular cases of the intersection ellipsoidal uncertainty set.

The main aim is to derive semidefinite programming relaxation results without requiring the Slater constraint qualification. Our study reveals that a conical linear program with a coupled sum-of-squares polynomial and linear matrix inequality constraints serves as our SDP relaxation problem inasmuch as the conical program can equivalently be reformulated as a semidefinite linear program. Under a mild well-posedness condition, we establish that the SDP relaxation is *tight* in the sense that the optimal values of the robust SOS-convex polynomial program and its relaxation problem are equal. We prove this by employing the strong hyperplane separation and the convex-concave minimax theorem of convex analysis, together with SOS-convexity of polynomials and a special variable transformation of a bounded intersection ellipsoidal set.

We also show, under a general regularity condition, that the SDP relaxation is *exact* in the sense that the relaxation problem not only shares the same optimal value with the robust SOS-convex polynomial program but also attains its optimum, extending the corresponding known results in the robust optimization literature to the general class of intersection ellipsoidal uncertainty. The regularity condition is shown to be satisfied whenever the Slater constraint qualification holds. Our exact SDP relaxation result extends and unifies the corresponding relaxation results of the literature (see e.g., [21, 20]) because many commonly used uncertainty sets can be expressed as an intersection ellipsoidal uncertainty set.

The outline of the paper is as follows. Section 2 presents a tight SDP relaxation for the robust SOS-convex polynomial program. Section 3 provides relaxation results with solution attainment under a general qualification condition. Section 4 concludes with a discussion on further research on SDP relaxation schemes for solving robust convex polynomial programs.

2. Tight SDP relaxations

We begin this section by fixing the notation and preliminaries that will be used throughout the paper. The symbol I_n stands for the *identity* $(n \times n)$ *matrix* with $n \in \mathbb{N} := \{1, 2, \ldots\}$. A symmetric $(n \times n)$ matrix A is said to be *positive semi-definite*, denoted by $A \succeq 0$, whenever $x^\top A x \geq 0$ for all $x \in \mathbb{R}^n$. The notation $\mathbb{R}^n$ signifies the Euclidean space whose norm is denoted by $\|\cdot\|$ for each $n \in \mathbb{N}$. The *inner product in* $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^\top y$ for all $x, y \in \mathbb{R}^n$. As usual, the *nonnegative orthant* of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ := \{y := (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_i \geq 0, i = 1, \ldots, n\}$. Denote by $\mathbb{R}[x]$ the ring of polynomials in x with real coefficients. The polynomial $f \in \mathbb{R}[x]$ is a *sum-of-squares polynomial* (cf. [26]) if there exist polynomials $f_j \in \mathbb{R}[x], j = 1, \ldots, r$ such that $f = \sum_{j=1}^r f_j^2$. The set of all sum-of-squares polynomials in x with degree at most d is denoted $\Sigma^2_d[x]$. Recall that a real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *SOS-convex* (see e.g., [11, 1]) if the polynomial $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $F(x, y) := f(x) - f(y) - \nabla f(y)^\top (x - y)$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, is a sum-of-squares polynomial. Clearly, the class of SOS-convex polynomials is a subclass of convex polynomials and it includes in particular all convex quadratic functions and all convex separable polynomials.

Model problem. We consider the *robust* SOS-convex polynomial program

$$\text{(P)} \qquad \inf_{x\in\mathbb{R}^n} \{f(x) \mid p_j^0(x) + \sum_{i=1}^{s_j} t_j^i p_j^i(x) \leq 0,\ \forall (t_j^1, \ldots, t_j^{s_j}) \in T_j,\ j \in \mathcal{Q}\},$$

where $\mathcal{Q} := \{1, \ldots, q\}$, $f : \mathbb{R}^n \to \mathbb{R}$ is an SOS-convex polynomial, $p_j^0, p_j^i : \mathbb{R}^n \to \mathbb{R}$, $i \in \mathcal{S}_j := \{1, \ldots, s_j\}$, $j \in \mathcal{Q}$, are polynomials, and T_j, $j \in \mathcal{Q}$, are prescribed *uncertainty* sets. We assume that, for each fixed $t_j := (t_j^1, \ldots, t_j^{s_j}) \in T_j$, the function

$$g_j(x,t_j) := p_j^0(x) + \sum_{i=1}^{s_j} t_j^i p_j^i(x) \tag{1}$$

is an SOS-convex polynomial for $j \in \mathcal{Q}$.

Geometry of the uncertainty sets. The uncertainty sets $T_j, j \in \mathcal{Q}$, are assumed to be *bounded* intersection ellipsoidal (see [4] for a more general concept), described by

$$T_j := \{t_j \in \mathbb{R}^{s_j} \mid t_j^\top M_j^l t_j + (a_j^l)^\top t_j + b_j^l \leq 0, l \in \mathcal{Q}_j\}, \tag{2}$$

where $\mathcal{Q}_j := \{1,\ldots,q_j\}, a_j^l \in \mathbb{R}^{s_j}, b_j^l \in \mathbb{R}, l \in \mathcal{Q}_j, j \in \mathcal{Q}$ and $M_j^l \succeq 0, l \in \mathcal{Q}_j, j \in \mathcal{Q}$ are symmetric $(s_j \times s_j)$ matrices. These uncertainty sets are closed convex sets and possess a broad spectrum of infinite convex sets including polytopes, balls and ellipsoids that commonly appear in robust optimization [3, 6].

Well-Posedness. We say that the problem (P) is *well-posed* if the set $\Omega := \{(r,z_1,\ldots,z_q) \in \mathbb{R}\times\mathbb{R}^q \mid \exists x \in \mathbb{R}^n,\ f(x) \leq r,\ g_j(x,t_j) \leq z_j,\ \forall t_j \in T_j,\ j \in \mathcal{Q}\}$ is closed.

It is easy to check using the standard sequence arguments that if either f or $g_j(\cdot,t_j)$ for some $t_j \in T_j, j \in \mathcal{Q}$ is *coercive* on $\mathbb{R}^n$ then Ω is closed; i.e., the problem (P) is well-posed. Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *coercive* if $\liminf_{||x||\to\infty} f(x) = +\infty$. The coercivity of a polynomial can easily check numerically as shown in [18]. The interested reader is referred to [23] for other criteria ensuring the well-posedness of (P).

The following proposition gives another sufficient condition for well-posedness in the case where Ω has a representation in the sense that

$$\Omega = \{(r,z_1,\ldots,z_q) \in \mathbb{R}\times\mathbb{R}^q \mid \exists x \in \mathbb{R}^n,\ f(x) \leq r,\ g_{ji}(x) \leq z_j,\ i \in \mathcal{L}_j,\ j \in \mathcal{Q}\}.$$

for some convex polynomials $g_{ji}, i \in \mathcal{L}_j := \{1,\ldots,\ell_j\}, j \in \mathcal{Q}$. Note that such a representation holds for (P) if in particular, each T_j is a singleton or more generally, each T_j is a polytope for $j \in \mathcal{Q}$ (see Corollary 2.5 below).

Proposition 2.1. (Sufficient conditions for well-posedness) *For the problem* (P)*, assume that there exist convex polynomials $g_{ji}, i \in \mathcal{L}_j, j \in \mathcal{Q}$, such that*

$$\Omega = \{(r,z_1,\ldots,z_q) \in \mathbb{R}\times\mathbb{R}^q \mid \exists x \in \mathbb{R}^n,\ f(x) \leq r,\ g_{ji}(x) \leq z_j,\ i \in \mathcal{L}_j,\ j \in \mathcal{Q}\}.$$

Then, the problem (P) *is well-posed.*

Proof. We need to show that Ω is closed. Take an arbitrary sequence $\{(r^k,z_1^k,\ldots,z_q^k)\} \subset \mathbb{R}^{q+1}$ with $(r^k,z_1^k,\ldots,z_q^k) \in \Omega$ for all $k \in \mathbb{N}$. Assume that the sequence $\{(r^k,z_1^k,\ldots,z_q^k)\}$ converges to some $(\bar r,\bar z_1,\ldots,\bar z_q) \in \mathbb{R}^{q+1}$ as $k \to \infty$. Then, for each $k \in \mathbb{N}$, there exists $x^k \in \mathbb{R}^n$ such that $f(x^k) \leq r^k$ and $g_{ji}(x^k) \leq z_j^k, i \in \mathcal{L}_j,\ j \in \mathcal{Q}$.

Now, let $f_0(r, z_1, \ldots, z_q) := (r - \bar{r})^2 + \sum_{j=1}^{q}(z_j - \bar{z}_j)^2$ and put

$$\nu := \inf_{\substack{(x,r,z_1,\ldots,z_q)\\ \in\mathbb{R}^{n+q+1}}} \Big\{ f_0(r, z_1, \ldots, z_q) \mid f(x) - r \leq 0, g_{ji}(x) - z_j \leq 0, i \in \mathcal{L}_j, j \in \mathcal{Q} \Big\}.$$

We see that $0 \leq \nu \leq f_0(r^k, z_1^k, \ldots, z_q^k)$ for all $k \in \mathbb{N}$. In addition,

$$f_0(r^k, z_1^k, \ldots, z_q^k) = (r^k - \bar{r})^2 + \sum_{j=1}^{q}(z_j^k - \bar{z}_j)^2 \to 0 \quad \text{as} \quad k \to \infty,$$

and so, $\nu = 0$. The Frank-Wolfe theorem (cf. [2, Theorem 3]), asserts that there exists $(x^*, r^*, z_1^*, \ldots, z_q^*) \in \mathbb{R}^{n+q+1}$ such that $f(x^*) - r^* \leq 0$, $g_{ji}(x^*) - z_j^* \leq 0$, $i \in \mathcal{L}_j$, $j \in \mathcal{Q}$, and $f_0(r^*, z_1^*, \ldots, z_q^*) = 0$. It entails that $(\bar{r}, \bar{z}_1, \ldots, \bar{z}_q) = (r^*, z_1^*, \ldots, z_q^*) \in \Omega$, which concludes that Ω is closed. So, the problem (P) is well-posed. $\Box$

The well-posedness of the problem (P) guarantees the existence of solutions for a robust convex polynomial program, which can be regarded as an extension of the Frank-Wolfe theorem (cf. [2, Theorem 3]) from a finite number of inequality constraints to a union of finitely many compact sets of inequality constraints.

Proposition 2.2. (Solution attainment under well-posedness) *Assume that the problem* (P) *is well-posed. If f is bounded from below on the nonempty set $C := \{x \in \mathbb{R}^n \mid g_j(x, t_j) \leq 0, \forall t_j \in T_j, j \in \mathcal{Q}\}$, then f attains its minimum on C; i.e.,* argmin(P) $\neq \emptyset$.

Proof. Let $r := \inf_{x\in C} f(x)$. By our assumptions, $r \in \mathbb{R}$ and the sequence $\{(r + \frac{1}{k}, 0)\}_{k\in\mathbb{N}} \subset \Omega$. Since Ω is closed, it ensures that $(r + \frac{1}{k}, 0) \to (r, 0) \in \Omega$ as $k \to \infty$. Hence, there exists $\bar{x} \in \mathbb{R}^n$ with $g_j(\bar{x}, t_j) \leq 0$ for all $t_j \in T_j, j \in \mathcal{Q}$ such that $f(\bar{x}) \leq r$. This entails that $f(\bar{x}) = r$, which shows that $\bar{x} \in$ argmin(P). $\Box$

From now on, for each $j \in \mathcal{Q}$ and $l \in \mathcal{Q}_j$, we use the notation $W_j^l := (W_j^{l,1}, \ldots, W_j^{l,s_j})$ to denote a decomposition factor of M_j^l, i.e., $M_j^l = (W_j^l)^\top W_j^l$, and let $a_j^l := (a_j^{l,1}, \ldots, a_j^{l,s_j})$. Then, the intersection ellipsoidal uncertainty sets T_j, $j \in \mathcal{Q}$ in (2) admit linear matrix inequality representations as

$$\begin{aligned} t_j \in T_j \quad &\Leftrightarrow \quad t_j^\top M_j^l t_j + (a_j^l)^\top t_j + b_j^l \leq 0, \quad l \in \mathcal{Q}_j, \\ &\Leftrightarrow \begin{pmatrix} I_{s_j} & W_j^l t_j \\ (W_j^l t_j)^\top & -(a_j^l)^\top t_j - b_j^l \end{pmatrix} \succeq 0, \quad l \in \mathcal{Q}_j, \end{aligned} \tag{3}$$

where the last equivalence holds by the Schur complement (cf. [5, Lemma 4.2.1]).

The following theorem establishes a characterization of optimality for the problem (P) in terms of an asymptotic sum-of-squares condition and linear matrix inequalities (LMIs). Note that the sum-of-squares condition can also be equivalently expressed as LMIs.

Theorem 2.3. (LMI characterization of robust solutions) *Assume that the problem* (P) *is well-posed. Let* $\bar{x} \in \mathbb{R}^n$ *be a feasible point of problem* (P)*. Then,* $\bar{x}$ *is an optimal solution of problem* (P) *if and only if for any* $\epsilon > 0$*, there exist* $\mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, j \in \mathcal{Q}, i \in \mathcal{S}_j$ *such that*

$$f + \sum_{j=1}^{q} \left(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\right) - f(\bar{x}) + \epsilon \in \Sigma_d^2[x], \tag{4}$$

$$\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q}, \tag{5}$$

where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$.

Proof. [$\Longrightarrow$] Assume that $\bar{x} \in \mathbb{R}^n$ is an optimal solution of (P). Let $\epsilon > 0$, and let $G_j(x) := \max_{t_j \in T_j} g_j(x, t_j),\ x \in \mathbb{R}^n$ for $j \in \mathcal{Q}$. Then, we see that each G_j is a convex function finite on $\mathbb{R}^n$ because $g_j(\cdot, t_j)$ is a convex polynomial for each $t_j \in T_j$, $g_j(x, \cdot)$ is an affine function for each $x \in \mathbb{R}^n$, and T_j is a compact set.

[**Abstract Lagrangian condition via strong separation**] We first show that there exists $\lambda := (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}_+^q$ such that

$$\inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{q} \lambda_j G_j(x) \right\} - f(\bar{x}) + \epsilon > 0. \tag{6}$$

As the problem (P) is well-posed, the convex set

$$\Omega = \{(r, z_1, \ldots, z_q) \in \mathbb{R}^{1+q} \mid \exists x \in \mathbb{R}^n,\ f(x) \le r,\ G_j(x) \le z_j, j \in \mathcal{Q}\}$$

is closed and so, the convex set $\widetilde{\Omega} := \Omega + \{(\epsilon - f(\bar{x}), 0)\}$ is closed as well. Since $\bar{x}$ is an optimal solution of (P), we can deduce that $(0, 0) \notin \widetilde{\Omega}$.

The strong separation theorem (see e.g., [30, Theorem 2.2]) guarantees that there exists $0 \ne (\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^q$, where $\lambda := (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^q$, such that

$$\inf \left\{ \lambda_0(r + \epsilon - f(\bar{x})) + \sum_{j=1}^{q} \lambda_j z_j \mid (r, z_1, \ldots, z_q) \in \Omega \right\} > 0.$$

This ensures that $\lambda_0 \ge 0$ and $\lambda \in \mathbb{R}_+^q$, and there exists $\delta_0 > 0$ such that

$$\lambda_0(f(x) - f(\bar{x}) + \epsilon) + \sum_{j=1}^{q} \lambda_j G_j(x) > \delta_0,\ \forall x \in \mathbb{R}^n. \tag{7}$$

If $\lambda_0 = 0$, then we assert by (7) that $\sum_{j=1}^{q} \lambda_j G_j(\bar{x}) > \delta_0 > 0$. This contradicts the fact that $\bar{x}$ is a feasible point of problem (P). So, we can assume without loss of generality that $\lambda_0 = 1$ and hence (6) holds.

[**Deriving positive polynomial condition via a minimax theorem**] Now, letting $T := \prod_{j=1}^{q} T_j \subset \mathbb{R}^{s_1+\dots+s_q}$, it holds that T is a convex compact set. Recalling the definition of $G_j, j = 1, \dots, q$, we get by (6) that

$$\inf_{x\in\mathbb{R}^n} \max_{(t_1,\dots,t_q)\in T} \left\{ f(x) + \sum_{j=1}^{q} \lambda_j g_j(x, t_j) \right\} - f(\bar{x}) + \epsilon > 0. \tag{8}$$

Let $F\colon \mathbb{R}^n \times \mathbb{R}^{s_1+\dots+s_q} \to \mathbb{R}$ be defined by $F(x,t) := f(x) + \sum_{j=1}^{q} \lambda_j g_j(x, t_j)$ for $x \in \mathbb{R}^n, t := (t_1, \dots, t_q) \in \mathbb{R}^{s_1+\dots+s_q}$. Then, F is a convex function in variable x and affine function in variable t. So, we apply the classical minimax theorem (see e.g., [34, Theorem 4.2]) to assert that

$$\inf_{x\in\mathbb{R}^n} \max_{t\in T} F(x,t) = \max_{t\in T} \inf_{x\in\mathbb{R}^n} F(x,t),$$

and hence, by (8),

$$\max_{(t_1,\dots,t_q)\in T} \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{q} \lambda_j g_j(x, t_j) \right\} - f(\bar{x}) + \epsilon > 0.$$

It follows that there exists $(\bar{t}_1, \dots, \bar{t}_q) \in T$ such that

$$f(x) + \sum_{j=1}^{q} \lambda_j g_j(x, \bar{t}_j) - f(\bar{x}) + \epsilon > 0, \quad \forall x \in \mathbb{R}^n. \tag{9}$$

[**Positivity to sum-of-squares condition via SOS-convexity**] Consider the function $\sigma_0\colon \mathbb{R}^n \to \mathbb{R}$ given by $\sigma_0(x) := f(x) + \sum_{j=1}^{q} \lambda_j g_j(x, \bar{t}_j) - f(\bar{x}) + \epsilon$ for $x \in \mathbb{R}^n$. Under our assumption, σ_0 is an SOS-convex polynomial on $\mathbb{R}^n$, and thus, by definition, the polynomial

$$H(x,y) := \sigma_0(x) - \sigma_0(y) - \nabla\sigma_0(y)^\top (x-y), (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{10}$$

is a sum-of-squares polynomial. By virtue of (9), it holds that $\sigma_0(x) > 0$ for all $x \in \mathbb{R}^n$; i.e., σ_0 is bounded from below on $\mathbb{R}^n$, and therefore, due to the Frank-Wolfe theorem (cf. [2, Theorem 3]), we find $x^* \in \mathbb{R}^n$ such that $\sigma_0(x^*) \le \sigma_0(x)$ for all $x \in \mathbb{R}^n$. It follows that $\nabla\sigma_0(x^*) = 0$. So, $\sigma_0 = H(\cdot, x^*) + \sigma_0(x^*) \in \Sigma^2[x]$, because this is a sum of the sum-of-squares polynomial $H(\cdot, x^*)$ and the positive constant $\sigma_0(x^*)$.

[**Employing a variable transformation for LMI characterization**] Consider any $j \in \mathcal{Q}$. On account of (3), the relation $\bar{t}_j := (\bar{t}_j^1, \ldots, \bar{t}_j^{s_j}) \in T_j$ means that

$$\begin{pmatrix} I_{s_j} & W_j^l \bar{t}_j \\ (W_j^l \bar{t}_j)^\top & -(a_j^l)^\top \bar{t}_j - b_j^l \end{pmatrix} \succeq 0, \quad l \in \mathcal{Q}_j. \tag{11}$$

By letting

$$A_j^{l,0} := \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix}, \quad A_j^{l,i} := \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix}, \; i \in \mathcal{S}_j, \; l \in \mathcal{Q}_j, \tag{12}$$

we conclude that (11) amounts to $A_j^{l,0} + \sum\limits_{i=1}^{s_j} \bar{t}_j^i A_j^{l,i} \succeq 0, l \in \mathcal{Q}_j$. Setting $\mu_j^0 := \lambda_j \geq 0, j \in \mathcal{Q}$ and $\mu_j^i := \mu_j^0 \bar{t}_j^i \in \mathbb{R}, j \in \mathcal{Q}, i \in \mathcal{S}_j$, we will verify that (5) is valid, or equivalently,

$$\mu_j^0 A_j^{l,0} + \sum_{i=1}^{s_j} \mu_j^i A_j^{l,i} \succeq 0, \; l \in \mathcal{Q}_j, \; j \in \mathcal{Q}. \tag{13}$$

To see this, let $j \in \mathcal{Q}$ and $l \in \mathcal{Q}_j$. If $\mu_j^0 = 0$, then $\mu_j^i = 0$ for all $i \in \mathcal{S}_j$, and hence, (13) holds trivially. If $\mu_j^0 \neq 0$, then

$$\mu_j^0 A_j^{l,0} + \sum_{i=1}^{s_j} \mu_j^i A_j^{l,i} = \mu_j^0 \left(A_j^{l,0} + \sum_{i=1}^{s_j} \frac{\mu_j^i}{\mu_j^0} A_j^{l,i} \right) = \mu_j^0 \left(A_j^{l,0} + \sum_{i=1}^{s_j} \bar{t}_j^i A_j^{l,i} \right) \succeq 0,$$

showing (13) holds, too.

By taking into account the definitions of functions g_j, $j \in \mathcal{Q}$ in (1), we obtain that $\mu_j^0 g_j(x, \bar{t}_j) = \mu_j^0 p_j^0(x) + \sum\limits_{i=1}^{s_j} \mu_j^i p_j^i(x), \; j \in \mathcal{Q}$ for all $x \in \mathbb{R}^n$. This entails that

$$\sum_{j=1}^{q} \lambda_j g_j(x, \bar{t}_j) = \sum_{j=1}^{q} \left[\mu_j^0 p_j^0(x) + \sum_{i=1}^{s_j} \mu_j^i p_j^i(x) \right] \tag{14}$$

for all $x \in \mathbb{R}^n$. Combining now (14) with the definition of σ_0 gives us that

$$f + \sum_{j=1}^{q} \left(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i \right) - f(\bar{x}) + \epsilon = \sigma_0,$$

which in particular shows that $\sigma_0 \in \Sigma^2_d[x]$ with $d = \max\{\deg(f),\ \deg(p^0_j), \deg(p^i_j), i \in \mathcal{S}_j,\ j \in \mathcal{Q}\}$, i.e., (4) holds. So, our conclusion follows.

[$\Longleftarrow$] Let $\epsilon > 0$. Assume that there exist $\mu^0_j \in \mathbb{R}_+, \mu^i_j \in \mathbb{R}, j \in \mathcal{Q}, i \in \mathcal{S}_j$ such that (4) and (5) hold. Note further that (5) is nothing else but

$$\mu^0_j A^{l,0}_j + \sum_{i=1}^{s_j} \mu^i_j A^{l,i}_j \succeq 0,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q}, \tag{15}$$

where $A^{l,i}_j, i \in \{0\} \cup \mathcal{S}_j, l \in \mathcal{Q}_j, j \in \mathcal{Q}$ are defined as in (12).

[**Obtaining LMI condition via the inverse variable transformation**] Consider any $j \in \mathcal{Q}$. By the compactness of T_j, we assert by (15) that if $\mu^0_j = 0$, then $\mu^i_j = 0$ for all $i \in \mathcal{S}_j$. Assume on the contrary that $\mu^0_j = 0$ but there exists $i_0 \in \mathcal{S}_j$ with $\mu^{i_0}_j \neq 0$. In this case, (15) gives $\sum\limits_{i=1}^{s_j} \mu^i_j A^{l,i}_j \succeq 0,\ l \in \mathcal{Q}_j$. Let $\bar{t}_j := (\bar{t}^1_j, \dots, \bar{t}^{s_j}_j) \in T_j$. This is equivalent to $A^{l,0}_j + \sum\limits_{i=1}^{s_j} \bar{t}^i_j A^{l,i}_j \succeq 0,\ l \in \mathcal{Q}_j$. Then, for $l \in \mathcal{Q}_j$, it holds

$$A^{l,0}_j + \sum_{i=1}^{s_j} (\bar{t}^i_j + \gamma\mu^i_j) A^{l,i}_j = \left(A^{l,0}_j + \sum_{i=1}^{s_j} \bar{t}^i_j A^{l,i}_j \right) + \gamma \sum_{i=1}^{s_j} \mu^i_j A^{l,i}_j \succeq 0 \ \text{ for all } \ \gamma > 0.$$

It guarantees that $\bar{t}_j + \gamma(\mu^1_j, \dots, \mu^{s_j}_j) \in T_j$ for all $\gamma > 0$, which contradicts the fact that T_j is a compact set. So, our claim must be true.

Next, take $\hat{t}_j := (\hat{t}^1_j, \dots, \hat{t}^{s_j}_j) \in T_j$ and define $\tilde{t}_j := (\tilde{t}^1_j, \dots, \tilde{t}^{s_j}_j)$ with

$$\tilde{t}^i_j := \begin{cases} \hat{t}^i_j & \text{if } \mu^0_j = 0, \\ \frac{\mu^i_j}{\mu^0_j} & \text{if } \mu^0_j \neq 0, \end{cases} \qquad i \in \mathcal{S}_j.$$

Then, for $l \in \mathcal{Q}_j$, it can be checked that

$$A^{l,0}_j + \sum_{i=1}^{s_j} \tilde{t}^i_j A^{l,i}_j = \begin{cases} A^{l,0}_j + \sum\limits_{i=1}^{s_j} \hat{t}^i_j A^{l,i}_j & \text{if } \mu^0_j = 0, \\ \frac{1}{\mu^0_j}(\mu^0_j A^{l,0}_j + \sum\limits_{i=1}^{s_j} \mu^i_j A^{l,i}_j) & \text{if } \mu^0_j \neq 0, \end{cases}$$

which shows that $A^{l,0}_j + \sum\limits_{i=1}^{s_j} \tilde{t}^i_j A^{l,i}_j \succeq 0$, and so, $\tilde{t}_j \in T_j$. Therefore, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \mu^0_j p^0_j(x) + \sum_{i=1}^{s_j} \mu^i_j p^i_j(x) =& \mu^0_j p^0_j(x) + \sum_{i=1}^{s_j} (\mu^0_j \tilde{t}^i_j) p^i_j(x) \\ =& \mu^0_j \big[p^0_j(x) + \sum_{i=1}^{s_j} \tilde{t}^i_j p^i_j(x) \big] = \mu^0_j g_j(x, \tilde{t}_j), \end{aligned}$$

where we note that if $\mu_j^0 = 0$, then $\mu_j^i = 0$ for all $i \in \mathcal{S}_j$ as shown above.

So, due to (4), we find $\sigma_0 \in \Sigma_d^2[x]$ such that

$$f(x) = \sigma_0(x) - \sum_{j=1}^{q} \mu_j^0 g_j(x, \tilde{t}_j) + f(\bar{x}) - \epsilon, \quad \forall x \in \mathbb{R}^n. \tag{16}$$

[**Guaranteeing optimality from LMI condition**] Now, let $\hat{x} \in \mathbb{R}^n$ be an arbitrary feasible point of problem (P). Then, it follows that $\sum_{j=1}^{q} \mu_j^0 g_j(\hat{x}, \tilde{t}_j) \le 0$ due to $\mu_j^0 \ge 0$ and $g_j(\hat{x}, \tilde{t}_j) \le 0$ for all $j \in \mathcal{Q}$. Keeping in mind the nonnegativity of sum-of-squares polynomials, evaluating (16) at $\hat{x}$, we arrive at

$$f(\hat{x}) \ge f(\bar{x}) - \epsilon.$$

As $\epsilon > 0$ was chosen arbitrarily, it entails that $f(\bar{x}) \le f(\hat{x})$. So, $\bar{x}$ is an optimal solution of problem (P). $\square$

SDP Relaxation Problem. We now consider a relaxation problem in terms of sum-of-squares and linear matrix inequalities for the robust SOS-convex polynomial optimization problem (P) as follows:

$$\begin{aligned}
\sup_{(r,\mu_j^0,\mu_j^i)} \Big\{ r \;| f + \sum_{j=1}^{q} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - r \in \Sigma_d^2[x],\ r \in \mathbb{R}, \\
\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0, \\
\mu_j^0 \in \mathbb{R}_+,\ \mu_j^i \in \mathbb{R},\ i \in \mathcal{S}_j,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q} \Big\},
\end{aligned} \tag{SDP}$$

where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$.

Observe that each σ_0 of degree $k := 2d$ in n variables can be expressed as (see e.g.,[32, Theorem 3.39] or [26, Proposition 2.1]): $\sigma_0 := X^\top B X$ and $B \succeq 0$, where $X := (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d)^\top$ and B is a symmetric matrix of order $e(d,n) := \begin{pmatrix} n+d \\ d \end{pmatrix}$. Then, for $\mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, r \in \mathbb{R}$, the constraint

$$f + \sum_{j=1}^{q} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - r \in \Sigma_d^2[x]$$

can be equivalently written as $B(\mu_j^0, \mu_j^i, r, \alpha_l) \succeq 0$, where $\alpha_l \in \mathbb{R}$ is a new variable for $l \in \mathbb{N}$ satisfying $0 \le l \le [e(d,n)]^2$. So, the problem (SDP) can be converted into a semidefinite linear program of the form:

$$\sup_{(r,\mu_j^0,\mu_j^i)} \Big\{ r \,|B(\mu_j^0,\mu_j^i,r,\alpha_l)\succeq 0,\ r\in\mathbb{R},$$

$$\mu_j^0\begin{pmatrix} I_{s_j} & 0\\ 0 & -b_j^l\end{pmatrix}+\sum_{i=1}^{s_j}\mu_j^i\begin{pmatrix} 0 & W_j^{l,i}\\ (W_j^{l,i})^\top & -a_j^{l,i}\end{pmatrix}\succeq 0,\ \mu_j^0\in\mathbb{R}_+,$$

$$\mu_j^i\in\mathbb{R},\ i\in\mathcal{S}_j,\ l\in\mathcal{Q}_j,\ j\in\mathcal{Q},\ \alpha_l\in\mathbb{R},\ 0\le l\le[e(d,n)]^2\Big\},$$

which can be solved efficiently.

The next theorem shows that if the robust SOS-convex polynomial optimization problem (P) has a solution, then the optimal value of the relaxation problem (SDP) coincides with the optimal value of problem (P), whenever it is well-posed.

Theorem 2.4. (Tight SDP relaxation) *For the problem* (P)*, let* argmin(P)$\neq\emptyset$. *If the problem* (P) *is well-posed, then*

$$\min(\mathrm{P})=\sup_{(r,\mu_j^0,\mu_j^i)}\Big\{ r\,|f+\sum_{j=1}^{q}\big(\mu_j^0p_j^0+\sum_{i=1}^{s_j}\mu_j^ip_j^i\big)-r\in\Sigma_d^2[x],\ r\in\mathbb{R},$$

$$\mu_j^0\begin{pmatrix} I_{s_j} & 0\\ 0 & -b_j^l\end{pmatrix}+\sum_{i=1}^{s_j}\mu_j^i\begin{pmatrix} 0 & W_j^{l,i}\\ (W_j^{l,i})^\top & -a_j^{l,i}\end{pmatrix}\succeq 0, \tag{17}$$

$$\mu_j^0\in\mathbb{R}_+,\ \mu_j^i\in\mathbb{R},\ i\in\mathcal{S}_j,\ l\in\mathcal{Q}_j,\ j\in\mathcal{Q}\Big\},$$

or equivalently, $\min(\mathrm{P})=\sup(\mathrm{SDP})$.

Proof. Let $\bar{x}\in$ argmin(P) and let

$$f^*:=\sup_{(r,\mu_j^0,\mu_j^i)}\Big\{ r\,|f+\sum_{j=1}^{q}\big(\mu_j^0p_j^0+\sum_{i=1}^{s_j}\mu_j^ip_j^i\big)-r\in\Sigma_d^2[x],\ r\in\mathbb{R},$$

$$\mu_j^0\begin{pmatrix} I_{s_j} & 0\\ 0 & -b_j^l\end{pmatrix}+\sum_{i=1}^{s_j}\mu_j^i\begin{pmatrix} 0 & W_j^{l,i}\\ (W_j^{l,i})^\top & -a_j^{l,i}\end{pmatrix}\succeq 0,$$

$$\mu_j^0\in\mathbb{R}_+,\ \mu_j^i\in\mathbb{R},\ i\in\mathcal{S}_j,\ l\in\mathcal{Q}_j,\ j\in\mathcal{Q}\Big\},$$

where $d=\max\{\deg(f),\deg(p_j^0),\deg(p_j^i),i\in\mathcal{S}_j,j\in\mathcal{Q}\}$.

[**Proving $f^*\le f(\bar{x})$ by using variable transformations**] We will verify that

$$f^*\le f(\bar{x}). \tag{18}$$

If the feasible set of problem (SDP) is empty, then $f^* = -\infty$, and in this case, (18) holds trivially. Now, let (r, μ_j^0, μ_j^i) be a feasible point of problem (SDP). Then, there exist $r \in \mathbb{R}, \mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, i \in \mathcal{S}_j, j \in \mathcal{Q}, \sigma_0 \in \Sigma_d^2[x]$ such that

$$f + \sum_{j=1}^{q} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - r = \sigma_0, \tag{19}$$

$$\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0,\ l \in \mathcal{Q}_j, \tag{20}$$

where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$. Due to (20), we employ the inverse variable transformation of LMI as in the proof of Theorem 2.3 to find $\tilde{t}_j \in T_j, j \in \mathcal{Q}$ such that

$$\mu_j^0 g_j(x, \tilde{t}_j) = \mu_j^0 p_j^0(x) + \sum_{i=1}^{s_j} \mu_j^i p_j^i(x),\ j \in \mathcal{Q}, \quad \forall x \in \mathbb{R}^n,$$

and therefore, we get by (19) that

$$f(x) = \sigma_0(x) - \sum_{j=1}^{q} \mu_j^0 g_j(x, \tilde{t}_j) + r, \quad \forall x \in \mathbb{R}^n. \tag{21}$$

Note here that $g_j(\bar{x}, \tilde{t}_j) \le 0, j \in \mathcal{Q}$ inasmuch as $\bar{x}$ is a feasible point of problem (P). Keeping in mind the nonnegativity of sum-of-squares polynomials, estimating (21) at $\bar{x}$, we obtain that $f(\bar{x}) \ge r$, which proves that (18) is valid.

[**Employing Theorem 2.3 to prove** $f^* \ge f(\bar{x})$] Let $\epsilon > 0$. Since $\bar{x}$ is an optimal solution of problem (P), where the problem (P) is well-posed, we invoke Theorem 2.3 to assert that there exist $\sigma_0 \in \Sigma_d^2[x]$, where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$, and $\mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, j \in \mathcal{Q}, i \in \mathcal{S}_j$ such that

$$\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q},$$

and

$$f + \sum_{j=1}^{q} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - r_0 = \sigma_0, \tag{22}$$

where $r_0 := f(\bar{x}) - \epsilon$. It shows that (r_0, μ_j^0, μ_j^i) is a feasible point of problem (SDP). Hence, $f^* \ge r_0 = f(\bar{x}) - \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily, we conclude that $f^* \ge f(\bar{x})$. It together with (18) confirms that $f^* = f(\bar{x})$, i.e., (17) holds. The proof of the theorem is complete. $\square$

The next corollary says that the above tight SDP relaxation result is *always* valid for the robust SOS-convex polynomial optimization problem (P) in a particular setting, where the uncertainty sets $T_j,\ j\in\mathcal{Q}$ are *polytopes.* We consider the uncertainty sets $T_j,\ j\in\mathcal{Q}$, given in (2) with $M_j^l=0,\ l\in\mathcal{Q}_j$, $j\in\mathcal{Q}$, i.e.,

$$T_j := \big\{t_j\in\mathbb{R}^{s_j} \mid (a_j^l)^\top t_j + b_j^l \le 0, l\in\mathcal{Q}_j\big\}. \tag{23}$$

Corollary 2.5. (Qualification-free relaxation) *Consider the problem* (P) *with* $T_j,\ j\in\mathcal{Q}$, *given in* (23). *Let* $\operatorname{argmin}(\mathrm{P})\neq\emptyset$. *Then,*

$$\begin{aligned}\min(\mathrm{P}) = \sup_{(r,\mu_j^0,\mu_j^i)} \Big\{ r \mid\ & f+\sum_{j=1}^{q}\big(\mu_j^0 p_j^0+\sum_{i=1}^{s_j}\mu_j^i p_j^i\big) - r\in\Sigma_d^2[x],\ r\in\mathbb{R},\\ & \mu_j^0\begin{pmatrix} I_{s_j} & 0\\ 0 & -b_j^l\end{pmatrix}+\sum_{i=1}^{s_j}\mu_j^i\begin{pmatrix} 0 & 0\\ 0 & -a_j^{l,i}\end{pmatrix}\succeq 0, \qquad (24)\\ & \mu_j^0\in\mathbb{R}_+,\ \mu_j^i\in\mathbb{R},\ i\in\mathcal{S}_j,\ l\in\mathcal{Q}_j,\ j\in\mathcal{Q}\Big\},\end{aligned}$$

where $d=\max\{\deg(f),\deg(p_j^0),\deg(p_j^i), i\in\mathcal{S}_j, j\in\mathcal{Q}\}$.

Proof. Since $T_j,\ j\in\mathcal{Q}$, are polytopes, we may assume that $T_j:=\operatorname{conv}\{w_j^1,\dots, w_j^{m_j}\}$, where $w_j^i\in\mathbb{R}^s, i\in\mathcal{M}_j:=\{1,\dots,m_j\},\ j\in\mathcal{Q}$ are fixed. In addition, as $g_j(x,\cdot)$ is an affine function for each $x\in\mathbb{R}^n$, we assert that $\max\limits_{t_j\in T_j} g_j(x,t_j) = \max\{g_j(x,w_j^i)\mid i\in\mathcal{M}_j\},\ x\in\mathbb{R}^n$ for $j\in\mathcal{Q}$. In this circumstance, it holds that

$$\begin{aligned}\Omega :&= \{(r,z_1,\dots,z_q)\in\mathbb{R}\times\mathbb{R}^q \mid \exists x\in\mathbb{R}^n,\ f(x)\le r,\ g_j(x,t_j)\le z_j,\\ &\qquad\qquad \forall t_j\in T_j,\ j\in\mathcal{Q}\}\\ &= \{(r,z_1,\dots,z_q)\in\mathbb{R}\times\mathbb{R}^q \mid \exists x\in\mathbb{R}^n,\ f(x)\le r,\ g_j(x,w_j^i)\le z_j,\\ &\qquad\qquad i\in\mathcal{M}_j,\ j\in\mathcal{Q}\}.\end{aligned}$$

Then, we get by Proposition 2.1 that the problem (P) is well-posed. Now, the conclusion follows from Theorem 2.4. □

In passing we observe, in particular, that if T_j is a singleton for $j\in\mathcal{Q}$, then Corollary 2.5 collapses to [21, Corollary 3.1].

Let us now provide a simple example, which verifies our optimality characterization and tight SDP relaxation for a robust SOS-convex polynomial program, where the Slater constraint qualification fails.

Example 2.6. (Tight SDP relaxation) Consider the robust SOS-convex polynomial optimization problem:

$$\begin{aligned}(\mathrm{EP1})\qquad \inf_{x\in\mathbb{R}} \big\{ f(x) := x^4-2x+1 \mid\ & g_1(x,t):=x^4+t_1x^2\le 0,\\ & g_2(x,t):=t_2x^4-1\le 0,\ \forall t:=(t_1,t_2)\in T\big\},\end{aligned}$$

where $T := \{t \in \mathbb{R}^2 \mid t_1^2 + t_2^2 \leq 1,\ t_1 \geq 0,\ t_2 \geq 0\}$. The problem (EP1) can be expressed in terms of problem (P), where $T_1 := T_2 := T$, $M_j^1 := I_2$, $a_j^1 := (0,0)$, $b_j^1 := -1$, $M_j^l := 0$, $a_j^2 := (-1,0)$, $a_j^3 := (0,-1)$, $b_j^l := 0$, $l = 2,3, j = 1,2$, and $p_1^0(x) := x^4$, $p_1^1(x) := x^2$, $p_1^2(x) := 0$, $p_2^0(x) := -1$, $p_2^1(x) := 0$, $p_2^2(x) := x^4$, $x \in \mathbb{R}$.

We see that the problem (EP1) is well-posed as the objective function f is coercive on $\mathbb{R}$. (Note here that the Slater constraint qualification does not hold.) It can be easily checked that $\bar{x} := 0$ is an optimal solution of problem (EP1) with the optimal value $\min(\text{EP1}) = 1$.

In this setting, we choose decomposition factors of M_j^l as $W_j^1 := I_2$, $W_j^l := 0$, $l = 2,3,\ j = 1,2$. Let us consider the SDP relaxation problem

$$
\begin{aligned}
f^* := \sup_{(r,\mu_j^0,\mu_j^i)} \Big\{ & r \mid f + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{2} \mu_j^i p_j^i\big) - r \in \Sigma_4^2[x], && (25)\\
& \mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu_j^1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mu_j^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \succeq 0, \\
& \mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_j^1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0, && \text{(E1-1)}\\
& \mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_j^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0, \\
& r \in \mathbb{R},\ \mu_j^0 \in \mathbb{R}_+,\ \mu_j^i \in \mathbb{R},\ i = 1,2,\ j = 1,2 \Big\}.
\end{aligned}
$$

Let (r, μ_j^0, μ_j^i) be a feasible point of problem (E1-1). From (25), there exists $\sigma_0 \in \Sigma_4^2[x]$ such that

$$
f + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{2} \mu_j^i p_j^i\big) - r = \sigma_0. \tag{26}
$$

Then, the equality $f(\bar{x}) + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0(\bar{x}) + \sum_{i=1}^{2} \mu_j^i p_j^i(\bar{x})\big) - r = \sigma_0(\bar{x})$ gives us that $1 - \mu_2^0 - r = \sigma_0(0) \geq 0$, which guarantees that $r \leq 1$. Hence, $f^* \leq 1$.

Take any $\epsilon \in (0, \frac{1}{2}]$. By choosing $r := 1 - \epsilon$, $\mu_1^0 := \frac{1}{\epsilon^2}$, $\mu_1^2 := 0$, $\mu_1^1 := \frac{2}{\epsilon}$, $\mu_2^0 := \mu_2^1 := \mu_2^2 := 0$ and $\sigma_0(x) := (\frac{\sqrt{\epsilon^2+1}}{\epsilon}x^2)^2 + (\sqrt{\frac{2}{\epsilon}}x - \sqrt{\frac{\epsilon}{2}})^2 + (\sqrt{\frac{\epsilon}{2}})^2, x \in \mathbb{R}$, we see that

$$f+\sum_{j=1}^{2}\big(\mu_j^0p_j^0+\sum_{i=1}^{2}\mu_j^ip_j^i\big)-r=\sigma_0\in\Sigma_4^2[x],$$
$$\mu_j^0\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\mu_j^1\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}+\mu_j^2\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\succeq 0,$$
$$\mu_j^0\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\mu_j^1\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\succeq 0,$$
$$\mu_j^0\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\mu_j^2\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\succeq 0,$$
$$r\in\mathbb{R},\ \mu_j^0\in\mathbb{R}_+,\ \mu_j^i\in\mathbb{R},\ i=1,2,\ j=1,2.$$

This shows that (r,μ_j^0,μ_j^i) is a feasible point of problem (E1-1), and therefore, $f^*\geq r=1-\epsilon$. Consequently, we arrive at the conclusion that $f^*=1=\min\,$(EP1) as proved by (17).

Since $\sigma_0\in\Sigma_4^2[x]$, there exists a symmetric (3×3) matrix B such that $\sigma_0=X^\top BX$ and $B\succeq 0$, where $X:=(1,x,x^2)$ (see e.g. [32, Lemma 3.33]). Letting

$$B:=\begin{pmatrix}B_1&B_2&B_3\\B_2&B_4&B_5\\B_3&B_5&B_6\end{pmatrix},$$

we derive from (26) that $B_1=1-\mu_2^0-r$, $B_2:=-1$, $2B_3+B_4=\mu_1^1$, $B_5=0$, $B_6=1+\mu_1^0+\mu_2^2$. Putting $B_3:=\mu\in\mathbb{R}$, we see that $B_4=\mu_1^1-2\mu$.

Then, the problem (E1-1) becomes the following semi-definite program (SDP):

$$\sup\Big\{r\mid\begin{pmatrix}1-\mu_2^0-r&-1&\mu\\-1&\mu_1^1-2\mu&0\\\mu&0&1+\mu_1^0+\mu_2^2\end{pmatrix}\succeq 0,\tag{E1-2}$$
$$\begin{pmatrix}\mu_1^0&0&\mu_1^1&0&0&0\\0&\mu_1^0&\mu_1^2&0&0&0\\\mu_1^1&\mu_1^2&\mu_1^0&0&0&0\\0&0&0&\mu_1^1&0&0\\0&0&0&0&\mu_1^2&0\\0&0&0&0&0&\mu_1^0\end{pmatrix}\succeq 0,\quad\begin{pmatrix}\mu_2^0&0&\mu_2^1&0&0&0\\0&\mu_2^0&\mu_2^2&0&0&0\\\mu_2^1&\mu_2^2&\mu_2^0&0&0&0\\0&0&0&\mu_2^1&0&0\\0&0&0&0&\mu_2^2&0\\0&0&0&0&0&\mu_2^0\end{pmatrix}\succeq 0,$$
$$r\in\mathbb{R},\ \mu_j^0\in\mathbb{R},\ \mu_j^i\in\mathbb{R},\ \mu\in\mathbb{R},\ i=1,2,\ j=1,2\Big\}.$$

Using the Matlab toolbox CVX (see e.g. [16]), we solve the SDP problem (E1-2). The solver returns an approximate optimal value +0.99949 for the true optimal value 1 of the robust problem (EP1).

In fact, the "sup" in the relaxation problem (E1-1) is *not* attained. In other words, in this setting, the sum-of-squares condition (4) does not hold with $\epsilon := 0$. To see this, we assume on the contrary that there exist $\sigma_0 \in \Sigma_4^2[x]$ and $\mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, j = 1, 2, i = 1, 2$ such that

$$f + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{2} \mu_j^i p_j^i\big) - f(\bar{x}) = \sigma_0.$$

By rearranging this, we obtain that $\big(1+\mu_1^0+\mu_2^2\big)x^4 + \mu_1^1 x^2 - 2x - \mu_2^0 = \sigma_0(x) \geq 0$ for each $x \in \mathbb{R}$. It follows that, for each $x \in \mathbb{R}$,

$$\big(1 + \mu_1^0 + \mu_2^2\big)x^4 \geq -\mu_1^1 x^2 + 2x + \mu_2^0 \geq 2x - \mu_1^1 x^2. \tag{27}$$

Considering $x := \frac{1}{k}$, where $k \in \mathbb{N}$, we deduce from (27) that

$$1 + \mu_1^0 + \mu_2^2 \geq k^2(2k - \mu_1^1),$$

and so,

$$1 + \mu_1^0 + \mu_2^2 \geq 2k - \mu_1^1 \tag{28}$$

for k large enough as the inequality $k^2(2k - \mu_1^1) \geq 2k - \mu_1^1$ is valid for all $k \in \mathbb{N}$ satisfying $k \geq \frac{\mu_1^1}{2}$. Now, letting $k \to \infty$ in (28), we arrive at a contradiction.

3. Tight SDP relaxations with solution attainment

In this section, we present an *exact* SDP relaxation in the sense that the relaxation problem (SDP) not only shares the same optimal value with the robust SOS-convex polynomial program (P) but also attains its optimum. This exact SDP relaxation holds whenever (P) satisfies a Karush-Kuhn-Tucker (KKT) qualification condition which is, in particular, guaranteed by the Slater constraint qualification.

KKT Qualification Condition. We say that the problem (P) satisfies the *KKT qualification condition* (KKT-QC) if for each optimal solution $\bar{x} \in \mathbb{R}^n$, there exist $\mu_j^0 \in \mathbb{R}_+, \mu_j^i \in \mathbb{R}, j \in \mathcal{Q}, i \in \mathcal{S}_j$ such that

$$f + \sum_{j=1}^{q} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - f(\bar{x}) \in \Sigma_d^2[x], \tag{29}$$

$$\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q}, \tag{30}$$

where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$.

The next proposition says that the problem (P) satisfies the above KKT qualification condition whenever the Slater constraint qualification holds, i.e., there exists $\hat{x} \in \mathbb{R}^n$ such that

$$g_j(\hat{x}, t_j) < 0, \ \forall t_j \in T_j, \ j \in \mathcal{Q}. \tag{31}$$

Although the Slater constraint qualification is easy to check in practice, as Example 3.4 below shows, the KKT qualification condition may hold while the Slater constraint qualification fails.

Proposition 3.1. (Sufficiency for KKT-QC) *If the Slater constraint qualification* (31) *holds, then the problem* (P) *satisfies the* (KKT-QC)*.*

Proof. Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of problem (P). The conclusion will follow if we show that there exist $\mu_j^0 \in \mathbb{R}_+$, $\mu_j^i \in \mathbb{R}$, $j \in \mathcal{Q}$, $i \in \mathcal{S}_j$ such that

$$f + \sum_{j=1}^{q} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - f(\bar{x}) \in \Sigma_d^2[x], \tag{32}$$

$$\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0, \ l \in \mathcal{Q}_j, j \in \mathcal{Q}, \tag{33}$$

where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$. If we put $G_j(x) := \max\limits_{t_j \in T_j} g_j(x, t_j)$, $x \in \mathbb{R}^n$ for $j \in \mathcal{Q}$, then the Slater condition (31) ensures that

$$G_j(\hat{x}) < 0, \ j \in \mathcal{Q}. \tag{34}$$

Since $\bar{x}$ is an optimal solution of problem (P), it entails that the set

$$\{x \in \mathbb{R}^n \mid f(x) - f(\bar{x}) < 0, \ G_j(x) < 0, \ j \in \mathcal{Q}\} = \emptyset.$$

Applying the classical alternative theorem in convex analyis (cf. [33, Theorem 21.1]), we find $\lambda_0 \geq 0$, $\lambda_j \geq 0$, $j \in \mathcal{Q}$, not all zero, such that

$$\lambda_0\big(f(x) - f(\bar{x})\big) + \sum_{j=1}^{q} \lambda_j G_j(x) \geq 0, \quad \forall x \in \mathbb{R}^n. \tag{35}$$

Due to (34), it stems from (35) that $\lambda_0 \neq 0$ and therefore, there is no loss of generality in assuming that $\lambda_0 = 1$. So, we have

$$\inf_{x \in \mathbb{R}^n} \big\{f(x) + \sum_{j=1}^{q} \lambda_j G_j(x)\big\} \geq f(\bar{x}).$$

Proceeding similarly as in the proof of Theorem 2.3 we find $\bar{t}_j := (\bar{t}_j^1, \ldots, \bar{t}_j^{s_j}) \in T_j$, $j \in \mathcal{Q}$, such that

$$f(x) + \sum_{j=1}^{q} \lambda_j g_j(x, \bar{t}_j) - f(\bar{x}) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Now, let $\sigma_0 \colon \mathbb{R}^n \to \mathbb{R}$ be given by $\sigma_0(x) := f(x) + \sum_{j=1}^q \lambda_j g_j(x, \bar{t}_j) - f(\bar{x})$ for $x \in \mathbb{R}^n$. By following similar lines as in the proof of Theorem 2.3, we arrive at the conclusion that (32) and (33) hold. $\square$

Theorem 3.2. (Exact SDP relaxation) *For the problem* (P)*, let* argmin(P)$\neq \emptyset$. *If the problem* (P) *satisfies the* (KKT-QC)*, then* min (P) = max (SDP).

Proof. Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of problem (P). Observe first that the upper boundedness of the optimal value for the problem (SDP) holds by following the same lines as in the proof of Theorem 2.4; i.e., we obtain that

$$f^* \leq f(\bar{x}), \tag{36}$$

where

$$\begin{aligned} f^* := \sup_{(r,\mu_j^0,\mu_j^i)} \Big\{ r \mid & f + \sum_{j=1}^q \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - r \in \Sigma_d^2[x], r \in \mathbb{R}, \\ & \mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0, \\ & \mu_j^0 \in \mathbb{R}_+,\ \mu_j^i \in \mathbb{R},\ i \in \mathcal{S}_j,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q} \Big\}. \end{aligned}$$

Since the problem (P) satisfies the (KKT-QC), we find $\sigma_0 \in \Sigma_d^2[x]$, where $d = \max\{\deg(f), \deg(p_j^0), \deg(p_j^i), i \in \mathcal{S}_j, j \in \mathcal{Q}\}$, and $\mu_j^0 \in \mathbb{R}_+$, $\mu_j^i \in \mathbb{R}$, $j \in \mathcal{Q}$, $i \in \mathcal{S}_j$ such that

$$\mu_j^0 \begin{pmatrix} I_{s_j} & 0 \\ 0 & -b_j^l \end{pmatrix} + \sum_{i=1}^{s_j} \mu_j^i \begin{pmatrix} 0 & W_j^{l,i} \\ (W_j^{l,i})^\top & -a_j^{l,i} \end{pmatrix} \succeq 0,\ l \in \mathcal{Q}_j,\ j \in \mathcal{Q}$$

and $f + \sum_{j=1}^q \big(\mu_j^0 p_j^0 + \sum_{i=1}^{s_j} \mu_j^i p_j^i\big) - \bar{r} = \sigma_0$, where $\bar{r} := f(\bar{x})$. Then, $(\bar{r}, \mu_j^0, \mu_j^i)$ is a feasible point of problem (SDP), which entails that $f^* \geq \bar{r} = f(\bar{x})$. This together with (36) gives us that $\bar{r} = f(\bar{x}) = f^*$. Therefore, we conclude that $(\bar{r}, \mu_j^0, \mu_j^i)$ is an optimal solution of problem (SDP) and min (P) = max (SDP) is valid, which completes the proof of the theorem. $\square$

We now consider a particular case of the robust SOS-convex polynomial problem (P), where the uncertainty sets are *restricted ellipsoidal* uncertainty sets (see e.g., [20, 15]); i.e.,

$$T_j := \big\{ t_j := (t_j^1, \ldots, t_j^{s_j}) \in \mathbb{R}^{s_j} \mid t_j^i \geq 0, i \in \mathcal{M}_j, ||t_j|| \leq 1 \big\}, \tag{37}$$

where $\mathcal{M}_j := \{1, \ldots, m_j\}$ and $1 \leq m_j \leq s_j$.

As a consequence of our results, we obtain an exact SDP relaxation for a robust SOS-convex polynomial program as follows.

Corollary 3.3. (Exact SDP relaxation under restricted ellipsoidal uncertainty) *Consider the problem* (P) *with* T_j, $j = 1, \ldots, q$, *given by* (37). *Assume that* $\operatorname{argmin}(\mathrm{P}) \neq \emptyset$. *If the problem* (P) *satisfies the* (KKT-QC), *then* $\min(\mathrm{P}) = \max(\mathrm{SDP})$.

Proof. Observe that the uncertainty sets T_j, $j \in \mathcal{Q}$ in (37) can be expressed in terms of (2), where $q_j := m_j + 1$, $M_j^l := 0$, $a_j^l := (0, \ldots, \underbrace{-1}_{l\text{th}}, \ldots, 0)$, $b_j^l := 0$ for $l \in \mathcal{M}_j$ and $M_j^{m_j+1} := I_{s_j}$, $a_j^{m_j+1} := 0$, $b_j^{m_j+1} := -1$. So, the desired conclusion follows from Theorem 3.2. $\square$

It is worth noting that the exact SDP relaxation result obtained in Corollary 3.3 was given in [20, Theorem 3.2] under the Slater constraint qualification. Similarly, by employing the polytopic uncertainty sets described in (23), one can easily derive an exact SDP relaxation, established in [20, Theorem 3.1], from Theorem 3.2.

Let us provide a simple example, which shows how the optimal value of a robust SOS-convex polynomial problem can be found from its relaxation problem in the case, where the Slater constraint qualification fails.

Example 3.4. (Exact SDP relaxation without the Slater condition) Consider a robust SOS-convex polynomial optimization problem of the form:

$$\inf_{x \in \mathbb{R}} \big\{ f(x) := x^4 - 1 \,|g_1(x,t) := x^4 - t_1 x^2 \leq 0,$$
$$g_2(x,t) := t_2 x^4 - 1 \leq 0, \ \forall t := (t_1, t_2) \in T \big\}, \tag{EP2}$$

where $T := \{t \in \mathbb{R}^2 \mid t_1^2 + t_2^2 \leq 1, t_1 \leq 0, t_2 \geq 0\}$. The problem (EP2) can be expressed in terms of problem (P), where $T_1 := T_2 := T$, $M_j^1 := I_2, a_j^1 := (0,0), b_j^1 := -1, M_j^l := 0, a_j^2 := (1,0), a_j^3 := (0,-1), b_j^l := 0, l = 2,3, j = 1,2$, and $p_1^0(x) := x^4, p_1^1(x) := -x^2, p_1^2(x) := 0, p_2^0(x) := -1, p_2^1(x) := 0, p_2^2(x) := x^4, x \in \mathbb{R}$. It can be easily checked that $\bar{x} := 0$ is an optimal solution of problem (EP2) with the optimal value $\min(\mathrm{EP2}) = -1$.

In this framework, we select decomposition factors of M_j^l as $W_j^1 := I_2, W_j^l := 0, l = 2,3, j = 1,2$. By choosing $\mu_j^i := 0, i = 0,1,2, j = 1,2$ and $\sigma_0(x) := x^4, x \in \mathbb{R}$, we see that

$$f + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{2} \mu_j^i p_j^i \big) - f(\bar{x}) = \sigma_0 \in \Sigma_4^2[x],$$

$$\mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu_j^1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mu_j^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \succeq 0,$$

$$\mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_j^1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \succeq 0,$$

$$\mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_j^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0,$$
$$r \in \mathbb{R},\ \mu_j^0 \in \mathbb{R}_+,\ \mu_j^i \in \mathbb{R},\ i = 1,2,\ j = 1,2.$$

This shows that the problem (EP2) satisfies the (KKT-QC). (Note here that the Slater constraint qualification does not hold.)

Now, we use the SDP relaxation scheme to verify the optimal value min (EP2) $=$ -1. Namely, we consider the relaxation problem

$$\max_{(r,\mu_j^0,\mu_j^i)} \Big\{ r \;|f + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{2} \mu_j^i p_j^i\big) - r \in \Sigma_4^2[x], \tag{38}$$
$$\mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu_j^1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mu_j^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \succeq 0,$$
$$\mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_j^1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \succeq 0, \tag{E2-1}$$
$$\mu_j^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_j^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0,$$
$$r \in \mathbb{R},\ \mu_j^0 \in \mathbb{R}_+,\ \mu_j^i \in \mathbb{R},\ i = 1,2,\ j = 1,2 \Big\}.$$

From (38), there exists $\sigma_0 \in \Sigma_4^2[x]$ such that

$$f + \sum_{j=1}^{2} \big(\mu_j^0 p_j^0 + \sum_{i=1}^{2} \mu_j^i p_j^i\big) - r = \sigma_0. \tag{39}$$

By $\sigma_0 \in \Sigma_4^2[x]$, we find a symmetric (3×3) matrix B such that $\sigma_0 = X^\top B X$ and $B \succeq 0$, where $X := (1, x, x^2)$ (see e.g.,[32, Lemma 3.33]). Letting

$$B := \begin{pmatrix} B_1 & B_2 & B_3 \\ B_2 & B_4 & B_5 \\ B_3 & B_5 & B_6 \end{pmatrix},$$

we derive from (39) that $B_1 = -1 - \mu_2^0 - r$, $B_2 := 0$, $2B_3 + B_4 = -\mu_1^1$, $B_5 = 0$, $B_6 = 1 + \mu_1^0 + \mu_2^2$. Putting $B_3 := \mu \in \mathbb{R}$, we see that $B_4 = -\mu_1^1 - 2\mu$.

Then, the problem (E2-1) becomes the following semi-definite program (SDP):

$$\max \Big\{ r \mid \begin{pmatrix} -1 - \mu_2^0 - r & 0 & \mu \\ 0 & -\mu_1^1 - 2\mu & 0 \\ \mu & 0 & 1 + \mu_1^0 + \mu_2^2 \end{pmatrix} \succeq 0, \tag{E2-2}$$

$$\begin{pmatrix} \mu_1^0 & 0 & \mu_1^1 & 0 & 0 & 0 \\ 0 & \mu_1^0 & \mu_1^2 & 0 & 0 & 0 \\ \mu_1^1 & \mu_1^2 & \mu_1^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_1^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_1^0 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} \mu_2^0 & 0 & \mu_2^1 & 0 & 0 & 0 \\ 0 & \mu_2^0 & \mu_2^2 & 0 & 0 & 0 \\ \mu_2^1 & \mu_2^2 & \mu_2^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_2^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2^0 \end{pmatrix} \succeq 0,$$

$$r \in \mathbb{R},\ \mu_j^0 \in \mathbb{R},\ \mu_j^i \in \mathbb{R},\ \mu \in \mathbb{R},\ i = 1,2,\ j = 1,2 \Big\}.$$

Using the Matlab toolbox CVX (see e.g. [16]) we solve the SDP problem (E2-2). The solver returns the following output:

```
 number of iterations   =  8
 primal objective value =  7.16885241e-12
 dual   objective value = -1.14492559e-09
 gap := trace(XZ)       = 2.46e-09
 relative gap           = 2.46e-09
 actual relative gap    = 1.15e-09
 rel. primal infeas (scaled problem)   = 2.19e-11
 rel. dual     "        "       "      = 3.09e-10
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.4e+00, 1.8e+01, 3.6e+01
 norm(A), norm(b), norm(C) = 6.0e+00, 2.4e+00, 2.0e+00
 Total CPU time (secs)  = 0.79
 CPU time per iteration = 0.10
 termination code       =  0
 DIMACS: 2.6e-11  0.0e+00  3.1e-10  0.0e+00  1.2e-09  2.5e-09
-------------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): -1
```

It verifies that the exact SDP relaxation holds. That is, the relaxation problem (E2-1) attains its optimal solution and $\max\,(\text{E2-1}) = -1 = \min\,(\text{EP2})$.

4. Conclusion and further work

In this paper we have presented a tight SDP relaxation for a subclass of robust convex polynomial programs, called robust SOS-convex polynomial programs, with intersection ellipsoidal data uncertainty and have shown that the optimal values of the robust SOS-convex polynomial program and its SDP relaxation problem are equal under the well-posedness condition. We have also provided a general regularity condition under which the SDP relaxation problem attains its optimal solution.

Also, it is known that tight SDP relaxation may not be possible for the class of *robust polynomial* programs. One approach to treating these robust programs is to develop a convergent sequence of conic linear programming relaxations [9, 10, 17, 19] using hybrid constraints of the form considered in the paper, extending the relaxation scheme of [22] for the ordinary convex polynomial programs. This would form an interesting topic for further research.

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