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Uniform Rotundity with Respect to Finite-Dimensional Subspaces

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We introduce the notion of uniform rotundity of a normed space with respect to a finite dimensional subspace as a generalization of uniform rotundity in a direction. We discuss several characterizations of this property and obtain a series of new classes of normed spaces which in a natural way generalize normed spaces that are uniformly rotund in every direction. Indeed for each positive integer k we get normed spaces that are uniformly rotund with respect to every k-dimensional subspace (URE$_k$) with $k = 1$ reducing to uniform rotundity in every direction. Also URE$_k$ implies URE$_{k+1}$ but not conversely. We show that URE$_k$ spaces turn out to be exactly those in which the Chebyshev center of a nonempty bounded set is either empty or is of dimension at most $k-1$ thus extending a well known result of Garkavi. These spaces have normal structure which is a sufficient condition for fixed property for nonexpansive maps on weakly compact convex sets. In addition, there is a common fixed point in the self-Chebyshev center of a weakly compact convex set for the collection of all isometric selfmaps on the set. Uniform rotundity with respect to a finite dimensional subspace is defined based on Sullivan's notion of k-uniform rotundity in the same fashion as uniform rotundity in a direction is based on Clarkson's uniform rotundity. But a characterization of the same in terms of Milman's modulus of k-uniform rotundity is also discussed.

Keywords: Uniform rotundity, finite-dimensional subspaces, k-uniform rotundity, multi-dimensional volumes, Chebyshev centers, asymptotic centers.

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1. Introduction

Given a real normed space X and a nonempty bounded subset K of X, let

$$R(x, K) = \sup\{\|x - y\| : y \in K\}, \quad x \in X.$$

If A is any nonempty set in X, then $R(A, K) = \inf\{R(x, K) : x \in A\}$ is said to be the Chebyshev radius of K relative to A and the set $C(A, K) = \{x \in A :$

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$R(x,K) = R(A,K)\}$ is called the Chebyshev center of K in A. In particular $R(X,K)$ and $C(X,K)$ are called the Chebyshev radius and Chebyshev center of K respectively while $R(K,K)$ and $C(K,K)$ are called the self-Chebyshev radius and self-Chebyshev center of K respectively. We denote $R(K,K)$ by $R(K)$ and $C(K,K)$ by $C(K)$. For any A and K as taken above, $C(A,K)$ is bounded. Since the map $x \to R(x,K)$ is continuous and convex, the set $C(A,K)$ is closed if A is closed and is convex if A is convex. Throughout this article, we consider only real normed spaces which we simply refer to as normed spaces.

In [11], it is proved that a normed space is uniformly rotund in every direction if and only if the Chebyshev center of any nonempty bounded set contains at most one point. From [8], we know further that if a normed space is uniformly rotund in every direction, then for any nonempty bounded set K and nonempty convex set A, the convex set $C(A,K)$ has at most one point. In [16] a generalization of Chebyshev centers called the asymptotic centers was introduced. For a nested sequence (K_n) (i. e. $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{Z}^+$) of nonempty bounded subsets of X, let

$$R(x,(K_n)) = \lim_{n\to\infty} R(x,K_n), \quad x \in X.$$

If A is any nonempty set in X, then $R(A,(K_n)) = \inf\{R(x,(K_n)) : x \in A\}$ is said to be the asymptotic radius of (K_n) relative to A and $C(A,(K_n)) = \{x \in A : R(x,(K_n)) = R(A,(K_n))\}$ is said to be the asymptotic center of (K_n) in A. If X is uniformly rotund in every direction then for any nonempty convex set A and nested sequence (K_n) of bounded sets, the convex set $C(A,(K_n))$ has at most one point [16].

In [14], it is proved that in a normed space that is k-UR, for any nonempty convex set A and a nested sequence (K_n) of bounded sets, the convex set $C(A,(K_n))$ is either empty or is of dimension at most $k-1$. *By dimension of a nonempty convex set C we mean the dimension of the linear space span$(C-x)$ for any $x \in C$.* In [28], it is proved that the same is true if the set of all directions in which a normed space is not uniformly rotund is contained in a countable union of $(k-1)$-dimensional subspaces. As a consequence of this property it is observed in [28] that in such spaces every isometric selfmap on a weakly compact convex set K maps the set $C(K)$ onto itself and there is a common fixed point in $C(K)$ for the collection of all isometries from K into K.

The main purpose of this article is to give a characterization analogous to that in [11] of normed spaces in which for a given positive integer k, the Chebyshev center of any nonempty bounded set is either empty or is of dimension at most $k-1$. We introduce the notion of uniform rotundity with respect to a k-dimensional subspace based on the idea of k-uniform rotundity [27]. We shall prove that a normed space is uniformly rotund with respect to every k-dimensional subspace if and only if the Chebyshev center of any nonempty bounded set is either empty or is of dimension at most $k-1$. Also we show that in this case for any nonempty convex set A and nested sequence (K_n) of bounded sets, the convex set $C(A,(K_n))$ is either empty or is of dimension at most $k-1$. From the above characterization it follows that uniform rotundity with respect to every k-dimensional subspace implies uniform rotundity with respect to every $(k+1)$-dimensional subspace. However, we shall give examples to show that the converse is not true. Also k-uniform rotundity implies uniform rotundity with respect to every k-dimensional subspace but the converse is not true. While the former implies reflexivity in a Banach space, the latter does not for any k. We shall construct an equivalent norm on l_∞ to illustrate this. We shall also show that even a reflexive space that is uniformly rotund with respect to every k-dimensional subspace need not be n-uniformly rotund for any n.

We would like to note that the concept of uniform rotundity with respect to finite dimensional subspaces considered in this paper is different from a concept with a similar name considered in [1] and [9]. In these papers, the authors study uniform rotundity in every direction of a subspace of a normed space. But we shall show that uniform rotundity with respect to a finite-dimensional subspace as we consider it is weaker than uniform rotundity in every direction of the subspace.

Before defining uniform rotundity with respect to a finite-dimensional subspace we recall some well-known geometric notions. Let X be a normed space.

1. [4] X is said to be *uniformly rotund* (UR) if, for every $\epsilon > 0$,
$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| \geq \epsilon\right\} > 0.$$
2. [11] X is said to be *uniformly rotund in the direction of* $z \neq 0$ (UR_z) if, for every $\epsilon > 0$,
$$\delta_X(\epsilon, z) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| \geq \epsilon, x-y \in \mathit{span}\{z\}\right\} > 0\,.$$
3. [27] X is said to be *k-uniformly rotund* (k-UR) if, for every $\epsilon > 0$,
$$\delta_X^k(\epsilon) = \inf\left\{1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : x_1, ..., x_{k+1} \in S_X, V(x_1, ..., x_{k+1}) \geq \epsilon\right\} > 0$$
where $V(x_1,\ldots,x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_1, \ldots, f_k \in B_{X^*}\right\}.$

The value $V(x_1,\ldots,x_{k+1})$ is said to be the k-dimensional volume enclosed by the vectors $x_1,\ldots,x_{k+1}$ and was introduced by Silverman in [22]. By Hahn

Banach theorem, it is easy to verify that $V(x_1, \ldots, x_{k+1}) > 0$ if and only if the dimension of $\text{aff}\{x_1, \ldots, x_{k+1}\}$ is k. For more information on multidimensional volumes one can refer to [2] and [12].

We first note that for any $z \neq 0$

$$\delta_X(\epsilon, z) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \begin{array}{l} x, y \in S_X, \|x-y\| \geq \epsilon, \\ span\{x-y\} = span\{z\} \end{array} \right\}.$$

Here, we consider line segments with end points in S_X whose lengths are at least ϵ and whose affine hulls(the corresponding lines) have the same linear translation namely $span\{z\}$. Since for any $y \in span\{z\}$, $y \neq 0$, $\delta(\epsilon, y) = \delta(\epsilon, z)$, we put

$$\delta(\epsilon, span\{z\}) = \delta(\epsilon, z)$$

and extend this idea to a k-dimensional subspace Y of X using Sullivan's k-uniform rotundity as given below.

Definition 1.1. For a k-dimensional subspace Y of a normed space X we define for each $\epsilon > 0$,

$$\delta_X(\epsilon, Y) = \inf \left\{ 1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in S_X, V(x_1, ..., x_{k+1}) \geq \epsilon, \\ span\{x_1 - x_{k+1}, ..., x_k - x_{k+1}\} = Y \end{array} \right\}.$$

The map $\epsilon \to \delta(\epsilon, Y)$ is said to be the modulus of rotundity of X with respect to Y. The space X is said to be uniformly rotund with respect to Y if $\delta_X(\epsilon, Y) > 0$ for all $\epsilon > 0$.

We would like to note that it is also correct to say that a normed space X is k-uniformly rotund with respect to a k-dimensional subspace Y if $\delta_X(\epsilon, Y) > 0$ for all $\epsilon > 0$. We choose to say X is uniformly rotund with respect to Y for the sake of convenience.

Here, we consider k-simplices with vertices in S_X whose volumes are at least ϵ and whose affine hulls have the same linear translation Y. Given $k \in \mathbb{Z}^+$, if X is uniformly rotund with respect to every k-dimensional subspace, we write X is URE_k. Uniform rotundity in every direction is equivalent to URE_1.

We shall also show that X is uniformly rotund with respect to a given k-dimensional subspace Y if and only if, for every $\epsilon > 0$,

$$\Delta_X(\epsilon, Y) = \inf_{\|x\|=1} \max_{\|y\|=\epsilon, y \in Y} \{\|x+y\|\} - 1 > 0.$$

2. URE_k spaces and Chebyshev centers

A normed space X is said to be rotund(strictly convex) if the unit sphere does not contain a nontrivial line-segment. A generalization of rotundity introduced in [23] by Singer is k-rotundity. A k-rotund(in short, k-R) normed space is that which does not contain a non-trivial k-simplex in its unit sphere.

It can be verified that the volume map $(x_1, \ldots, x_{k+1}) \to V(x_1, \ldots, x_{k+1})$ on X^{k+1} is continuous with respect to the product topology. Proposition 2.1 follows directly from the definitions and the continuity of the volume map.

Proposition 2.1. *Let $k \in \mathbb{Z}^+$. If X is k-UR, then X is URE$_k$. If X is URE$_k$, then X is k-R. If X is finite-dimensional, then k-UR, URE$_k$ and k-R are equivalent.*

Lemma 2.2. *Let $x_1, \ldots, x_{k+1} \in X$ and $y_i = \sum_{j=1}^{k+1} \alpha_{ij} x_j$, where each $\alpha_{ij} \in \mathbb{R}$ and $\sum_{j=1}^{k+1} \alpha_{ij} = 1$ for each $i \in \{1, \ldots, k+1\}$. Then for any $f_1, \ldots, f_k \in X^*$,*

$$\begin{bmatrix} 1 & \cdots & 1 \\ f_1(y_1) & \cdots & f_1(y_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(y_1) & \cdots & f_k(y_{k+1}) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{bmatrix} \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{k+1,1} \\ \alpha_{12} & \cdots & \alpha_{k+1,2} \\ \vdots & \vdots & \vdots \\ \alpha_{1,k+1} & \cdots & \alpha_{k+1,k+1} \end{bmatrix}.$$

Hence $$V(y_1, \ldots, y_{k+1}) = |\det[\alpha_{ij}]| V(x_1, \ldots, x_{k+1}).$$

In particular if each $\alpha_{ij} \geq 0$, then the matrix $[\alpha_{ij}]$ is a stochastic matrix (column sums are 1) and therefore $|\det[\alpha_{ij}]| \leq 1$, which implies

$$V(y_1, \ldots, y_{k+1}) \leq V(x_1, \ldots, x_{k+1}).$$

If $w = \sum_{i=1}^{k+1} \lambda_i x_i$ with $\sum_{i=1}^{k+1} \lambda_i = 1$, then for any $j \in \{1, \ldots, k+1\}$,

$$V(x_1, \ldots, x_{j-1}, w, x_{j+1}, \ldots, x_{k+1}) = |\lambda_j| V(x_1, \ldots, x_{k+1}).$$

Lemma 2.3. *Let $x_1, \ldots, x_{k+1} \in B_X$.*

(1) *For any $w \in co\{x_1, \ldots, x_{k+1}\}$, we have* $\dfrac{1 - \|w\|}{k+1} \leq 1 - \dfrac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\|.$

(2) *For any $w \in B_X$ and $w \in aff\{x_1, \ldots, x_{k+1}\}$, we have*

$$\frac{(1 - \|w\|) V(x_1, \ldots, x_{k+1})}{(k+1)^{\frac{k+3}{2}}} \leq 1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\|.$$

Proof. Let $w = \lambda_1 x_1 + \cdots + \lambda_{k+1} x_{k+1}$ where $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{R}$ and $\sum_{i=1}^{k+1} \lambda_i = 1$. Let $\lambda_j = max\{\lambda_1, \ldots, \lambda_{k+1}\}$. Then $\frac{1}{k+1} \leq \lambda_j$. So

$$x_j = \frac{1}{\lambda_j} w - \sum_{i \neq j} \frac{\lambda_i}{\lambda_j} x_i \,.$$

Using this we get,

$$\frac{1}{k+1}\sum_{i=1}^{k+1} x_i = \frac{1}{k+1}(x_j + \sum_{i\neq j} x_i) = \frac{1}{(k+1)\lambda_j}\left(w + \sum_{i\neq j}(\lambda_j - \lambda_i)x_i\right),$$

where $\frac{1}{(k+1)\lambda_j} > 0$ and $\lambda_j - \lambda_i \geq 0$ for $i \neq j$ and $\frac{1}{(k+1)\lambda_j}(1 + \sum_{i\neq j}(\lambda_j - \lambda_i)) = 1$.

So, $$\frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| \leq \frac{1}{(k+1)\lambda_j}\|w\| + \frac{1}{(k+1)\lambda_j}\sum_{i\neq j}(\lambda_j - \lambda_i),$$

which implies $$\frac{1-\|w\|}{(k+1)\lambda_j} \leq 1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\|.$$

If $w \in \operatorname{co}\{x_1, \ldots, x_{k+1}\}$, then $\lambda_j \leq 1$ and so

$$\frac{1-\|w\|}{k+1} \leq 1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\|,$$

proving (1). From [2] we know that if $x_1, \ldots, x_{k+1} \in B_X$, then

$$V(x_1, \ldots, x_{k+1}) \leq (k+1)^{\frac{k+1}{2}}.$$

If $V(x_1, \ldots, x_{k+1}) = 0$, there is nothing to prove. So let $V(x_1, \ldots, x_{k+1}) > 0$. If $w \in B_X$ and $w \in \operatorname{aff}\{x_1, \ldots, x_{k+1}\}$, by Lemma 2.2

$$\lambda_j = \frac{V(w, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k+1})}{V(x_1, \ldots, x_{k+1})} \leq \frac{(k+1)^{\frac{k+1}{2}}}{V(x_1, \ldots, x_{k+1})}.$$

Thus $$\frac{(1-\|w\|)V(x_1, \ldots, x_{k+1})}{(k+1)^{\frac{k+3}{2}}} \leq 1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\|. \qquad \square$$

Remark 2.4. An easy consequence of Lemma 2.3 is that given $k \in \mathbb{Z}^+$, every k-dimensional normed space is k-R and hence URE$_k$ and k-UR. To see this suppose X is k-dimensional and let $x_1, \ldots, x_{k+1}$ be affinely independent elements in S_X. Then $\operatorname{aff}\{x_1, \ldots, x_{k+1}\} = X$ and so $0 \in \operatorname{aff}\{x_1, \ldots, x_{k+1}\}$. Now using part (2) of Lemma 2.3, we get $\frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| < 1$. Thus X is k-R and hence URE$_k$ and k-UR. This fact can also be deduced from Singer's characterizations of k-rotundity in [23].

Proposition 2.5. *A normed space X is uniformly rotund with respect to a k-dimensional subspace Y if and only if for each $\epsilon > 0$,*

$$\inf\left\{1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ span\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y \end{array}\right\} > 0.$$

Proof. It is clear that if for each $\epsilon > 0$,

$$\inf\left\{1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ span\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y \end{array}\right\} > 0,$$

then X is uniformly rotund with respect to Y.

Suppose that there exists $\lambda > 0$ such that

$$\inf\left\{1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \lambda, \\ span\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y \end{array}\right\} = 0.$$

We first show that for any $x_1, \ldots, x_{k+1} \in B_X$, there exist $y_1, \ldots, y_{k+1} \in S_X$ such that $x_1, \ldots, x_{k+1} \in \text{co}\{y_1, \ldots, y_{k+1}\}$. If $\frac{1}{k+1}\sum_{j=1}^{k+1} x_j \in S_X$, then put $y_i = x_i$ for all $i \in \{1, \ldots, k+1\}$. Instead if $\frac{1}{k+1}\|\sum_{j=1}^{k+1} x_j\| < 1$ then consider the set

$$\left\{\|(1-\mu)\frac{1}{k+1}\sum_{j=1}^{k+1} x_j + \mu x_i\| : \mu \geq 0\right\}$$

for each $i \in \{1, \ldots, k+1\}$. It is an unbounded interval and for $0 \leq \mu < 1$,

$$\|(1-\mu)\frac{1}{k+1}\sum_{j=1}^{k+1} x_j + \mu x_i\| < 1.$$

So there exists $\eta_i \geq 1$ such that $\|(1-\eta_i)\frac{1}{k+1}\sum_{i=1}^{k+1} x_i + \eta_i x_i\| = 1$. Let

$$y_i = (1-\eta_i)\frac{1}{k+1}\sum_{j=1}^{k+1} x_j + \eta_i x_i.$$

Now $$x_i = \frac{1}{\eta_i} y_i + \left(1 - \frac{1}{\eta_i}\right)\frac{1}{k+1}\sum_{j=1}^{k+1} x_j, \quad i \in \{1, \ldots, k+1\}.$$

It follows that $$\left(\sum_{i=1}^{k+1} \frac{1}{\eta_i}\right)\frac{1}{k+1}\sum_{j=1}^{k+1} x_j = \sum_{i=1}^{k+1} \frac{1}{\eta_i} y_i.$$

Now, $$x_i = \frac{1}{\eta_i} y_i + \left(1 - \frac{1}{\eta_i}\right)\frac{1}{\left(\sum_{j=1}^{k+1} \frac{1}{\eta_j}\right)}\sum_{j=1}^{k+1} \frac{1}{\eta_j} y_j, \quad i \in \{1, \ldots, k+1\}.$$

Thus $x_1, \ldots, x_{k+1} \in \text{co}\{y_1, \ldots, y_{k+1}\}$. So by Lemma 2.2

$$V(y_1, \ldots, y_{k+1}) \geq V(x_1, \ldots, x_{k+1}) \geq \lambda.$$

Also, $\operatorname{aff}(y_1, \dots, y_{k+1}) = \operatorname{aff}(x_1, \dots, x_{k+1})$ which implies

$$span\{y_1 - y_{k+1}, \dots, y_k - y_{k+1}\} = span\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\} = Y.$$

Using part (2) of Lemma 2.3, we get

$$\frac{1}{(k+1)^{\frac{k+3}{2}}} \left(1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} y_i\|\right) \lambda \leq 1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\|$$

from which it follows that $\delta_X(\lambda, Y) = 0$. So X is not uniformly rotund with respect to Y. □

Theorem 2.6 (Silverman[22]). *Let Y be a k-dimensional subspace of X. Fix a basis $\{y_1, \dots, y_k\} \subset S_X$ of Y. If $x_1, \dots, x_{k+1} \in X$ such that for each $i \in \{1, ..., k\}$, $x_i - x_{k+1} = \alpha_{i1} y_1 + \cdots + \alpha_{ik} y_k$, where $\alpha_{ij} \in \mathbb{R}$ for all $i, j \in \{1, ..., k\}$, then $V(x_1, \dots, x_{k+1}) = |\det(\alpha_{ij})| V(0, y_1, \dots, y_k)$.*

Theorem 2.7. *Let X be a normed space, let $k \in \mathbb{Z}^+$ and let $y_1, \dots, y_k \in S_X$ be linearly independent. Then the statements* (1), (2) *and* (3) *below are equivalent.*

(1) *There exists a bounded subset K of X such that $C(X, K)$ contains $co\{0, y_1, \dots, y_k\}$.*
(2) *X is not uniformly rotund with respect to $Y = span\{y_1, \dots, y_k\}$.*
(3) *There exists a bounded subset K of X such that $C(X, K)$ contains $co\{\pm y_1, \dots, \pm y_k\}$.*

Proof. It is enough to prove that (1) ⇒ (2) and (2) ⇒ (3).

(1) ⇒ (2): Suppose there exists a bounded subset K of X such that $C(X, K)$ contains $\operatorname{co}\{0, y_1, \dots, y_k\}$. Put $y_{k+1} = 0$ and $w = \frac{1}{k+1} \sum_{i=1}^{k+1} y_i$. Then $R(w, K) = R(X, K)$. Given $\delta \in (0, 1)$, choose $x \in K$ such that

$$(1 - \delta) R(X, K) < \|w - x\|.$$

Then we have

$$1 - \delta < \frac{\|w - x\|}{R(X, K)} \leq \frac{1}{k+1} \left\| \sum_{i=1}^{k+1} \frac{(y_i - x)}{R(X, K)} \right\|.$$

For $i \in \{1, \dots, k+1\}$, we have $\|y_i - x\| \leq R(X, K)$. By Proposition 2.5, it follows that X is not uniformly rotund with respect to Y.

(2) ⇒ (3): Suppose that X is not uniformly rotund with respect to Y. It is easy to see that given a bounded subset K of X and $\alpha > 0$, $R(X, \alpha K) = \alpha R(X, K)$ and $C(X, \alpha K) = \alpha C(X, K)$. So it is enough to prove that there exists a

bounded subset K of X such that $\mathrm{co}\{\pm qy_1, \ldots, \pm qy_k\} \subseteq C(X,K)$ for some $q > 0$. Choose $\lambda > 0$ such that $\delta_X(\lambda, Y) = 0$. Let

$$K = \left\{ \frac{1}{k+1} \sum_{i=1}^{k+1} x_i : \begin{array}{l} x_1, \ldots, x_{k+1} \in S_X, \\ V(x_1, \ldots, x_{k+1}) \geq \lambda, \\ span\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y \end{array} \right\}.$$

Then $K \subseteq B_X$, $R(0,K) = 1$ and $diam(K) = 2$. Since $diam(K) \leq 2R(X,K)$, we have $R(X,K) = 1$. Let $w \in K$. Then we have $w = \frac{1}{k+1} \sum_{i=1}^{k+1} x_i$ for some $x_1, \ldots, x_{k+1} \in S_X$ such that $V(x_1, \ldots, x_{k+1}) \geq \lambda$ and moreover $span\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y$. Let $x_i - x_{k+1} = \alpha_{i1} y_1 + \cdots + \alpha_{ik} y_k$, where $\alpha_{ij} \in \mathbb{R}$ for $i, j \in \{1, \ldots, k\}$. By Cramer's rule we have,

$$y_j = \frac{c_{1j}(x_1 - x_{k+1}) + \cdots + c_{kj}(x_k - x_{k+1})}{\det(\alpha_{ij})}$$

where c_{ij} is the cofactor of α_{ij} for each $i, j \in \{1, \ldots, k\}$.

Since any two norms on a finite-dimensional space are equivalent, there exists $M > 0$ such that $|\alpha_1| + \cdots + |\alpha_k| \leq M \|\alpha_1 y_1 + \cdots + \alpha_k y_k\|$ for all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. It can be verified that $|c_{ij}| \leq (2M)^{k-1}$ for all i, j and so $\sum_{i=1}^{k} |c_{ij}| \leq k(2M)^{k-1}$. Put $M_0 = k(2M)^{k-1}$ and $V_Y = V(0, y_1, \ldots, y_k)$. By Theorem 2.6, $V(x_1, \ldots, x_{k+1}) = |det(\alpha_{ij})| V_Y$. Let $q = \frac{\lambda}{(k+1)V_Y M_0}$. Fix $j \in \{1, \ldots, k\}$. Then

$$\begin{aligned} w - qy_j &= \frac{1}{k+1} \sum_{i=1}^{k+1} x_i - \frac{q}{det(\alpha_{ij})} \sum_{i=1}^{k} c_{ij} \left(x_i - x_{k+1}\right) \\ &= \sum_{i=1}^{k} \left(\frac{1}{k+1} - \frac{q}{det(\alpha_{ij})} c_{ij} \right) x_i + \left(\frac{1}{k+1} + \frac{q}{det(\alpha_{ij})} \sum_{i=1}^{k} c_{ij} \right) x_{k+1}. \end{aligned}$$

By the choice of q it is clear that the coefficient of each x_i in the above expression is nonnegative. Also

$$\sum_{i=1}^{k} \left(\frac{1}{k+1} - \frac{q}{det(\alpha_{ij})} c_{ij} \right) + \left(\frac{1}{k+1} + \frac{q}{det(\alpha_{ij})} \sum_{i=1}^{k} c_{ij} \right) = 1.$$

Thus, $w - qy_j \in \mathrm{co}\{x_1, \ldots, x_{k+1}\}$. So $\|w - qy_j\| \leq 1$. Hence $R(qy_j, K) = 1$. Since $K = -K$, we get $R(-qy_j, K) = 1$, and therefore $\mathrm{co}\{\pm qy_1, \ldots, \pm qy_k\} \subseteq C(X,K)$. □

Corollary 2.8. *Given a positive integer k, a normed space X is URE_k if and only if the Chebyshev center of every nonempty bounded set is either empty or is of dimension at most $k - 1$.*

Proof. Suppose K is a nonempty bounded subset of X. If $x_1, \dots, x_{k+1} \in C(X,K)$ and $V(x_1, \dots, x_{k+1}) > 0$, then $0, x_1 - x_{k+1}, \dots, x_k - x_{k+1} \in C(X,K) - x_{k+1} = C(X, K - x_{k+1})$. It follows from Theorem 2.7 that X is not uniformly rotund with respect to $span\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\}$.

If X is not uniformly rotund with respect to some k-dimensional subspace then by Theorem 2.7, there exists a bounded set whose Chebyshev center is of dimension at least k.

Thus X is URE_k if and only if the Chebyshev center of every nonempty bounded set is either empty or is of dimension at most $k-1$. □

Corollary 2.9 (Garkavi [11])**.** *A normed space X is uniformly rotund in every direction if and only if the Chebyshev center of every nonempty bounded set contains at most one point.*

Corollary 2.10. *If a normed space X is uniformly rotund with respect to a finite-dimensional subspace Y, then X is uniformly rotund with respect to every finite-dimensional subspace containing Y. In particular, if X is uniformly rotund in a direction $z \neq 0$, then X is uniformly rotund with respect to every finite-dimensional subspace containing z.*

Proof. Suppose X is not uniformly rotund with respect to the k-dimensional subspace $Y = span\{y_1, \dots, y_k\}$. Then by Theorem 2.7, there exists a nonempty bounded set K in X such that $co\{\pm y_1, \dots, \pm y_k\} \subseteq C(X,K)$. For any $u \in Y$, there exist $\lambda_1, \dots, \lambda_k \geq 0$ and $c_1, \dots, c_k \in \{-1, 1\}$ such that $u = \lambda_1 c_1 y_1 + \cdots + \lambda_k c_k y_k$. So $u = (\lambda_1 + \cdots + \lambda_k)v$ for some $v \in co\{\pm y_1, \dots, \pm y_k\}$. Thus if Z is any subspace of Y, then there exist $z_1, \dots, z_m \in co\{\pm y_1, \dots, \pm y_k\}$ such that $Z = span\{z_1, \dots, z_m\}$. Now $co\{0, z_1, \dots, z_m\} \subset C(X,K)$. Hence X is not uniformly rotund with respect to Z. □

Corollary 2.11. *If X is uniformly rotund with respect to all k-dimensional subspaces except countably many, then X is uniformly rotund with respect to every $(k+1)$-dimensional subspace of X.*

Proof. This follows directly from Corollary 2.10 as every $(k+1)$-dimensional space contains uncountably many k-dimensional subspaces. □

Corollary 2.12. *Let k be a positive integer. If X is URE_k, then X is URE_{k+1}.*

In [28], it is proved that if the set of all directions in which a normed space X is not uniformly rotund is contained in a countable union of k-dimensional subspaces, then asymptotic centers relative to convex sets are of dimension at most k. Here, we have the following.

Corollary 2.13. *If the set of all directions in which a normed space X is not uniformly rotund is contained in a countable union of k-dimensional subspaces, then X is URE_{k+1}.*

The following chart of implications where each arrow gives the direction of implication shows the position of the new classes of normed spaces, namely, URE_k, among the known classes of spaces.

$$\begin{array}{ccccccccc}
\mathrm{R} & \rightarrow & 2\text{-R} & \rightarrow & \cdots & k\text{-R} & \rightarrow & (k+1)\text{-R} & \cdots \\
\uparrow & & \uparrow & & \cdots & \uparrow & & \uparrow & \cdots \\
\mathrm{URE}_1 & \rightarrow & \mathbf{URE_2} & \rightarrow & \cdots & \mathbf{URE_k} & \rightarrow & \mathbf{URE_{(k+1)}} & \cdots \\
\uparrow & & \uparrow & & \cdots & \uparrow & & \uparrow & \cdots \\
\mathrm{UR} & \rightarrow & 2\text{-UR} & \rightarrow & \cdots & k\text{-UR} & \rightarrow & (k+1)\text{-UR} & \cdots
\end{array}$$

3. A common fixed point property for isometries in URE_k spaces

In [3], Brodskii and Milman introduced a geometric property called normal structure for nonempty convex sets in normed spaces.

Definition 3.1. ([3]) A nonempty convex subset A of a normed space is said to have normal structure if for every bounded convex subset K of A with $diam(K) > 0$, there exists $a \in K$ such that $R(a, K) < diam(K)$. Such a point is called a nondiametral point of K.

Alternately, A has normal structure if for every bounded convex subset K of A with $diam(K) > 0$, it is true that $C(K) \neq K$.

Theorem 3.2. (Brodskii and Milman[3]) *Every nonempty compact convex set of a normed space has normal structure.*

Theorem 3.3. (Brodskii and Milman[3]) *Let K be a nonempty weakly compact convex subset of a normed space. If K has normal structure, then the collection of all surjective isometric selfmaps on K has a common fixed point.*

Theorem 3.4. (Kirk[13]) *Let K be a nonempty weakly compact convex subset of a normed space. If K has normal structure, then every nonexpansive selfmap on K has a fixed point.*

In [7], it is observed that in Corollary 2.9, $C(X, K)$ may be replaced with $C(K)$. Similarly in Theorem 2.8, $C(X, K)$ may be replaced with $C(K)$. The proof of Theorem 3.5 is essentially the same as that of Theorem 2.7.

Theorem 3.5. *Let X be a normed space. Given $k \in \mathbb{Z}^+$ and $y_1, \ldots, y_k \in S_X$ linearly independent, X is not uniformly rotund with respect to $Y = span\{y_1, \ldots, y_k\}$ if and only if there exists a bounded convex subset K of X such that $C(K)$ contains $co\{\pm y_1, \ldots, \pm y_k\}$.*

Proof. To prove the only-if part let $\delta_X(\lambda, Y) = 0$ for some $\lambda > 0$ and choose M as in Theorem 2.7. Put $M_0 = \max\{k(2M)^{k-1}, (2M)^k\}$. Consider the set

$$K = \mathrm{co}\left\{ \pm q y_j, \frac{1}{k+1}\sum_{i=1}^{k+1} x_i : \begin{array}{l} j \in \{1,\dots,k\} \\ x_1,\dots,x_{k+1} \in S_X, \\ V(x_1,\dots,x_{k+1}) \geq \lambda, \\ span\{x_1 - x_{k+1},\dots,x_k - x_{k+1}\} = Y \end{array} \right\}$$

where $q = \frac{\lambda}{(k+1)V_Y M_0}$. Then $q \leq \frac{1}{2}$, and therefore $\|qy_i \pm qy_j\| \leq 1$. The rest of the proof is the same as in Theorem 2.7. □

So we have the following.

Corollary 3.6. *Given a positive integer k, a normed space X is URE$_k$ if and only if for every nonempty bounded convex set K in X, $C(K)$ is either empty or is of dimension at most $k-1$.*

Corollary 3.7. *If a normed space X is URE$_k$ for some positive integer k, then X has normal structure.*

Proof. From [3] we know that every compact convex set has a nondiametral point. Hence every totally bounded convex set has a nondiametral point. If K is any bounded convex subset of X, it follows from Corollary 3.6 that $C(K)$ is totally bounded. If K is not totally bounded, then clearly $C(K) \neq K$. Thus X has normal structure. □

Corollary 3.8 (Sangeetha and Veeramani[28])**.** *If the set of all directions in which a normed space X is not uniformly rotund is contained in a countable union of k-dimensional subspaces, then for every nonempty bounded convex set K in X, $C(K)$ is either empty or is of dimension at most k.*

In normed spaces that are URE$_k$ for some positive integer k, like the Chebyshev centers, the asymptotic center of a nested sequence of nonempty bounded sets relative to a convex set is either empty or is of dimension at most $k-1$. To prove this we make use of a sequential characterization of uniform rotundity with respect to a finite-dimensional subspace which is an extension of a characterization of uniformly rotundity in a direction proved in [11] by Garkavi. The technique used here is the same as that used in [11].

Proposition 3.9. *Given a normed space space X, the following are equivalent.*

(1) *X is uniformly rotund with respect to a k-dimensional subspace Y.*

(2) *If $(x_1^{(n)}),\dots,(x_{k+1}^{(n)})$ are sequences in X such that*

$$span\{x_1^{(n)} - x_{k+1}^{(n)},\dots,x_k^{(n)} - x_{k+1}^{(n)}\} = Y \quad \text{for } n \in \mathbb{Z}^+ \text{ and}$$
$$\lim_{n\to\infty} \|x_i^{(n)}\| = \lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i^{(n)}\| = 1 \quad \text{for} \quad i \in \{1,\dots,k+1\},$$

then $\lim_{n\to\infty} V(x_1^{(n)},\dots,x_{k+1}^{(n)}) = 0$.

Proof. It is easy to see that (2) $\Rightarrow$ (1). So we prove that (1) $\Rightarrow$ (2). Suppose that X is uniformly rotund with respect to a k-dimensional subspace Y. Let $(x_1^{(n)}), \dots, (x_{k+1}^{(n)})$ be sequences in X such that $span\{x_1^{(n)} - x_{k+1}^{(n)}, \dots, x_k^{(n)} - x_{k+1}^{(n)}\} = Y$ for $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} \|x_i^{(n)}\| = \lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i^{(n)}\| = 1$ for $i \in \{1, \dots, k+1\}$. If for $n \in \mathbb{Z}^+$, we let $M_n = \max\{\|x_1^{(n)}\|, \dots, \|x_{k+1}^{(n)}\|\}$, then $\frac{\|x_1^{(n)}\|}{M_n}, \dots, \frac{\|x_{k+1}^{(n)}\|}{M_n} \leq 1$ and $\lim_{n\to\infty} M_n = 1$. So $\lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} \frac{1}{M_n} x_i^{(n)}\| = 1$. Also

$$span\left\{\frac{x_1^{(n)} - x_{k+1}^{(n)}}{M_n}, \dots, \frac{x_k^{(n)} - x_{k+1}^{(n)}}{M_n}\right\} = Y$$

for all $n \in \mathbb{Z}^+$. Since X is uniformly rotund with respect to Y we have by Proposition 2.5,

$$\lim_{n\to\infty} \frac{V(x_1^{(n)}, \dots, x_{k+1}^{(n)})}{M_n^k} = 0,$$

which implies $\lim\limits_{n\to\infty} V(x_1^{(n)}, \dots, x_{k+1}^{(n)}) = 0$. □

Proposition 3.10. *Let K be a convex subset of X and (K_n) a nested sequence of bounded subsets of X. If $x_1, \dots, x_{k+1} \in C(K, (K_n))$ such that $V(x_1, \dots, x_{k+1}) > 0$, then X is not uniformly rotund with respect to span$\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\}$.*

Proof. Let $x_1, \dots, x_{k+1} \in C(K, (K_n))$ and $w = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$. Then we have $w \in C(K, (K_n))$. For each $n \in \mathbb{Z}^+$, choose $w_n \in K_n$ such that $R(w, K_n) - \frac{1}{n} < \|w - w_n\|$. Then

$$\lim_{n\to\infty} \|w - w_n\| = \lim_{n\to\infty} \frac{\|x_1 - w_n\| + \cdots + \|x_{k+1} - w_n\|}{k+1} = R(K, (K_n)).$$

It is not difficult to see that for each $i \in \{1, \dots, k+1\}$, $\lim_{n\to\infty} \|x_i - w_n\| = R(K, (K_n))$. For $n \in \mathbb{Z}^+$, put $a_i^{(n)} = x_i - w_n$ for $i \in \{1, \dots, k+1\}$. Then we have $V(a_1^{(n)}, \dots, a_{k+1}^{(n)}) = V(x_1, \dots, x_{k+1})$ and $\text{span}\{a_1^{(n)} - a_{k+1}^{(n)}, \dots, a_k^{(n)} - a_{k+1}^{(n)}\} = \text{span}\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\}$. Hence by Proposition 3.9, X is not uniformly rotund with respect to $\text{span}\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\}$. □

Corollary 3.11. *If X is a normed space that is URE$_k$, then the asymptotic center of any nested sequence of nonempty bounded sets in a convex set is either empty or is of dimension at most $k-1$. In particular, the Chebyshev center of any nonempty bounded set in a convex set is either empty or is of dimension at most $k-1$.*

Corollary 3.12 (Lim [16], Day [8]). *If X is a normed space that is uniformly rotund in every direction, then the asymptotic center of any nested sequence* *of nonempty bounded sets in a convex set contains at most one point. In particular, the Chebyshev center of any nonempty bounded set in a convex set contains at most one point.*

Corollary 3.13 (Kirk [14])**.** *If X is a normed space that is k-UR, then the asymptotic center of any nested sequence of nonempty bounded sets in a convex set is either empty or is of dimension at most $k-1$.*

In [17], Lim et al. proved the following.

Theorem 3.14 (Lim et al. [17])**.** *If K is a weakly compact convex subset of normed space X and K has normal structure, then for any isometric selfmap T on K, the set $C(K,(T^n(K)))$ is invariant under T and the fixed points of T in $C(K,(T^n(K)))$(which exist by Theorem 3.4) belong to $C(K)$. Hence if X is uniformly rotund in every direction then the unique element of $C(K)$ is fixed by every isometry from K into K.*

In [17], the authors asked whether in general if K is a weakly compact convex set with normal structure and $T\colon K \to K$ is an isometry, then $C(K)$ is invariant under T. Another question raised in [17] is whether there is a common fixed point for the collection of all isometric self maps on a weakly compact convex set with normal structure. In [21], it is shown the answer to the first question is negative.

But in [28], it is proved that the answers to both questions are positive if the set of all directions in which a normed space is not uniformly rotund is contained in a countable union of k-dimensional subspaces. This is essentially due to the fact that asymptotic centers of nested sequences of bounded sets in weakly compact convex sets of such spaces are compact.

Theorem 3.15. *Let X be a normed space and let K be a nonempty weakly compact convex subset of X. Let T be an isometry that maps K into K. Suppose that $C(K)$ is contained in a compact subset A of K such that $T(A) \subseteq A$. Then T maps $C(K)$ onto $C(K)$. If $K_1 = C(K)$ and $K_{n+1} = C(K_n)$ for all $n \in \mathbb{Z}^+$, then $\bigcap_{n=1}^{\infty} K_n$ contains exactly one point and this point is fixed by T.*

Proof. Suppose that $C(K)$ is contained in a compact subset A of K such that $T(A) \subseteq A$. Note that for any $x \in K$, $R(x,K) = R(Tx,TK) \leq R(Tx,K)$. Let $x \in C(K)$. Then $\{T^n x : n \in \mathbb{Z}^+\} \subseteq A$. Hence $(T^n x)$ has a Cauchy subsequence. Given $\epsilon > 0$, choose $j,k \in \mathbb{Z}^+, j < k$ such that $\left\|T^j x - T^k x\right\| < \epsilon$. Since T is an isometry, it follows that $\left\|x - T^{k-j} x\right\| < \epsilon$. Thus $x \in \overline{T(K)} = T(K)$. So $C(K) \subseteq T(K)$. Now T^{-1} is an isometry on $C(K)$ and for any $x \in C(K)$ we have $R(T^{-1}x,K) \leq R(x,K)$. Thus $T^{-1}C(K) \subseteq C(K)$. Since $C(K)$ is compact, we have $T^{-1}(C(K)) = C(K)$. Hence $T(C(K)) = C(K)$.

Note that the sequence $(diam(K_n))$ is monotonically decreasing and for each $n \in \mathbb{Z}^+$, $R(K_{n+1}) \leq diam(K_{n+1}) \leq R(K_n)$.

So we have, $\lim_{n\to\infty} diam(K_n) = \lim_{n\to\infty} R(K_n)$. Since $C(K)$ is compact, (K_n) is a nested sequence of compact sets of X. Hence $\lim_{n\to\infty} diam(K_n) = diam(\bigcap_{n=1}^{\infty} K_n)$. Let $x \in \bigcap_{n=1}^{\infty} K_n$. Then for each $n \in \mathbb{Z}^+$, $R(x, K_n) = R(K_n)$ and there exists $y_n \in K_n$ such that $\|x - y_n\| = R(K_n)$. Choose a convergent subsequence (y_{n_r}) of (y_n). Let $y = \lim_{r\to\infty} y_{n_r}$. Now, $\|x - y\| = \lim_{r\to\infty} \|x - y_{n_r}\| = \lim_{r\to\infty} R(K_{n_r}) = diam(\bigcap_{n=1}^{\infty} K_n)$ and $y \in \bigcap_{n=1}^{\infty} K_n$ which means x is a diametral point of $\bigcap_{n=1}^{\infty} K_n$. Thus the nonempty compact convex set $\bigcap_{n=1}^{\infty} K_n$ has no non-diametral point and so it is a singleton say $\{c\}$. Since $TK_n = K_n$ for each $n \in \mathbb{Z}^+$, it follows that $Tc = c$. □

Using Theorem 3.14, the following easy observation was made in [28].

Lemma 3.16. (Sangeetha and Veeramani [28]) *If K is a nonempty weakly compact convex subset of normed space and K has normal structure, then for any isometric selfmap T on K, $C(K) \subseteq C(K, (T^n(K)))$.*

Combining Theorem 2.8, Lemma 3.16 and Theorem 3.15, we have the following

Corollary 3.17. *Let X be a normed space that is URE_k and let K be a nonempty weakly compact convex subset of X. Then every isometry that maps K into K maps $C(K)$ onto $C(K)$ and there exists a point in $C(K)$ that is fixed by every isometry from K into K.*

Corollary 3.18. (Sangeetha and Veeramani [28]) *Let the set of all directions in which a normed space X is not uniformly rotund be contained in a countable union of k-dimensional subspaces. If K is a nonempty weakly compact convex subset of X, then every isometry that maps K into K maps $C(K)$ onto $C(K)$ and there exists a point in $C(K)$ that is fixed by every isometry from K into K.*

4. Uniform rotundity with respect to finite-dimensional subspaces in product spaces

In [8], the following characterization of normed spaces that are uniformly rotund in every direction is given.

Theorem 4.1 (Day, James, Swaminathan[8])**.** *A normed space X is uniformly rotund in every direction if and only if whenever $(x_n), (y_n)$ are sequences in B_X such that $\lim_{n\to\infty} \frac{\|x_n+y_n\|}{2} = 1$ and $(x_n - y_n)$ is convergent, then $\lim_{n\to\infty}(x_n - y_n) = 0$.*

We give a generalized version of this result in the next theorem.

Theorem 4.2. *A normed space X is not uniformly rotund with respect to a k-dimensional subspace Y if and only if there exist sequences $(x_1^{(n)}), \ldots, (x_{k+1}^{(n)})$ in B_X such that $\lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i^{(n)}\| = 1$ and $\lim_{n\to\infty} x_i^{(n)} - x_{k+1}^{(n)} = z_i \in Y$ for each $i \in \{1, \ldots, k\}$ where $z_1, \ldots, z_k$ are linearly independent.*

Proof. Suppose that there exist $(x_1^{(n)}), \dots, (x_{k+1}^{(n)})$ in B_X such that

$$\lim_{n\to\infty} \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i^{(n)}\| = 1, \ \ \lim_{n\to\infty} x_i^{(n)} - x_{k+1}^{(n)} = z_i \in Y$$

for each $i \in \{1, \dots, k\}$ and $z_1, \dots, z_k$ are linearly independent. Using part (1) of Lemma 2.3, it is easy to see that for each $i \in \{1, \dots, k\}$, $\lim_{n\to\infty} \|x_i^{(n)}\| = 1$. Let $y_1^{(n)} = x_{k+1}^{(n)} + z_1, \ \dots, \ y_k^{(n)} = x_{k+1}^{(n)} + z_k, \ y_{k+1}^{(n)} = x_{k+1}^{(n)}$. Now

$$\lim_{n\to\infty} \|y_i^{(n)} - x_i^{(n)}\| = 0, \quad i \in \{1, \dots, k+1\}.$$

So $\lim_{n\to\infty} \|y_i^{(n)}\| = 1$ for $i \in \{1, \dots, k+1\}$. Also we have,

$$\lim_{n\to\infty} \|\sum_{i=1}^{k+1} y_i^{(n)} - \sum_{i=1}^{k+1} x_i^{(n)}\| = 0.$$

Therefore, $\lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} y_i^{(n)}\| = 1$. Besides,

$$span\{y_1^{(n)} - y_{k+1}^{(n)}, \dots, y_k^{(n)} - y_{k+1}^{(n)}\} = span\{z_1, \dots, z_k\} = Y$$

and

$$V(y_1^{(n)}, \dots, y_{k+1}^{(n)}) = V(z_1, \dots, z_k, 0) > 0$$

for all $n \in \mathbb{Z}^+$. Hence by Proposition 3.9, X is not uniformly rotund with respect to Y.

Suppose that X is not uniformly rotund with respect to a k-dimensional subspace Y. Fix a basis $\{y_1, \dots, y_k\}$ of Y. By Proposition 2.5 there exist sequences $(x_1^{(n)}), \dots, (x_{k+1}^{(n)})$ in B_X such that

$$span\{x_1^{(n)} - x_{k+1}^{(n)}, \dots, x_k^{(n)} - x_{k+1}^{(n)}\} = Y \text{ and } 0 < \lambda \le V(x_1^{(n)}, \dots, x_{k+1}^{(n)})$$

for all $n \in \mathbb{Z}^+$ satisfying $\lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i^{(n)}\| = 1$. Since $(x_1^{(n)} - x_{k+1}^{(n)}), \dots,$ $(x_k^{(n)} - x_{k+1}^{(n)})$ are bounded sequences in Y, we may assume without loss of generality that they are convergent. Let $\lim_{n\to\infty} x_i^{(n)} - x_{k+1}^{(n)} = z_i$. For each $i, j \in \{1, \dots, k\}$ and $n \in \mathbb{Z}^+$ let $\alpha_{ij}^{(n)}, \alpha_{ij} \in \mathbb{R}$ such that

$$x_i^{(n)} - x_{k+1}^{(n)} = \alpha_{i1}^{(n)} y_1 + \cdots + \alpha_{ik}^{(n)} y_k \ \text{ and } \ z_i = \alpha_{i1} y_1 + \cdots + \alpha_{ik} y_k.$$

Then $V(x_1^{(n)}, \dots, x_{k+1}^{(n)}) = |\det(\alpha_{ij}^{(n)})| V(0, y_1, \dots, y_k)$ by Theorem 2.6. Also $\lim_{n\to\infty} \alpha_{ij}^{(n)} = \alpha_{ij}$. So $\lim_{n\to\infty} \det(\alpha_{ij}^{(n)}) = \det(\alpha_{ij})$ and it follows that

$$0 < \frac{\lambda}{V(0, y_1, \dots, y_k)} \le |\det(\alpha_{ij})|.$$

Thus $z_1, \dots, z_k$ are linearly independent. $\square$

Now we consider uniform rotundity with respect to finite-dimensional subspaces in products of normed spaces.

Definition 4.3 (Day[6]**).** Let S be a nonempty set. A full function space is a Banach space X of real-valued functions on S such that if $f \in X$ and $g\colon S \to \mathbb{R}$ satisfies $|g(s)| \leq |f(s)|$ for all $s \in S$ then $g \in X$ and $\|g\| \leq \|f\|$.

It can be verified that the spaces l_p for $1 \leq p \leq \infty$ are full function spaces.

Let $\{N_s : s \in S\}$ be a nonempty collection of normed spaces. Given $x \in \prod_{s\in S} N_s$, let $F_x\colon S \to \mathbb{R}$ be defined by $F_x(s) = \|x(s)\|$ for all $s \in S$ and if $F_x \in X$, let $\|x\| = \|F_x\|$. Then $P_X(N_s) = (\{x \in \prod_{s\in S} N_s : F_x \in X\}, \|\cdot\|)$ is a normed space which is complete if and only if each N_s is a Banach space[6].

Theorem 4.4. *Let $\{N_s : s \in S\}$ be a collection of normed spaces and let X be a full function space over S. Suppose that for any $a \in X$ such that $0 \leq a(s)$ for all $s \in S$, the order interval*

$$[0, a] = \{f\colon S \to \mathbb{R} : 0 \leq f(s) \leq a(s) \text{ for all } s \in S\}$$

is compact in X and that X is uniformly rotund in every direction. Given a k-dimensional subspace $Y = span\{y_1, \ldots, y_k\}$ of $P_X(N_s)$, if for some $s_0 \in S$, $y_1(s_0), \ldots, y_k(s_0)$ are linearly independent and N_{s_0} is uniformly rotund with respect to $span\{y_1(s_0), \ldots, y_k(s_0)\}$, then $P_X(N_s)$ is uniformly uniformly rotund with respect to Y.

Proof. Let $x_1, \ldots, x_{k+1} \in S_X$ such that $span\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y$. Let $x_i - x_{k+1} = \alpha_{i1} y_1 + \cdots + \alpha_{ik} y_k$ for $i \in \{1, \ldots, k\}$. For each $i \in \{1, \ldots, k\}$,

$$\begin{aligned}|F_{x_i}(s) - F_{x_{k+1}}(s)| &= |\|x_i(s)\| - \|x_{k+1}(s)\|| \leq \|x_i(s) - x_{k+1}(s)\| \\ &= \|\alpha_{i1} y_1(s) + \cdots + \alpha_{ik} y_k(s)\| \leq |\alpha_{i1}|\|y_1(s)\| + \cdots + |\alpha_{ik}|\|y_k(s)\| \\ &\leq |\alpha_{i1}| F_{y_1}(s) + \cdots + |\alpha_{ik}| F_{y_k}(s).\end{aligned}$$

Since Y is a finite-dimensional space, we may choose $M > 0$ such that

$$\max\{|\lambda_1|, \ldots, |\lambda_k|\} \leq M\|\lambda_1 y_1 + \cdots + \lambda_k y_k\|$$

for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Then $|\alpha_{ij}| \leq 2M$ for all $i, j \in \{1, \ldots, k\}$ which implies

$$|F_{x_i}(s) - F_{x_{k+1}}(s)| \leq 2M(F_{y_1} + \cdots + F_{y_k})(s).$$

We also have $\left|F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i} - F_{x_{k+1}}(s)\right| \leq \dfrac{k}{k+1} 2M(F_{y_1} + \cdots + F_{y_k})(s)$. So,

$$F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i} - F_{x_{k+1}}, F_{x_1} - F_{x_{k+1}}, \ldots, F_{x_k} - F_{x_{k+1}} \in [-a, a],$$

where $a = 2M(F_{y_1} + \cdots + F_{y_k})$. Also we note that

$$\frac{1}{2}\left\|\frac{1}{k+1}\sum_{i=1}^{k+1} x_i + x_{k+1}\right\| \leq \frac{1}{2}\|F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i} + F_{x_{k+1}}\|,$$

and

$$\frac{1}{2}\|x_i + x_{k+1}\| \leq \frac{1}{2}\|F_{x_i} + F_{x_{k+1}}\| \quad i \in \{1, \ldots, k\}.$$

Now let $(x_1^{(n)}), \ldots, (x_{k+1}^{(n)})$ be sequences in $S_{P_X(N_s)}$ such that

$$span\{x_1^{(n)} - x_{k+1}^{(n)}, \ldots, x_k^{(n)} - x_{k+1}^{(n)}\} = Y$$

for all $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} \frac{1}{k+1}\| \sum_{i=1}^{k+1} x_i^{(n)}\| = 1$. Then $(F_{x_1^{(n)}}), \ldots, (F_{x_{k+1}^{(n)}})$ are sequences in S_X and $(F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{(n)}})$ is a sequence in B_X such that

$$\lim_{n\to\infty} \frac{1}{2}\|F_{x_i^{(n)}} + F_{x_{k+1}^{(n)}}\| = 1, \quad i \in \{1, \ldots, k\},$$

$$\lim_{n\to\infty} \frac{1}{2}\|F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{(n)}} + F_{x_{k+1}^{(n)}}\| = 1$$

and $F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{(n)}} - F_{x_{k+1}^{(n)}}, F_{x_1^{(n)}} - F_{x_{k+1}^{(n)}}, \ldots, F_{x_k^{(n)}} - F_{x_{k+1}^{(n)}} \in [-a, a]$ for all $n \in \mathbb{Z}^+$.

Since by assumption, $[-a, a]$ is compact, we assume that these sequences are convergent. In addition, since the sequences $(\|x_1^{(n)}(s_0)\|), \ldots, (\|x_{k+1}^{(n)}(s_0)\|)$ and $(\frac{1}{k+1}\| \sum_{i=1}^{k+1} x_i^{(n)}(s_0)\|)$ are bounded we assume these are also convergent. There is no loss of generality in assuming these sequences are convergent as it is enough to prove that a subsequence of $V(x_1^{(n)}, \ldots, x_{k+1}^{(n)})$ converges to 0.

Since X is uniformly rotund in every direction, we have for $i \in \{1, \ldots, k\}$, by Theorem 4.1,

$$\lim_{n\to\infty} F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{(n)}} - F_{x_{k+1}^{(n)}} = \lim_{n\to\infty} F_{x_i^{(n)}} - F_{x_{k+1}^{(n)}} = 0.$$

So $\lim_{n\to\infty} F_{\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{(n)}}(s_0) - F_{x_{k+1}^{(n)}}(s_0) = \lim_{n\to\infty} F_{x_i^{(n)}}(s_0) - F_{x_{k+1}^{(n)}}(s_0) = 0$, i.e.

$$\lim_{n\to\infty} \|\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{(n)}(s_0)\| - \|x_{k+1}^{(n)}(s_0)\| = \lim_{n\to\infty} \|x_i^{(n)}(s_0)\| - \|x_{k+1}^{(n)}(s_0)\| = 0.$$

Thus

$$\lim_{n\to\infty} \|x_i^{(n)}(s_0)\| = \lim_{n\to\infty} \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i^{(n)}(s_0)\| = r$$

for $i \in \{1, \ldots, k+1\}$. If $x_i^{(n)} - x_{k+1}^{(n)} = \alpha_{i1}^{(n)} y_1 + \cdots + \alpha_{ik}^{(n)} y_k$ for $i \in \{1, \ldots, k\}$ and $n \in \mathbb{Z}^+$ where each $\alpha_{ij}^{(n)} \in \mathbb{R}$, then

$$\frac{V(x_1^{(n)}, \ldots, x_{k+1}^{(n)})}{V(0, y_1, \ldots, y_k)} = \frac{V(x_1^{(n)}(s_0), \ldots, x_{k+1}^{(n)}(s_0))}{V(0, y_1(s_0), \ldots, y_k(s_0))} = |\det(\alpha_{ij}^{(n)}|.$$

If $r=0$, then $\lim_{n\to\infty}\det(\alpha_{ij}^{(n)})=0$ and so $\lim_{n\to\infty} V(x_1^{(n)}(s_0),\ldots,x_{k+1}^{(n)}(s_0)) = 0$. If $r\neq 0$, since B_{s_o} is uniformly rotund with respect to $span\{y_1(s_0),\ldots,y_k(s_0)\}$, it follows by Proposition 3.9 that $\lim_{n\to\infty} V(x_1^{(n)}(s_0),\ldots,x_{k+1}^{(n)}(s_0)) = 0$.

Hence $\lim_{n\to\infty} V(x_1^{(n)},\ldots,x_{k+1}^{(n)}) = 0$. $\square$

Corollary 4.5 (Smith[25]**).** *Let $\{N_s : s\in S\}$ be a collection of normed spaces and let X be a full function space over S. Suppose that for any $a\in X$ such that $0\le a(s)$ for all $s\in S$ the order interval,*

$$[0,a] = \{f\colon S\to\mathbb{R} : 0\le f(s)\le a(s) \text{ for all } s\in S\}$$

is compact in X and that X is uniformly rotund in every direction. Given $y\in P_X(N_s)$, $y\neq 0$ if for some $s_0\in S$, $y(s_0)\neq 0$ and N_{s_0} is uniformly rotund in the direction of $y(s_0)$, then $P_X(N_s)$ is uniformly rotund in the direction of y.

Corollary 4.6. *Let $\{N_j : j\in\mathbb{Z}^+\}$ be a collection of normed spaces. Fix $k\in\mathbb{Z}^+$. Suppose that one of the spaces is URE$_k$ and all others are URE$_1$. then $X = P_{l_2}(N_j)$ is URE$_k$.*

Proof. We assume without loss of generality that N_1 is URE$_k$ and N_j is URE$_1$ for $j\ge 2$. Let $y_1,\ldots,y_k$ be linearly independent elements of X. If for some $i\in\{1,\ldots,k\}$ and $j\ge 2$, $y_i(j)\neq 0$, then N_j is uniformly rotund with respect to $span\{y_i(j)\}$ and hence X is uniformly rotund with respect to $span\{y_i\}$ by Corollary 4.5. This implies X is uniformly rotund with respect to $span\{y_1,\ldots,y_k\}$ by Corollary 2.10.

If on the other hand, $y_i(j) = 0$ for all $i\in\{1,\ldots,k\}$ and $j\ge 2$, then clearly $y_1(1),\ldots,y_k(1)$ are linearly independent in N_1. Since N_1 is uniformly rotund with respect to $span\{y_1(1),\ldots,y_k(1)\}$, by Theorem 4.4, we have X is uniformly rotund with respect to $span\{y_1,\ldots,y_k\}$. Thus X is URE$_k$. $\square$

5. Examples

Example 5.1. Let $B_1 = (\mathbb{R}^k,\|\cdot\|_\infty)$, $B_2 = P_{l_2}(\mathbb{R}^n,\|\cdot\|_n)_{n=1}^\infty$ and $X = P_{l_2^2}(B_i)$. Then X is URE$_k$, X is not $(k-1)$-R and X is not n-UR for any $n\in\mathbb{Z}^+$.

As observed in Remark 2.4, B_1 is k-UR and from Corollary 4.5, we know that B_2 is URE$_1$. Using Corollary 4.6, it follows that X is URE$_k$. It is easy to verify that B_1 is not $(k-1)$-R and from [5], we know that B_2 cannot be renormed to be uniformly rotund. From [27] we know that n-UR spaces are superreflexive and so can be renormed to be uniformly rotund. Thus B_2 is not n-UR for any $n\in\mathbb{Z}^+$. Since B_1 and B_2 are subspaces of X, it follows that X is not $(k-1)$-R and X is not n-UR for any $n\in\mathbb{Z}^+$.

Next we shall see examples of Banach spaces $Y_2, Y_3, \ldots$ of sequences of real numbers such that for each $k \geq 2$, Y_k is not uniformly rotund in countably infinitely many $(k-1)$-dimensional subspaces.

Example 5.2. Let $\mathbb{R}^\omega$ be the collection of all sequences of real numbers. For $n \geq 2$, let

$$\|x\|_{(2,n)} = \left(\sum_{1 \leq i < j \leq n} \|(x_i, x_j)\|_\infty^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

If $Y_2 = \{y \in \mathbb{R}^\omega : \sum_{n=2}^{\infty} \frac{1}{n^2({}^nC_2)} \|(y_1, \ldots, y_n)\|_{(2,n)}^2 < \infty\}$ and for all $y \in Y_2$, $\|y\|_{(2)} = \left(\sum_{n=2}^\infty \frac{1}{n^2({}^nC_2)} \|(y_1, \ldots, y_n)\|_{(2,n)}\right)^{\frac{1}{2}}$, then $(Y_2, \|\cdot\|_{(2)})$ is uniformly rotund with respect to all one-dimensional subspaces except *span*$\{e_i\}$ for all $i \in \mathbb{Z}^+$. Hence Y_2 is URE$_2$ but not URE$_1$.

In general, given $k \geq 2$ and $n \geq k$, let

$$\|x\|_{(k,n)} = \left(\sum_{1 \leq j_1 < \cdots < j_k \leq n} \|(x_{j_1}, \ldots, x_{j_k})\|_\infty^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

If

$$Y_k = \left\{ y \in \mathbb{R}^\omega : \sum_{n=k}^{\infty} \frac{1}{n^2({}^nC_k)} \|(y_1, \ldots, y_n)\|_{(k,n)}^2 < \infty \right\}$$

and $\|y\|_{(k)} = \left(\sum_{n=k}^\infty \frac{1}{n^2({}^nC_k)} \|(y_1, \ldots, y_n)\|_{(k,n)}\right)^{\frac{1}{2}}$ for all $y \in Y_k$, then $(Y_k, \|\cdot\|_{(k)})$ is uniformly rotund with respect to all $(k-1)$-dimensional subspaces except those spanned by the elements of $\{e_i : i \in \mathbb{Z}^+\}$. Hence Y_k is URE$_k$ but not URE$_{(k-1)}$.

Explanation: We first show that $(\mathbb{R}^n, \|\cdot\|_{(2,n)})$ is uniformly rotund with respect to all finite-dimensional subspaces except *span*$\{e_1\}, \ldots,$ *span*$\{e_n\}$. Note that $\|\cdot\|_{(2,2)} = \|\cdot\|_\infty$ and it is easy to see that it is not uniformly rotund only with respect to the two subspaces *span*$\{e_1\}$ and *span*$\{e_2\}$. For $n \geq 3$, let $\mathbb{Y}_n = P_{l_2^n}(X_i)_{i=1}^n$ where l_2^n denotes the n-dimensional real linear space with $\|\cdot\|_2$ norm and $X_i = (\mathbb{R}^2, \|\cdot\|_\infty)$ for $i \in \{1, \ldots, n\}$. From Corollary 4.5 it follows that given $z \in \mathbb{Y}_n$, for some $k \in \{1, \ldots, n\}$, if both the coordinates of z_k are nonzero, then $\mathbb{Y}_n$ is uniformly rotund with respect to *span*$\{z\}$. Consider the map $\phi_n \colon \mathbb{R}^n \to \mathbb{Y}_{{}^nC_2}$ given by

$$\phi_n(x_1, \ldots, x_n) = ((x_1, x_2), \ldots, (x_1, x_n), (x_2, x_3), \ldots, (x_2, x_n), \ldots, (x_{n-1}, x_n)).$$

Since this map is linear and injective, $\|\cdot\|_{(2,n)} \colon \mathbb{R}^n \to \mathbb{R}$ given by

$$\|x\|_{(2,n)} = \left(\sum_{i=1}^{{}^nC_2} \|\phi_n(x)(i)\|_\infty^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n$$

is a norm. Suppose $z \in \mathbb{R}^n$ and $z \notin \bigcup_{k=1}^n span\{e_n\}$. Then there exist $i, j \in \{1, \dots, n\}$, $i < j$ such that $z_i \neq 0$ and $z_j \neq 0$. By definition, (z_i, z_j) is a coordinate of $\phi_n(z)$. So $\mathbb{Y}_{{}^nC_2}$ and hence $\phi_n(\mathbb{R}^n)$ is uniformly rotund with respect to $span\{\phi_n(z)\}$. Since $\phi_n \colon (\mathbb{R}^n, \|x\|_{(2,n)}) \to \mathbb{Y}_{{}^nC_2}$ is a linear isometry it follows that $(\mathbb{R}^n, \|\cdot\|_{(2,n)})$ is uniformly rotund with respect to $span\{z\}$. Fix $j \in \{1, \dots, n\}$. If $x = e_1 + \cdots + e_{j-1} + e_j + e_{j+1} + \cdots + e_n$ and $y = e_1 + \cdots + e_{j-1} - e_j + e_{j+1} + \cdots + e_n$, then $\frac{1}{2}(x+y) = e_1 + \cdots + e_{j-1} + 0 + e_{j+1} + \cdots + e_n$ and

$$\|x\|_{(2,n)} = \|y\|_{(2,n)} = \left\| \frac{x+y}{2} \right\|_{(2,n)} = \sqrt{{}^nC_2}$$

while $x - y = 2e_j$. Hence $(\mathbb{R}^n, \|\cdot\|_{(2,n)})$ is not uniformly rotund relative to $span\{e_1\}, \dots, span\{e_n\}$.

Now for each $n \geq 2$, let $\mathbb{B}_{(2,n)} = (\mathbb{R}^n, \frac{1}{n\sqrt{{}^nC_2}}\|\cdot\|_{(2,n)})$. Then $\mathbb{B}_{(2,n)}$ is uniformly rotund with respect to all one-dimensional subspaces other than $span\{e_1\}, \dots,$ $span\{e_n\}$. Consider the set $Y_2 = \{y \in \mathbb{R}^\omega : \sum_{n=2}^\infty \frac{1}{n^2({}^nC_2)} \|(y_1, ..., y_n)\|_{(2,n)}^2 < \infty\}$. With the canonical definitions of addition and scalar multiplication, this set forms a linear space. The map $\phi \colon Y_2 \to P_{l_2}(\mathbb{B}_{(2,n)})_{n=2}^\infty$ given by $\phi(y) = ((y_1, y_2),$ $(y_1, y_2, y_3), \dots)$ for all $y \in Y_2$ is an injective linear map. So, $\|\cdot\|_{(2)} \colon Y_2 \to \mathbb{R}$, given by

$$\|y\|_{(2)} = \left(\sum_{n=2}^\infty \frac{1}{n^2({}^nC_2)} \|(y_1, \dots, y_n)\|_{(2,n)}^2 \right)^{\frac{1}{2}}, \quad y \in Y_2,$$

is a norm. Note that any sequence of elements in $\{-1, 0, 1\}$ belongs to Y_2. Fix $j \in \mathbb{Z}^+$. Let $x = e_1 + \cdots + e_{j-1} + e_j + e_{j+1} + \cdots$ and $y = e_1 + \cdots + e_{j-1} - e_j + e_{j+1} + \cdots$. Then, $\frac{1}{2}(x+y) = e_1 + \cdots + e_{j-1} + 0 + e_{j+1} + \cdots$ and

$$\|x\|_{(2)} = \|y\|_{(2)} = \left\| \frac{x+y}{2} \right\|_{(2)} = \left(\sum_{n=2}^\infty \frac{1}{n^2} \right)^{\frac{1}{2}}.$$

Therefore $(Y_2, \|\cdot\|_{(2)})$ is not uniformly rotund relative to $span\{e_j\}$.

Given $z \in Y_2$, suppose that z has two nonzero coordinates namely z_m, z_n where $m < n$. Then, $(z_1, \dots, z_n)$ has two distinct nonzero coordinates which implies $\mathbb{B}_{(2,n)}$ is uniformly rotund in the direction of $(z_1, \dots, z_n)$ and hence by Corollary 4.5, $P_{l_2}(\mathbb{B}_n)_{n=2}^\infty$ is uniformly rotund in the direction of $\phi(z)$. Since ϕ is a linear isometry, it follows that $(Y_2, \|\cdot\|_{(2)})$ is uniformly rotund in the direction of z.

The general case follows using similar arguments and Corollary 2.10. We give the steps to be followed briefly. Let $n \geq k \geq 2$

Step 1: $\|\cdot\|_{(k,k)} = \|\cdot\|_\infty$ in $\mathbb{R}^k$.

Step 2: The space $P_{l_2^n}(X_i)_{i=1}^n$ where $X_i = (\mathbb{R}^k, \|\cdot\|_\infty)$ for $i \in \{1, \dots, n\}$ is uniformly rotund with respect to $span\{(z_1, \dots, z_n)\}$, if for some $i \in \{1, \dots, n\}$, all the coordinates of the k-tuple z_i are nonzero by Corollary 4.5.

Step 3: Let $X_i = (\mathbb{R}^k, \|\cdot\|_\infty)$ for $i \in \{1\dots, {}^nC_k\}$. Consider the map $\phi_n \colon \mathbb{R}^n \to P_{l_2^{{}^nC_k}}(X_i)_{i=1}^{{}^nC_k}$ such that for each $i \in \{1\dots, {}^nC_k\}$, $\phi_n(x)(i) = (x_{p_1}, \dots, x_{p_k})$ for some $p_1, \dots, p_k \in \{1, \dots, n\}$ satisfying $p_1 < \dots < p_k$ and for $i < j$, if $\phi_n(x)(i) = (x_{p_1}, \dots, x_{p_k})$ and $\phi_n(x)(j) = (x_{q_1}, \dots, x_{q_k})$, then $(p_1, \dots, p_k) \prec (q_1, \dots, q_k)$ where $\prec$ is the dictionary order on k-tuples of integers. This map is linear and injective. So, $\|\cdot\|_{(k,n)}$ defined above on $\mathbb{R}^n$ is a norm.

Step 4: If $z \in \mathbb{R}^n$ has atleast k distinct nonzero coordinates, then $(\mathbb{R}^n, \|\cdot\|_{(k,n)})$ is uniformly rotund with respect to $span\{z\}$.

Step 5: The space $(\mathbb{R}^n, \|\cdot\|_{(k,n)})$ for $n \geq k$ is not uniformly rotund with respect to the $(k-1)$-dimensional subspaces spanned by the elements of $\{e_1, \dots, e_n\}$. To see this, let $m_1, \dots, m_{k-1} \in \{1, \dots, n\}$ be distinct and let $x_i = e_1 + \dots + e_n - e_{m_i}, \quad i \in \{1, \dots, k-1\}$, $x_k = e_1 + \dots + e_n$. Then

$$\|x_1\|_{(k,n)} = \dots = \|x_k\|_{(k,n)} = \frac{1}{k}\|\sum_{i=1}^{k} x_i\|_{(k,n)} = \sqrt{{}^nC_k}.$$

Step 6: Thus $(\mathbb{R}^n, \|\cdot\|_{(k,n)})$ is uniformly rotund with respect to all $(k-1)$-dimensional subspaces other than those spanned by the elements of $\{e_1, \dots, e_n\}$.

Step 7: For each $n \geq k$, let $\mathbb{B}_{(k,n)} = \left(\mathbb{R}^n, \frac{1}{n\sqrt{{}^nC_k}}\|\cdot\|_{(k,n)}\right)$. The map $\phi \colon Y_k \to P_{l_2}(\mathbb{B}_{(k,n)})_{n=k}^\infty$ given by $\phi(y) = ((y_1, \dots, y_k), (y_1, \dots, y_{k+1}), \dots)$ for $y \in Y_k$ is an injective linear map. So $\|\cdot\|_{(k)}$ defined above on Y_k is a norm.

Step 8: The space $(Y_k, \|\cdot\|_{(k)})$ is uniformly rotund with respect to z if z has k distinct nonzero coordinates.

Step 9: Let $i_1, \dots, i_{k-1} \in \mathbb{Z}^+$ be distinct. Put $x_j = (1,1,1,1,\dots) - e_{i_j}$ for $j \in \{1, \dots, k-1\}$ and $x_k = (1,1,1,1,\dots)$. Then $V(x_1, \dots, x_{k+1}) > 0$ and

$$\|x_1\|_{(k)} = \dots = \|x_k\|_{(k)} = \frac{1}{k}\|\sum_{j=1}^{k} x_j\|_{(k)} = \left(\sum_{n=k}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}}.$$

Thus $(Y_k, \|\cdot\|_{(k)})$ is not uniformly rotund with respect to $span\{e_{i_1}, \dots, e_{i_{k-1}}\}$.

Step 10: Thus $(Y_k, \|\cdot\|_{(k)})$ is uniformly rotund with respect to all $(k-1)$-dimensional subspaces other than those spanned by the elements of $\{e_n : n \in \mathbb{Z}^+\}$.

For each $k \geq 2$, $(Y_k, \|y\|_{(k)})$ is a closed subspace of $P_{l_2}(\mathbb{B}_{(k,n)})_{n=k}^{\infty}$. From [5] we know that, for any $1 < p < \infty$, the l_p product of reflexive spaces is reflexive. Hence $(Y_k, \|y\|_{(k)})$ is a reflexive Banach space that is uniformly rotund with respect all but countably infinitely many $(k-1)$-dimensional subspaces. So it is URE$_k$ but not URE$_{(k-1)}$(In fact, it is not even $(k-1)$-R). It is not clear whether Y_k is k-UR or not.

Remark 5.3. Suppose $y \in \mathbb{R}^{\omega}$ is bounded. Then for each $n \in \mathbb{Z}^+$,

$$\|(y_1, \dots, y_n)\|_{(k,n)} \leq \sqrt{{}^nC_k}\|y\|_{\infty}.$$

Hence $\sum_{n=k}^{\infty} \frac{1}{n^2({}^nC_k)}\|(y_1, \dots, y_n)\|^2_{(k,n)} \leq \|y\|^2_{\infty} \sum_{n=k}^{\infty} \frac{1}{n^2}$. This shows that for each $k \geq 2$, Y_k contains all bounded sequences and the inclusion map from l_{∞} into $(Y_k, \|y\|_{(k)})$ is continuous. By the open mapping theorem and the reflexivity of $(Y_k, \|y\|_{(k)})$, it is clear that Y_k also contains unbounded elements of $\mathbb{R}^{\omega}$.

However the continuity of the inclusion map mentioned in Remark 5.3 enables us to renorm l_{∞} so as to be URE$_k$ for any $k \in \mathbb{Z}^+$. Thereby we get an example of a nonreflexive Banach space that is URE$_k$ but not URE$_{(k-1)}$. We use the following result originally due to Zizler[29] and modified slightly by Smith[24].

Proposition 5.4 (Zizler [29]; Smith [24]**).** *Let X and Y be Banach spaces. Let $F: X \to Y$ be a linear, continuous map. For $x \in X$, let $|||x||| = \sqrt{\|x\|^2_X + \|Fx\|^2_Y}$. Let $z \in X$, $Fz \neq 0$. If Y is uniformly rotund in the direction of Fz, then $(X, |||.|||)$ is uniformly rotund in the direction of z.*

Using the Proposition 5.4 we have the following example.

Example 5.5. Let m be the set of all bounded sequences of real numbers and define $|||y|||_k = \sqrt{\|y\|^2_{\infty} + \|y\|^2_{(k)}}$ for $y \in m$. Then $(m, ||| \cdot |||_k)$ is URE$_k$ but not $(k-1)$-R. In particular it is uniformly rotund with respect to all $(k-1)$-dimensional subspaces except those spanned by the elements of $\{e_i : i \in \mathbb{Z}^+\}$.

We note that there exist normed spaces that are rotund but do not have normal structure as shown in [26]. This proves the existence of rotund normed spaces that are not URE$_k$ for any positive integer k.

In [29], it is proved that the space c_0 of all sequences that converge to 0 with $\|\cdot\|_{\infty}$ norm is not uniformly rotund in any direction and in [8], it is proved that if I is an uncountable set, then $c_0(I) = \{x : I \to \mathbb{R} : \text{For each } \epsilon > 0, |x(i)| < \epsilon \text{ for all except finitely many values of } i\}$ with $\|\cdot\|_{\infty}$ norm cannot be renormed to be uniformly rotund in every direction. In the next two theorems we say more about these spaces.

Theorem 5.6. *The space c_0 of all sequences of real numbers that converge to 0 with $\|\cdot\|_{\infty}$ is not uniformly rotund with respect to any finite-dimensional subspace.*

Proof. Let $k \in \mathbb{Z}^+$ and let $z_1, \ldots, z_k$ be linearly independent elements of c_0 each of norm 1. For each $n \in \mathbb{Z}^+$, choose $j_n \in \mathbb{Z}^+$ with $|z_1(j_n)|, \ldots, |z_k(j_n)| < \frac{1}{n}$. Let $x_i^{(n)} = e_{j_n} - z_i$ for $i \in \{1, \ldots, k\}$ and $x_{k+1}^{(n)} = e_{j_n}$ Let $i \in \{1, \ldots, k\}$. Then

$$\|x_i^{(n)}\| \geq |x_i^{(n)}(j_n)| = |e_{j_n}(j_n) - z_i(j_n)| \geq |e_{j_n}(j_n)| - |z_i(j_n)| \geq 1 - \frac{1}{n},$$

$$|x_i^{(n)}(j)| = |z_i(j)| \leq 1 \quad \forall\ j \neq j_n$$

$$|x_i^{(n)}(j_n)| = |e_{j_n}(j_n) - z_i(j_n)| \leq |e_{j_n}(j_n)| + |z_i(j_n)| \leq 1 + \frac{1}{n}$$

So, $\|x_i^{(n)}\| \leq 1 + \frac{1}{n}$. Similarly we have,

$$1 - \frac{k}{(k+1)n} \leq \frac{\left\|x_1^{(n)} + \cdots + x_{k+1}^{(n)}\right\|}{k+1} = \left\|e_{j_n} - \frac{z_1 + \cdots + z_k}{k+1}\right\| \leq 1 + \frac{k}{(k+1)n}.$$

Thus we have

$$\lim_{n\to\infty} \|x_i^{(n)}\| = 1 = \lim_{n\to\infty} \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i^{(n)}\| \quad \text{for } i \in \{1, \ldots, k+1\}.$$

However, $V(x_1^{(n)}, \ldots, x_{k+1}^{(n)}) = V(z_1, \ldots, z_k, 0)$ for all $n \in \mathbb{Z}^+$. Hence, we can conclude from Proposition 3.9 that c_0 is not uniformly rotund with respect to $span\{z_1, \ldots, z_k\}$. □

Theorem 5.7. *Let I be an uncountable set. Then $c_0(I) = \{x\colon I \to \mathbb{R} :$ for each $\epsilon > 0$, $|x(i)| < \epsilon$ for all but finitely many values of $i\}$ with $\|\cdot\|_\infty$ norm is not isomorphic to a Banach space that is URE$_k$ for any positive integer k.*

Proof. Fix a positive integer k. Let $\|\cdot\|$ be a norm on $c_0(I)$ equivalent to $\|\cdot\|_\infty$. Let $r = \sup\{\|x\| : \|x\|_\infty = 1\}$. Choose a sequence (u_n) such that $\|u_n\|_\infty = 1$ for all n and $\lim_{n\to\infty} \|u_n\| = r$. Since the set $\bigcup_{n=1}^{\infty}\{i \in I : u_n(i) \neq 0\}$ is countable, we may choose k distinct elements $i_1, \ldots, i_k \in I$ such that $u_n(i_j) = 0$ for all $n \in \mathbb{Z}^+$ and $j \in \{1, \ldots, k\}$. Let $y_j(i_j) = 1$ and $y_j(i) = 0$ for $i \neq i_j$ for each $j \in \{1, \ldots, k\}$. Then $y_1, \ldots, y_k$ are linearly independent elements of $c_0(I)$. We show that $(c_0(I), \|\cdot\|)$ is not uniformly rotund with respect to $Y = span\{y_1, \ldots, y_k\}$. Let

$$x_1^{(n)} = u_n + y_1, \ldots, x_k^{(n)} = u_n + y_k,\ x_{k+1}^{(n)} = u_n - (y_1 + \ldots + y_k).$$

Then $\|x_i^{(n)}\|_\infty = 1$ and so $\|x_i^{(n)}\| \leq r$ for all $i \in \{1, \ldots, k+1\}$, $n \in \mathbb{Z}^+$. Also

$$\lim_{n\to\infty} \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i^{(n)}\| = \lim_{n\to\infty} \|u_n\| = r$$

and $span\{x_1^{(n)} - x_{k+1}^{(n)}, \ldots, x_k^{(n)} - x_{k+1}^{(n)}\} = Y$. But

$$V(x_1^{(n)}, \ldots, x_{k+1}^{(n)}) = V(y_1, \ldots, y_k, -(y_1 + \ldots + y_k)) > 0$$

for all $n \in \mathbb{Z}^+$. □

The above result also follows from Corollary 3.7 and a result proved in [15].

6. An equivalent formulation using Milman's modulus

In [19], the following modulus of finite dimensional uniform rotundity was introduced and studied.

$$\Delta_X^{(k)}(\epsilon) = \inf_{Y \in \mathcal{S}_k(X)} \inf_{\|x\|=1} \max_{\|y\|=\epsilon, y \in Y} \{\|x + y\|\} - 1, \quad \epsilon > 0$$

where $\mathcal{S}_k(X)$ is the collection of all k-dimensional subspaces of X. In [19] and [10] it is established that X is uniformly rotund if and only if $\Delta_X^{(1)}(\epsilon) > 0$ for every $\epsilon > 0$. Following the introduction of k-uniform rotundity in [27], it is proved in [12] that X is 2-uniformly rotund if and only if $\Delta_X^{(2)}(\epsilon) > 0$ for every $\epsilon > 0$ and in [18] the same is proved for any positive integer k.

Given a finite-dimensional subspace Y of X let,

$$\Delta_X(\epsilon, Y) = \inf_{\|x\|=1} \max_{\|y\|=\epsilon, y \in Y} \{\|x + y\|\} - 1$$

for every $\epsilon > 0$. A careful examination of the proof in [18] of the result that X is k-uniformly rotund if and only if $\Delta_X^{(k)}(\epsilon) > 0$ for every $\epsilon > 0$ shows us that X is uniformly rotund with respect to a finite-dimensional subspace Y if and only if $\Delta_X(\epsilon, Y) > 0$. We shall show this below.

Lemma 6.1 (Geremia and Sullivan [12]). *If X is a normed space and $x_1, \ldots, x_{k+1} \in X$, then $V(x_1, \ldots, x_{k+1}) \geq dist(x_1, aff\{x_2, \ldots, x_{k+1}\}) V(x_2, \ldots, x_{k+1})$.*

Proposition 6.2 (Geremia and Sullivan [12], Lin [18]).
Let X be a normed space of dimension at least k for some $k \in \mathbb{Z}^+$ and $\epsilon > 0$. Then there exist $v_1, \ldots, v_k$ such that $\|v_1\| = \cdots = \|v_k\| = \|v_1 + \cdots + v_k\| = \epsilon$ and $dist(v_i, span\{v_1, \ldots, v_{i-1}\}) \geq \frac{\epsilon}{3}$ for $i \in \{2, \ldots, k\}$.

Using essentially the same techniques used in [12] and [18] we observe that in Proposition 6.2 the quantity $\frac{\epsilon}{3}$ can be replaced with $\frac{\epsilon}{2}$. We give a proof of this claim for the sake of completeness. Following [12] we use an observation of Phelps.

Lemma 6.3 (Phelps [20]). *Let X be a normed space and $\epsilon > 0$. If $f, g \in S_{X^*}$ such that $f^{-1}\{0\} \bigcap B_X \subset g^{-1}[\frac{-\epsilon}{2}, \frac{\epsilon}{2}]$, then $\|f + g\| \leq \epsilon$ or $\|f - g\| \leq \epsilon$.*

Proposition 6.4. *Let X be a normed space of dimension at least k for some $k \in \mathbb{Z}^+$ and $\epsilon > 0$. Then there exist $v_1, \dots, v_k$ such that $\|v_1\| = \dots = \|v_k\| = \|v_1 + \dots + v_k\| = \epsilon$ and $dist(v_i, span\{v_1, \dots, v_{i-1}\}) \geq \frac{\epsilon}{2}$ for $i \in \{2, \dots, k\}$.*

Proof. We may assume that $k \geq 2$. Let $u \in X$ and $\|u\| = \epsilon$. Then consider the map $x \to \|u + x\|$ on $\{x \in X : \|x\| = \epsilon\}$. Since the map is continuous and the domain is connected, there exists $v \in X$ such that $\|v\| = \epsilon$ and $\|u + v\| = \epsilon$. Also $\|u - v\| = \|2v - (u + v)\| \geq \|2v\| - \|u + v\| = \epsilon$. For any $0 < r < 1$, $\|u + v\| > r\epsilon$ and $\|u - v\| > r\epsilon$. Now applying Lemma 6.3 on the evaluation functionals $f \to \frac{1}{\epsilon} f(u)$ and $f \to \frac{1}{\epsilon} f(v)$ on X^*, we can see that there exists $f_r \in B_{X^*}$ such that $f_r(u) = 0$ and $|f_r(v)| > \frac{r\epsilon}{2}$. So for any $\alpha \in \mathbb{R}$, $\|v - \alpha u\| \geq |f_r(v - \alpha u)| = |f_r(v)| > \frac{r\epsilon}{2}$. Thus $dist(v, span\{u\}) \geq \frac{r\epsilon}{2}$ for any $0 < r < 1$ which gives $dist(v, span\{u\}) \geq \frac{\epsilon}{2}$.

Choose $x_1, ..., x_k \in S_X$ such that $d(x_i, span\{x_1, ..., x_{i-1}\}) = 1$ for $i \in \{2, ..., k\}$. Starting with $v_1 = \epsilon x_1$, choose $v_2, \dots, v_k$ such that $v_i \in span\{v_1 + \dots + v_{i-1}, x_i\}$, $\|v_i\| = \|v_1 + \dots + v_{i-1} + v_i\| = \epsilon$ and $dist(v_i, span\{v_1 + \dots + v_{i-1}\}) \geq \frac{\epsilon}{2}$. The linear independence of v_i and $v_1 + \dots + v_{i-1}$ shows that $v_i = \alpha_i(v_1 + \dots + v_{i-1}) + \beta_i x_i$ for some $\beta_i \neq 0$. Now

$$dist(v_i, span\{v_1, \dots, v_{i-1}\}) = |\beta_i| dist(x_i, span\{v_1, \dots, v_{i-1}\})$$
$$= |\beta_i| dist(x_i, span\{x_1, \dots, x_{i-1}\}) = |\beta_i| \geq dist(v_i, span\{v_1 + \dots + v_{i-1}\}) \geq \frac{\epsilon}{2}.$$

Before proceeding further we need the following notation. For each $\epsilon > 0$, let

$$\delta_X^{(B)}(\epsilon, Y) = \inf \left\{ 1 - \frac{1}{k+1} \| \sum_{i=1}^{k+1} x_i \| : \begin{array}{l} x_1, ..., x_{k+1} \in B_X, V(x_1, ..., x_{k+1}) \geq \epsilon, \\ span\{x_1 - x_{k+1}, ..., x_k - x_{k+1}\} = Y \end{array} \right\}.$$

Theorem 6.5. *Let X be a finite-dimensional normed space. If Y is a k-dimensional subspace of X, then*

$$\delta_X^{(B)} \left(\frac{(k+1)\epsilon^k}{2^k (1 + \Delta_X(\epsilon, Y))^k}, Y \right) \leq \frac{\Delta_X(\epsilon, Y)}{1 + \Delta_X(\epsilon, Y)}$$

for $\epsilon > 0$ and

$$\Delta_X \left(\frac{\epsilon}{(k+1)^{k+1}(1 - \delta_X(\epsilon, Y))}, Y \right) \leq \frac{\delta_X(\epsilon, Y)}{1 - \delta_X(\epsilon, Y)}$$

for $\epsilon > 0$ such that $\delta_X(\epsilon, Y) < 1$.

Proof. Since X is finite-dimensional,

$$\Delta_X(\epsilon, Y) = \max_{\|y\| = \epsilon, y \in Y} \{\|x_0 + y\|\} - 1$$

for some $x_0 \in X$, $\|x_0\| = 1$. By Proposition 6.4, there exist $v_1, \dots, v_k \in Y$ such that $\|v_1\| = \dots = \|v_k\| = \|v_1 + \dots + v_k\| = \epsilon$ and $dist(v_i, span\{v_1, \dots, v_{i-1}\}) \geq \frac{\epsilon}{2}$ for $i \in \{2, \dots, k\}$. Put $v_{k+1} = -(v_1 + \dots + v_k)$. For $i \in \{1, \dots, k+1\}$, $\|x_0 + v_i\| - 1 \leq \Delta_X(\epsilon, Y)$. So if $a = \frac{1}{1+\Delta_X(\epsilon,Y)}$ and $x_i = a(x_0 + v_i)$, then $\|x_i\| \leq 1$. Now using Lemma 6.1, we have $V(x_1, \dots, x_{k+1}) = a^k V(v_1, \dots, v_{k+1}) = a^k(k+1)V(0, v_1, \dots, v_k) \geq a^k(k+1)\frac{\epsilon^k}{2^k}$. Also $span\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\} = span\{v_1, \dots, v_k\} = Y$. Thus we have

$$\delta_X^{(B)}(a^k(k+1)\frac{\epsilon^k}{2^k}, Y) \leq 1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| = 1 - a,$$

i. e.,

$$\delta_X^{(B)}\left(\frac{(k+1)\epsilon^k}{2^k(1+\Delta_X(\epsilon,Y))^k}, Y\right) \leq \frac{\Delta_X(\epsilon,Y)}{1+\Delta_X(\epsilon,Y)}.$$

To prove the second inequality let $\delta_X(\epsilon, Y) < 1$. Choose $x_1, \dots, x_{k+1} \in S_X$ such that $V(x_1, \dots, x_{k+1}) \geq \epsilon$, $span\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\} = Y$ and

$$\delta_X(\epsilon, Y) = 1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\|.$$

Also choose $f_1, \dots, f_k \in B_{X^*}$ such that

$$V(x_1, \dots, x_{k+1}) = \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix}$$

For $i \in \{1, \dots, k\}$, let

$$v_i = \frac{1}{(k+1)^{k+1}(1-\delta(\epsilon,Y))} \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_{i-1}(x_1) & \cdots & f_{i-1}(x_{k+1}) \\ x_1 & \cdots & x_{k+1} \\ f_{i+1}(x_1) & \cdots & f_{i+1}(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix}$$

Using a simple column operation it can be seen that each v_i is a linear combination of $x_1 - x_{k+1}, \dots, x_k - x_{k+1}$ and $f_i(v_i) = \frac{V(x_1,\dots,x_{k+1})}{(k+1)^{k+1}(1-\delta_X(\epsilon,Y))}$ while $f_j(v_i) = 0$ for $j \neq i$. Hence $v_1, \dots, v_k$ are linearly independent and $span\{v_1, \dots, v_k\} = Y$.

Fix $u = \frac{x_1+\cdots+x_{k+1}}{\|x_1+\cdots+x_{k+1}\|}$. Using the same arguments as in [18], it can be proved that if $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$\|a_1 v_1 + \cdots + a_k v_k\| = \frac{\epsilon}{(k+1)^{k+1}(1-\delta_X(\epsilon, Y))}$$

then

$$\|u + a_1 v_1 + \cdots + a_k v_k\| \leq \frac{1}{1-\delta_X(\epsilon, Y))}.$$

It follows that $\Delta_X \left(\frac{\epsilon}{(k+1)^{k+1}(1-\delta_X(\epsilon, Y))}, Y \right) \leq \frac{\delta_X(\epsilon, Y)}{1-\delta_X(\epsilon, Y)}$. □

Corollary 6.6. *A normed space X is uniformly rotund with respect to a finite-dimensional subspace Y if and only if $\Delta_X(\epsilon, Y) > 0$ for every $\epsilon > 0$.*

Proof. Note that

$$\Delta_X(\epsilon, Y) = \inf\{\Delta_Z(\epsilon, Y) : Y \subset Z \subset X, Z \text{ is finite-dimensional}\}$$

$$\delta_X(\epsilon, Y) = \inf\{\delta_Z(\epsilon, Y) : Y \subset Z \subset X, Z \text{ is finite-dimensional}\}$$

$$\delta_X^{(B)}(\epsilon, Y) = \inf\{\delta_Z^{(B)}(\epsilon, Y) : Y \subset Z \subset X, Z \text{ is finite-dimensional}\}$$

Let Z be a finite-dimensional subspace of X that contains the k-dimensional subspace Y. Since $\Delta_Z(\epsilon, Y) \leq \epsilon$ and the function $\delta_Z(\cdot, Y)$ is monotonically increasing the first inequality in Theorem 6.5 implies

$$\delta_Z^{(B)} \left(\frac{(k+1)\epsilon^k}{2^k(1+\epsilon)^k}, Y \right) \leq \Delta_Z(\epsilon, Y).$$

Hence

$$\delta_X^{(B)} \left(\frac{(k+1)\epsilon^k}{2^k(1+\epsilon)^k}, Y \right) \leq \Delta_X(\epsilon, Y).$$

Further, the second inequality in Theorem 6.5 implies

$$\frac{1}{1+\epsilon} \Delta_Z \left(\frac{\epsilon}{(k+1)^{k+1}}, Y \right) \leq \delta_Z(\epsilon, Y),$$

and hence $$\frac{1}{1+\epsilon} \Delta_X \left(\frac{\epsilon}{(k+1)^{k+1}}, Y \right) \leq \delta_X(\epsilon, Y).$$ □

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