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Reduced and Minimally Convex Pairs of Sets

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We study algebraic properties of convex sets. In the first section we present one of our main results which is an algebraic order cancellation law of convex sets. Then we study properties of convex pairs and reduced pairs mainly within the general frame of commutative semigroups which satisfy the order cancellation law.

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1. Introduction

Pairs of compact convex sets arise in quasidifferential calculus of V. F. Demyanov and A. M. Rubinov as sub- and superdifferentials of quasidifferentiable functions [2]. These are directional differentiable functions, whose directional derivative can be written as a difference of two sublinear functions. The general framework for the investigation of differences of sublinear functions is the Minkowski-Rådström-Hörmander space over a real topological vector space X of classes of pairs of nonempty compact convex sets of X by considering pairs

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of subdifferentials instead of sublinear functions [10, 15]. In particular we are interested in special classes of pairs of subdifferentials, such as minimal representants, convex pairs or reduced pairs.

2. Pairs of convex sets

The quasidifferential calculus of V. F. Demyanov and A. M. Rubinov [2, 3], which is based on formal rules for pairs of subdifferentials (called sub- and superdifferential) stimulated the interest in studying pairs of nonempty compact convex sets in a locally convex vector space in other respects.

The research started with the work of L. Hörmander [10] and H. Rådström [17] who investigated for a locally convex vector space X the linear space of all classes of equivalent pairs of nonempty bounded closed convex subsets. This space is known as the *Minkowski-Rådström-Hörmander lattice* and can be defined in the following way: Denote by $\mathcal{B}(X)$ (resp. $\mathcal{K}(X)$) the set of all nonempty bounded closed (resp. compact) convex subsets of X and introduce on the set $\mathcal{B}^2(X) = \mathcal{B}(X) \times \mathcal{B}(X)$ of all pairs of nonempty bounded closed convex subsets the following relation

$$(A, B) \sim (C, D) \quad \Longleftrightarrow \quad A \dotplus D \ = \ B \dotplus C,$$

where

$$A \dotplus B \ = \ \mathrm{cl}\{x = a + b \mid a \in A \ \text{ and } \ b \in B\}$$

is the closure of the Minkowski sum of A and B. Two pairs (A, B) and (C, D) which satisfy this relation are called *equivalent* and $\mathcal{B}^2(X)/_\sim$ is the linear space of all classes of equivalent pairs. The element of $\mathcal{B}^2(X)/_\sim$ which contains the pair (A, B) is denoted by $[A, B] \in \mathcal{B}^2(X)/_\sim$. Analogously we use the notations $\mathcal{K}^2(X)$ and $\mathcal{K}^2(X)/_\sim$ for the case of compact convex sets.

A key for the following investigations is the *order cancellation law,* which was first proved by H. Rådström [17] for nonempty compact convex sets and for nonempty bounded closed convex sets by R. Urbański [21] (see also Theorem 3.2.1 [15]).

In 1966 A.G.Pinsker [16] introduced the following ordering on $\mathcal{B}^2(X)/_\sim$ namely: $[A, B] \preceq [C, D] \quad \Longleftrightarrow \quad A \dotplus D \ \subset \ B \dotplus C$, which is independent of the special choice of representatives, because of the order cancellation law. Endowed with this ordering the Minkowski-Rådström-Hörmander lattice is a vector lattice.

Let us recall that for $A, B \in \mathcal{B}(X)$ the *Minkowski* (*Pontryagin, Hukuhara*) *difference* of the sets A and B is given by:

$$A \ \dot{-} \ B = \{x \in X \mid B + x \subset A\} = \bigcap_{b \in B} (A - b)$$

and commonly called *Pontryagin difference* of A and B.

The quasidifferential calculus gave also rise to the investigation of *inclusion minimal representant* for elements of the Minkowski-Rådström-Hörmander lattice of classes of equivalent pairs of nonempty bounded closed convex sets [21]. To be more precise: Let us introduce the following order

$$(A,B) \leqslant (C,D) \quad \Longleftrightarrow \quad A \subset C \quad \text{and} \quad B \subset D.$$

Then $(A_0, B_0) \in [A,B]$ is an *inclusion minimal representant* of $[A,B]$ if it is a minimal element in $[A,B]$. We call (A_0,B_0) a *minimal pair.*

We will call a set $A \in \mathcal{B}(X)$ a *summand* of $B \in \mathcal{B}(X)$ if there exists a set $C \in \mathcal{B}(X)$ such that $A \dotplus C = B$.

Obviously the following characterization of inclusion minimal representant follows from the order cancellation law:

The pair $(A_0,B_0) \in [A,B]$ is an inclusion minimal representant if and only if there exists no proper bounded closed convex subset $K \subset A_0 \dotplus B_0$ such that A_0 is a summand of K and B_0 is a summand of K.

Related to this observation is the notion of reduced pairs which was introduced by Ch. Bauer and R. Schneider [1, 18].

Definition 2.1. A pair (A,B) is *reduced* if and only if

$$[A,B] = \{(A\dotplus C, B\dotplus C) \mid C \in \mathcal{B}(X)\}$$

or equivalently: (A,B) is reduced if and only if $A+B$ is a summand of C whenever A and B are summands of C.

All reduced pairs of sets are minimal, but not all minimal pairs are reduced.

The following notation will be used: For a continuous linear functional $f\colon X \to \mathbb{R}$, i.e. $f \in X^*$, and a compact convex subset $K \subset X$ of a locally convex space X the set

$$H_f(K) = \left\{ z \in A \mid f(z) \;=\; \max_{y \in K} f(y) \right\}$$

is called the (maximal) *face* of K with respect to f.

For finite dimensional spaces Ch. Bauer characterized all reduced pairs of polytopes, namely: Let A and B be two polytopes in $\mathbb{R}^n$, then an edge (one-dimensional face) k of A and an edge l of B is called *equiparallel* if $k = H_f(A)$ and $l = H_f(B)$ for some linear functional $f \in (\mathbb{R}^n)'$. Then Ch. Bauer proved:

Theorem 2.2. *Let (A,B) be a pair of polytopes in $\mathbb{R}^n$. The pair (A,B) is reduced if and only if A and B have no equiparallel edges.*

Minimal pairs of nonempty bounded closed convex sets have many interesting properties which are studied in a series of papers [4, 5, 6, 7, 8, 12, 13, 14, 15, 19]. For instance in the 2-dimensional case, equivalent minimal pairs of compact convex sets are uniquely determined up to translations [5, 19], which is not longer true for the 3-dimensional case. A continuous family of equivalent minimal pairs of compact convex sets which are not related by a translation for different indices is given in [12]. J. Grzybowski and R. Urbański [8] showed under the assumption of the continuum hypothesis, that if there exist any two equivalent minimal pairs of compact convex sets which are not related by a translation, then there exists already a continuous family of equivalent minimal pairs which are also not related by translations.

3. The order cancellation law and its generalizations

The order cancellation law plays crucial role in nonsmooth analysis. First we state two variants of the order cancellation law: the standard one (olc) and nonstandard (olc'). In both cases the set B is nonempty bounded and C is convex. In the first case C has to be closed and in the latter one B has to be compact and finite dimensional. In this section we present some generalizations of nonclasic order cancellation law, we also show that the order cancellation law can hold true for some cases when neither B nor C is closed. We present certain properties when the order cancellation law cannot hold true.

Proposition 3.1. ([6, 15, 21]) *If A, B and C are subsets of Hausdorff topological vector space X, B is nonempty bounded and C is closed and convex, then*

$$A + B \subset B \dot{+} C \quad \Longrightarrow \quad A \subset C. \tag{ocl}$$

An excelent example of the strength of the last proposition is a short proof of the Riesz theorem [20].

Theorem 3.2. *Hausdorff topological vector space is locally compact if and only if it is finite dimensional.*

Proof. Let X be a locally compact topological vector space. Without loss of generality we can assume that there exists a totally bounded neighborhood U of origin. Let V be an open neighborhood of origin such that $V + V \subset U$. The family $\{x + V | x \in \overline{U}\}$ is an open cover of the compact set $\overline{U}$. Hence there exists a finite set F such that $\overline{U} \subset F + V$. Therefore,

$$\overline{U} + V \subset F + V + V \subset \mathrm{conv} F + \overline{U}.$$

Since conv$F$ is closed and convex and $\overline{U}$ is bounded, by law of cancellation we get $V \subset \mathrm{conv} F$. □

Proposition 3.3. ([6]) *If A, B and C are subsets of $\mathbb{R}^n$, B is nonempty compact and C is convex, then*

$$A + B \subset B + C \quad \Longrightarrow \quad A \subset C. \tag{ocl$'$}$$

As proved in [6], Proposition 3.1 is equivalent to the following equality

$$(B\dot{+}C)\dot{-}B = C \tag{ocl-eq}$$

for all closed convex $C \subset X$ and bounded $B \subset X$ and in a similar way, Proposition 3.3 is equivalent to the equality

$$(B + C)\dot{-}B = C \tag{ocl'-eq}$$

for all convex $C \subset \mathbb{R}^n$ and compact $B \subset \mathbb{R}^n$.

Corollary 3.4. *If A, B and C are subsets of a Hausdorff topological vector space X, B is nonempty finite dimensional and compact and C is convex, then*

$$A + B \subset B + C \implies A \subset C.$$

The corollary follows from Proposition 3.3 and from the fact that for each $x \in X$ we have $A \cap (x + Y) + B \subset B + C \cap (x + Y)$, where Y is a finite dimensional subspace of Y.

Although the next result follows from corollary 3.4, we give however, due to its importance, an elementary direct proof.

Proposition 3.5. (Algebraic order cancellation law) *If A, B and C are subsets of a vector space X, B is nonempty finite and C is convex, then*

$$A + B \subset B + C \implies A \subset C.$$

Proof. Let $a \in A$ and $(b_i) \subset B$ and $(c_i) \subset C$ be sequences satisfying equalities $a + b_i = b_{i+1} + c_i$, $i \in \{1, \dots, n\}$. There exist $1 \leqslant i < j \leqslant n + 1$ such that $b_i = b_j$. By adding i-th, $(i + 1)$-th, ..., $(j - 1)$-th equalities we obtain $(j - i)a + \sum_{k=i}^{j-1} b_k = \sum_{k=i+1}^{j} b_k + \sum_{k=i}^{j-1} c_k$. Then $a = \frac{1}{j-i} \sum_{k=i}^{j-1} c_k \in C$. □

Also the following proposition follows from the Corollary 3.4. Again, we give an elementary proof of it.

Proposition 3.6. *If A, B and C are subsets of a vector space X, B is a polytope and C is convex, then*

$$A + B \subset B + C \implies A \subset C.$$

Proof. Let $b_i, i = 1, \dots, s$ be all vertices of B and $a \in A$. For all i there exist $b'_i \in B$ and $c_i \in C$ such that $a + b_i = b'_i + c_i$, $i \in \{1, \dots, s\}$. If $0 \notin \operatorname{conv}_{i=1}^{s}(b_i - b'_i)$ then for some linear functional f on X we have $f(b_i - b'_i) < 0$ for all i. Notice that here continuity of f has no significance. There exists a vertex b_k such that $f(b_k) = \max f(B)$. Hence $f(b_k) \geqslant f(b'_k)$, and $f(b_k - b'_k) \geqslant 0$. This contradicts the assumption that $0 \notin \operatorname{conv}_{i=1}^{s}(b_i - b'_i) = \operatorname{conv}_{i=1}^{s}(c_i - a)$. Then 0 belongs to the convex hull of $\{c_i - a \,|\, i = 1, \dots, s\}$. Hence $0 \in C - a$. □

The following proposition shows that the order law of cancellation, which concerns a bounded set B and a convex set C can be true in certain cases when neither B is closed nor C. For example if $B = [0,1)$ and $C = (0,1]$ are intervals in $\mathbb{R}$ then $A + [0,1) \subset [0,1) + (0,1]$ implies $A \subset (0,1]$.

Proposition 3.7. *Let B and C be subsets of $\mathbb{R}^n$, B be bounded and C convex. If $H_f(B) = H_f(\overline{B})$ or $H_f(C) = H_f(\overline{C})$ for all non-zero linear functionals f on $\mathbb{R}^n$ then*

$$C + B \dot{-} B = C.$$

Proof. It is enough to prove that $B \subset B + C$ implies that $0 \in C$. Assume that $B \subset B + C$ and $0 \notin C$. We have $0 + B \subset B \dot{+} \overline{C}$. By Proposition 3.1 we obtain $0 \in \overline{C}$. There exists a non-zero functional f such that $0 \in H_f(\overline{C})$. We have $\sup f(B) \leqslant \sup f(B + C) = \sup f(B) + \sup f(C) = \sup f(B) + \sup f(\overline{C}) = \sup f(B) + 0$. Hence $H_f(B) \subset H_f(B) + H_f(C)$. Obviously, $H_f(C) \neq H_f(\overline{C})$. Then by assumption $H_f(B) = H_f(\overline{B})$. Since $H_f(B)$ is closed (and compact), by Proposition 3.3 we obtain $0 \in H_f(C) \subset C$. □

The following proposition shows that the order law of cancellation, which concerns a bounded set B and a convex set C cannot be true if B is open and C is not closed.

Proposition 3.8. *Let X be a normed space, B and C be subsets of X, where B is bounded and open and C is convex. Then*

$$C + B \dot{-} B = \overline{C}.$$

Proof. Let $x \in \overline{C}$. Take any $b \in B$. Since B is open, the set $b - B$ is an open neigborhood of the origin. Then $x + b - B$ is a neighborhood of x and for some $c \in C$, we have $c \in x + b - B$. Hence $x + b \in c + B$. Since b is any element of B, we obtain $x + B \subset B + C$. Therefore, $\overline{C} \subset C + B \dot{-} B$. By Proposition 3.1 we obtain $\overline{C} = C + B \dot{-} B$. □

Proposition 3.5 implies the next one.

Proposition 3.9. *If A, B and C are subsets a Hausdorff topological vector space X, B is nonempty compact and C is open convex, then*

$$A + B \subset B + C \implies A \subset C.$$

Proof. Let $a \in A$. The compact set $a + B$ is contained in a union of open sets $B + C = \bigcup_{b \in B}(b + C)$. Then there exists a finite set $B' \subset B$ such that $a + B \subset \bigcup_{b \in B'}(b + C) = B' + C$. Hence $a + B' \subset B' + C$, and by Proposition 3.5 we have $a \in C$. □

Corollary 3.10. *If A, B and C are subsets a Hausdorff topological vector space X, B is nonempty compact and C is convex, then for all open convex U*

$$A + B \subset B + C \implies A + U \subset C + U.$$

Proposition 3.11. *Let X be a Hausdorff topological vector space and $A, B, C \subset X$ be subsets such that B is compact and C is convex. Let there exists such a sequence $(f_n) \subset X^*$ that $\bigcap_{n=1}^{\infty} \ker f_n = \{0\}$. Denote $F_n(x) = (f_1(x), \ldots, f_n(x))$. Assume that for any $a \in X$ if $F_n^{-1}(F_n(a)) \cap C \neq \emptyset$ for all n then $a \in C$. Then the condition $A + B \subset B + C$ implies that $A \subset C$.*

Proof. Let $x \in A$. Then $x + B \subset B + C$, and $B \subset B + (C - x)$. Hence $F_n(B) \subset F_n(B) + F_n(C - x)$, for all n where $F_n(B)$ is compact and $F_n(C - x)$ is convex. Applying Proposition 3.3 we obtain $0 \in F_n(C - x)$. Hence $F_n(x) \in F_n(C)$. Then $F_n^{-1}(F_n(x)) \cap C \neq \emptyset$, and, by assumption, $x \in C$. □

Example 3.12. Let S be a vector space of all real sequences and let $X \subset S$ be a vector subspace of S. Suppose that there is given a linear topology $\tau \subset 2^X$ such that the functionals $f_n \colon X \to \mathbb{R}$, $f_n(x) = x_n$ are continuous. Let $C = \{x \in X \,|\, x_i < 0 \text{ for all } i\}$ be the negative ortant of X. Then for any $A, B \subset X$ if B is compact then the condition $A + B \subset C + B$ implies $A \subset C$. The set C need not to be open. For example it is not open if $X = l^1$.

4. Reduced pairs in commutative semigroups

The family $\mathcal{B}(X)$ of all nonempty bounded convex subsets of a topological vector space X endowed with the *operation* " $\dot{+}$ " given by: $A \dot{+} B = \mathrm{cl}\{x = a + b \mid a \in A \text{ and } b \in B\}$ and the ordering induced by the inclusion is an *ordered commutative semigroup which satisfies the order cancellation law.* Let us recall that according to Bauer and Schneider (see [1] and [18]) a pair $(A, B) \in \mathcal{B}(X)$ is reduced if and only if $A \dot{+} B$ is a summand of C whenever A and B are summands of C. We call such a pair s-*reduced.* For this pair the following set equality holds true:

$$(A \dot{+} \mathcal{B}(X)) \cap (B \dot{+} \mathcal{B}(X)) = A \dot{+} B \dot{+} \mathcal{B}(X).$$

We say that a pair (A, B) is q-*reduced* if and only if

$$[A, B] = [(A, B)] = \{(A \dot{+} C, B \dot{+} C) \mid C \in \mathcal{B}(X)\}, \quad \text{or}$$

$$[A, B] = (A, B) \dot{+} \mathcal{B}(X).$$

In this case s-reducibility and q-reducibility are equivalent [15] thanks to the order law of cancellation. However, in the following we study pairs of sets such that one set can be unbounded i.e. it belongs to the family $\mathcal{C}(X)$ of all nonempty closed convex subsets of X.

This leads us to a general consideration of reduced pairs in commutative semigroups. Therefore we define:

Let $S = (S, +)$ be a commutative semigroup. Assume that $T \subset S$ is a subsemigroup of S. We say that S satisfies the *cancellation law with respect to T* if for all $b \in T, a, c \in S$ we have

$$a + b = c + b \ \text{ implies } \ a = c. \tag{lc$T$}$$

Notice that if S satisfies the cancellation law with respect to finite subsemigroup T then T is a subgroup.

The pair $(s, s') \in S \times S$ is called s-*reduced* if $s+s'$ is a summand of r whenever s and s' are summands of r $((s+S) \cap (s'+S) = s+s'+S)$. The relation $(s_1, s_2) \sim (s_1', s_2') \Longleftrightarrow s_1 + s_2' = s_1' + s_2$ is a relation of equivalence in $(S \times T) \cup (T \times S)$ [6]. The pair (s, s') is called q-*reduced* if and only if for a quotient class of $(s, s') \in (S \times T) \cup (T \times S)$ we have $[s, s'] = \{(s + v, s' + v) \mid v \in T\}$ or $[s, s'] = (s, s') + T$.

The next example shows that s-reducibility and q-reducibility are independent. If $S = T$ then both properties coincide.

Example 4.1. (a) Let $S = \mathcal{C}(\mathbb{R}^2), T = \mathcal{B}(\mathbb{R}^2), s = \{0\} \times [0, \infty) \in S, t = [0, 1] \times \{0\} \in T, t' = \{(x_1, x_2) \in \mathbb{R}^2 | \, x_1 \geqslant 0, x_2 \geqslant 0, x_1 + x_2 \leqslant 1\} \in T$. The pairs (s, t) and (s, t') are equivalent. Moreover both pairs are s-reduced. However, since for no $t'' \in T$ the equality $t' = t + t''$ holds true, the pair (s, t) is not q-reduced. Neither (s, t') is q-reduced.

(b) Let $\mathbb{Z}[i]$ be the ring of Gaussian integers $S = \mathbb{Z}[i] \setminus \{0\}$ and $T = \mathbb{Z} \setminus \{0\}$. The set S is a multiplicative semigroup and T is its subsemigroup. The pair $(1 - i, 2)$ is q-reduced but it is not s-reduced because both $1 - b$ and 2 divide 2, which is not divisible by $2(1 - i)$.

In order to prove that q-reduced pair (A, B) with unbounded A has a singleton set B we need three following facts.

Proposition 4.2. *Let X be a Hausdorff topological vector space $A \subset X$ be closed and $B \subset X$ be compact. Then the skeleton $\bigcup_{t \in [0,1]} (tB + (1 - t)A)$ is a closed subset of X.*

Proof. Let x be a limit of a net $x_\alpha = t_\alpha b_\alpha + (1 - t_\alpha) a_\alpha$, where $t_\alpha \in [0, 1]$, $a_\alpha \in A$ and $b_\alpha \in B$. Since the interval $[0, 1]$ and the set B are compact there exists a subnet x_β such that $\lim t_\beta = t_0$ and $\lim b_\beta = b_0 \in B$. If $t_0 = 1$ then $x = b_0 \in B$. If $t_0 < 1$ then the net $a_\beta = \frac{x_\beta - t_\beta b_\beta}{1 - t_\beta}$ is convergent to some element a_0 of A. Then $x = t_0 b_0 + (1 - t_0) a_0 \in \bigcup_{t \in [0,1]} (tB + (1 - t)A)$. □

Corollary 4.3. *Let X be a Hausdorff topological vector space $A, B \in \mathcal{B}(X)$ and B be compact. Then the convex hull $\operatorname{conv}(A \cup B) = \bigcup_{t \in [0,1]} (tB + (1 - t)A)$ is closed and an element of $\mathcal{B}(X)$.*

Proof. The corollary follows from Proposition 4.2 and from the fact that $\operatorname{conv}(A \cup B) = \bigcup_{t \in [0,1]} (tB + (1 - t)A)$. □

The next proposition follows from Theorem 4.5.11 in [15], which is also quoted in this paper latter as Theorem 5.2.

Proposition 4.4. *Let X be a Hausdorff topological vector space $A, B \in \mathcal{B}(X)$. Then the union $(A \dot{+} \overline{\mathrm{conv}}(A \cup B)) \cup (B \dot{+} \overline{\mathrm{conv}}(A \cup B))$ is convex and*

$$(A \dot{+} \overline{\mathrm{conv}}(A \cup B)) \cap (B \dot{+} \overline{\mathrm{conv}}(A \cup B)) = A \dot{+} B.$$

Theorem 4.5. ([9]) *Let $A, B, A \cup B \in \mathcal{B}(X)$, $A \not\subset B, B \not\subset A$ and $A \cap B$ is a summand of $A \cup B$. Then $A = A \cap B + [a, 0]$, $B = A \cap B + [0, b]$ for a some $a, b \in X$ such that 0 belongs to the relative interior of the segment $[a, b]$.*

Proposition 4.6. *Let $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ be a* q*-reduced pair with unbounded set A. If A contains a ray then B is a singleton.*

Proof. Let $a + \mathbb{R}_+ \cdot v$ be a ray contained in A and b be a fixed point in B. Denote $B_n = \mathrm{conv}(B \cup \{b + nv\})$. For sufficiently large n the set B is a proper subset of B_n. By Proposition 4.2 we have $B_n \in \mathcal{B}(X)$. Moreover $A + B_n = \mathrm{conv}((A+B) \cup (A+b+nv)) = \mathrm{conv}((A+B) \cup ((A+nv)+b)) \subset A+B$. Then $A \dot{+} B_n = A \dot{+} B$. Since (A, B) is q-reduced, there exists $C \in \mathcal{B}(X)$ such that $A = A \dot{+} C, B_n = B \dot{+} C$.

By Proposition 4.4 a pair $(B \dot{+} B_n, B_n + b + nv)$ is convex and $(B \dot{+} B_n) \cap (B_n + b + nv) = B + b + nv$. Moreover, $B + b + nv$ is a summand of $(B \dot{+} B_n) \cup (B_n + b + nv) = 2B_n = 2B \dot{+} 2C$. Hence, applying Theorem 4.5, we obtain $B \dot{+} B_n = B + b + nv + [c, 0]$ and $B_n + b + nv = B + b + nv + [0, d]$ where $0 \in \mathrm{relint}[c, d]$. Cancelling B we have $B_n = b + nv + [c, 0]$. Therefore, the set B_n is a segment containing a segment $[b, b + nv]$. Hence $B \subset B_n$ is a singleton or a closed segment parallel to the vector v. In the latter case $B = [b_0, b_0 + tv]$, where $t > 0$. Observe that $(A, \{b_0\}) \in [A, B]$. Hence $B = \{b_0\}$. $\square$

Lemma 4.7. *Let $s \in S, t_1, t_2 \in T$. The pair $(s, t_1 + t_2)$ is* q*-reduced if and only if the pairs $(s, t_i), i = 1, 2$ are* q*-reduced.*

Proof. Let the pair $(s, t_1 + t_2)$ be q-reduced and let $(s, t_1) \sim (r, u)$ for a some $r \in S, v \in T$. Then $s + u = r + t_1$, and we have $t_1 + t_2 + r = s + t_2 + u$. Hence $(s, t_1 + t_2) \sim (r, t_2 + u)$, and we obtain $r = s + v$ and $t_2 + u = t_1 + t_2 + v$ for a some $v \in T$. Now by the law of cancelation we have $r = s + v$ and $u = t_1 + v$. Hence the pair (s, t_1) is q-reduced.

Now, assume that the pairs $(s, t_i), i = 1, 2$ are q-reduced. Let $(s, t_1 + t_2) \sim (r, u)$. Then $s + u = t_1 + t_2 + r$. Hence $(s, t_1) \sim (r + t_2, u)$, and we have $u = t_1 + v$ and $r + t_2 = s + v$ for some $v \in T$. Then $(s, t_2) \sim (r, v)$. Therefore, $r = s + w$ and $v = t_2 + w$ for a some $w \in T$. Hence $u = t_1 + t_2 + w$ and $r + t_2 = s + t_2 + w$. Now, by the law of cancelation we have $r = s + w$ and $u = t_1 + t_2 + w$. Hence the pair $(s, t_1 + t_2)$ is q-reduced. $\square$

Proposition 4.8. *Let (X, τ) be a Hausdorff topological vector space. For every pair $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ the following properties are equivalent:*
(a) (A, B) *is* q*-reduced.*
(b) (B, A) *is* q*-reduced.*

(c) *For every* $a, b \in X$ *the pair* $(A + a, B + b)$ *is* q-*reduced.*
(d) *For every* $\alpha \in \mathbb{R}$ *the pair* $(\alpha A, \alpha B)$ *is* q-*reduced.*
(e) *For every* $\alpha, \beta \in \mathbb{R}$, $\alpha\beta \geqslant 0$ *the pair* $(\alpha A, \beta B)$ *is* q-*reduced.*

Proof. (a) $\Leftrightarrow$ (b): This follows immediately from the definition.

(a) $\Leftrightarrow$ (c): This follows from the equality $(A, B) = (A + a - a, B + b - b)$ and Lemma 4.7.

(a) $\Leftrightarrow$ (d): Let $\alpha \in \mathbb{R}, \alpha \neq 0$. We observe that $\mathcal{B}(X) = \{\alpha C | C \in \mathcal{B}(X)\}$.

(d) $\Leftrightarrow$ (e): Let $\alpha, \beta \in \mathbb{R}, \alpha\beta \geqslant 0$. If $\alpha = 0$ then a pair $(\alpha A, \beta B)$ is q-reduced. If $\alpha \neq 0$ then for some natural number n we have $n|\alpha| \geqslant |\beta|$. By Lemma 4.7 a pair $(\alpha A, n\alpha B)$ is q-reduced. Applying Lemma 4.7 again we obtain that pairs $(\alpha A, \beta B)$ and $(\alpha A, (n\alpha - \beta)B)$ are q-reduced. $\square$

For s-reducibility the last proposition holds for (a)-(d). From Lemma 4.7 it also follows:

Proposition 4.9. *Let* $(A, B_i) \in \mathcal{C}(X) \times \mathcal{B}(X)$, *where* $i = 1, \ldots, k$. *Then the pairs* (A, B_i) *are* q-*reduced for all* i *if and only if the pair* $(A, \sum_{i=1}^{k} B_i)$ *is* q-*reduced.*

The following proposition is a corollary from the previous one.

Proposition 4.10. *Let* $A_i, i = 1, \ldots, k$ *and* $B_j, j = 1, \ldots, l$ *be convex polyhedra. Then the pairs* (A_i, B_j) *are* q-*reduced for all* i *and* j *if and only if the pair* $(\sum_{i=1}^{k} A_i, \sum_{j=1}^{l} B_j)$ *is* q-*reduced.*

Proposition 4.10 also can be proved from Bauer's criterion [1] for reduced pairs of polytopes.

5. Reduced convex and minimally convex pairs

Now we will come back to the ordered semigroup $(\mathcal{C}(X), \dot{+}, \subset)$ of all nonempty closed convex subsets of a topological vector space X, endowed with the Minkowski sum and orderd by inclusion and define a new class of convex pairs.

A pair $(A, B) \in \mathcal{C}^2(X)$ is called a *convex pair* if the set $A \cup B$ is a convex. A convex pair $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ is called *minimal convex* if for any equivalent convex pair $(C, D) \in \mathcal{C}(X) \times \mathcal{B}(X)$ the relation $(C, D) \leqslant (A, B)$ implies $(C, D) = (A, B)$. The pair $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ is called *reduced convex* if for every convex pair $(C, D) \sim (A, B)$ there exist $K \in \mathcal{B}(X)$ such that $C = A \dot{+} K$ and $D = B \dot{+} K$.

Theorem 5.1. [6] *Let* X *be a Hausdorff topological vector space and* $A, B \in \mathcal{C}(X)$. *Then the following statements are equivalent:*
(a) *The set* $A \cup B$ *is convex.*
(b) *The set* $A \cap B$ *separates the sets* A *and* B.

(c) $C \dot{+} A \cap B = (A \dot{+} C) \cap (B \dot{+} C)$ *for every segment* C *in* X.
(d) $C \dot{+} A \cap B = (A \dot{+} C) \cap (B \dot{+} C)$ *for every convex set* C.
(e) $C + A \cap B = (A + C) \cap (B + C)$ *for every convex set* C.

For bounded convex sets, the following theorem ([15], Theorem 4.5.11) holds true.

Theorem 5.2. *Let* X *be a Hausdorff topological vector space and* $A, B \in \mathcal{B}(X)$. *Then the following statements are equivalent:*
(a) *The set* $A \cup B$ *is convex.*
(b) *The set* $\overline{\mathrm{conv}}(A \cup B)$ *is a summand of the set* $A \dot{+} B$.
(c) $A \dot{+} B = \overline{\mathrm{conv}}(A \cup B) \dot{+} A \cap B$.

Theorem 5.3. *Let* X *be a Hausdorff topological vector space,* $A, B, F \in \mathcal{C}(X)$, $A \cap B \in \mathcal{B}(X)$, *and* $A \cup B$ *be convex. Then the inclusion*

$$A \cap B \subset (A \dot{+} F) \cup (B \dot{+} F) \ \text{ implies } \ A \cap B \subset A \dot{+} F \ \text{ or } \ A \cap B \subset B \dot{+} F.$$

Proof. Suppose that $A \cap B \not\subset A \dot{+} F$ and $A \cap B \not\subset B \dot{+} F$. Then there exist $p_0, q_0 \in A \cap B$ such that

$$p_0 \in (A \dot{+} F) \setminus (B \dot{+} F) \ \text{ and } \ q_0 \in (B \dot{+} F) \setminus (A \dot{+} F).$$

Let

$$[p_1, q_1] = [p_0, q_0] \cap (A \dot{+} F) \cap (B \dot{+} F),$$

where p_0 is closer to p_1 than to q_1 in the segment $[p_0, q_0]$. Note that this intersection is not empty, since $[p_0, q_0] \subset A \cap B \subset (A \dot{+} F) \cup (B \dot{+} F)$. By Theorem 5.1 (d)

$$(A \dot{+} F) \cap (B \dot{+} F) = A \cap B \dot{+} F \subset (A \dot{+} 2F) \cup (B \dot{+} 2F).$$

Now, suppose that $p_1 \in B \dot{+} 2F$, then

$$r_1 = \frac{p_0 + p_1}{2} \in \frac{A \cap B \dot{+} B \dot{+} 2F}{2} \subset B \dot{+} F.$$

On the other hand

$$r_1 \in \frac{A \dot{+} F \dot{+} A \dot{+} F}{2} = A \dot{+} F.$$

From this it follows that $r_1 \in (A \dot{+} F) \cap (B \dot{+} F)$. This contradicts the definition of p_1. Therefore $p_1 \in (A \dot{+} 2F) \setminus (B \dot{+} 2F)$. Similarly $q_1 \in (B \dot{+} 2F) \setminus (A \dot{+} 2F)$. In the same time $[p_1, q_1] \subset (A \dot{+} F) \cap (B \dot{+} F) = A \cap B \dot{+} F$.

This way we construct a sequence $[p_n, q_n] \subset A \cap B \dot{+} nF$ for every $n \in \mathbb{N}$. Since $\frac{1}{n} q_n \in \frac{1}{n}(A \cap B) \dot{+} F$ and the set $A \cap B$ is bounded, we get $0 \in F$. This contradicts the assumption. $\square$

Corollary 5.4. *Let X be a Hausdorff topological vector space, $A, B, F \in \mathcal{C}(X), A \cap B \in \mathcal{B}(X)$, and let $A \cup B$ be a convex set. Then*

$$(A \cup B \dotplus F) \dot{-} A \cap B = [(A \dotplus F) \dot{-} A \cap B] \cup [(B \dotplus F) \dot{-} A \cap B].$$

Lemma 5.5. *Let X be a Hausdorff topological vector space, $A, B, F \in \mathcal{C}(X)$, $A \cap B \in \mathcal{B}(X)$, $A \cup B$ be a convex set, and let $A_0 = (A \dotplus F) \dot{-} A \cap B$, $B_0 = (B \dotplus F) \dot{-} A \cap B$. Then* (i) *$A_0 \cap B_0 = F$,* (ii) *$A_0 \cup B_0 \in \mathcal{C}(X)$,* (iii) *$(A_0 \dotplus B) \cup (B_0 \dotplus A) = F \dotplus A \cup B$,* (iv) *$(A_0 \dotplus B) \cap (B_0 \dotplus A) \dotplus F = F \dotplus A_0 \cup B_0 \dotplus A \cap B$ and* (v) *$A_0 \cup B_0 \dotplus A \cap B \subset A \cup B \dotplus F$.*

Proof. Denote $A_0 = (A \dotplus F) \dot{-} A \cap B$, $B_0 = (B \dotplus F) \dot{-} A \cap B$. Then

$$\begin{aligned} A_0 \cap B_0 &= ((A \dotplus F) \dot{-} A \cap B) \cap ((B \dotplus F) \dot{-} A \cap B) \\ &= (A \dotplus F) \cap (B \dotplus F) \dot{-} A \cap B \\ &= (A \cap B \dotplus F) \dot{-} A \cap B = F. \end{aligned}$$

Now we show that F separates the sets A_0 and B_0. Let $x \in A_0$ and $y \in B_0$. The by the definition $x + A \cap B \subset A \dotplus F$ and $y + A \cap B \subset B \dotplus F$. Hence $A \cap B \subset (A \dotplus F + [-x, -y]) \cap (B \dotplus F + [-x, -y]) = A \cap B \dotplus F + [-x, -y]$. Now from the order cancellation law we have $0 \in F + [-x, -y]$. Hence $[x, y] \cap F \neq \emptyset$ and $A_0 \cup B_0$ is a convex set.

By the definition of A_0 and B_0, we get

$$\begin{aligned} A_0 &\subset (A \dotplus B \dotplus F) \dot{-} (A \cap B \dotplus B) = (A \cup B \dotplus F) \dot{-} B, \\ B_0 &\subset (A \dotplus B \dotplus F) \dot{-} (A \cap B \dotplus A) = (A \cup B \dotplus F) \dot{-} A. \end{aligned}$$

Therefore

$$A \cup B \dotplus F \subset (A_0 \dotplus B) \cup (B_0 \dotplus A) \quad \subset \quad A \cup B \dotplus F.$$

Hence

$$(A_0 \dotplus B) \cup (B_0 \dotplus A) = A \cup B \dotplus F.$$

This yields

$$\begin{aligned} F \dotplus (A_0 \dotplus B) \cap (B_0 \dotplus A) &\subset A_0 \cap B_0 \dotplus A_0 \cup B_0 \dotplus A \cap B \\ &= A_0 \dotplus B_0 \dotplus A \cap B \\ &\subset (A_0 \dotplus B \dotplus F) \cap (B_0 \dotplus A \dotplus F). \end{aligned}$$

Now we observe

$$\begin{aligned} A_0 \cup B_0 \dotplus A \cap B &= (A_0 \dotplus A \cap B) \cup (B_0 \dotplus A \cap B) \\ &\subset (A \dotplus F) \cup (B \dotplus F) = A \cup B \dotplus F. \end{aligned}$$

□

Example 5.6. Let $X = \mathbb{R}^2$ and $A = (-1,1) \vee (0,2) \vee (1,1) \vee (1,-1) \vee (0,-2) \vee (-1,-1)$, $B = (1,1) \vee (2,0) \vee (1,-1) \vee (-1,-1) \vee (-2,0) \vee (-1,1)$. Given $A_0 = (A \dotplus F) \dot{-} A \cap B = \{0\}$, $B_0 = (B \dotplus F) \dot{-} A \cap B = \{0\}$ and $F = \{0\}$. Then $A \cap B = A_0 \cup B_0 \dotplus A \cap B \subset A \cup B = A \cup B \dotplus F$. Hence in Lemma 5.5 the inclusion $A_0 \cup B_0 \dotplus A \cap B \subset A \cup B \dotplus F$ is essential.

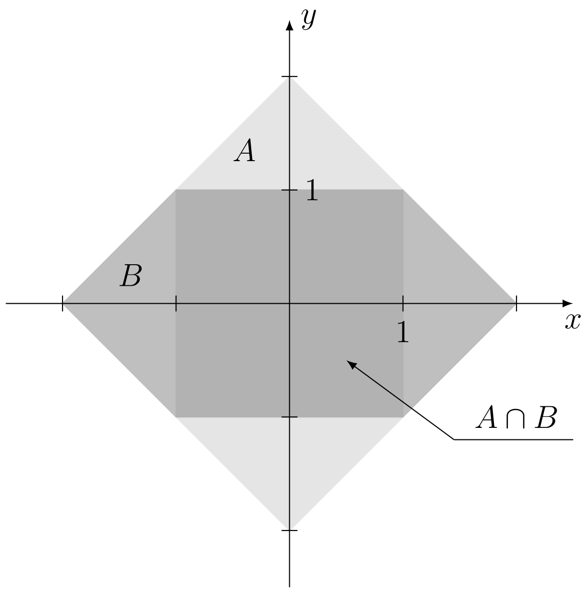

Figure 5.6

Theorem 5.7. *Let X be a Hausdorff topological vector space, $(A,B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ be a convex pairs and let $(G, F) \sim (A \cup B, A \cap B)$ for $(G, F) \in \mathcal{C}(X) \times \mathcal{B}(X)$. Then there exists a convex pair (A_0, B_0) such that $A_0 \cap B_0 = F$, $A_0 \cup B_0 = G$ and $(A_0, B_0) \sim (A, B)$. Moreover, $A_0 = (A \dotplus F) \dot{-} A \cap B$, $B_0 = (B \dotplus F) \dot{-} A \cap B)$.*

Proof. Denote $A_0 = (A \dotplus F) \dot{-} A \cap B$, $B_0 = (B \dotplus F) \dot{-} A \cap B$. Since $F \dotplus A \cup B = A_0 \cup B_0 \dotplus A \cap B$, from Corollary 5.4 and Lemma 5.5 we have

$$F = A_0 \cap B_0 \text{ and } G = A_0 \cup B_0 = (A \cup B \dotplus F) \dot{-} A \cap B.$$

Now from Lemma 5.5 we obtain

$$(A_0 \dotplus B) \cap (B_0 \dotplus A) = (A_0 \dotplus B) \cup (B_0 \dotplus A).$$

Hence $(A, B) \sim (A_0, B_0)$. $\square$

Lemma 5.8. *Let $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ and $A \cup B$ be a convex set. Then:*
(a) *If the pair (A, B) is* q*-reduced then the pair $(A \cup B, A \cap B)$ is* q*-reduced.*
(b) *If the pair $(A \cup B, A \cap B)$ is* q*-reduced then the pair (A, B) is reduced convex.*

Proof. (a) Let (A, B) be a q-reduced pair and let $(A \cup B, A \cap B) \sim (G, F)$. Then from Theorem 5.7 there exists $(A_0, B_0) \sim (A, B)$ such that $A_0 \cap B_0 = F$

and $A_0 \cup B_0 = G$. Hence $A_0 = A \dot{+} C, B_0 = B \dot{+} C$ for a some $C \in \mathcal{B}(X)$. This implies $F = A_0 \cap B_0 = (A \dot{+} C) \cap (B \dot{+} C) = A \cap B \dot{+} C$ and $G = A_0 \cup B_0 = (A \dot{+} C) \cup (B \dot{+} C) = A \cup B \dot{+} C$. Hence the pair $(A \cup B, A \cap B)$ is q-reduced.

(b) Now, assume that the pair $(A \cup B, A \cap B)$ is q-reduced. Let $(A, B) \sim (C, D)$, where $C \cup D$ is convex. Then $(A \cup B, A \cap B) \sim (C \cup D, C \cap D)$. Hence there exists $K \in \mathcal{B}(X)$ such that $C \cap D = A \cap B \dot{+} K$ and $C \cup D = A \cup B \dot{+} K$. But $A \cap B \dot{+} K \dot{+} A = C \cap D \dot{+} A = A \cap B \dot{+} C$ and $A \cap B \dot{+} K \dot{+} B = C \cap D \dot{+} B = A \cap B \dot{+} D$. Hence by the law of cancellation we have $C = A \dot{+} K$ and $D = B \dot{+} K$. $\square$

Example 5.9. Let $X = \mathbb{R}^2$ and $A_0 = (-1, 0) \vee (1, 0)$, $B_0 = (0, -1) \vee (0, 1)$, $C = A_0 \vee B_0$. Given $A = A_0 + C$ and $B_0 + C$. Then the pair $(A \cap B, A \cup B)$ is q-reduced. Hence from Lemma 5.8(b) the pair (A, B) is reduced convex. Now we observe that the pair (A_0, B_0) is q-reduced but not reduced convex.

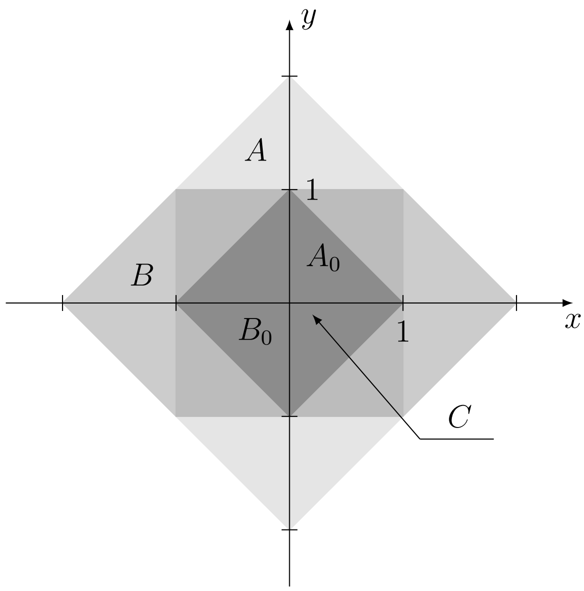

Figure 5.9

Theorem 5.10. *Let X be a Hausdorff topological vector space. A convex pair $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ is minimal convex if and only if $(A \cup B, A \cap B)$ is minimal.*

Proof. *Necessity*: Given any pair $(G, F) \sim (A \cup B, A \cap B)$ with $F \subset A \cap B$. By Theorem 5.7 for $A_0 = (A \dot{+} F) \dot{-} A \cap B$, $B_0 = (B \dot{+} F) \dot{-} A \cap B$ we have $(A_0, B_0) \sim (A, B), A_0 \cap B_0 = F$ and $A_0 \cup B_0 = G$. Thus $A_0 = (A \dot{+} F) \dot{-} A \cap B \subset (A \dot{+} A \cap B) \dot{-} A \cap B = A$ analogously $B_0 \subset B$. Hence $A_0 = A, B_0 = B$ and we have $G = A \cup B$ and $F = A \cap B$.

Sufficiency: Let (A', B') be a convex pair equivalent to (A, B) such that $A' \subset A$ and $B' \subset B$. Obviously, $A \cup B \dot{+} B' = (A \dot{+} B') \cup (B \dot{+} B') = (A' \dot{+} B) \cup (B \dot{+} B') = B \dot{+} A' \cup B'$. By convexity of pairs of sets $B \dot{+} A' \cap B' =$

$(A' \dot{+} B) \cap (B \dot{+} B') = (A \dot{+} B') \cap (B \dot{+} B') = A \cap B \dot{+} B'$. Adding obtained equalities and cancelling $B \dot{+} B'$ we obtain $(A \cup B, A \cap B) \sim (A' \cup B', A' \cap B')$. Since $(A \cup B, A \cap B)$ is minimal, $A' \cup B' = A \cup B, A' \cap B' = A \cap B$. Hence $B = B'$ and $A = A'$. $\square$

Let us define

$$\begin{aligned} \mathcal{M} &= \{(A,B) \in \mathcal{C}(X) \times \mathcal{B}(X) |\ (A,B) \text{ is minimal}\}, \\ \mathcal{NM} &= \mathcal{C}(X) \times \mathcal{B}(X) \setminus \mathcal{M}, \\ \mathcal{R} &= \{(A,B) \in \mathcal{C}(X) \times \mathcal{B}(X) |\ (A,B) \text{ is q-reduced}\}, \\ \mathcal{NR} &= \mathcal{C}(X) \times \mathcal{B}(X) \setminus \mathcal{R}. \end{aligned}$$

The following theorem shows that amounts of minimal, nonminimal, reduced and nonreduced pairs of sets are equal.

Theorem 5.11. *Let X be a nontrivial topological vector space. Then*

$$\operatorname{card} \mathcal{M} = \operatorname{card} \mathcal{NM} = \operatorname{card} \mathcal{R} = \operatorname{card} \mathcal{NR}.$$

Proof. Let $C \in \mathcal{B}(X)$ be such that $\operatorname{card} C > 1$. Then define the mapping $T \colon \mathcal{M} \to \mathcal{C}(X) \times \mathcal{B}(X)$ with $T(A,B) = (A \dot{+} C, B \dot{+} C)$. Since for every $(A,B) \in \mathcal{M}$ the pair $T(A,B)$ is not minimal, $T(\mathcal{M}) \subset \mathcal{NM}$ and hence $T \colon \mathcal{M} \to \mathcal{NM}$. Moreover, the mapping T is injective. Therefore $\operatorname{card} \mathcal{M} \leqslant \operatorname{card} \mathcal{NM}$. On the other hand, for every $A \in \mathcal{C}(X)$ the pair $(A, \{0\})$ is minimal. Hence $\operatorname{card} \mathcal{M} \geqslant \operatorname{card} \mathcal{C}(X) = \operatorname{card} \mathcal{C}^2(X)$ since for every infinite set S the equality $\operatorname{card} \mathrm{S} = \operatorname{card} \mathrm{S}^2$ holds true. Hence

$$\operatorname{card} \mathcal{C}^2(X) = \operatorname{card} \mathcal{C}(X) \leqslant \operatorname{card}\ \mathcal{M} \leqslant \operatorname{card} \mathcal{NM} \leqslant \operatorname{card} \mathcal{C}^2(X),$$

and we have $\operatorname{card} \mathcal{M} = \operatorname{card} \mathcal{NM} = \operatorname{card} \mathcal{C}(X)$.

Now we observe that $\mathcal{R} \subset \mathcal{M}$. Moreover for every $A \in \mathcal{C}(X)$ the pair $(A, \{0\})$ is q-reduced. Hence $\operatorname{card} \mathcal{C}(X) \leqslant \operatorname{card} \mathcal{R} \leqslant \operatorname{card} \mathcal{M} = \operatorname{card} \mathcal{C}(X)$. $\square$

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