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Asymptotic Planes and Closedness Conditions for Linear Images and Vector Sums of Sets

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We obtain new closedness conditions for linear images and vector sums of sets in finite-dimensional vector spaces and illustrate their relations with the existing results on the subject. These new conditions use the concept of asymptotic plane and are applicable to the case of arbitrary (not only convex) sets.

Keywords: Asymptotic cone, asymptotic plane, closedness condition, lineality space, linear image, recession cone, vector sum.

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1. Introduction

It is well known that some operations on closed convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$ (such as linear transformation, vector addition, conic hull, and convex hull) fail to preserve their closedness. At the same time, closedness of resulting sets plays an important role in convex analysis, since it allows one to keep lower semi-continuity of functions and to assure the existence of solutions to various extremum problems.

In this regard, a variety of sufficient conditions providing a desired closedness of sets were obtained, especially after their treatment in the book [11, Section 9]. These conditions are usually formulated for the case of convex sets in terms of recession cones and lineality spaces. For instance, closedness of linear images of convex cones is studied in [3, 4, 10], while combinations of linear images and sums of closed convex sets are described in [13, 14]. Another concept, asymptotic cone, originated in [5] and extensively presented in the book [2, Chapter 2], is used in the study of closedness conditions for arbitrary closed sets (possibly nonconvex).

A connection between closedness conditions for convex sets and their asymptotic properties was observed in [7, 8] within the study of continuous convex

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sets. (We recall that, in equivalent terms, a closed convex set $K \subset \mathbb{R}^n$ of dimension n is *continuous* if it has neither boundary halflines nor asymptotic halflines.) Later, it was proved in [1] that a closed convex set $K \subset \mathbb{R}^n$ with closed linear images cannot have asymptotic planes. An assertion from [2, Proposition 2.4.2] states that every linear image of a continuous convex set $K \subset \mathbb{R}^n$ is closed.

In this paper, we obtain new results which further specify the relation between closedness of linear images and vector sums of sets and asymptotic properties of these sets. These results are formulated in terms of *asymptotic planes*, defined in [8].

The outline of the paper is as follows. In Section 2, some notation, terminology, and preliminary results on recession cones and lineality spaces of arbitrary sets are described. A closedness criterion for linear images of sets in $\mathbb{R}^n$ is given in Section 3. In Section 4, we consider a closedness criterion for vector sums of arbitrary sets in $\mathbb{R}^n$. Both Sections 3 and 4 describe the relations between the obtained results and the existing sufficient conditions elaborated in [1, 2, 5, 7, 8, 11].

2. Notation and preliminaries

We follow standard terminology and notation (see, e. g., [11] and [12]). For instance, the closure and (linear) span of a set $X \subset \mathbb{R}^n$ are denoted, respectively, by $\operatorname{cl} X$ and $\operatorname{span} X$; the symbol $\varnothing$ stands for the empty set, while o denotes the zero vector of $\mathbb{R}^n$. The elements of $\mathbb{R}^n$ are called vectors or points, and real numbers are called scalars. Given sets X and Y in $\mathbb{R}^n$, their vector sum $X + Y$ is defined by

$$X + Y := \{x + y : x \in X, y \in Y\}.$$

For a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$, the subspaces

$$\operatorname{null} f := \{x \in \mathbb{R}^n : f(x) = o\} \quad \text{and} \quad \operatorname{rng} f := f(\mathbb{R}^n)$$

are called the null space and the range of f, respectively. Given a vector $x \in \mathbb{R}^n$, its norm (length) is denoted $\|x\|$.

A translate $L := u + S$ of a subspace $S \subset \mathbb{R}^n$ by a vector $u \in \mathbb{R}^n$ is called a *plane* (or an affine variety); L is expressible in the form $L = v + S$, with $v \in \mathbb{R}^n$, if and only if $v \in L$ (see [11, Theorem 1.2]). We observe that a plane can be of any dimension between 0 and n inclusively. A set $C \subset \mathbb{R}^n$ is called a *cone* (with apex o) provided $\lambda x \in C$ whenever $x \in C$ and $\lambda \geq 0$. (This definition implies the inclusion $o \in C$, although an option to consider improper apices of C can be beneficial, a shown in [9]). It can be easily proved that a cone C is convex if and only if $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \geq 0$. Subspaces S and T of $\mathbb{R}^n$ are called *complementary* if $S + T = \mathbb{R}^n$ and $S \cap T = \{o\}$, and a linear transformation $p\colon \mathbb{R}^n \to \mathbb{R}^n$ is the *projection* on S along T provided

$p(x) = y$ for all $x \in \mathbb{R}^n$, where $x = y + z$ is a unique representation of x as a sum of vectors $y \in S$ and $z \in T$.

Following [11, Section 8], we define the *recession cone* of a nonempty set $X \subset \mathbb{R}^n$, denoted $\operatorname{rec} X$, as the set of all vectors $e \in \mathbb{R}^n$ such that $x + \lambda e \in X$ whenever $x \in X$ and $\lambda \geq 0$ (unlike [11], we do not assume the convexity of X). In a standard way, the *lineality space* of a nonempty set $X \subset \mathbb{R}^n$ is defined by

$$\operatorname{lin} X := \operatorname{rec} X \cap (-\operatorname{rec} X).$$

We will need the following assertion from [11, Theorem 8.2].

Proposition 2.1. ([11]) *Let $K \subset \mathbb{R}^n$ be a nonempty convex set. A vector e in $\mathbb{R}^n$ belongs to* $\operatorname{rec}(\operatorname{cl} K)$ *if and only if e is the limit of a sequence $\lambda_1 x_1, \lambda_2 x_2, \ldots$, where all $x_1, x_2, \ldots$ belong to K and all $\lambda_1, \lambda_2, \ldots$ are positive scalars tending to* 0. □

The auxiliary lemma below describes some properties of recession cones and lineality spaces, already known for the case of convex sets. We provide the proof for the sake of completeness.

Lemma 2.2. *Given nonempty sets $X, Y \subset \mathbb{R}^n$, the following assertions hold.*

(a) $\operatorname{rec} X$ *is a convex cone, and* $\operatorname{lin} X$ *is a subspace.*

(b) *The cone* $\operatorname{rec}(\operatorname{cl} X)$ *is closed. Furthermore,*

$$\operatorname{cl}(\operatorname{rec} X) \subset \operatorname{rec}(\operatorname{cl} X) \quad \textit{and} \quad \operatorname{cl}(\operatorname{lin} X) \subset \operatorname{lin}(\operatorname{cl} X).$$

(c) $\operatorname{rec} X + \operatorname{rec} Y \subset \operatorname{rec}(X + Y)$, $\operatorname{lin} X + \operatorname{lin} Y \subset \operatorname{lin}(X + Y)$, *and*

$$\operatorname{span}(\operatorname{rec} X \cap (-\operatorname{rec} Y)) \subset \operatorname{lin}(X + Y). \tag{1}$$

(d) *If the set X is not closed and $u \in \operatorname{cl} X \setminus X$, then the plane $u + \operatorname{lin} X$ lies in* $\operatorname{cl} X \setminus X$.

Proof. (a) Choose vectors $c, e \in \operatorname{rec} X$ and scalars $\lambda, \mu \geq 0$. Given a point $x \in X$, we have $x + \lambda c \in X$. Therefore,

$$x + (\lambda c + \mu e) = (x + \lambda c) + \mu e \in X,$$

which shows the inclusion $\lambda c + \mu e \in \operatorname{rec} X$. Hence $\operatorname{rec} X$ is a convex cone. Consequently, $\operatorname{lin} X$ is a convex cone as the intersection of $\operatorname{rec} X$ and $-\operatorname{rec} X$. Furthermore, being symmetric about o, the set $\operatorname{lin} X$ is a subspace of $\mathbb{R}^n$.

(b) Let a vector $e \in \mathbb{R}^n$ be the limit of a sequence of vectors $e_1, e_2, \ldots$ from $\operatorname{rec}(\operatorname{cl} X)$. Since $x + \lambda e_i \in \operatorname{cl} X$ whenever $x \in \operatorname{cl} X$, $\lambda \geq 0$, and $i \geq 1$, we obtain

$$x + \lambda e = \lim_{i \to \infty} (x + \lambda e_i) \in \operatorname{cl} X \quad \text{for all} \quad \lambda \geq 0.$$

Thus $e \in \operatorname{rec}(\operatorname{cl} X)$, which shows the closedness of $\operatorname{rec}(\operatorname{cl} X)$.

Choose a vector $c \in \operatorname{rec} X$, a point $x \in \operatorname{cl} X$, and a scalar $\lambda \geq 0$. Expressing x as the limit of a sequence of points $x_1, x_2, \ldots$ from X, we have $x_i + \lambda c \in X$, $i \geq 1$. Therefore,

$$x + \lambda c = \lim_{i \to \infty} (x_i + \lambda c) \in \operatorname{cl} X,$$

implying that $c \in \operatorname{rec}(\operatorname{cl} X)$. Thus $\operatorname{rec} X \subset \operatorname{rec}(\operatorname{cl} X)$. The desired inclusion

$$\operatorname{cl}(\operatorname{rec} X) \subset \operatorname{rec}(\operatorname{cl} X)$$

follows from the closedness of $\operatorname{rec}(\operatorname{cl} X)$. Furthermore,

$$\begin{aligned} \operatorname{cl}(\operatorname{lin} X) &= \operatorname{cl}(\operatorname{rec} X \cap (-\operatorname{rec} X)) \subset \operatorname{cl}(\operatorname{rec} X) \cap (-\operatorname{cl}(\operatorname{rec} X)) \\ &\subset \operatorname{rec}(\operatorname{cl} X) \cap (-\operatorname{rec}(\operatorname{cl} X)) = \operatorname{lin}(\operatorname{cl} X). \end{aligned}$$

(c) Choose vectors $c \in \operatorname{rec} X$ and $e \in \operatorname{rec} Y$. Let u be a point in $X + Y$. Then $u = x + y$ for suitable points $x \in X$ and $y \in Y$. For a scalar $\lambda \geq 0$, one has

$$u + \lambda(c + e) = (x + \lambda c) + (y + \lambda e) \in X + Y,$$

which gives $c + e \in \operatorname{rec}(X + Y)$. Thus $\operatorname{rec} X + \operatorname{rec} Y \subset \operatorname{rec}(X + Y)$.

For the second inclusion, we observe that

$$A \cap B + C \cap D \subset (A + C) \cap (B + D)$$

for any choice of nonempty sets A, B, C, and D in $\mathbb{R}^n$. Consequently,

$$\begin{aligned} \operatorname{lin} X + \operatorname{lin} Y &= \operatorname{rec} X \cap (-\operatorname{rec} X) + \operatorname{rec} Y \cap (-\operatorname{rec} Y) \\ &\subset (\operatorname{rec} X + \operatorname{rec} Y) \cap (-\operatorname{rec} X + (-\operatorname{rec} Y)) \\ &\subset \operatorname{rec}(X + Y) \cap (-\operatorname{rec}(X + Y)) = \operatorname{lin}(X + Y). \end{aligned}$$

For the inclusion (1), choose any vector $e \in \operatorname{rec} X \cap (-\operatorname{rec} Y)$. Let u be a point in $X + Y$. Then $u = x + y$ for suitable points $x \in X$ and $y \in Y$. We assert that $u + \lambda e \in X + Y$ whenever $\lambda \in \mathbb{R}$. Indeed, if $\lambda \geq 0$, then

$$u + \lambda e = (x + \lambda e) + y \in X + Y.$$

Similarly, if $\lambda \leq 0$, then

$$u + \lambda e = x + (y + (-\lambda)(-e)) \in X + Y.$$

Summing up, $e \in \operatorname{lin}(X + Y)$. Therefore,

$$\operatorname{rec} X \cap (-\operatorname{rec} Y) \subset \operatorname{lin}(X + Y).$$

Since $\operatorname{lin}(X + Y)$ is a subspace of $\mathbb{R}^n$, the last inclusion implies (1).

(d) Choose in X a sequence $u_1, u_2, \ldots$ of points convergent to u. Since $u_i + \operatorname{lin} X \subset X$ for all $i \geq 1$, the limit plane $u + \operatorname{lin} X$ lies in $\operatorname{cl} X$. Assume for a moment that $u + \operatorname{lin} X$ meets X at a point v. Then

$$u = u + o \in u + \operatorname{lin} X = v + \operatorname{lin} X \subset X,$$

contrary to the choice of u. Hence $(u + \operatorname{lin} X) \cap X = \varnothing$. $\square$

Standard arguments of convex analysis result in the lemma below.

Lemma 2.3. *For nonempty convex sets K and M in $\mathbb{R}^n$, the following assertions hold.*

(a) *If $K \subset M$, then* $\operatorname{rec} K \subset \operatorname{rec}(\operatorname{cl} K) \subset \operatorname{rec}(\operatorname{cl} M)$.

(b) $\operatorname{lin}(\operatorname{cl} K + \operatorname{cl} M) \subset \operatorname{lin}(\operatorname{cl}(K + M))$. □

3. Closedness conditions for linear images

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and X be a set in $\mathbb{R}^n$. Then $f(\operatorname{cl} X) \subset \operatorname{cl} f(X)$ because f is a continuous mapping on $\mathbb{R}^n$. Various conditions that guarantee the equality $f(\operatorname{cl} X) = \operatorname{cl} f(X)$ are called *closedness conditions*. Since this equality holds if f is one-to-one or X is bounded, the task to find such conditions becomes non-trivial only in the case when f is not one-to-one and X is unbounded.

The first result in this regard was obtained in [11, Theorem 9.1] for the case of convex sets. Afterwards, various assertions on closedness of linear images of convex cones were proved in [3, 4, 10]. Similar results in terms of asymptotic cones of arbitrary closed sets are summarized in [2, Chapter 2].

Proposition 3.1. ([11]) *Given a convex set $K \subset \mathbb{R}^n$ and a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$, the equality $f(\operatorname{cl} K) = \operatorname{cl} f(K)$ holds provided*

$$\operatorname{rec}(\operatorname{cl} K) \cap \operatorname{null} f \subset \operatorname{lin}(\operatorname{cl} K). \tag{2}$$

Theorem 3.4 below uses the concept of asymptotic plane in establishing a closedness criterion for the equality $f(\operatorname{cl} X) = \operatorname{cl} f(X)$. Following [8], we say that a plane $L \subset \mathbb{R}^n$ is *asymptotic* to a set $X \subset \mathbb{R}^n$ if the conditions below are satisfied:

$$\operatorname{cl} X \cap L = \varnothing \quad \text{and} \quad \delta(X, L) := \inf\{\|x - y\| : x \in X,\ y \in L\} = 0.$$

We will need the following auxiliary theorem of proper interest.

Theorem 3.2. *For a nonempty set $X \subset \mathbb{R}^n$ and a subspace $S \subset \mathbb{R}^n$, the following assertions hold.*

(a) $X + S \subset \operatorname{cl} X + S \subset \operatorname{cl}(X + S) = \operatorname{cl}(\operatorname{cl} X + S)$. *Consequently, the set $\operatorname{cl} X + S$ is closed if and only if $\operatorname{cl} X + S = \operatorname{cl}(X + S)$.*

(b) *The set $\operatorname{cl} X + S$ is not closed if and only if there is a translate of S which is asymptotic to X.*

(c) *If the set $\operatorname{cl} X + S$ is not closed, then $\operatorname{cl}(X + S)$ is a disjoint union of the nonempty sets $\operatorname{cl} X + S$ and S_X, where S_X denotes the union of all translates of S which are asymptotic planes of X.*

Proof. Assertion (a) immediately follows from standard properties of closure, and assertion (b) is proved in [12, Lemma 2.51].

(c) By assertion (b), $S_X \neq \varnothing$. First, we state that

$$(\mathrm{cl}\, X + S) \cap S_X = \varnothing.$$

Indeed, assume for a moment the existence of a point $p \in (\mathrm{cl}\, X + S) \cap S_X$. Then $p = y + z$ for suitable points $y \in \mathrm{cl}\, X$ and $z \in S$. By the definition of S_X, there is a plane $L \subset S_X$ which is a translate of S, contains p, and is asymptotic to X. The first two conditions give $L = p + S$, while the third one implies $\mathrm{cl}\, X \cap L = \varnothing$. On the other hand, $y = p - z \in p + S = L$, a contradiction.

Next, we are going to prove that

$$\mathrm{cl}\,(X + S) \subset (\mathrm{cl}\, X + S) \cup S_X.$$

Clearly, this inclusion is equivalent to the following:

$$\mathrm{cl}\,(X + S) \setminus (\mathrm{cl}\, X + S) \subset S_X. \tag{3}$$

By assertion (a), $\mathrm{cl}\,(X + S) \setminus (\mathrm{cl}\, X + S) \neq \varnothing$. Choose any point

$$q \in \mathrm{cl}\,(X + S) \setminus (\mathrm{cl}\, X + S)$$

and consider the plane $M = q + S$. We state that M is asymptotic to X. For this, we need to verify the conditions $\mathrm{cl}\, X \cap M = \varnothing$ and $\delta(X, M) = 0$. Indeed, assume for a moment that $\mathrm{cl}\, X \cap M \neq \varnothing$ and choose a point $z \in \mathrm{cl}\, X \cap M$. Then $q - z \in M - M = S$, which gives the inclusion

$$q = z + (q - z) \in \mathrm{cl}\, X + S,$$

contrary to the choice of q. For the equality $\delta(X, M) = 0$, choose in $X + S$ a sequence $q_1, q_2, \ldots$ of points which converge to q. Let $q_i = u_i + v_i$, where $u_i \in X$ and $v_i \in S$, $i \geq 1$. Then $q - v_i \in q + S = M$ and

$$\begin{aligned} \delta(X, M) &\leq \lim_{i \to \infty} \|u_i - (q - v_i)\| \\ &= \lim_{i \to \infty} \|(q_i - v_i) - (q - v_i)\| = \lim_{i \to \infty} \|q_i - q\| = 0. \end{aligned}$$

Summing up, M is an asymptotic plane of X. Thus $q \in M \subset S_X$, implying the inclusion (3).

Finally, it remains to show that

$$(\mathrm{cl}\, X + S) \cup S_X \subset \mathrm{cl}\,(X + S).$$

Due to assertion (a), it suffices to prove the inclusion $S_X \subset \mathrm{cl}\,(X + S)$. Choose any point $t \in S_X$. Then there is a plane $N \subset \mathbb{R}^n$ which is a translate of S, contains t, and is asymptotic to X. Clearly, $N = t + S$. Furthermore, there is a sequence $z_1, z_2, \ldots$ of points from X such that $\lim_{i \to \infty} \delta(z_i, N) = 0$. Denote by y_i the orthogonal projection of z_i on N, $i \geq 1$. Then $\|z_i - y_i\| = \delta(z_i, N)$. Put $t_i := z_i + (t - y_i)$, $i \geq 1$. Since $t - y_i \in N - N = S$, one has $t_i \in z_i + S \subset X + S$.

Furthermore,

$$\lim_{i\to\infty} \|t - t_i\| = \lim_{i\to\infty} \|z_i - y_i\| = \lim_{i\to\infty} \delta(z_i, N) = 0.$$

Consequently, $t \in \mathrm{cl}\,(X + S)$. Summing up, $S_X \subset \mathrm{cl}\,(X + S)$. □

The following example illustrates Theorem 3.2.

Example 3.3. Let C be a circular solid cone in $\mathbb{R}^3$, given by

$$C := \{(x, y, z) : \sqrt{x^2 + y^2} \le z\}.$$

It is easy to see that, for a subspace S of $\mathbb{R}^3$, the sum $C + S$ is nonclosed if and only if S has dimension one and $C \cap S$ is a boundary halfline of C. For instance, if S is the line given by

$$S := \{(x, y, z) : x = z, y = 0\},$$

then $C + S$ is a nonclosed set, which is the union of S and the open halfspace

$$W := \{(x, y, z) : x < z\}.$$

Consider the plane

$$H := \{(x, y, z) : x = z\}.$$

It is easy to see that H supports C such that $C \cap H = C \cap S$. A simple argument shows that a translate L of S is asymptotic to C if and only if L lies in $H \setminus S$. Clearly,

$$\mathrm{cl}\,(C + S) = \mathrm{cl}\, W = \{(x, y, z) : x \le z\},$$

which is a closed halfspace. This set is a disjoint union of $C + S$ and $H \setminus S$, where $H \setminus S$ is the union of all asymptotic lines of C parallel to S (that is, $H \setminus S = S_C$ in terms of Theorem 3.2).

Theorem 3.4. *For a nonempty set $X \subset \mathbb{R}^n$ and a linear transformation $f \colon \mathbb{R}^n \to \mathbb{R}^m$, the following assertions hold.*

(a) $f(X) \subset f(\mathrm{cl}\, X) \subset \mathrm{cl}\, f(X) = \mathrm{cl}\, f(\mathrm{cl}\, X)$. *Consequently, the set $f(\mathrm{cl}\, X)$ is closed if and only if $f(\mathrm{cl}\, X) = \mathrm{cl}\, f(X)$.*

(b) *The set $f(\mathrm{cl}\, X)$ is closed if and only if the sum $\mathrm{cl}\, X + \mathrm{null}\, f$ is closed.*

(c) *The set $f(\mathrm{cl}\, X)$ is not closed if and only if there is a translate of* $\mathrm{null}\, f$ *which is asymptotic to X.*

(d) *If the set $f(\mathrm{cl}\, X)$ is not closed, then $\mathrm{cl}\, f(X)$ is a disjoint union of nonempty sets $f(\mathrm{cl}\, X)$ and $f(N_X)$, where N_X denotes the union of all translates of* $\mathrm{null}\, f$ *which are asymptotic planes of X.*

Proof. Assertion (a) follows from standard properties of continuous functions, and assertion (b) is proved in [3].

(c) This assertion follows from Theorem 3.2 and assertion (b) above.

(d) By assertion (c), $N_X \neq \varnothing$. First, we state that

$$f(\operatorname{cl} X) \cap f(N_X) = \varnothing.$$

Indeed, assume for a moment the existence of a point $p \in f(\operatorname{cl} X) \cap f(N_X)$. Then $p = f(q) = f(t)$ for suitable points $q \in \operatorname{cl} X$ and $t \in N_X$. By the definition of N_X, the plane $M := t + \operatorname{null} f$ lies in N_X and is asymptotic to X. In particular, $\operatorname{cl} X \cap M = \varnothing$. On the other hand, the equality

$$f(q - t) = f(q) - f(t) = o$$

implies the inclusion $q - t \in \operatorname{null} f$. Consequently,

$$q = t - (t - q) \in t + \operatorname{null} f = M,$$

a contradiction with $\operatorname{cl} X \cap M = \varnothing$. Next, we are going to prove that

$$\operatorname{cl} f(X) \subset f(\operatorname{cl} X) \cup f(N_X).$$

This inclusion is equivalent to the following:

$$\operatorname{cl} f(X) \setminus f(\operatorname{cl} X) \subset f(N_X). \tag{4}$$

By assertion 1), $\operatorname{cl} f(X) \setminus f(\operatorname{cl} X) \neq \varnothing$. Choose any point $u \in \operatorname{cl} f(X) \setminus f(\operatorname{cl} X)$. Let T be a subspace of $\mathbb{R}^n$ complementary to $\operatorname{null} f$. Denote by Y the projection of $\operatorname{cl} X$ on T along $\operatorname{null} f$. Then

$$\operatorname{cl} X + \operatorname{null} f = Y + \operatorname{null} f,$$

which gives

$$f(\operatorname{cl} X) = f(\operatorname{cl} X + \operatorname{null} f) = f(Y + \operatorname{null} f) = f(Y).$$

From linear algebra we know that the the restriction of f on T, defined by $h(x) := f(x)$, $x \in T$, is an invertible linear transformation between T and $\operatorname{rng} f$. Clearly, $h(Y) = f(Y)$. Because $f(X)$ lies in the subspace $\operatorname{rng} f$, which is a closed set, we have $\operatorname{cl} f(X) \subset \operatorname{rng} f$. Hence there is a point $z \in T$ such that $f(z) = h(z) = u$ (see Fig. 3.1). Consider the plane $L := z + \operatorname{null} f$. Obviously, $u = f(z) \in f(L)$.

We assert that $L \subset N_X$. For this, we first observe that $\operatorname{cl} X \cap L = \varnothing$. Indeed, otherwise choosing a point $v \in \operatorname{cl} X \cap L$, we will have $L = v + \operatorname{null} f$. Then

$$u = f(z + \operatorname{null} f) = f(L) = f(v + \operatorname{null} f) = f(v) \in f(\operatorname{cl} X),$$

contrary to the choice of u in $\operatorname{cl} f(X) \setminus f(\operatorname{cl} X)$.

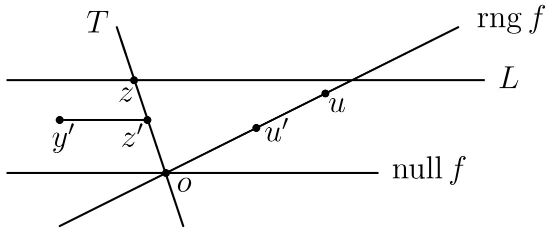

Figure 3.1: An illustration for the proof of Theorem 3.4.

Next, we intend to show that the plane L is asymptotic to X. For this, choose any scalar $\varepsilon > 0$. Because $u \in \operatorname{cl} f(X)$, there is a point $u' \in f(X)$ such that $\|u' - u\| < \varepsilon/\beta$, where β is a Lipschitz constant of h^{-1}. Choose a point $y' \in X$ with the property $f(y') = u'$, and denote by z' the projection of y' on T along $\operatorname{null} f$. Clearly, $h(z') = f(z') = f(y') = u'$ and

$$y' + \operatorname{null} f = z' + \operatorname{null} f.$$

Consequently, $\delta(y', L) = \delta(z', L)$, which gives

$$\begin{aligned} \delta(X, L) &\le \delta(y', L) = \delta(z', L) \le \|z' - z\| \\ &= \|h^{-1}(u' - u)\| \le \beta\|u' - u\| < \varepsilon. \end{aligned}$$

Hence $\delta(X, L) = 0$, which implies that L is asymptotic to X. So,

$$u = f(z) \in f(L) \subset f(N_X),$$

and the inclusion (4) is proved.

Finally, we are going to prove the opposite inclusion

$$f(\operatorname{cl} X) \cup f(N_X) \subset \operatorname{cl} f(X).$$

Indeed, the inclusion $f(\operatorname{cl} X) \subset \operatorname{cl} f(X)$ was already mentioned in 1). For the second one, $f(N_X) \subset \operatorname{cl} f(X)$, choose any point $w \in N_X$ and denote by P a translate of $\operatorname{null} f$ which contains w. Because

$$P = w + \operatorname{null} f \subset N_X,$$

the plane P is asymptotic to X. According to Lemma 2.2, $P \subset \operatorname{cl}(X + \operatorname{null} f)$. Hence, for a given $\varepsilon > 0$, there is a point $y \in X + \operatorname{null} f$ such that $\|w - y\| < \varepsilon/\alpha$, where α is a Lipschitz constants of f. Let $y := u + v$, where $u \in X$ and $v \in \operatorname{null} f$. Then $f(y) = f(u) \in f(X)$ and

$$\|f(w) - f(u)\| = \|f(w) - f(y)\| \le \alpha\|w - y\| < \varepsilon.$$

Summing up, $f(w) \in \operatorname{cl} f(X)$, which gives the inclusion $f(N_X) \subset \operatorname{cl} f(X)$. $\square$

Theorems 3.4 implies a corollary, proved in [12, Theorem 2.52].

Corollary 3.5. ([12]) *For a nonempty set $X \subset \mathbb{R}^n$ and a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$, the following conditions are equivalent.*

(a) $f(\mathrm{cl}\, X) = \mathrm{cl}\, f(X)$.

(b) *No translate of* $\mathrm{null}\, f$ *is an asymptotic plane of* X. □

The following example illustrates the difference between Proposition 3.1 and Corollary 3.5 even for the case of convex sets.

Example 3.6. Consider the closed convex set $K := \{(x, y) : x, y \geq 0\}$ in $\mathbb{R}^2$. Let f denote the orthogonal projection of $\mathbb{R}^2$ on its x-axis. Clearly, $\mathrm{rec}\, K = K$, $\mathrm{lin}\, K = \{o\}$, and $\mathrm{null}\, f$ is the y-axis of $\mathbb{R}^2$. Furthermore,

$$\mathrm{rec}\, K \cap \mathrm{null}\, f = \{(0, y) : y \geq 0\}$$

is the positive halfline of the y-axis. Hence the condition (2) in Proposition 3.1 is not satisfied, while the set $f(K)$ is closed. The reason for this, according to Corollary 3.5, is that no translate of the subspace $\mathrm{null}\, f$ is asymptotic to K. □

The next corollary refines a result from [1] (which states that a closed convex set in $\mathbb{R}^n$ with closed linear images cannot have asymptotic planes) and a result from [2, Proposition 2.4.2] (which states that every linear image of a continuous convex set in $\mathbb{R}^n$ is closed).

Corollary 3.7. *If $X \subset \mathbb{R}^n$ is a nonempty set, then the following conditions are equivalent.*

(a) $f(\mathrm{cl}\, X) = \mathrm{cl}\, f(X)$ *for every linear transformation* $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

(b) $p(\mathrm{cl}\, X) = \mathrm{cl}\, p(X)$ *for every linear projection* $p\colon \mathbb{R}^n \to \mathbb{R}^n$.

(c) X *has no asymptotic planes.* □

Example 3.6 shows that condition (2) in Proposition 3.1 is stronger than condition (b) in Corollary 3.5. The following theorem confirms this fact in a more general setting.

Theorem 3.8. *Assume the existence of a translate of a subspace $S \subset \mathbb{R}^n$ which is an asymptotic plane of a nonempty convex set $K \subset \mathbb{R}^n$. Then*

$$\mathrm{rec}\,(\mathrm{cl}\, K) \cap S \not\subset \mathrm{lin}\,(\mathrm{cl}\, K).$$

Consequently, $S \not\subset \mathrm{lin}\,(\mathrm{cl}\, K)$.

Proof. Let a translate $L := v + S$ of S be an asymptotic plane of K. Since $\delta(K, L) = 0$, there is a sequence $x_1, x_2, \ldots$ of points from K satisfying the condition $\lim_{i\to\infty} \delta(x_i, L) = 0$. We observe that this sequence is unbounded. Indeed, assume for a moment that $x_1, x_2, \ldots$ is bounded. Then it contains a subsequence $x'_1, x'_2, \ldots$ which converges to a point $x \in \mathrm{cl}\, K$. Consequently,

$$\delta(x, L) = \lim_{i\to\infty} \delta(x'_i, L) = 0.$$

Hence $x \in L$, contrary to the condition $\mathrm{cl}\, K \cap L = \varnothing$.

By a compactness argument, there is a subsequence $z_1, z_2, \dots$ of $x_1, x_2, \dots$ such that $\lim_{i\to\infty} \|z_i\| = \infty$ and the unit vectors $e_i := z_i/\|z_i\|$, $i \ge 1$, converge to a unit vector e. By Proposition 2.1, $e \in \operatorname{rec}(\operatorname{cl} K)$. Since

$$\begin{aligned}\delta(e, S) &= \lim_{i\to\infty} \delta(e_i, S) = \lim_{i\to\infty} \frac{\delta(z_i, S)}{\|z_i\|} = \lim_{i\to\infty} \frac{\delta(z_i, L - v)}{\|z_i\|} \\ &\le \lim_{i\to\infty} \frac{\delta(z_i, L) + \|v\|}{\|z_i\|} = 0,\end{aligned}$$

one has $e \in S$. Summing up, $e \in \operatorname{rec}(\operatorname{cl} K) \cap S$.

By Lemma 2.2, $\operatorname{rec}(\operatorname{cl} K)$ is a closed convex cone, whence so is the set

$$C := \operatorname{rec}(\operatorname{cl} K) \cap S.$$

Assume, for contradiction, that $C \subset \operatorname{lin}(\operatorname{cl} K)$. Under this assumption, we assert that C is a subspace. Indeed, if $x \in C$, then

$$x \in C \subset \operatorname{lin}(\operatorname{cl} K) = \operatorname{rec}(\operatorname{cl} K) \cap (-\operatorname{rec}(\operatorname{cl} K)),$$

implying that $-x \in \operatorname{rec}(\operatorname{cl} K) \cap S = C$. Therefore, C is a subspace as a convex cone symmetric about o.

Denote by T a subspace of $\mathbb{R}^n$ complementary to C. The inclusion $C \subset \operatorname{lin}(\operatorname{cl} K)$ implies that every plane $C_i := z_i + C$, $i \ge 1$, lies in $\operatorname{cl} K$. Because the planes C_i and T are translates of complementary subspaces C and T, the set $C_i \cap T$ is a singleton, say u_i (see [12, Theorem 1.11]). Since C_i is a translate of the subspace C which lies in S, and since both vectors u_i and z_i belong to C_i, one has $\delta(u_i, L) = \delta(z_i, L)$.

By the construction, the sequence $u_1, u_2, \dots$ lies to $\operatorname{cl} K \cap T$. We assert that this sequence $u_1, u_2, \dots$ is bounded. Indeed, repeating otherwise the above argument (with u_i instead of z_i), we would obtain a recession unit vector of $\operatorname{cl} K$ that belongs to $C \cap T$, which is impossible because of $C \cap T = \{o\}$. Hence the sequence $u_1, u_2, \dots$ contains a subsequence $u_{i_1}, u_{i_2}, \dots$ that converges to a point $u \in \operatorname{cl} K \cap T$. Finally,

$$\delta(u, L) = \lim_{r\to\infty} \delta(u_{i_r}, L) = \lim_{i\to\infty} \delta(z_{i_r}, L) = 0,$$

since $z_{i_1}, z_{i_2}, \dots$ is a subsequence of $x_1, x_2, \dots$ which converges to L. So, $u \in L$, in contradiction with $\operatorname{cl} K \cap L = \varnothing$. Summing up, $C \not\subset \operatorname{lin}(\operatorname{cl} K)$. □

A similar approach can be developed for the case of asymptotic cones of arbitrary sets in $\mathbb{R}^n$. Following [5], we say that the *asymptotic cone* of a nonempty set $X \subset \mathbb{R}^n$, denoted X_∞, consists of the vectors $e \in \mathbb{R}^n$ which are limits of sequence $\lambda_1 x_1, \lambda_2 x_2, \dots,$ where $x_1, x_2, \dots$ belong to X and $\lambda_1, \lambda_2, \dots$ are positive scalars tending to 0. Clearly, $(\operatorname{cl} X)_\infty = X_\infty$. As shown in [6] (also see [2, Corollary 2.3.2]), the image of a closed set $X \subset \mathbb{R}^n$ under a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is closed provided

$$X_\infty \cap \operatorname{null} f = \{o\}.$$

Obvious modification in the proof of Theorem 3.8 result in the following corollary.

Corollary 3.9. *If there is a a translate of a subspace $S \subset \mathbb{R}^n$ which is an asymptotic plane of a nonempty set $X \subset \mathbb{R}^n$, then $X_\infty \cap S \neq \{o\}$.* □

4. Closedness conditions for sums

If X and Y are sets in $\mathbb{R}^n$, then $\operatorname{cl} X + \operatorname{cl} Y \subset \operatorname{cl}(X+Y)$. *Closedness conditions* here are those which guarantee the equality

$$\operatorname{cl} X + \operatorname{cl} Y = \operatorname{cl}(X+Y). \tag{5}$$

Our next result illustrates how Theorem 3.4 generates a closedness criterion for sums of arbitrary sets. Let $\mathbb{R}^n \times \mathbb{R}^n$ denote the Cartesian product of $\mathbb{R}^n$ and $\mathbb{R}^n$, and $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by

$$\varphi(x_1, x_2) := x_1 + x_2 \quad \text{for all} \quad x_1, x_2 \in \mathbb{R}^n.$$

Clearly, $\varphi(X \times Y) = X + Y$ for any choice of sets X and Y in $\mathbb{R}^n$. Furthermore, $\operatorname{null} \varphi$ is the n-dimensional subspace of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ given by

$$\operatorname{null} \varphi = \{(x, -x) : x \in \mathbb{R}^n\}. \tag{6}$$

Theorem 4.1. *For nonempty sets X and Y in $\mathbb{R}^n$, the following holds.*

(a) $X + Y \subset \operatorname{cl} X + \operatorname{cl} Y \subset \operatorname{cl}(X+Y) = \operatorname{cl}(\operatorname{cl} X + \operatorname{cl} Y)$. *Consequently, the set $\operatorname{cl} X + \operatorname{cl} Y$ is closed if and only if the equality (5) holds.*

(b) *The set $\operatorname{cl} X + \operatorname{cl} Y$ is not closed if and only if there is a translate of the subspace (6) which is asymptotic to $X \times Y \subset \mathbb{R}^n \times \mathbb{R}^n$.*

(c) *If the set $\operatorname{cl} X + \operatorname{cl} Y$ is not closed, then $\operatorname{cl}(X+Y)$ is a disjoint union of nonempty sets $\operatorname{cl} X + \operatorname{cl} Y$ and $\varphi(N_{X\times Y})$, where $N_{X\times Y}$ denotes the union of all translates of the subspace (6) which are asymptotic to $X \times Y$.*

Proof. Assertion (a) immediately follows from standard properties of closure.

(b) and (c): Clearly, $\operatorname{cl}(X \times Y) = \operatorname{cl} X \times \operatorname{cl} Y$. Hence

$$\varphi(\operatorname{cl}(X \times Y)) = \varphi(\operatorname{cl} X \times \operatorname{cl} Y) = \operatorname{cl} X + \operatorname{cl} Y,$$
$$\operatorname{cl} \varphi(X \times Y) = \operatorname{cl}(X+Y).$$

This argument and assertion (a) show that the set $\operatorname{cl} X + \operatorname{cl} Y$ is not closed if and only if

$$\varphi(\operatorname{cl}(X \times Y)) \neq \operatorname{cl} \varphi(X \times Y).$$

Therefore, Theorem 3.4 implies both assertions (b) and (c). □

Corollary 4.2. *Let X and Y be nonempty sets in $\mathbb{R}^n$. Then the equality* (5) *holds if and only if no translate of the subspace* (6) *is asymptotic to $X \times Y$.* $\square$

The following example illustrates Theorem 4.1.

Example 4.3. Consider the closed convex sets

$$X := \{(x, y) : x > 0,\ xy \geq 1\} \quad \text{and} \quad Y := \{(x, 0) : x \leq 0\}.$$

Clearly, $X + Y = \{(x, y) : y > 0\}$ is the open halfplane in $\mathbb{R}^2$, and $X \times Y$ is a closed convex set in $\mathbb{R}^4$, given by $X \times Y = \{(x, y, z, 0) : x > 0,\ y \geq x^{-1},\ z \leq 0\}$. According to Theorem 4.1, there is a translate of the 2-dimensional subspace $\operatorname{null} \varphi = \{(x, y, -x, -y)\}$ of $\mathbb{R}^4$ which is asymptotic to $X \times Y$. We assert that, in fact, $\operatorname{null} \varphi$ is asymptotic to $X \times Y$. Indeed, the sets $X \times Y$ and $\operatorname{null} \varphi$ are disjoint (if a point $(x, y, -x, -y)$ belonged to $X \times Y$, then $y = 0$, contrary to $y \geq x^{-1}$). For the equality $\delta(X \times Y, \operatorname{null} \varphi) = 0$, choose a $\varepsilon > 0$ and a scalar $x > \varepsilon^{-1}$. Then

$$p := (x, x^{-1}, -x, 0) \in X \times Y, \quad q := (x, x^{-1}, -x, -x^{-1}) \in \operatorname{null} \varphi,$$
$$\delta(X \times Y, \operatorname{null} \varphi) \leq \|p - q\| = x^{-1} < \varepsilon.$$

In a similar way, the intersection of $\operatorname{null} \varphi$ and the 3-dimensional subspace

$$S := \operatorname{span}(X \times Y) = \{(x, y, z, 0) : x, y, z \in \mathbb{R}\}$$

is the line $l := \{(x, 0, -x, 0) : x \in \mathbb{R}\}$, which is asymptotic to $X \times Y$. Furthermore, given a scalar $a \in \mathbb{R}$, the translate $l_a := (a + x, 0, -x, 0)$ of l is the asymptotic line of $X \times Y$ such that

$$\varphi(l_a) = (a, 0) \in \operatorname{cl}(X + Y) \setminus (X + Y).$$

We also observe that the set $X \times Y$ has a large variety of asymptotic lines which are related to the asymptotic lines of X. For instance, these are

$$\{(x, 0, a, 0) : x \in \mathbb{R}\}, \quad \{(0, y, a, 0) : y \in \mathbb{R}\}, \quad a \leq 0.$$

Theorem 4.1 gives another proof of a result from [8] (formulated there for the case of closed convex sets).

Theorem 4.4. *If K_1 and K_2 are nonempty convex sets in $\mathbb{R}^n$, neither having an asymptotic plane, then*

$$\operatorname{cl} K_1 + \operatorname{cl} K_2 = \operatorname{cl}(K_1 + K_2).$$

Proof. According to Theorem 4.1, it suffices to show that the set $K_1 \times K_2$ has no asymptotic plane. For this, express $K_1 \times K_2$ as

$$K_1 \times K_2 = (K_1 \times \mathbb{R}^n) \cap (\mathbb{R}^n \times K_2).$$

Assume for a moment that $K_1 \times K_2$ has an asymptotic plane L. Then one of the sets $K_1 \times \mathbb{R}^n$ and $\mathbb{R}^n \times K_2$, say $K_1 \times \mathbb{R}^n$, has an asymptotic plane $F \subset L$ (see [8]). Clearly, the n-dimensional subspace $V := \{o\} \times \mathbb{R}^n$ satisfies the inclusion

$$V \subset \operatorname{lin}(\operatorname{cl} K_1 \times \mathbb{R}^n).$$

Therefore, no translate of F lies in V, as follows from Theorem 3.8. This argument implies that the orthogonal projection G of F on the subspace $\mathbb{R}^n \times \{o\}$ of $\mathbb{R}^n \times \mathbb{R}^n$ has positive dimension. As easily seen, G is an asymptotic plane of $K_1 \times \{o\} = K_1$, contrary to the assumption. $\square$

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