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Conic James' Compactness Theorem

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The following result is proved:

Let A be a convex bounded non weakly relatively compact subset of a Banach space E. We fix a convex weakly compact subset D of E which does not contain the origin.

Then there is a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} and $g_0^* \in \mathrm{co}_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell_\infty(A)$ satisfying that for all $a \in A$,

$$\liminf_{n\geq 1} x_n^*(a) \leq h(a) \leq \limsup_{n\geq 1} x_n^*(a),$$

we have that $g_0^* - h$ does not attain its supremum on A and $(g_0^* - h)(d) > 0$ for every $d \in D$.

Keywords: James' compactness theorem, weakly compact set.

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1. Introduction

The well-known James' theorem [4] claims that *a bounded closed convex set C in a Banach space E is weakly compact if and only if every $x^* \in E^*$ attains its supremum on C.*

The aim of this paper is to complete the answer given in [1] to the following question raised by Delbaen:

Question 1.1. *Let E be a Banach space and A be a bounded, convex and closed subset of E with $0 \notin A$. Let us assume that for every $x^* \in E^*$ with*

$$\inf x^*(A) > 0,$$

the infimum of x^* on A is attained. Is the set A weakly compact?

For Banach spaces with a *weak** convex-block compact dual unit ball we gave a positive answer in [1] that was called One-Sided James' Theorem. Thus in

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particular for Banach spaces with *weak** sequentially compact dual unit ball. For arbitrary Banach spaces W. Moors has recently given a positive answer with an argument that reduces the problem to study the compactness of the level sets of an associated proper function, [6, 7]. In [1] an extension of James compactness result was presented for Bananch spaces with *weak** sequentially compact dual unit ball, namely when the hypothesis above $\inf x^*(A) > 0$ is replaced by requiring that $x^*(D) > 0$ for some fixed weakly compact set of directions D. The aim of the present paper is to give a proof of this last result for arbitrary Banach spaces, solving the question posed in (Question 6.(ii) of [1]). It reads as follows:

Theorem 1.2. *Let E be a Banach space and D be a weakly compact subset of E with $0 \notin D$. If A is a bounded subset of E such that every $x^* \in E^*$ with $x^*(D) > 0$ attains its supremum on A, then A is weakly relatively compact.*

Our Theorem 1.2 improves the fundamental James Compactness' Theorem [4] with new constraints on the linear forms involved inside it. Our motivation was Question 1 asked to me by F. Delbaen for the Banach space L^1. We have found a positive solution in [1]. In the course of our answer a *related statement*, which contains results obtained some years ago by the author to answer a question of W. Schachermayer (see Theorem A.1 of [5] and Remark 2.4 of Section 2) was inside the basis of our approach. For a description of matters look to [8, 9, 2] where applications to convex analysis and mathematical finance are obtained. Nevertheless no proof of our *related statement* for arbitrary Banach spaces was known. A variational form of James' Compactness Theorem for arbitrary Banach spaces was obtained in [8, 11] and W.Moors has recently described it with a different reduction argument to the classical James compactness theorem, see [6]. In the present paper we present a proof of the *related statement*, our Theorem 1.2, in full generality and we very much hope more applications in convex analysis and financial mathematics will come soon. An up to date account on James' compactness theorem can be found in [2].

Acknowledgement: I am very grateful for fruitful discussions with R. Haydon on the separable case of Section 2. In the very beginning we thought to have a counterexample for Delbaen's question in every non reflexive Banach space, so we began to look for *related positive results* as the one I finally present here.

1.1. Notation and terminology

Most of our notation and terminology are standard and can be found in our standard references for Banach spaces [3].

Unless otherwise stated, E will denote a Banach space with the norm $\|\cdot\|$. Given a subset S of a vector space, we write $\mathrm{co}\,(S)$, to denote its *convex hull.* If $(E, \|\cdot\|)$ is a normed space then E^* denotes its *topological dual.* If S is a subset of E^*, then $\sigma(E, S)$ denotes the *topology of pointwise convergence* on S.

Dually, if S is a subset of E, then $\sigma(E^*, S)$ is the topology for E^* of pointwise convergence on S. In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (ω) and weak* (ω^*) topologies respectively.

Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$ for the evaluation of x^* at x. If $x \in E$ and $\delta > 0$ we denote by $B(x,\delta)$ (or $B[x,\delta]$) the open (resp. closed) *ball centered at x of radius δ*. To simplify, we will simply write $B_E := B[0,1]$; and the *unit sphere* $\{x \in E\colon \|x\| = 1\}$ will be denoted by S_E. An element $x^* \in E^*$ is *norm-attaining* if there is $x \in B_E$ with $x^*(x) = \|x^*\|$. The set of norm-attaining functionals of E is normally denoted by $NA(E)$.

If (x_n) is a bounded sequence in the Banach space E we denote by

$$\mathrm{co}_\sigma = \{\sum_{n=1}^{\infty} \xi_n x_n : \xi_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \xi_n = 1\}$$

the *σ-convex hull* of the sequence (x_n)

2. The separable case

In this section we present a proof of our main result for the separable but unbounded case. Of course it follows from our results in [1]. Nevertheless it should be of interest for deeply understanding of matters. We are going to apply the following result, see Theorem 3 in [9] or Corollary 10.6 in [2]

Theorem 2.1. (Inf-liminf theorem in $\mathbb{R}^X$) *Assume that X is a nonempty set, $\{f_n\}_{n\geq 1}$ is a pointwise bounded sequence in $\mathbb{R}^X$ and Y is a subset of X with the property*

$$\textit{for every } g \in \mathrm{co}_{\sigma_p}\{f_n\colon n \geq 1\} \textit{ there exists } y \in Y \textit{ with } g(y) = \inf_X(g),$$

where

$$\mathrm{co}_{\sigma_p}\{f_n\colon\ n \geq 1\} := \left\{\sum_{n=1}^{\infty} \lambda_n f_n :\ \textit{for all } n \geq 1,\ \lambda_n \geq 0 \textit{ and } \sum_{n=1}^{\infty} \lambda_n = 1\right\},$$

and the functions $\sum_{n=1}^{\infty} \lambda_n f_n \in \mathbb{R}^X$ above are pointwise defined on X, i.e. for every $x \in X$ the absolutely convergent series

$$\sum_{n=1}^{\infty} \lambda_n f_n(x)$$

defines the function $\sum_{n=1}^{\infty} \lambda_n f_n : X \to \mathbb{R}$. Then

$$\inf_X \left(\liminf_n f_n\right) = \inf_Y \left(\liminf_n f_n\right).$$

Lemma 2.2. *Let E be a separable Banach space and $A \subset E$ a closed and convex subset. Let x_0^{**} in $E^{**} \setminus E$ be a weak*-cluster point of A and $D \subset E$ such that $0 \notin \overline{\mathrm{co}(A \cup D)}^{\|\cdot\|}$. Then there is a weak*-convergent sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} and $0 < \alpha < \beta$ such that*

$$|\langle x_n^*, x_0^{**}\rangle| < \alpha \tag{1}$$

whenever $n \geq 1$, and

$$\lim_n \langle x_n^*, x\rangle > \beta > \alpha \tag{2}$$

for any $x \in \overline{\mathrm{co}(A \cup D)}^{\|\cdot\|}$.

Proof. The compact segment line $[-x_0^{**}, +x_0^{**}]$ can be strictly separated from the closed convex set $C := \overline{\mathrm{co}(A \cup D)}$ in E^{**} by Hahn-Banach Theorem. So there exists a continuous linear functional $x^{***} \in B_{E^{***}}$ satisfying

$$x^{***}(C) > \beta > \alpha > x^{***}([-x_0^{**}, +x_0^{**}])$$

for some $0 < \alpha < \beta$. Let us fix a countable dense subset $\{d_n : n \in \mathbb{N}\}$ of E and let us consider, for every $n \geq 1$, the set

$$V_n := \left\{y^{***} \in E^{***} : |y^{***}(x_0^{**})| < \alpha,\ |(y^{***} - x^{***})(d_i)| < \frac{1}{n},\ i = 1, 2, \ldots, n\right\}$$

which is a weak*-open neighborhood of x^{***}. Goldstein's theorem permits us to pick up $x_n^* \in B_{E^*} \cap V_n$ for every $n \geq 1$. We will have the sequence $\{x_n^*\}_{n\geq 1}$ which clearly satisfies

$$\lim_n \langle x_n^*, d_p\rangle = \langle x^{***}, d_p\rangle$$

for every $p \in \mathbb{N}$. The density of $\{d_1, \cdots, d_n, \cdots\}$ in E together with the equicontinuity of the sequence $\{x_n^*\}_{n\geq 1}$ give us the weak*-convergence of the sequence, thus we see that

$$\lim_n \langle x_n^*, x\rangle = \langle x^{***}, x\rangle$$

for all $x \in E$, and $|\langle x_n^*, x_0^{**}\rangle| < \alpha$ for every $n \in \mathbb{N}$. So the sequence $\{x_n^*\}_{n\geq 1}$ satisfies the lemma and the proof is complete. □

Theorem 2.3. *Let E be a separable Banach space and A be closed and convex set. Let us fix a relatively weakly compact set of directions $D \subset B_E \setminus \{0\}$ such that $0 \notin \overline{\mathrm{co}(A \cup D)}^{\|\cdot\|}$ and for every $x^* \in E^*$ we have that*

$$\inf\{x^*(a) : a \in A\}$$

is attained whenever $x^*(d) > 0$ for every $d \in D$. Then A is weak-closed in E^{**} and $A \cap rB_E$ is a weakly compact set for every $r > 0$*

Proof. If $A \cap B_E$ is not weakly relatively compact there is a weak*-cluster point $x_0^{**} \in E^{**} \setminus E$ of $A \cap B_E$. Since $0 \notin \overline{\mathrm{co}((A \cap B_E) \cup D)}$ our Lemma 2.2 applies to provide us with a weak*-convergent sequence $\{x_n^*\}_{n\geq 1}$ to x_0^* in B_{E^*} and numbers $\beta > \alpha > 0$ satisfying (1) and (2) for $\overline{\mathrm{co}((A \cap B_E) \cup D)}$. Since the set of directions D is weakly relatively compact and we have

$$x_0^* \in \overline{\mathrm{co}\{x_n^* : n \geq m\}}^{\tau(E^*,E)}$$

where $\tau(E^*, E)$ is the Mackey topology of the dual pair $\langle E^*, E\rangle$, i.e the topology of uniform convergence on weakly compact convex subsets of E, it follows that $x_0^* \in \overline{\mathrm{co}\{x_n^* : n \geq m\}}^{p_D}$ for all $m \in \mathbb{N}$ where p_D is the seminorm

$$p_D(x^*) := \sup\{|\langle x^*, d\rangle| : d \in D\}.$$

So we can find a sequence of finite subsets of integers (F_n), and convex combinations of scalars $\sum_{i\in F_n} \lambda_i^n = 1, 0 < \lambda_i^n \leq 1$ such that the sequence $y_n^* := \sum_{i\in F_n} \lambda_i^n x_i^*$ converges uniformly to x_0^* on D. Since $x_0^*(d) > \beta$ for all $d \in D$ by (2) we may and do assume $y_n^*(d) > \beta > 0$ for all $d \in D$ and $n \in \mathbb{N}$. Thus y^* attains its infimum on A for every $y^* \in \mathrm{co}_{\sigma_p}\{y_n^* : n \geq 1\}$. Inf-liminf Theorem 2.1 can be applied here to obtain that

$$\inf_{\overline{A}^{\sigma(E^{**},E^*)}} \liminf_n y_n^* = \inf_A \liminf_n y_n^*$$

which contradicts (1) and (2) since both inequalities are valid for (y_n^*) instead of (x_n^*). Indeed we have:

$$\inf_A \liminf_n y_n^* = \inf_A \lim_n y_n^* = \inf_A x_0^* \geq \beta$$

and

$$\inf_{\overline{A}^{\sigma(E^{**},E^*)}} \liminf_n y_n^* \leq \liminf_n y_n^*(x_0^{**}) \leq \alpha < \beta. \qquad \square$$

Remark 2.4. Let us remark here that Theorem A.1 in [5] is a consequence of the former result in separable Banach spaces. Indeed, if we set A equal to the epigraph of the penalty function V there, it is enough to consider the singleton set $D = \{0, 1\}$ to get the result.

3. Conic James' construction

We present in this section the main construction for our proof of Theorem 1.2. A careful analysis of ideas of R. C. James [4], M. Ruiz Galán and S. Simons [10] leads us to find an appropriate (conic-sided) non attaining linear form. Indeed we have the following general statement:

Theorem 3.1. *Let A be a convex bounded subset of a Banach space E such that the set*

$$\{a^{**} \in \overline{A}^{\sigma(E^{**},E^{*})} \backslash E : \textit{there exists } \{a_n\}_{n\geq 1} \textit{ in } A \textit{ with } a^{**} \in \overline{\{a_n : n \geq 1\}}^{w^*}\}$$

is non–void. Let us fix a convex weakly compact subset D of E which does not contain the origin.

Then there is a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} and $g_0^* \in co_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell_\infty(A)$ satisfying that for all $a \in A$,

$$\liminf_{n\geq 1} x_n^*(a) \leq h(a) \leq \limsup_{n\geq 1} x_n^*(a),$$

we have that $g_0^* - h$ does not attain its supremum on A and $(g_0^* - h)(d) > 0$ for every $d \in D$.

Proof. By hypothesis we can fix a point $x_0^{**} \in \overline{A}^{\sigma(E^{**},E^{*})} \backslash E$ and $\{x_n\}_{n\geq 1}$ a sequence in A with $x_0^{**} \in \overline{\{x_n : n \geq 1\}}^{\sigma(E^{**},E^{*})}$. The line segment $[-x_0^{**}, x_0^{**}]$ does not meet the $\sigma(E, E^*)$-compact set D since the origin is not inside D. Let us take $x^* \in B_{E^*}$ which strictly separates both sets, we will have:

$$x^*([-x_0^{**}, x_0^{**}]) = [-\mu, +\mu] \quad \text{and} \quad x^*(D) = [a, b],$$

where μ, a, b are real numbers and $[-\mu, \mu] \cap [a, b] = \emptyset$. Without loss of generality we assume that

$$-\mu \leq 0 \leq \mu < \beta < \alpha < a < b$$

for α and β real numbers chosen by the strict separation provided with the linear form x^*.

Without loss of generality we may and do assume the fact that

$$x^*(x_n) < \beta \text{ for every } n \in \mathbb{N}. \tag{3}$$

Another Hahn-Banach application, now in the duality (E^{**}, E^{***}), provides us a linear form $z^{***} \in B_{E^{***}}$ such that $z^{***}(x_0^{**}) > 0$ but $z^{***}(E) = \{0\}$. Let us choose $\lambda > 0$ such that $\lambda z^{***}(x_0^{**}) + x^*(x_0^{**}) > \alpha$

Let us consider the linear functional

$$x^{***} := x^* + \lambda z^{***}$$

and we will have

$$x^{***}(D) = x^*(D) \geq a > \alpha > 0, x^{***}(x_0^{**}) = \lambda z^{***}(x_0^{**}) + x^*(x_0^{**}) > \alpha > 0 \tag{4}$$

Let us remind the reader that we also have by (3)

$$x^{***}(x_n) = x^*(x_n) < \beta \tag{5}$$

for every $n \in \mathbb{N}$. Without loss of generality, taking $\frac{1}{\|x^{***}\|}x^{***}$, $\frac{1}{\|x^{***}\|}\alpha$ and $\frac{1}{\|x^{***}\|}\beta$ instead of x^{***}, α and β if necessary, we may assume the former inequalities (4) and (5) with $x^{***} \in B_{E^{***}}$.

By Goldstine and Mackey-Arens theorems we can construct a bounded sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} satisfying

$$\text{for all } p \geq 1, \qquad \lim_{n\geq 1} x_n^*(x_p) = x^{***}(x_p) = x^*(x_p),$$

and

$$\lim_{n\geq 1} x_0^{**}(x_n^*) = x^{***}(x_0^{**}), \qquad \lim_{n\geq 1}(x_n^*(d)) = x^{***}(d) = x^*(d)$$

uniformly on $d \in D$. Then

$$\text{for all } p \geq 1, \text{ there is } n_p \text{ such that } \beta > x_n^*(x_p) \text{ for } n \geq n_p, \tag{6}$$

and without loss of generality we may assume that:

$$x_0^{**}(x_n^*) > \alpha, x_n^*(d) > \alpha, \forall n \in \mathbb{N} \text{ and } d \in D \tag{7}$$

by uniform convergence on the weak compact set D.

Let us note that, given a $\sigma(E^*, E)$–cluster point x_0^* of the sequence $\{x_n^*\}_{n\geq 1}$, we have that

$$x_0^{**}(x_0^*) \leq \beta, \tag{8}$$

because $x_0^{**} \in \overline{\{x_p : p \geq 1\}}^{w^*}$ and for all $p \geq 1$, $x_0^*(x_p) \leq \beta$ by (6).

We use Pryce's arguments (see [10] Lemma 9,c; Proposition 10.14 and Theorem 10.15 in [2]) to find a subsequence $\left\{x_{n_k}^* : k \geq 1\right\}$ of $\{x_n^*\}_{n\geq 1}$ such that we get for all $h_0 \in \mathrm{co}_\sigma \{x_n^* : n \geq 1\}$

$$\sup_A \left(h_0 - \limsup_{k\geq 1} x_{n_k}^*\right) = \sup_A \left(h_0 - \liminf_{k\geq 1} x_{n_k}^*\right). \tag{9}$$

Let us now observe that for x_0^{**} we have by (7) that

$$\forall h_0 \in \mathrm{co}_\sigma \left\{x_{n_k}^* : k \geq 1\right\}, \qquad x_0^{**}(h_0) > \alpha. \tag{10}$$

We fix a $\sigma(E^*, E)$-cluster point x_0^* of the sequence $\left\{x_{n_k}^* : k \geq 1\right\}$; then it follows that for all $a \in A$,

$$\limsup_{k\geq 1} x_{n_k}^*(a) \geq x_0^*(a) \geq \liminf_{k\geq 1} x_{n_k}^*(a)$$

and thus, for all $a \in A$,

$$\left(h_0(a) - \liminf_{k\geq 1} x_{n_k}^*(a)\right) \geq (h_0 - x_0^*)(a)) \geq \left(h_0(a) - \limsup_{k\geq 1} x_{n_k}^*(a)\right).$$

Therefore, in view of (9) we deduce that for all h_0,

$$\sup_A\left(h_0-\limsup_{k\geq1}x^*_{n_k}\right)=\sup_A\left(h_0-\liminf_{k\geq1}x^*_{n_k}\right)=\sup_A\left(h_0-x^*_0\right).$$

Let us observe that for $h_0\in\mathrm{co}_\sigma\left\{x^*_{n_k}:k\geq1\right\}$ we have:

$$\sup_A\left(h_0-x^*_0\right)=\sup_{\overline{A}^{w^*}}\left(h_0-x^*_0\right)\geq\left(x^{**}_0(h_0)-x^{**}_0(x^*_0)\right)=\langle x^{**}_0,h_0-x^*_0\rangle$$

and

$$\langle x^{**}_0,h_0-x^*_0\rangle>\alpha-\beta>0,$$

by (8) and (10), as needed to apply [5, Corollary 8] and obtain a sequence $\{g^*_i\}_{i\geq1}$ with $g_i\in\mathrm{co}_\sigma\{x^*_{n_k}:k\geq i\}$ and $g^*_0\in\mathrm{co}_\sigma\left\{g^*_i:i\geq1\right\}$ such that for all $\tilde g\in\ell_\infty(A)$ with

$$\liminf_{i\geq1}g^*_i\leq\tilde g\leq\limsup_{i\geq1}g^*_i\text{ on }A$$

we have that

$$(g^*_0-\tilde g)\text{ does not attain its supremum on }A.$$

Let us observe that $\tilde g$ does coincide with x^{***} on D and that g_i are σ-convex combinations of linear forms of the sequence $\{x^*_n\}$. If we may assume in the construction that $x^*_n(d)>x^{***}(d)$ for all $d\in D$ and every $n\in\mathbb{N}$, then we should have $g_i(d)>x^{***}(d)$ for all $d\in D$, therefore $(g^*_0-\tilde g)(d)>0$ for every $d\in D$ and the proof should be over. In particular, for every $\sigma(E^*,E)$–cluster point of the sequence $\{g^*_i\}_{i\geq1}$, let us say $\tilde g^*$, we have that $g^*_0-\tilde g^*\in E^*$ does not attain its supremum on A but $(g^*_0-\tilde g*)(d)>0$ for every $d\in D$.

We finish with the proof of our claim:

It is possible to choose x^*_n above such that $x^*_n(d)>x^{***}(d)$ for every $d\in D$.

We set

$$\epsilon_n:=2\sup\{|x^{***}(d)-x^*_n(d)|:d\in D\}$$

and

$$c_n:=\sup\{\frac{x^{***}(d)}{x^{***}(d)-\epsilon_n}:d\in D\}$$

that is well defined because without loss of generality we may assume that

$$\epsilon_n<a=\inf x^{***}(D)$$

for every n.

The function $f(t,\epsilon):=\frac{t}{t-\epsilon}$ is decreasing in t for $t>\epsilon$ and $\epsilon>0$ fixed, and $\lim_{\epsilon\to0}f(t,\epsilon)=1$ for every $t>0$ fixed too. If we set $\hat x^*_n:=c_nx^*_n$, we will have that the sequence $\hat x^*_n$ verifies our claim since ϵ_n goes to zero and c_n to 1 when n goes to infinity. Indeed the proof is complete. □

Now we are able to give the proof of our Theorem 1.2:

Proof. The Eberlein-Smulian Theorem, see Theorem 3.109 in [3], says that a subset K of the Banach space E is relatively weakly compact if, and only if, every sequence (x_n) in K has a $\sigma(E, E^*)$-cluster point in E. Since bounded subsets of E are relatively $\sigma(E^{**}, E^*)$-compact subsets of the bidual E^{**}, it follows that the convex hull $\mathrm{co}(A)$ satisfies the hypothesis of Theorem 3.1 when it is not relatively weakly compact. Let us argue by contradiction assuming it, thus we will find a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} and $g_0^* \in \mathrm{co}_\sigma\{x_n^* : n \geq 1\}$ such that for all $h \in \ell_\infty(\mathrm{co}(A))$ satisfying that for all $x \in \mathrm{co}(A)$,

$$\liminf_{n\geq 1} x_n^*(x) \leq h(x) \leq \limsup_{n\geq 1} x_n^*(x),$$

we will have that

$$g_0^* - h \text{ does not attain its supremum on } \mathrm{co}(A).$$

and $(g_0^* - h)(d) > 0$ for every $d \in D$.

The $\sigma(E^*, E)$-compactness of the dual ball B_{E^*} permits us to find $h_0^* \in B_{E^*}$ which is a $\sigma(E^*, E)$-cluster point of the sequence $\{x_n^*\}_{n\geq 1}$. Since

$$\liminf_{n\geq 1} x_n^*(x) \leq h_0^*(x) \leq \limsup_{n\geq 1} x_n^*(x)$$

for every $x \in E$, it follows that the functional $g_0^* - h_0^*$ does not attains its supremum on $\mathrm{co}(A)$, and therefore not in A. But $(g_0^* - h_0^*)(D) > 0$ as we have proved above, which is a contradiction with our hypothesis and the proof is complete. □

Theorem 1.2 extends Theorem 10 of [1] for arbitrary Bananch spaces solving a question asked in that paper. It can be derived from the recent results in [6] too. Nevertheless we present here our direct approach with Theorem 3.1, our main new result here indeed, with a precise control in the construction of constrained linear functionals which do not attain supremum, so it could help for solving future constrained problems, too.

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