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Valadier-like Formulas for the Supremum Function I*

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We generalize and improve the original characterization given by Valadier [19, Theorem 1] of the subdifferential of the pointwise supremum of convex functions, involving the subdifferentials of the data functions at nearby points. We remove the continuity assumption made in that work and obtain a general formula for such a subdifferential. In particular, when the supremum is continuous at some point of its domain, but not necessarily at the reference point, we get a simpler version which gives rise to the Valadier formula. Our starting result is the characterization given in [11, Theorem 4], which uses the ε-subdifferential at the reference point.

Keywords: Pointwise supremum function, convex functions, Fenchel subdifferential, Valadier-like formulas.

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1. Introduction

Consider a family $f_t \colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ of extended real-valued convex functions indexed by an arbitrary index set T, and defined in a locally convex

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topological vector space X. Under the continuity of the supremum function

$$f := \sup_{t\in T} f_t \tag{1}$$

at a point $x \in X$, a remarkable pioneering result due to Valadier [19] states that the subdifferential of f at x is completely characterized by means of the subdifferentials of data functions f_t at nearby points; more precisely

$$\partial f(x) = \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t\in T_\varepsilon(x),\ p(y-x)\leq\varepsilon} \partial f_t(y) \right\}, \tag{2}$$

where $\mathcal{P}$ is the family of continuous seminorms in X, and

$$T_\varepsilon(x) := \{t \in T \mid f_t(x) \geq f(x) - \varepsilon\}; \tag{3}$$

when $f(x) = +\infty$, we have $f_t(x) = +\infty$ for all $t \in T_\varepsilon(x)$. We also use the notation

$$T(x) := \{t \in T \mid f_t(x) = f(x)\}. \tag{4}$$

There have been many contributions to this problem, in various settings, depending on the structure of the space X, the algebraic and topological properties of the index set T, the behavior of the function $f(x,t) := f_t(x)$, etc. See, for instance, [2, 3, 4, 6, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21] and references therein, among many others. First results on the subject are reported and commented in [14, Section 6 and comments in paragraph 4° of Bibliographical notes, page 69].

More recently, using the machinery of ε-subdifferentials, and under the following condition

$$\text{cl}\, f = \sup_{t\in T} \text{cl}\, f_t, \tag{5}$$

involving the lower semicontinuous (lsc, for short) envelopes, it has been established in [11, Theorem 4] (see also [9]) that, for every $x \in X$,

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \text{N}_{L\cap \text{dom}\, f}(x) \right\}, \text{ where} \tag{6}$$

$$\mathcal{F}(x) := \{L \subset X \mid L \text{ is a finite-dimensional linear subspace with } x \in L\}. \tag{7}$$

When T is finite and $T(x) = T$, the last formula gives rise to ([2]; see, also, [11, Corollary 12])

$$\partial f(x) = \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \bigcup_{t\in T(x)} \partial_\varepsilon f_t(x) \right\}, \tag{8}$$

being the first result of this kind that does not require any continuity conditions.

Both approaches are based on enlargements of ∂f_t, using either the exact subdifferentials at nearby points or the ε-subdifferentials at the reference point. They are comparable because: (i) both formulas coincide when the f_t's are affine, (ii) both enlargements are related in the Banach spaces setting thanks to the Brøndsted-Rockafellar theorem.

The main purpose of the present paper consists of generalizing and improving formula (2). We drop the continuity assumption and obtain general formulas for $\partial f(x)$, in the settings of Banach spaces (16) and locally convex spaces (24), both of them involving enlargements of ∂f_t, which use the exact subdifferentials of the f_t's at nearby points.

The paper is organized as follows. After Section 2, devoted to notation and preliminaries, Section 3 provides the free-continuity extension of (2) in a Banach space (16), via the application of Brøndsted-Rockafellar theorem. Our characterization is comparable with the main limiting-like results given in [16]. The validity limitation of this result outside the Banach setting is illustrated in this section by means of some examples. In Section 4 we establish the second free-continuity formula for ∂f (24) in the setting of locally convex spaces. Section 5 is devoted to deriving formula (38), which constitutes a simpler version of (16) and (24), which is valid only when we are confined to the case in which the supremum function f is finite and continuous at some point. This last section finishes with the derivation of (2) from our formula (38).

In a forthcoming paper we establish the corresponding counterparts of the formulas provided in the current paper to the case in which the index set T is compact and the functions $t \to f_t(z)$, $z \in \operatorname{dom} f$, are upper semicontinuous on $T_\varepsilon(x)$. This will generalize the second theorem in [19].

2. Notation and preliminaries

In this paper X stands for a (real) separated locally convex (lcs, shortly) space, whose topological dual space is denoted by X^* and, unless otherwise stated, is endowed with the weak*-topology. Hence, the (X, X^*) forms a dual topological pair by means of the canonical bilinear form $\langle x, x^* \rangle = \langle x^*, x \rangle := x^*(x)$, $(x, x^*) \in X \times X^*$. The zero vectors in all the involved spaces are denoted by θ, and the convex, closed and balanced neighborhoods of θ are called θ-neighborhoods. The families of such θ-neighborhoods in X and in X^* are denoted by $\mathcal{N}_X$ and $\mathcal{N}_{X^*}$, respectively, whereas for $x \in X$ and $x^* \in X^*$ we denote $\mathcal{N}_x := x + \mathcal{N}_X$ and $\mathcal{N}_{x^*} := x^* + \mathcal{N}_{X^*}$.

Given a nonempty set A in X (or in X^*), by $\operatorname{co} A$, $\operatorname{cone} A$, and $\operatorname{aff} A$ we denote the *convex hull,* the *conic hull*, and the *affine hull* of A, respectively. Moreover, $\operatorname{int} A$ is the *interior* of A, and $\operatorname{cl} A$ and $\overline{A}$ are indistinctly used for denoting the *closure* of A (*weak*-closure* if $A \subset X^*$). Thus, $\overline{\operatorname{co}}A := \operatorname{cl}(\operatorname{co} A)$, $\overline{\operatorname{aff}}A := \operatorname{cl}(\operatorname{aff} A)$, etc. We use $\operatorname{ri} A$ to denote the (topological) *relative interior* of A (i.e., the interior of A in the topology relative to $\overline{\operatorname{aff}}A$). Associated with a nonempty subset A of X, we consider the (one-sided) *polar* and the *orthogonal* of A defined, respectively, by

$$A^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 1 \text{ for all } x \in A\},$$

and

$$A^\perp := \{x^* \in X^* \mid \langle x^*, x\rangle = 0 \text{ for all } x \in A\}.$$

For $B \subset X$ and $\Omega \subset \mathbb{R}$ we define

$$A + B := \{a + b \mid a \in A,\ b \in B\} \text{ and } \Omega A := \{\lambda a \mid \lambda \in \Omega,\ a \in A\}, \tag{9}$$

with the conventions

$$\Omega\emptyset = A + \emptyset = \emptyset. \tag{10}$$

Throughout the paper we apply the following two lemmas concerning the second product in (9). Let us introduce the set

$$\Delta_k := \{(\lambda_1, \cdots, \lambda_k) \mid \lambda_i > 0,\ \sum\nolimits_{i=1}^{k} \lambda_i = 1\}.$$

Lemma 2.1. *If Ω is a compact interval such that $0 \notin \Omega$, then*

$$\Omega(\overline{\text{co}}A) = \overline{\text{co}}(\Omega A).$$

Proof. The conclusion follows from the algebraic equality $\Omega(\text{co}\, A) = \text{co}(\Omega A)$ together with its consequence $\Omega(\overline{\text{co}}A) = \text{cl}(\Omega(\text{co}\, A)) = \overline{\text{co}}(\Omega A)$, both being true thanks to the condition $0 \notin \Omega$. □

Lemma 2.2. *Suppose that $(\Lambda_\varepsilon)_{\varepsilon>0}$ is a non-increasing family of closed sets in $\mathbb{R}$ (i.e. $\varepsilon' < \varepsilon$ implies $\Lambda_{\varepsilon'} \subset \Lambda_\varepsilon$) such that $\bigcap_{\varepsilon>0} \Lambda_\varepsilon = \{1\}$. Let $(A_\varepsilon)_{\varepsilon>0}$ be another non-increasing family of closed sets in X (or in X^*). Then*

$$\bigcap\nolimits_{\varepsilon>0} \Lambda_\varepsilon A_\varepsilon = \bigcap\nolimits_{\varepsilon>0} A_\varepsilon.$$

Proof. The inclusion "$\supset$" is obvious as $1 \in \Lambda_\varepsilon$ entails $A_\varepsilon \subset \Lambda_\varepsilon A_\varepsilon$. Let us prove the inclusion "$\subset$".

If $a \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon A_\varepsilon$, we have $a = \lambda_\varepsilon a_\varepsilon$ with $\lambda_\varepsilon \in \Lambda_\varepsilon$ and $a_\varepsilon \in A_\varepsilon$, for all $\varepsilon > 0$. Since $\lim_{\varepsilon\downarrow 0} \lambda_\varepsilon = 1$ we can assume that $\lambda_\varepsilon > 0$. For each $\varepsilon > 0$ the net $(a_\delta)_{0<\delta<\varepsilon}$ is contained in A_ε, and due to the closedness of A_ε, $\lim_{\delta\downarrow 0} a_\delta = \lim_{\delta\downarrow 0}(\lambda_\delta)^{-1} a = a \in A_\varepsilon$, and we are done. □

We say that a function $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is *proper* if its *(effective) domain*, $\text{dom}\,\varphi := \{x \in X \mid \varphi(x) < +\infty\}$, is nonempty. We say that φ is *convex* (*lower semicontinuous* or *lsc*, for short, respectively) if its *epigraph,* $\text{epi}\,\varphi := \{(x, \lambda) \in X \times \mathbb{R} \mid \varphi(x) \leq \lambda\}$, is convex (closed, respectively). We shall denote by $\Lambda(X)$ the set of all the proper convex functions defined on X and by $\Gamma_0(X)$ the functions in $\Lambda(X)$ which are lsc. The *lsc envelope* of φ is the function $\operatorname{cl}\varphi$ such that $\operatorname{epi}(\operatorname{cl}\varphi) = \operatorname{cl}(\operatorname{epi}\varphi)$.

If $\varepsilon \geq 0$, the *ε-subdifferential* of φ at a point x where $\varphi(x)$ is finite is the weak*-closed convex set

$$\partial_\varepsilon \varphi(x) := \{x^* \in X^* \mid \varphi(y) - \varphi(x) \geq \langle x^*, y - x\rangle - \varepsilon \text{ for all } y \in X\}.$$

If $\varphi(x) \notin \mathbb{R}$, then we set $\partial_\varepsilon \varphi(x) := \emptyset$. In particular, for $\varepsilon = 0$ we get the *Fenchel subdifferential* of φ at x, $\partial\varphi(x) := \partial_0\varphi(x)$. When $\partial\varphi(x) \neq \emptyset$, we know that

$$\varphi(x) = (\operatorname{cl}\varphi)(x) \text{ and } \partial\varphi(x) = \partial(\operatorname{cl}\varphi)(x). \tag{11}$$

The *support function,* the *indicator function* and the *Minkowski gauge* of $A \subset X$ are, respectively, defined as

$$\begin{gathered}\sigma_A(x^*) := \sup\{\langle x^*, a\rangle \mid a \in A\}, \text{ for } x^* \in X^*, \text{ and } \sigma_\emptyset \equiv -\infty, \qquad (12)\\ \mathrm{I}_A(x) := 0, \text{ if } x \in A; \ +\infty, \text{ if } x \in X \setminus A, \text{ and}\\ p_A(x) := \inf\{\lambda \geq 0 : x \in \lambda A\},\end{gathered}$$

with $\inf \emptyset = +\infty$. If $U \in \mathcal{N}_X$ (or $U \in \mathcal{N}_{X^*}$), then p_U is a continuous seminorm and

$$U = \{x \in X \mid p_U(x) \leq 1\}. \tag{13}$$

(Remember that U is closed.)

If A is convex, $x \in X$ and $\varepsilon \geq 0$, we define the *ε-normal set* to A at x as

$$\mathrm{N}_A^\varepsilon(x) := \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq \varepsilon \text{ for all } y \in A\}, \text{ if } \ x \in A; \ \ \emptyset, \text{ otherwise.}$$

If $\varepsilon = 0$, we omit the reference to ε and write $\mathrm{N}_A(x)$.

We shall frequently use the following well-known relation, for any set $A \subset X$

$$\bigcap_{U \in \mathcal{N}_X} \operatorname{cl}(A + U) = \operatorname{cl}(A). \tag{14}$$

The same relation is true for $A \subset X^*$ replacing $\mathcal{N}_X$ by $\mathcal{N}_{X^*}$.

Lemma 2.3. *Let A be a set in X^*. Then for every $x \in X$*

$$\bigcap_{L \in \mathcal{F}(x)} \operatorname{cl}\left(A + L^\perp\right) = \operatorname{cl}(A).$$

If X is a normed space, then

$$\bigcap_{\varepsilon > 0} \operatorname{cl}(A + \varepsilon B_*) = \operatorname{cl}(A),$$

where B_* is the closed unit ball in X^*.

Proof. Due to the fact that the family of sets

$$\{x^* \in X^* \mid |\langle x_i, x^* \rangle| \leq \delta,\ i \in \overline{1,k}\},\quad x_i \in X,\ i \in \overline{1,k},\quad \delta > 0\}$$

is a basis of θ-neighborhoods of the weak*-topology, and observing that

$$\{x_i \in X,\ i \in \overline{1,k};\ x\}^\perp \subset \{x^* \in X^* \mid |\langle x_i, x^* \rangle| \leq \delta,\ i \in \overline{1,k}\},$$

(14) allows us to write

$$\mathrm{cl}\,(A) \subset \bigcap_{L \in \mathcal{F}(x)} \mathrm{cl}\left(A + L^\perp\right) \subset \bigcap_{U \in \mathcal{N}} \mathrm{cl}\,(A + U) = \mathrm{cl}\,(A)\,.$$

Now, assume that X is a normed space. Since the topology on the norm in X^* is stronger than every compatible topology in X^*, we get, using (14) in X^*,

$$\mathrm{cl}\,(A) \subset \bigcap_{\varepsilon>0} \mathrm{cl}\,(A + \varepsilon B_*) \subset \bigcap_{U \in \mathcal{N}_{X^*}} \mathrm{cl}\,(A + U) = \mathrm{cl}\,(A). \qquad \square$$

3. The case of Banach spaces

Our purpose in this section is to provide a first direct consequence of formula (6), which yields a free-continuity extension of (2) in a Banach space $(X, \|\cdot\|)$. This is accomplished via a straightforward application of the Brøndsted-Rockafellar theorem. We also illustrate by examples the validity limitation of this approach outside the Banach setting. We need the following previous simple lemma.

Lemma 3.1. *Given a convex function $\varphi\colon X \to \overline{\mathbb{R}}$, for every $x \in \operatorname{dom}\varphi$ we have*

$$\bigcap_{L \in \mathcal{F}(x)} \partial(\varphi + \mathrm{I}_L)(x) = \partial\varphi(x).$$

For a function $\varphi \in \Gamma_0(X)$, we recall here the mapping $\breve{\partial}^\varepsilon\varphi\colon X \rightrightarrows X^*$ defined for $x \in \operatorname{dom}\varphi$ as (see, e.g., [8, (7) in p. 1268])

$$\breve{\partial}^\varepsilon\varphi(x) := \left\{ y^* \in X^* \;\middle|\; \begin{array}{l} \exists y \in B_\varepsilon(x),\ \text{such that } |\varphi(y) - \varphi(x)| \leq \varepsilon, \\ y^* \in \partial\varphi(y) \text{ and } |\langle y^*, y - x\rangle| \leq \varepsilon\} \end{array} \right\}, \tag{15}$$

which constitutes an enlargement of the Fenchel subdifferential, and that can also be written as

$$\breve{\partial}^\varepsilon\varphi(x) = \bigcup_{y \in B_\varphi(x,\varepsilon)} \partial\varphi(y) \cap \mathrm{N}^\varepsilon_{\{y\}}(x),$$

where $B_\varphi(x,\varepsilon) := \{y \in B_\varepsilon(x) \mid |\varphi(y) - \varphi(x)| \leq \varepsilon\}$.

When $x \notin \operatorname{dom}\varphi$ we adopt the natural convention $\breve{\partial}^{\varepsilon}\varphi(x) := \emptyset$. It is clear that $\partial\varphi(x) \subset \breve{\partial}^{\varepsilon}\varphi(x)$ and, moreover, it can be easily checked that $\cap_{\varepsilon>0}\breve{\partial}^{\varepsilon}\varphi(x) = \partial\varphi(x)$. Let us emphasize here that the elements of $\breve{\partial}^{\varepsilon}\varphi(x)$ are exact Fenchel subgradients at nearby points to x, not ε-subgradients like in (6) and (8).

The following version of the Brøndsted-Rockafellar theorem (see, e.g. [22, Theorem 3.1.1]) is a key tool in our approach.

Lemma 3.2. *Let $\varphi \in \Gamma_0(X)$ and $x \in \operatorname{dom}\varphi$. Then, for every $\varepsilon > 0$ and $x^* \in \partial_{\varepsilon}\varphi(x)$, there exist $x_{\varepsilon} \in X$, $y_{\varepsilon}^* \in B_*$, and $\lambda_{\varepsilon} \in [-1,1]$ such that*

$$\|x_{\varepsilon} - x\| \leq \sqrt{\varepsilon}, \quad x_{\varepsilon}^* := x^* + \sqrt{\varepsilon}(y_{\varepsilon}^* + \lambda_{\varepsilon}x^*) \in \partial\varphi(x_{\varepsilon}),$$

$$|\langle x_{\varepsilon}^*, x_{\varepsilon} - x\rangle| \leq \varepsilon + \sqrt{\varepsilon}, \quad |\varphi(x_{\varepsilon}) - \varphi(x)| \leq \varepsilon + \sqrt{\varepsilon},$$

which implies that $x_{\varepsilon}^* \in \partial_{2\varepsilon}\varphi(x)$ and $x_{\varepsilon}^* \in \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}}\varphi(x)$.

Now we give the main result in this section.

Theorem 3.3. *Let $f_t \in \Gamma_0(X)$, $t \in T$, and $f = \sup_{t\in T} f_t$. Then for every $x \in X$*

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\operatorname{co}} \left\{ \bigcup_{t\in T_{\varepsilon}(x)} \breve{\partial}^{\varepsilon} f_t(x) + \mathrm{N}_{L\cap\operatorname{dom} f}(x) \right\}, \tag{16}$$

where $T_{\varepsilon}(x)$ and $\mathcal{F}(x)$ have been defined in (3) *and* (7)*, respectively.*

Proof. If $x \notin \operatorname{dom} f$ the formula holds trivially since both sets $\mathrm{N}_{L\cap\operatorname{dom} f}(x)$ and $\partial f(x)$ are empty and we apply (10); hence, we suppose that $x \in \operatorname{dom} f$.

Step 1. To prove the inclusion "$\supset$" we first show that, for $\varepsilon > 0$, we have

$$\bigcup_{t\in T_{\varepsilon}(x)} \breve{\partial}^{\varepsilon} f_t(x) \subset \partial_{3\varepsilon} f(x). \tag{17}$$

Indeed, fix $t \in T_{\varepsilon}(x)$ and pick a $y^* \in \breve{\partial}^{\varepsilon} f_t(x)$. By (15), there exists $y \in B_{\varepsilon}(x)$ with $|f_t(y) - f_t(x)| \leq \varepsilon$ such that $y^* \in \partial f_t(y)$ and $|\langle y^*, y - x\rangle| \leq \varepsilon$. Then for every $z \in X$

$$\begin{aligned} f(z) - f(x) &\geq f_t(z) - f_t(x) - \varepsilon \geq f_t(z) - f_t(y) - 2\varepsilon \geq \langle y^*, z - y\rangle - 2\varepsilon \\ &= \langle y^*, z - x\rangle + \langle y^*, x - y\rangle - 2\varepsilon \geq \langle y^*, z - x\rangle - 3\varepsilon, \end{aligned}$$

and we get $y^* \in \partial_{3\varepsilon} f(x)$.

Step 2. Now (15) implies, for every $L \in \mathcal{F}(x)$,

$$\begin{aligned} \bigcup_{t\in T_{\varepsilon}(x)} \breve{\partial}^{\varepsilon} f_t(x) + \mathrm{N}_{L\cap\operatorname{dom} f}(x) &\subset \partial_{3\varepsilon} f(x) + \mathrm{N}_{L\cap\operatorname{dom} f}(x) \\ &\subset \partial_{3\varepsilon}(f + \mathrm{I}_{L\cap\operatorname{dom} f})(x) = \partial_{3\varepsilon}(f + \mathrm{I}_L)(x), \end{aligned}$$

so that, by convexifying and intersecting over ε and L, and applying Lemma 3.1,

$$\bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left\{\bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^{\varepsilon} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right\} \subset \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \partial_{3\varepsilon}(f+\mathrm{I}_L)(x)$$
$$= \bigcap_{L\in\mathcal{F}(x)} \partial(f+\mathrm{I}_L)(x) = \partial f(x).$$

Step 3. To prove the inclusion "$\subset$" we pick an $x^* \in \partial_\varepsilon f_t(x)$ for $\varepsilon > 0$ and $t \in T_\varepsilon(x)$. Then by Lemma 3.2 there are $x_\varepsilon \in X$, $y^*_\varepsilon \in B_*$, and $\lambda_\varepsilon \in [-1,1]$ such that

$$\|x_\varepsilon - x\| \leq \sqrt{\varepsilon}, \quad x^*_\varepsilon := x^* + \sqrt{\varepsilon}(y^*_\varepsilon + \lambda_\varepsilon x^*) \in \partial f_t(x_\varepsilon),$$
$$|\langle x^*_\varepsilon, x_\varepsilon - x\rangle| \leq \varepsilon + \sqrt{\varepsilon}, \quad |f_t(x_\varepsilon) - f_t(x)| \leq \varepsilon + \sqrt{\varepsilon},$$

and $x^*_\varepsilon \in \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x)$. Assuming without loss of generality that $1 \leq 2(1-\sqrt{\varepsilon})$ (i.e., $\varepsilon < 1/4$), we have

$$x^* \in \frac{1}{1+\lambda_\varepsilon\sqrt{\varepsilon}} \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \frac{\sqrt{\varepsilon}}{1+\lambda_\varepsilon\sqrt{\varepsilon}} B_* \subset \Lambda_\varepsilon \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + 2\sqrt{\varepsilon} B_*,$$

where $\Lambda_\varepsilon := \left[\frac{1}{1+\sqrt{\varepsilon}}, \frac{1}{1-\sqrt{\varepsilon}}\right]$. Hence, taking into account Lemma 2.3, formula (6) leads us to

$$\begin{aligned}
\partial f(x) &\subset \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left\{\bigcup_{t\in T_\varepsilon(x)} \Lambda_\varepsilon \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + 2\sqrt{\varepsilon} B_* + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right\} \\
&\subset \bigcap_{\delta>0} \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left\{\bigcup_{t\in T_\varepsilon(x)} \Lambda_\varepsilon \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \delta B_* + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right\} \\
&= \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \bigcap_{\delta>0} \overline{\mathrm{co}}\left\{\bigcup_{t\in T_\varepsilon(x)} \Lambda_\varepsilon \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \delta B_* + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right\} \\
&= \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left\{\bigcup_{t\in T_\varepsilon(x)} \Lambda_\varepsilon \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right\}.
\end{aligned} \tag{18}$$

Step 4. Taking into account now that $\mathrm{N}_{L\cap \operatorname{dom} f}(x)$ is a cone we have

$$\Lambda_\varepsilon \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) = \Lambda_\varepsilon \left(\breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right),$$

and (18) yields

$$\begin{aligned}
\partial f(x) &\subset \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \Lambda_\varepsilon \left(\breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right) \right\} \\
&= \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \Lambda_\varepsilon \left(\bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right) \right\} \\
&= \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \Lambda_\varepsilon \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right\} \\
&= \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right\} \qquad (19)
\end{aligned}$$

where the second and the third equalities come from Lemma 2.1 and Lemma 2.2, respectively. Hence,

$$\begin{aligned}
\partial f(x) &\subset \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_{\varepsilon+\sqrt{\varepsilon}}(x)} \breve{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right\} \\
&= \bigcap_{\substack{\eta>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^{\eta} f_t(x) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right\}. \qquad \square
\end{aligned}$$

Remark 3.4. Observe that formula (16) remains valid if instead of $\breve{\partial}^\varepsilon f_t(x)$ we use the larger set

$$\widehat{\partial}^\varepsilon f_t(x) := \left\{ y^* \in X^* \;\middle|\; \begin{array}{l} \exists y \in X, \text{ such that } |f_t(y) - f_t(x)| \le \varepsilon, \\ y^* \in \partial f_t(y) \text{ and } |\langle y^*, y-x\rangle| \le \varepsilon \} \end{array} \right\}. \qquad (20)$$

The reason is that this set is also included in $\partial_{3\varepsilon} f(x)$ (see the proof of (17)).

If X is finite-dimensional, $\mathrm{N}_{\operatorname{dom} f}(x) \subset \mathrm{N}_{L\cap \operatorname{dom} f}(x)$ for all $L \in \mathcal{F}(x)$ and (16) collapses to

$$\partial f(x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^\varepsilon f_t(x) + \mathrm{N}_{\operatorname{dom} f}(x) \right\}. \qquad (21)$$

In the following corollary we use an alternative enlargement of the Fenchel subdifferential, which involves both subgradients at nearby points y and ε-subgradients at the nominal point x. This provides a characterization of $\partial f(x)$

which is at the same time a refinement of formulas (6) and (16). For instance, the ε-subgradients of the f_t's involved in (6) can be chosen in the range of ∂f_t at sufficiently f_t-graphically close points to x.

Corollary 3.5. *Under the assumptions of Theorem* 3.3 *we have, for every* $x \in X$,

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x) + \mathrm{N}_{L\cap \mathrm{dom}\, f}(x) \right\},$$

where

$$\widehat{\partial}^\varepsilon f_t(x) := \left\{ y^* \in X^* \;\middle|\; \begin{array}{l} \exists y \in B_\varepsilon(x) \text{ such that } |f_t(y) - f_t(x)| \le \varepsilon \\ \text{and } y^* \in \partial f_t(y) \cap \partial_{2\varepsilon} f_t(x) \end{array} \right\}.$$

Proof. Let us first check that $\breve{\partial}^\varepsilon f_t(x) \subset \widehat{\partial}^\varepsilon f_t(x)$.

If $x^* \in \breve{\partial}^\varepsilon f_t(x)$, then $x^* \in \partial f_t(y)$ for some $y \in B_\varepsilon(x)$ such that $|f_t(y) - f_t(x)| \le \varepsilon$ and $|\langle x^*, y - x\rangle| \le \varepsilon$. So, for every $z \in \mathrm{dom}\, f_t$,

$$\begin{aligned} \langle x^*, z - x\rangle &= \langle x^*, z - y\rangle + \langle x^*, y - x\rangle \\ &\le f_t(z) - f_t(y) + \varepsilon \le f_t(z) - f_t(x) + 2\varepsilon, \end{aligned}$$

and we deduce that $x^* \in \partial_{2\varepsilon} f_t(x)$; that is, $x^* \in \widehat{\partial}^\varepsilon f_t(x)$. Then, according to Theorem 3.3 we get the direct inclusion "$\subset$".

To check the converse inclusion "$\supset$" we prove now that

$$\bigcup_{t\in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x) \subset \partial_{3\varepsilon} f(x). \tag{22}$$

To this aim, take $x^* \in \bigcup_{t\in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x)$. Then, there is some $t_0 \in T_\varepsilon(x)$ and $y \in B_\varepsilon(x)$ such that $x^* \in \partial f_{t_0}(y) \cap \partial_{2\varepsilon} f_{t_0}(x)$, with $|f_{t_0}(y) - f_{t_0}(x)| \le \varepsilon$. Then for every $z \in \mathrm{dom}\, f$

$$\begin{aligned} \langle x^*, z - x\rangle &\le \langle x^*, z - y\rangle + \langle x^*, y - x\rangle \\ &\le f_{t_0}(z) - f_{t_0}(y) + f_{t_0}(y) - f_{t_0}(x) + 2\varepsilon \\ &= f_{t_0}(z) - f_{t_0}(x) + 2\varepsilon \le f(z) - f(x) + 3\varepsilon, \end{aligned}$$

that is, $x^* \in \partial_{3\varepsilon} f(x)$.

Thanks to (22) we write

$$\begin{aligned} \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x) + \mathrm{N}_{L\cap \mathrm{dom}\, f}(x) \right\} &\subset \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \partial_{3\varepsilon} f(x) + \mathrm{N}_{L\cap \mathrm{dom}\, f}(x) \right\} \\ &\subset \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x)}} \partial_{3\varepsilon}(f + \mathrm{I}_L)(x) = \partial f(x). \qquad \square \end{aligned}$$

The following example illustrates the need for the lower semicontinuity assumption in Theorem 3.3, as well as for the condition $|f_t(y) - f_t(x)| \leq \varepsilon$.

Example 3.6. *Let $X = \mathbb{R}$ and consider the functions $f_1, f_2 \colon X \to \mathbb{R}$ as*

$$f_1(x) = \begin{cases} +\infty, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ x, & \text{if } x > 0, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} -x, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ +\infty, & \text{if } x > 0. \end{cases}$$

So, $f = \max\{f_1, f_2\} = 1 + \mathrm{I}_{\{0\}}$ so that $\partial f(0) = \mathbb{R}$, but for small $\varepsilon > 0$

$$\breve{\partial}^{\varepsilon} f_1(0) = \partial f_1(0) = \emptyset, \ \breve{\partial}^{\varepsilon} f_2(0) = \partial f_2(0) = \emptyset,$$

and Theorem 3.3 fails.

Example 3.7. *Let X be any infinite-dimensional Banach space, let g be a non-continuous linear mapping, and define the functions $f_t \colon X \to \mathbb{R}$, $t \in T :=]0, +\infty[$, as*

$$f_t(x) := tg(x).$$

So, $f := \sup_{t \in T} f_t = \mathrm{I}_{[g \leq 0]}$ and we get $\partial f(\theta) = \mathrm{N}_{[g \leq 0]}(\theta) \neq \emptyset$. But $\partial f_t \equiv t\partial g \equiv \emptyset$, so that $\breve{\partial}^{\varepsilon} f_t(\theta) \equiv \emptyset$, for all $t \in T$, and Theorem 3.3 fails again.

The following example (e.g., [11]) shows that it is necessary to consider exact subdifferentials at nearby points.

Example 3.8. *Let $f_1, f_2 \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$f_1(x) = \begin{cases} -\sqrt{x}, & \text{if } x \geq 0, \\ +\infty, & \text{if } x < 0, \end{cases} \quad \text{and } f_2(x) = f_1(-x).$$

Then, $f := \max\{f_1, f_2\} = \mathrm{I}_{\{0\}}$ so that $\partial f(0) = \mathbb{R}$. But, $\partial f_1(0) = \partial f_2(0) = \emptyset$ and so, for every $\varepsilon > 0$, we have $T_\varepsilon(\theta) = \{1, 2\}$ and

$$\bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \left\{ (\partial f_1(0) \cup \partial f_2(0)) + \mathrm{N}_{\mathrm{dom}\, f}(0) \right\} = \emptyset.$$

So, in order to recover the whole set $\partial f(0)$, we need to consider the subdifferentials of f_1 and f_2 at nearby points. Indeed, for $\varepsilon < 1$ we have

$$\begin{aligned} \breve{\partial}^{\varepsilon} f_1(0) &= \left\{ y^* \in X^* \;\middle|\; \begin{array}{l} \exists\ |y| \leq \varepsilon \text{ such that } |f_1(y)| \leq \varepsilon, \\ y^* \in \partial f_1(y) \text{ and } |y^* y| \leq \varepsilon \} \end{array} \right\} \\ &= \left\{ -\frac{1}{2\sqrt{y}} \mid 0 < y \leq \varepsilon^2 \right\} = \left]-\infty, -\frac{1}{2\varepsilon}\right], \end{aligned}$$

and, similarly, $\breve{\partial}^{\varepsilon} f_2(0) = \left[\frac{1}{2\varepsilon}, +\infty\right[$. Since $T_\varepsilon(0) = \{1, 2\}$,

$$\bigcup_{t \in T_\varepsilon(0)} \breve{\partial}^{\varepsilon} f_t(0) + \mathrm{N}_{\mathrm{dom}\, f}(0) = \left]-\infty, -\frac{1}{2\varepsilon}\right] \cup \left[\frac{1}{2\varepsilon}, +\infty\right[+ \mathbb{R} = \mathbb{R}.$$

We close this section with the following example showing that, in general, Theorem 3.3 is not valid outside the Banach spaces setting.

Example 3.9. *Let X be a lcs space such that there exists a proper lsc convex function $g \in \Gamma_0(X)$ having an empty subdifferential everywhere. This is the case of some Fréchet spaces* ([17])*, and also of some non-complete normed spaces (actually certain subspaces of $l^2(\mathbb{N})$* [1]*).*

According to [17]*, we may suppose that $\theta \in \operatorname{dom} g$ and $g(\theta) = 0$. For $t \in T :=]0, +\infty[$, we define the function $f_t \in \Gamma_0(X)$ as*

$$f_t(x) := tg(x),$$

so that $f := \sup_{t\in T} f_t = \mathrm{I}_{[g\leq 0]}$. Since $\partial f_t \equiv t\partial f \equiv \emptyset$, the right-hand set in (16) *is empty, whereas*

$$\partial f(\theta) = \mathrm{N}_{[g\leq 0]}(\theta) \neq \emptyset.$$

4. The case of locally convex spaces

We give in this section a general free-continuity formula characterizing the subdifferential of the supremum function in the setting of a lcs X. In what follows, $\mathcal{P}$ is the family of continuous seminorms in X.

We adapt the mapping introduced in (15) to our current setting: given a convex function $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ and $x \in \operatorname{dom}\varphi$, for $p \in \mathcal{P}$ we denote

$$\breve{\partial}_p^{\varepsilon}\varphi(x) := \left\{ y^* \in X^* \;\middle|\; \begin{array}{l} \exists y \in X \text{ with } p(y-x) \leq \varepsilon,\ |\varphi(y) - \varphi(x)| \leq \varepsilon, \\ y^* \in \partial\varphi(y) \text{ and } |\langle y^*, y-x\rangle| \leq \varepsilon \end{array} \right\}. \tag{23}$$

Also now, when $x \notin \operatorname{dom}\varphi$, we set $\breve{\partial}_p^{\varepsilon}\varphi(x) = \emptyset$.

Remark 4.1. It is worth observing that $\breve{\partial}_p^{\varepsilon}\varphi(x)$ remains the same if we replace the inequality $|\langle y^*, y-x\rangle| \leq \varepsilon$ by $\langle y^*, y-x\rangle \leq \varepsilon$, thanks to the fact that $y^* \in \partial\varphi(y)$ and $|\varphi(y) - \varphi(x)| \leq \varepsilon$.

We give the main theorem of this section.

Theorem 4.2. *Let $f_t \in \Gamma_0(X)$, $t \in T$, and $f = \sup_{t\in T} f_t$. Then for every $x \in X$*

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0,\ p\in\mathcal{P} \\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L\cap \operatorname{dom} f}})(x) \right\}. \tag{24}$$

Proof. If $x \notin \operatorname{dom} f$, then $x \notin \operatorname{dom} f_t$ for $t \in T_\varepsilon(x)$; hence, $x \notin \operatorname{dom}(f_t + \mathrm{I}_{\overline{L\cap \operatorname{dom} f}})$ and the sets in both sides of (24) are empty.

Now we assume that $x \in \operatorname{dom} f$. First we verify the inclusion "$\supset$" in (24). We take $L \in \mathcal{F}(x)$ and $y^* \in \breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L\cap \operatorname{dom} f}})(x)$ for some $t \in T_\varepsilon(x)$. Let

$y \in \overline{L \cap \operatorname{dom} f}$ be such that $p(y-x) \leq \varepsilon$,

$$\left|(f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(y) - (f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(x)\right| = |f_t(y) - f_t(x)| \leq \varepsilon,$$

$y^* \in \partial(f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(y)$ and $|\langle y^*, y - x\rangle| \leq \varepsilon$. Then, for all $z \in L \cap \operatorname{dom} f$,

$$\langle y^*, z - x\rangle = \langle y^*, z - y\rangle + \langle y^*, y - x\rangle \leq f_t(z) - f_t(y) + \varepsilon \leq f(z) - f(x) + 3\varepsilon,$$

and we get $y^* \in \partial_{3\varepsilon}(f + \mathrm{I}_L)(x)$; that is, $\breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(x) \subset \partial_{3\varepsilon}(f + \mathrm{I}_L)(x)$. Thus, recalling Lemma 3.1,

$$\bigcap_{\substack{\varepsilon > 0,\, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(x) \right\} \subset \bigcap_{L \in \mathcal{F}(x)} \bigcap_{\varepsilon > 0} \partial_{3\varepsilon}(f + \mathrm{I}_L)(x)$$
$$= \bigcap_{L \in \mathcal{F}(x)} \partial(f + \mathrm{I}_L)(x) = \partial f(x).$$

To prove the inclusion "$\subset$" in (24) we shall proceed by steps.

Step 1. Observe that, due to the following relations

$$\partial_\varepsilon f_t(x) + \mathrm{N}_{L \cap \operatorname{dom} f}(x) = \partial_\varepsilon f_t(x) + \mathrm{N}_{\overline{L \cap \operatorname{dom} f}}(x) \subset \partial_\varepsilon(f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(x),$$

formula (6) implies that

$$\partial f(x) \subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon(f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}})(x) \right\}. \tag{25}$$

Step 2. Now, we fix $p \in \mathcal{P}$, $U \in \mathcal{N}_{X^*}$, $L \in \mathcal{F}(x)$, and $\varepsilon > 0$. Let $\|\cdot\|$ be a norm in L such that $p \leq \|\cdot\|$ in such a subspace, and take into account that $L^* \simeq X^*/L^\perp$ so that $\mathcal{N}_{X^*} + L^\perp$ is a basis of θ-neighborhoods in $X^*/L^\perp$ (see, e.g., [7]). Choose $\eta > 0$ such that

$$\eta + \sqrt{\eta} \leq \varepsilon \text{ and } \sqrt{\eta} B_{L^*} \subset U + L^\perp, \tag{26}$$

where B_{L^*} is the unit ball in L^*.

For the sake of notational brevity, we shall denote

$$g_t := f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}},\ t \in T,$$

where L is our fixed subspace.

Take $x^* \in \partial_\eta g_t(x)$, with $t \in T_\eta(x)$. Then $x^*_{|L} \in \partial_\eta g_{t_{|L}}(x)$, and by Proposition 3.2 we find $x_\eta \in L$ $(\subset X)$ and $v_\eta^* \in \partial g_{t_{|L}}(x_\eta)$ (hence, $x_\eta \in \overline{L \cap \operatorname{dom} f}$), together with $\lambda_\eta \in [-1, 1]$ and $y^* \in B_{L^*}$ (hence, $\sqrt{\eta} y^* \in \sqrt{\eta} B_{L^*} \subset U + L^\perp$), such that

$$p(x_\eta - x) \leq \|x_\eta - x\| \leq \sqrt{\eta} \leq \varepsilon, \tag{27}$$

$$v_\eta^* := x_{|L}^* + \sqrt{\eta}(y^* + \lambda_\eta x_{|L}^*) = (1 + \lambda_\eta \sqrt{\eta})x_{|L}^* + \sqrt{\eta}y^*,$$

$$\left|\left\langle v_\eta^*, x_\eta - x\right\rangle\right| \leq \eta + \sqrt{\eta} \leq \varepsilon,$$

and

$$|f_t(x_\eta) - f_t(x)| = |g_t(x_\eta) - g_t(x)| \leq \eta + \sqrt{\eta} \leq \varepsilon. \tag{28}$$

By the Hahn-Banach theorem we extend v_η^* to an $x_\eta^* \in X^*$; hence, in particular, we have $x_\eta^* \in \partial g_t(x_\eta)$ and $\left|\left\langle x_\eta^*, x_\eta - x\right\rangle\right| = \left|\left\langle v_\eta^*, x_\eta - x\right\rangle\right| \leq \varepsilon$, which together with (27) and (28) ensures that

$$x_\eta^* \in \breve{\partial}_p^\varepsilon g_t(x). \tag{29}$$

Hence we have proved that

$$\partial_\eta g_t(x) \subset \breve{\partial}_p^\varepsilon g_t(x), \text{ for all } t \in T_\eta(x).$$

Step 3. Moreover, for every $u \in L$ we obtain

$$\begin{aligned}
\langle (1 + \lambda_\eta \sqrt{\eta})x^*, u - x\rangle &= \left\langle (1 + \lambda_\eta \sqrt{\eta})x_{|L}^*, u - x\right\rangle \\
&= \left\langle (1 + \lambda_\eta \sqrt{\eta})x_{|L}^* - v_\eta^*, u - x\right\rangle + \left\langle v_\eta^*, u - x\right\rangle \\
&= \langle -\sqrt{\eta}y^*, u - x\rangle + \left\langle x_\eta^*, u - x\right\rangle \\
&= \left\langle x_\eta^* - \sqrt{\eta}y^*, u - x\right\rangle
\end{aligned}$$

that is, taking into account that $\sqrt{\eta}y^* \in \sqrt{\eta}B_{L^*} \subset U + L^\perp$ (by (26)), and using (29),

$$\begin{aligned}
(1 + \lambda_\eta \sqrt{\eta}) \langle x^*, u - x\rangle &= \left\langle x_\eta^* - \sqrt{\eta}y^*, u - x\right\rangle \\
&\leq \sigma_{\breve{\partial}_p^\varepsilon g_t(x) + U + L^\perp}(u - x).
\end{aligned} \tag{30}$$

Observe that (30) can be extended to $u \in X$, i.e. we also have

$$(1 + \lambda_\eta \sqrt{\eta}) \langle x^*, u - x\rangle \leq \sigma_{\breve{\partial}_p^\varepsilon g_t(x) + U + L^\perp}(u - x) \text{ for all } u \in X, \tag{31}$$

due to the fact that, for any $u \in X$,

$$\sigma_{\breve{\partial}_p^\varepsilon g_t(x) + U + L^\perp}(u - x) = \sup\{\langle u^* + \ell^*, u - x\rangle \mid u^* \in \breve{\partial}_p^\varepsilon g_t(x) + U,\ \ell^* \in L^\perp\},$$

and the expression on the right-hand is $+\infty$ when $u - x \notin L$, equivalently $u \notin L$.

Step 4. Now observe that

$$\breve{\partial}_p^\varepsilon g_t(x) + L^\perp \subset \breve{\partial}_p^\varepsilon g_t(x). \tag{32}$$

Indeed, if $z^* \in \breve{\partial}_p^{\varepsilon} g_t(x)$ and $u^* \in L^{\perp}$, then by (23) there exists $z \in X$ such that $p(z-x) \leq \varepsilon$, $|g_t(z) - g_t(x)| \leq \varepsilon$, $z^* \in \partial g_t(z)$ and $|\langle z^*, z-x\rangle| \leq \varepsilon$; in particular, $z \in \operatorname{dom} g_t \subset L$. Then,

$$
\begin{aligned}
z^* + u^* \in \partial g_t(z) + L^{\perp} &\subset \partial (g_t + \mathrm{I}_L)(z) \\
&= \partial (f_t + \mathrm{I}_{\overline{L \cap \operatorname{dom} f}} + \mathrm{I}_L)(z) = \partial g_t(z).
\end{aligned}
$$

As $|\langle z^* + u^*, z-x\rangle| = |\langle z^*, z-x\rangle| \leq \varepsilon$ we deduce that $z^* + u^* \in \breve{\partial}_p^{\varepsilon} g_t(x)$, yielding (32).

Step 5. Consequently, using (31) and (32), for all $z \in X$ (as $t \in T_\eta(x) \subset T_\varepsilon(x)$)

$$
\begin{aligned}
(1+\lambda_\eta \sqrt{\eta})\, \langle x^*, z-x\rangle &\leq \sigma_{\breve{\partial}_p^{\varepsilon} g_t(x)+U}(z-x) \leq \sigma_{\bigcup\limits_{t\in T_\eta(x)} \breve{\partial}_p^{\varepsilon} g_t(x)+U}(z-x) \\
&\leq \sigma_{\bigcup\limits_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x)+U}(z-x) = \sigma_{\overline{\mathrm{co}}\left\{\bigcup\limits_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x)+U\right\}}(z-x);
\end{aligned}
$$

therefore

$$
(1+\lambda_\eta \sqrt{\eta}) x^* \in \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x) + U \right\}
$$

and so, recalling that $x^* \in \partial_\eta g_t(x)$, with $t \in T_\eta(x)$,

$$
\bigcup_{t\in T_\eta(x)} \partial_\eta g_t(x) \subset \Lambda_\eta \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x) + U \right\}, \tag{33}
$$

where $\Lambda_\eta = \left[\frac{1}{1+\sqrt{\eta}}, \frac{1}{1-\sqrt{\eta}}\right]$.

Step 6. Now, from (33) and Lemma 2.1, which ensures that the set in the right-hand side above is closed and convex,

$$
\begin{aligned}
\bigcap_{\substack{\delta>0 \\ M\in\mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\delta(x)} \partial_\delta (f_t + \mathrm{I}_{\overline{M\cap \operatorname{dom} f}})(x) \right\} &\subset \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\eta(x)} \breve{\partial}_p^{\eta} g_t(x) \right\} \\
&\subset \Lambda_\eta \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x) + U \right\},
\end{aligned}
$$

so that, intersecting over $\varepsilon > 0$ (recall (25) and Lemma (2.2)),

$$
\partial f(x) \subset \bigcap_{\varepsilon>0} \Lambda_\eta \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x) + U \right\} = \bigcap_{\varepsilon>0} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon} g_t(x) + U \right\},
$$

which in turn gives us, by intersecting over U (recall (14)),

$$\partial f(x) \subset \bigcap_{\varepsilon>0} \overline{\mathrm{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \breve{\partial}_p^\varepsilon g_t(x) \right\}.$$

Finally, the inclusion follows since L and p were arbitrarily chosen. $\square$

Remark 4.3. (24) gives an alternative formula of (16) in the setting of lcs spaces. If the underlying space is Banach, all that can be derived from (24), by using Proposition 3.2, is

$$\breve{\partial}_p^\varepsilon (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x) \subset \breve{\partial}_p^\varepsilon f_t(y_1) + \mathrm{N}_{\overline{L \cap \mathrm{dom}\, f}}(y_2),$$

for some y_1 *and* y_2 *close to* x. In this sense, we can not directly deduce (16) from (24) (in Banach spaces).

Remark 4.4. Similar to the observation made in Remark 3.4, we can equivalently write formula (24) as

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0 \\ L \in \mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \overset{\frown}{\partial^\varepsilon} (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x) \right\},$$

where $\overset{\frown}{\partial^\varepsilon}$ is the mapping introduced in (20).

Remark 4.5. As in Corollary 3.5, we can show that

$$\breve{\partial}_p^\varepsilon (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x) \subset \widehat{\partial}_p^\varepsilon (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x),$$

where, for a function $\varphi \in \Lambda(X)$, we denote

$$\widehat{\partial}_p^\varepsilon \varphi(x) := \left\{ y^* \in X^* \;\middle|\; \begin{array}{l} \exists y \in X \text{ with } p(y-x) \le \varepsilon,\ |\varphi(y) - \varphi(x)| \le \varepsilon \\ \textit{and } y^* \in \partial\varphi(y) \cap \partial_{2\varepsilon}\varphi(x) \end{array} \right\}. \tag{34}$$

As a consequence of (34) we have that

$$\breve{\partial}_p^\varepsilon (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x) \subset \widehat{\partial}_p^\varepsilon (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x) \subset \partial_{2\varepsilon} (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x). \tag{35}$$

Then similar arguments to those used in Corollary 3.5 give rise to

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0 \\ L \in \mathcal{F}(x)}} \overline{\mathrm{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}_p^\varepsilon (f_t + \mathrm{I}_{\overline{L \cap \mathrm{dom}\, f}})(x) \right\}.$$

Once again the intersection over the L's in (24) can be removed in the finite-dimensional setting. In the following theorem we show that this is also the case in a more general setting.

Theorem 4.6. *Under the assumptions of Theorem* 4.2, *if* $\mathrm{ri}(\mathrm{dom}\, f) \neq \emptyset$ *and* $f_{|\mathrm{aff}(\mathrm{dom}\, f)}$ *is continuous on* $\mathrm{ri}(\mathrm{dom}\, f)$, *then for all* $x \in X$

$$\partial f(x) = \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(x) \right\}.$$

Proof. Fix $x \in \mathrm{dom}\, f$ and choose $L \in \mathcal{F}(x)$ such that $L \cap \mathrm{ri}(\mathrm{dom}\, f) \neq \emptyset$; hence by the accessibility lemma

$$\overline{L \cap \mathrm{dom}\, f} = L \cap \overline{\mathrm{dom}\, f}. \tag{36}$$

Next, given $\varepsilon > 0$, $p \in \mathcal{P}$ and $t \in T_\varepsilon(x)$, we show that

$$\breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L\cap\mathrm{dom}\, f}})(x) \subset \mathrm{cl}\left(\breve{\partial}_p^{2\varepsilon}(f_t + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(x) + L^\perp\right). \tag{37}$$

Take $y^* \in \breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L\cap\mathrm{dom}\, f}})(x)$ and let $y \in \overline{L \cap \mathrm{dom}\, f}$ $(\subset L)$ be such that $p(y-x) \leq \varepsilon$, $|f_t(y) - f_t(x)| \leq \varepsilon$, $|\langle y^*, y-x\rangle| \leq \varepsilon$, and $y^* \in \partial(f_t + \mathrm{I}_{\overline{L\cap\mathrm{dom}\, f}})(y)$.

Denote $h_t := f_t + \mathrm{I}_{\overline{\mathrm{dom}\, f}}$, $t \in T$. Since $\mathrm{dom}\, f \subset \mathrm{dom}\, f_t \cap \overline{\mathrm{dom}\, f} = \mathrm{dom}\, h_t \subset \overline{\mathrm{dom}\, f}$, we have that

$$\mathrm{ri}(\mathrm{dom}\, f) = \mathrm{ri}(\mathrm{dom}\, f_t \cap \overline{\mathrm{dom}\, f}) = \mathrm{ri}(\mathrm{dom}\, h_t) \text{ and } \mathrm{aff}(\mathrm{dom}\, h_t) = \mathrm{aff}(\mathrm{dom}\, f).$$

Consequently, because $h_{t|\mathrm{aff}(\mathrm{dom}\, g_t)} \leq f_{|\mathrm{aff}(\mathrm{dom}\, f)}$ the function $h_{t|\mathrm{aff}(\mathrm{dom}\, g_t)}$ is continuous on $\mathrm{ri}(\mathrm{dom}\, h_t)$ and so, according to [5, Theorem 15] (remember (36)),

$$y^* \in \partial(f_t + \mathrm{I}_{\overline{L\cap\mathrm{dom}\, f}})(y) = \partial(h_t + \mathrm{I}_L)(y) = \mathrm{cl}(\partial h_t(y) + L^\perp).$$

Let $U_0 \in \mathcal{N}_{X^*}$ be such that $\sigma_{U_0}(y-x) \leq \varepsilon$. Then for every $U \in \mathcal{N}_{X^*}$ such that $U \subset U_0$ there exists $z^* \in \partial h_t(y)$ such that $y^* \in z^* + L^\perp + U$ and so, as $x, y \in L$,

$$|\langle z^*, y-x\rangle| \leq |\langle y^*, y-x\rangle| + \sigma_U(y-x) \leq |\langle y^*, y-x\rangle| + \sigma_{U_0}(y-x) \leq 2\varepsilon,$$

showing that $z^* \in \breve{\partial}_p^{2\varepsilon} h_t(x)$ and so,

$$y^* \in z^* + L^\perp + U \subset \breve{\partial}_p^{2\varepsilon} h_t(x) + L^\perp + U.$$

This leads us to

$$\begin{aligned} \breve{\partial}_p^{\varepsilon}(f_t + \mathrm{I}_{\overline{L\cap\mathrm{dom}\, f}})(x) &\subset \bigcap\nolimits_{U\in\mathcal{N}_{X^*}, U\subset U_0} \left(\breve{\partial}_p^{2\varepsilon} h_t(x) + L^\perp + U\right) \\ &= \bigcap\nolimits_{U\in\mathcal{N}_{X^*}} \left(\breve{\partial}_p^{2\varepsilon} h_t(x) + L^\perp + U\right) = \mathrm{cl}\left(\breve{\partial}_p^{2\varepsilon} h_t(x) + L^\perp\right), \end{aligned}$$

and (37) follows.

Now, by applying (24), and using (37), we obtain

$$\begin{aligned}
\partial f(x) &= \bigcap_{\substack{\varepsilon>0,\ p\in\mathcal{P}\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left\{\bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_p^{\varepsilon}(f_t+\mathrm{I}_{\overline{L\cap\operatorname{dom} f}})(x)\right\}\\
&\subset \bigcap_{\substack{\varepsilon>0,\ p\in\mathcal{P}\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left(\bigcup_{t\in T_\varepsilon(x)} \mathrm{cl}\left(\breve{\partial}_p^{2\varepsilon}(f_t+\mathrm{I}_{\overline{\operatorname{dom} f}})(x)+L^{\perp}\right)\right)\\
&\subset \bigcap_{\substack{\varepsilon>0,\ p\in\mathcal{P}\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left(\mathrm{cl}\left(\bigcup_{t\in T_{2\varepsilon}(x)} \breve{\partial}_p^{2\varepsilon}(f_t+\mathrm{I}_{\overline{\operatorname{dom} f}})(x)+L^{\perp}\right)\right)\\
&= \bigcap_{\substack{\varepsilon>0,\ p\in\mathcal{P}\\ L\in\mathcal{F}(x)}} \overline{\mathrm{co}}\left(\bigcup_{t\in T_{2\varepsilon}(x)} \breve{\partial}_p^{2\varepsilon}(f_t+\mathrm{I}_{\overline{\operatorname{dom} f}})(x)+L^{\perp}\right)\\
&= \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}}\left(\bigcup_{t\in T_{2\varepsilon}(x)} \breve{\partial}_p^{2\varepsilon}(f_t+\mathrm{I}_{\overline{\operatorname{dom} f}})(x)\right),
\end{aligned}$$

where in the last equality we used Lemma 2.3. □

5. The continuous case

In this section we get reduced versions of the previous formulas, (16) and (24), when continuity assumptions are imposed. We work in the general setting of a lcs space X, where $\mathcal{P}$ is the family of continuous seminorms in X. The main result comes next.

Theorem 5.1. *Let $f_t \in \Lambda(X)$, $t \in T$, and $f = \sup_{t\in T} f_t$. Assume that f is finite and continuous at some point. Then for every $x \in X$*

$$\partial f(x) = \mathrm{N}_{\operatorname{dom} f}(x) + \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}}\left\{\bigcup_{t\in\overline{T}_\varepsilon(x)} \breve{\partial}_p^{\varepsilon}(\mathrm{cl}\, f_t)(x)\right\}, \tag{38}$$

where $\overline{T}_\varepsilon(x) := \{t \in T \mid (\mathrm{cl}\, f_t)(x) \geq f(x) - \varepsilon\}$.

Proof. We assume without loss of generality that $x = \theta \in \operatorname{dom} f$ and $f(\theta) = 0$. We fix $x^* \in \partial f(\theta)$ and let $x_0 \in \operatorname{dom} f$ be such that f is continuous at x_0, that is, there exist $W \in \mathcal{N}_X$ and $m \in \mathbb{R}_+$ such that

$$x_0 + W \subset \operatorname{dom} f \quad \text{and} \quad \sup_{t\in T,\ w\in W} f_t(x_0 + w) \leq m; \tag{39}$$

hence, all the f_t's are continuous at x_0 [22, Lemma 2.2.8].

Step1. Let us first suppose that all the f_t's are lsc, so that $\overline{T}_\varepsilon(x) = T_\varepsilon(x)$. Fix $\varepsilon \in]0,1[$ and $p \in \mathcal{P}$.

We introduce the continuous seminorm $\tilde{p} := \max\{p, p_W\}$, where p_W is the Minkowski gauge of W. Then, by Theorem 4.6,

$$\begin{aligned}\partial f(\theta) &= \bigcap_{\delta>0,\ q\in\mathcal{P}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\delta(\theta)} \breve{\partial}_q^\delta (f_t + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(\theta) \right\} \\ &\subset \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}_{\tilde{p}}^\varepsilon (f_t + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(\theta) \right\}. \end{aligned} \tag{40}$$

Step 2. Choose $V_0 \in \mathcal{N}_{X^*}$ satisfying

$$\sigma_{V_0}(x_0) \leq 1, \tag{41}$$

and take $V \in \mathcal{N}_{X^*}$ such that $V \subset V_0$; hence, $\sigma_V(x_0) \leq \sigma_{V_0}(x_0) \leq 1$, and by (40)

$$x^* \in \mathrm{co} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}_{\tilde{p}}^\varepsilon (f_t + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(\theta) \right\} + V.$$

In other words, there exist $k \in \mathbb{N}$, $(\lambda_1, \cdots, \lambda_k) \in \Delta_k$, $t_1, \cdots, t_k \in T_\varepsilon(\theta)$, and $z_i^* \in \breve{\partial}_{\tilde{p}}^\varepsilon (f_{t_i} + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(\theta)$, $i \in \overline{1,k}$, such that

$$x^* \in \lambda_1 z_1^* + \cdots + \lambda_k z_k^* + V.$$

Moreover, taking into account Moreau-Rockafellar sum rule (recall that the f_t's are continuous at $x_0 \in \mathrm{dom}\, f$), and using the definition of $\breve{\partial}_{\tilde{p}}^\varepsilon$, for each $i \in \overline{1,k}$ there exist $y_i \in \overline{\mathrm{dom}\, f}$, with $\tilde{p}(y_i) \leq \varepsilon$, and elements $y_i^* \in \partial f_{t_i}(y_i)$ and $v_i^* \in \mathrm{N}_{\overline{\mathrm{dom}\, f}}(y_i)$ such that $z_i^* = y_i^* + v_i^*$,

$$|\langle y_i^* + v_i^*, y_i \rangle| \leq \varepsilon, \tag{42}$$

and

$$|f_{t_i}(y_i) - f_{t_i}(\theta)| = \left| (f_{t_i} + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(y_i) - (f_{t_i} + \mathrm{I}_{\overline{\mathrm{dom}\, f}})(\theta) \right| \leq \varepsilon; \tag{43}$$

consequently, y_i and y_i^* satisfy the following relations

$$y_i^* \in \partial_{3\varepsilon} f(\theta) \text{ and } |\langle y_i^*, y_i \rangle| \leq \varepsilon, \tag{44}$$

and

$$y_i^* \in \breve{\partial}_{\tilde{p}}^\varepsilon f_{t_i}(\theta). \tag{45}$$

Indeed, for any $u \in \mathrm{dom}\, f$ the information we have above on y_i and y_i^* leads, on one hand, to

$$\begin{aligned}\langle y_i^*, u \rangle &= \langle y_i^*, u - y_i \rangle + \langle y_i^* + v_i^*, y_i \rangle + \langle v_i^*, -y_i \rangle \\ &\leq f_{t_i}(u) - f_{t_i}(y_i) + \varepsilon \leq f(u) - f_{t_i}(\theta) + 2\varepsilon \leq f(u) - f(\theta) + 3\varepsilon,\end{aligned}$$

which shows that $y_i^* \in \partial_{3\varepsilon} f(\theta)$. The second inequality in (44) also follows since

$$\langle y_i^*, -y_i \rangle \le f_{t_i}(\theta) - f_{t_i}(y_i) \le \varepsilon,$$

and, using (42),

$$\langle y_i^*, y_i \rangle = \langle y_i^* + v_i^*, y_i \rangle + \langle v_i^*, -y_i \rangle \le \varepsilon.$$

This shows that $|\langle y_i^*, y_i \rangle| \le \varepsilon$, which together with (43), and the relations $\tilde{p}(y_i) \le \varepsilon$ and $y_i^* \in \partial f_{t_i}(y_i)$, lead us to (45).

We also have

$$v_i^* \in \mathrm{N}^{2\varepsilon}_{\overline{\operatorname{dom} f}}(\theta). \tag{46}$$

Actually, for any $u \in \operatorname{dom} f$, by (42) and (44) we have

$$\begin{aligned} \langle v_i^*, u \rangle &= \langle v_i^*, u - y_i \rangle + \langle v_i^*, y_i \rangle \le \langle v_i^*, y_i \rangle \\ &= \langle v_i^* + y_i^*, y_i \rangle - \langle y_i^*, y_i \rangle \le 2\varepsilon, \end{aligned}$$

and this yields (46).

Also, since $p_W(y_i) \le \tilde{p}(y_i) \le \varepsilon$, we infer that $y_i \in \varepsilon W$ and so, using (39),

$$x_0 + y_i \in x_0 + \varepsilon W \subset x_0 + W \subset \operatorname{dom} f,$$

that is, since $v_i^* \in \mathrm{N}_{\overline{\operatorname{dom} f}}(y_i)$,

$$\langle v_i^*, x_0 \rangle = \langle v_i^*, (x_0 + y_i) - y_i \rangle \le 0. \tag{47}$$

Step 3. Now, we denote

$$y_V^* := \lambda_1 y_1^* + \cdots + \lambda_k y_k^*, \ v_V^* := \lambda_1 v_1^* + \cdots + \lambda_k v_k^*;$$

hence, by (44), (47) and (45),

$$y_V^* \in \partial_{3\varepsilon} f(\theta) \cap \operatorname{co} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \breve{\partial}^{\varepsilon}_{\tilde{p}} f_t(\theta) \right\}, \tag{48}$$

and

$$v_V^* \in \mathrm{N}^{2\varepsilon}_{\overline{\operatorname{dom} f}}(\theta), \ \ \langle v_V^*, x_0 \rangle \le 0, \tag{49}$$

at the same time as

$$v_V^* \in x^* - y_V^* + V, \tag{50}$$

entailing that, due to (41),

$$\begin{aligned} \langle y_V^*, x_0 \rangle &= \langle y_V^* + v_V^*, x_0 \rangle - \langle v_V^*, x_0 \rangle \ge \langle x^*, x_0 \rangle - \sigma_V(x_0) \\ &\ge \langle x^*, x_0 \rangle - \sigma_{V_0}(x_0) \ge \langle x^*, x_0 \rangle - 1. \end{aligned} \tag{51}$$

Consequently, since $y_V^* \in \partial_{3\varepsilon} f(\theta)$, we obtain that, for all $w \in W$,

$$\langle y_V^*, x_0 + w\rangle \leq f(x_0 + w) + 3\varepsilon \leq m + 3\varepsilon, \tag{52}$$

and so, because of (51), we find some $r > 0$ such that

$$\langle y_V^*, w\rangle \leq -\langle y_V^*, x_0\rangle + m + 3\varepsilon \leq -\langle x^*, x_0\rangle + 1 + m + 3\varepsilon \leq r; \tag{53}$$

that is, considering the natural partial order in $\mathcal{N}_{X^*}$, by Alaoglu-Bourbaki theorem the net $(y_V^*)_{V\in\mathcal{N}_{X^*},\, V\subset V_0}$ is contained in the weak*-compact set rW°. Hence, we may assume that $(y_V^*)_V$ weak*-converges to some element $y_\varepsilon^* \in X^*$ such that (recall (48))

$$y_\varepsilon^* \in rW^\circ \cap \overline{\text{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}_{\tilde{p}}^{\varepsilon} f_t(\theta) \right\} \subset rW^\circ \cap \overline{\text{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}_{\tilde{p}}^{\varepsilon} f_t(\theta) \right\}. \tag{54}$$

It also follows that the corresponding (sub)net $(v_V^*)_{V\in\mathcal{N}_{X^*},\, V\subset V_0}$ weak*-converges to the element $v_\varepsilon^* := x^* - y_\varepsilon^*$ and that $v_\varepsilon^* \in \mathrm{N}^{2\varepsilon}_{\overline{\mathrm{dom}\, f}}(\theta)$. Indeed, given $U_1 \in \mathcal{N}_{X^*}$, we choose $U_0 \in \mathcal{N}_{X^*}$ such that $U_0 + U_0 \subset U_1$. Since $U := v_\varepsilon^* + U_0 \in \mathcal{N}_{v_\varepsilon^*}$, we have $x^* - U \in \mathcal{N}_{y_\varepsilon^*}$ and so, there is some $V_1 \in \mathcal{N}_{X^*}$ such that $y_{V'}^* \in x^* - U$ for all $V' \subset V_1$; that is, $\theta \in y_{V'}^* - x^* + U$ and, using (50),

$$v_{V'}^* \in v_{V'}^* + y_{V'}^* - x^* + U \subset V' + U.$$

Then we get, since $V' \cap U_0 \subset V'$ and $U = v_\varepsilon^* + U_0$,

$$v_{V'}^* \in V' \cap U_0 + U = v_\varepsilon^* + V' \cap U_0 + U_0 \subset v_\varepsilon^* + U_1,$$

showing that $(v_V^*)_{V\in\mathcal{N}_{X^*},\, V\subset V_0}$ weak*-converges to $v_\varepsilon^* \in \mathrm{N}^{2\varepsilon}_{\overline{\mathrm{dom}\, f}}(\theta)$, by (49).

Step 4. Again, since $(y_\varepsilon^*)_\varepsilon \subset rW^\circ$ we may assume that the net $(y_\varepsilon^*)_\varepsilon$ weak*-converge to some y^*, which by (54) satisfies

$$y^* \in \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}_{\tilde{p}}^{\varepsilon} f_t(\theta) \right\}.$$

Moreover, since p was arbitrarily fixed in $\mathcal{P}$, we deduce that

$$y^* \in \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}_{\tilde{p}}^{\varepsilon} f_t(\theta) \right\}.$$

In addition, since $v_\varepsilon^* = x^* - y_\varepsilon^*$, by (49) it follows that $(v_\varepsilon^*)_\varepsilon$ also weak*-converges to

$$v^* = x^* - y^* \in \mathrm{N}_{\overline{\mathrm{dom}\, f}}(\theta) = \mathrm{N}_{\mathrm{dom}\, f}(\theta).$$

Consequently,

$$x^* = v^* + y^* \in \mathrm{N}_{\mathrm{dom}\, f}(\theta) + \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}^{\varepsilon}_{\tilde{p}} f_t(\theta) \right\},$$

yielding the inclusion "$\subset$".

Step 5. We consider the case when the f_t's are not necessarily lsc. Due to the continuity assumption on f, entailing the continuity of the f_t's at x_0, according to [11, Corollary 9(ii)] we have

$$\mathrm{cl}\, f = \sup_{t\in T} \mathrm{cl}\, f_t.$$

If $x \in \mathrm{dom}\, f$ is such that $\partial f(x) \neq \emptyset$, then $\partial f(x) = \partial(\mathrm{cl}\, f)(x)$ and $f(x) = (\mathrm{cl}\, f)(x)$. Thus, from Step 4 applied to the $(\mathrm{cl}\, f_t)$'s, we obtain

$$\partial f(x) = \partial(\mathrm{cl}\, f)(x) = \mathrm{N}_{\mathrm{dom}(\mathrm{cl}\, f)}(x) + \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in \overline{T}_\varepsilon(x)} \breve{\partial}^{\varepsilon}_{\tilde{p}} (\mathrm{cl}\, f_t)(x) \right\},$$

and the inclusion "$\subset$" follows since $\mathrm{N}_{\mathrm{dom}(\mathrm{cl}\, f)}(x) \subset \mathrm{N}_{\mathrm{dom}\, f}(x)$.

Step 6. We show the opposite inclusion "$\supset$". Given $x \in \mathrm{dom}\, f$ $(\subset \mathrm{dom}\, f_t)$, by (35) we have $\breve{\partial}^{\varepsilon}_{\tilde{p}}(\mathrm{cl}\, f_t)(x) \subset \partial_{2\varepsilon}(\mathrm{cl}\, f_t)(x)$. Then, since it can be easily proved that $\partial_{2\varepsilon}(\mathrm{cl}\, f_t)(x) \subset \partial_{3\varepsilon} f(x)$ for all $t \in \overline{T}_\varepsilon(x)$, we deduce that for all $\varepsilon > 0$ and $p \in \mathcal{P}$

$$\overline{\mathrm{co}} \left\{ \bigcup_{t\in \overline{T}_\varepsilon(x)} \breve{\partial}^{\varepsilon}_{\tilde{p}} (\mathrm{cl}\, f_t)(x) \right\} \subset \partial_{3\varepsilon} f(x).$$

Therefore

$$\begin{aligned} \mathrm{N}_{\mathrm{dom}\, f}(x) + \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in \overline{T}_\varepsilon(x)} \breve{\partial}^{\varepsilon}_{\tilde{p}} (\mathrm{cl}\, f_t)(x) \right\} &\subset \mathrm{N}_{\mathrm{dom}\, f}(x) + \bigcap_{\varepsilon>0} \partial_\varepsilon f(x) \\ &= \mathrm{N}_{\mathrm{dom}\, f}(x) + \partial f(x) = \partial f(x). \end{aligned}$$

□

Remark 5.2. As can be observed in the proof of Theorem 5.1, the continuity assumption on f used there can be weakened to the existence of small $\varepsilon > 0$ such that the function $\sup_{t\in\overline{T}_\varepsilon(x)} f_t$ is finite and continuous at some point.

The following corollary is straightforward from Theorem 5.1.

Corollary 5.3. *Let $f_t \in \Gamma_0(X)$, $t \in T$, and $f = \sup_{t\in T} f_t$. Assume that f is finite and continuous at some point. Then for every $x \in X$*

$$\partial f(x) = \mathrm{N}_{\operatorname{dom} f}(x) + \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\operatorname{co}} \left\{ \bigcup_{t\in T_\varepsilon(x)} \breve{\partial}^{\varepsilon}_{\tilde{p}} f_t(x) \right\}.$$

We obtain the classical result of Valadier [19, Theorem 1]:

Corollary 5.4. *Let $f_t \in \Lambda(X)$, $t \in T$, and $f = \sup_{t\in T} f_t$. Assume that f is continuous at $x \in \operatorname{dom} f$. Then*

$$\partial f(x) = \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\operatorname{co}} \left\{ \bigcup_{t\in T_\varepsilon(x),\ p(y-x)\le\varepsilon} \partial f_t(y) \right\}.$$

Proof. We assume again that $x = \theta \in \operatorname{dom} f$ and $f(\theta) = 0$. Since f is continuous at θ, by Theorem 5.1 we have

$$\begin{aligned}\partial f(\theta) &= \mathrm{N}_{\operatorname{dom} f}(\theta) + \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\operatorname{co}} \left\{ \bigcup_{t\in \overline{T}_\varepsilon(\theta)} \breve{\partial}^{\varepsilon}_{\tilde{p}} (\operatorname{cl} f_t)(\theta) \right\} \\ &= \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\operatorname{co}} \left\{ \bigcup_{t\in \overline{T}_\varepsilon(\theta)} \breve{\partial}^{\varepsilon}_{\tilde{p}} (\operatorname{cl} f_t)(\theta) \right\}. \end{aligned} \tag{55}$$

On one hand, the continuity of f at θ implies the continuity of all the f_t's at θ. So,

$$(\operatorname{cl} f_t)(\theta) = f_t(\theta) \text{ for all } t \in T,$$

and we get

$$\overline{T}_\varepsilon(\theta) = T_\varepsilon(\theta). \tag{56}$$

On the other hand, there exist $U \in \mathcal{N}_X$ and $m \ge 0$ such that, for all $u \in U$,

$$f_t(u) \le f(u) \le m, \tag{57}$$

and so, by [22, Lemma 2.2.8],

$$|f_t(u) - f_t(\theta)| \le m p_U(u). \tag{58}$$

This entails the continuity of the f_t's at the points of U. Hence, (55) together with (56) give us, for all $p \in \mathcal{P}$,

$$\partial f(\theta) \subset \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta)} \breve{\partial}^{\varepsilon}_{\tilde{p}} (\operatorname{cl} f_t)(\theta) \right\},$$

where $\tilde{p} := \max\{p, p_U\}$. Observe that if $y^* \in \breve{\partial}^{\varepsilon}_{\tilde{p}}(\mathrm{cl}\, f_t)(\theta)$, for $t \in T_\varepsilon(\theta)$, then there exists $y \in X$ satisfying $\tilde{p}(y) \leq \varepsilon$ such that $y^* \in \partial(\mathrm{cl}\, f_t)(y)$. But $p_U(y) \leq \varepsilon$ ensures that $y \in \varepsilon U \subset U$, and so, by (57), f_t is continuous at y so that $y^* \in \partial(\mathrm{cl}\, f_t)(y) = \partial f_t(y)$. This gives rise to

$$\partial f(\theta) \subset \bigcap_{\varepsilon>0} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta),\ p(y)\leq\varepsilon} \partial f_t(y) \right\},$$

and the inclusion "$\subset$" follows by intersecting over $p \in \mathcal{P}$.

To show the converse inclusion "$\supset$", we take

$$x^* \in \bigcap_{\varepsilon>0,\ p\in\mathcal{P}} \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta),\ p(y)\leq\varepsilon} \partial f_t(y) \right\}.$$

Then, given $\varepsilon \in \left]0, \frac{1}{2}\right[$ and $p \in \mathcal{P}$, we get

$$x^* \in \overline{\mathrm{co}} \left\{ \bigcup_{t\in T_\varepsilon(\theta),\ \tilde{p}(y)\leq\varepsilon} \partial f_t(y) \right\}, \tag{59}$$

where $\tilde{p} := \max\{p, p_U\}$ $(\in \mathcal{P})$, with U being defined as in (57). Observe that if $y^* \in \partial f_t(y)$ for some $t \in T_\varepsilon(\theta)$ and $y \in X$ such that $\tilde{p}(y) \leq \varepsilon$, then $y \in \varepsilon U \subset \frac{1}{2}U$ and, taking into account (58),

$$|f_t(y) - f_t(\theta)| \leq m p_U(y) \leq m\tilde{p}(y) \leq m\varepsilon,$$
$$|f_t(2y) - f_t(\theta)| \leq m p_U(2y) \leq 2m\tilde{p}(y) \leq 2m\varepsilon.$$

Consequently, for every $z \in \mathrm{dom}\, f$,

$$\begin{aligned}
\langle y^*, z\rangle &= \langle y^*, z - y\rangle + \langle y^*, y\rangle = \langle y^*, z - y\rangle + \langle y^*, 2y - y\rangle \\
&\leq f_t(z) - f_t(y) + f_t(2y) - f_t(y) \\
&= (f_t(z) - f_t(\theta)) + (f_t(\theta) - f_t(y)) + (f_t(2y) - f_t(\theta)) + (f_t(\theta) - f_t(y)) \\
&\leq f_t(z) - f_t(\theta) + 4m\varepsilon,
\end{aligned}$$

and we get, since $t \in T_\varepsilon(\theta)$,

$$\langle y^*, z\rangle \leq f(z) - f(\theta) + 4m\varepsilon + \varepsilon,$$

so that $y^* \in \partial_{(4m+1)\varepsilon} f(\theta)$. Thus, from (59) it follows that

$$x^* \in \bigcap_{0<\varepsilon<\frac{1}{2}} \partial_{(4m+1)\varepsilon} f(\theta) = \partial f(\theta),$$

and the proof is complete. □

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References

[1] J. Borwein, D. W. Tingley: On supportless convex sets, Proc. Amer. Math. Soc. 94(3) (1985) 471–476.

[2] A. Brøndsted: On the subdifferential of the supremum of two convex functions, Math. Scand. 31 (1972) 225–230.

[3] R. Correa, A. Jofré: Some properties of semismooth and regular functions in nonsmooth analysis, in: Recent Advances in System Modelling and Optimization (Santiago, 1984), Lecture Notes Control Inform. Sci. 87, Springer, Berlin (1986) 69–85.

[4] R. Correa, A. Seeger: Directional derivative of a minmax function, Nonlinear Anal. 9(1) (1985) 14–22.

[5] R. Correa, A. Hantoute, M. A. López: Weaker conditions for subdifferential calculus of convex functions, J. Funct. Anal. 271 (2016) 1177–1212.

[6] J. M. Danskin: The Theory of Max-Min and its Applications to Weapons Allocations Problems, Springer, New York (1967).

[7] M. Fabian, P. Habala, P. Hajek, V. Montesinos Santalucia, J. Pelant, V. Zizler: Functional Analysis and Infinite-Dimensional Geometry, Springer, New York (2001).

[8] J. Florence, M. Lassonde: Subdifferential estimate of the directional derivative, optimality criterion and separation principles, Optimization 62(9) (2013) 1267–1288.

[9] A. Hantoute: Subdifferential set of the supremum of lower semi-continuous convex functions and the conical hull intersection property, Top 14(2) (2006) 355–374.

[10] A. Hantoute, M. A. López: A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions, J. Convex Analysis 15(4) (2008) 831–858.

[11] A. Hantoute, M. A. López, C. Zălinescu: Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions, SIAM J. Optim. 19(2) (2008) 863–882.

[12] A. Hantoute, A. Svensson: General representation of δ-normal sets and normal cones to sublevel sets of convex functions, Set-Valued Var. Analysis 25(4) (2017) 651–678..

[13] A. D. Ioffe: A note on subdifferentials of pointwise suprema, Top 20 (2012) 456–466.

[14] A. D. Ioffe, V. L. Levin: Subdifferentials of convex functions (Russian), Trudy Moskov. Mat. Obsc. 26 (1972) 3–73, transl. in Trans. Moscow Math. Soc. 26 (1972).

[15] C. Li, K. F. Ng: Subdifferential calculus rules for supremum functions in convex analysis, SIAM J. Optim. 21(3) (2011) 782–797.

[16] O. López, L. Thibault: Sequential formula for subdifferential of upper envelope of convex functions, J. Nonlinear Convex Analysis 14(2) (2013) 377–388.

[17] R. T. Rockafellar: On the subdifferentiability of convex functions, Proc. Amer. Math. Soc. 16 (1965) 605–611.

[18] V. N. Solov'ev: The subdifferential and the directional derivatives of the maximum of a family of convex functions, Izv. Ross Akad. Nauk Ser. Mat. 65 (2001) 107–132.

[19] M. Valadier: Sous-différentiels d'une borne supérieure et d'une somme continue de fonctions convexes, C. R. Acad. Sci. Paris, Sér. A-B Math. 268 (1969) 39–42.

[20] M. Volle: Sous-différentiels d'une enveloppe supérieure de fonctions convexes, C. R. Acad. Sci., Paris Sér. I Math. 317 (1993) 845–849.

[21] M. Volle: On the subdifferential of an upper envelope of convex functions, Acta Math. Vietnam. 19 (1994) 137–148.

[22] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific Publishing Comp., River Edge (2002).