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Midsets and Voronoi Type Decomposition with Respect to Closed Convex Sets

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Let Ω_k denote the collection of all nonempty closed convex subsets of $\mathbb{R}^k$. We provide short proofs for the following: (i) $\{x \in \mathbb{R}^k : dist(x, A) = \varepsilon\}$ is a C^1-manifold of dimension $k-1$ for every $A \in \Omega_k \setminus \{\mathbb{R}^k\}$ and $\varepsilon > 0$, (ii) $\{x \in \mathbb{R}^k : dist(x, A) = dist(x, B)\}$ is a C^1-manifold of dimension $k-1$ for any two disjoint $A, B \in \Omega_k$. We also study the distance of points in $\mathbb{R}^k$ to finitely many closed convex sets. Let $k, n \geq 2$ and $A = \bigcup_{j=1}^n A_j$, where $A_1, \ldots, A_n \in \Omega_k$ are pairwise disjoint. We consider a Voronoi type decomposition of $\mathbb{R}^k$ and establish some topological properties of its 'conflict set'. Letting $X_p = \{x \in \mathbb{R}^k : |\{a \in A : \|x - a\| = dist(x, A)\}| = p\}$, we prove with the help of result (ii) stated above that $X_1 \cup X_2$ is a connected dense open subset of $\mathbb{R}^k$ and that $\overline{X_2} = \bigcup_{p=2}^n X_p$.

Keywords: Euclidean geometry, closed convex sets, Voronoi decomposition, midsets.

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1. Introduction

Our work is closely related to the theory of *Voronoi diagrams*. A Voronoi diagram is a partitioning of a plane (or more generally $\mathbb{R}^k$ or other suitable spaces) into regions based on the distance of points to specific subsets of the plane. It is a much studied concept, and we refer the reader to [10, 1] and the references therein for a basic introduction to the subject. In this article, we will restrict our attention to closed convex subsets of a Euclidean space, and all discussions are with respect to the Euclidean distance.

Let Ω_k denote the collection of all nonempty closed convex subsets of $\mathbb{R}^k$ for $k \in \mathbb{N}$. For $A \in \Omega_k$, let $d_A : \mathbb{R}^k \to \mathbb{R}$ be the distance function defined as $d_A(x) = \inf\{\|x - a\| : a \in A\}$. It is well-known that for each $A \in \Omega_k$ and $x \in \mathbb{R}^k$, there is a unique point $P_A(x) \in A$ (in fact, $P_A(x) \in \partial A$) such that $\|x - P_A(x)\| = d_A(x)$ and that the function $P_A : \mathbb{R}^k \to A$ (called the *nearest point mapping* of A) is continuous. For $A \in \Omega_k$ and $\varepsilon > 0$, let $S_\varepsilon(A) = \{x \in \mathbb{R}^k : d_A(x) = \varepsilon\}$ be the ε-boundary of A in $\mathbb{R}^k$. Also, if $A, B \in \Omega_k$ are disjoint,

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let $M(A,B)$ be the *midset* (also called the *equidistant set* or the *conflict set*) of A and B defined as $M(A,B) = \{x \in \mathbb{R}^k : d_A(x) = d_B(x)\}$. Since the distance function is continuous, the sets $S_\varepsilon(A)$ and $M(A,B)$ are closed in $\mathbb{R}^k$.

In the literature, the reader may find several notions related to the notion of a midset: *conflict set* (Siersma [15]), *central set, cut locus, and skeleton* (see Yomdin [21] and the references therein), *mediatrices* (Veerman and Bernhard [18]), *medial axis, symmetry set, and bisector* (see for example Bruce, Giblin, and Gibson [4]). These sets share various local properties.

The topology of the sets $S_\varepsilon(A)$ and $M(A,B)$ have been studied (see for instance, [7, 20, 11]) even for the general case where A, B are just closed sets (not necessarily convex). We recall some relevant results, and state them below only for the specific case of closed convex sets; the reader may refer to the original articles for more general versions of these results. Ferry [7] showed that $S_\varepsilon(A)$ is a $(k-1)$-dimensional topological submanifold of $\mathbb{R}^k$ for all sufficiently small $\varepsilon > 0$ when A is a finite polyhedron, and Loveland [11] obtained the same conclusion for all $\varepsilon > 0$ when A is a closed convex set. For disjoint members $A, B \in \Omega_k$, Wilker [20] proved that $M(A,B)$ is nonempty, connected, and has empty interior in $\mathbb{R}^k$, and Loveland [11] established that $M(A,B)$ is a $(k-1)$-dimensional topological submanifold of $\mathbb{R}^k$.

The Differential Geometry of the set $M(A,B)$ and its generalizations (by considering more than two sets) are studied in works such as [3, 15, 16, 17]. In the case of C^j-smooth ($j \geq 1$) hypersurfaces, van Manen showed among other things that the ε-boundary and the conflict set are also C^j-smooth (Theorem 1.1 and its proof in [16]). When A, B are sets in the plane with smooth boundaries, a formula for the curvature of $M(A,B)$ is obtained in [15]. The curvature and torsion are discussed in [17]. In [15] (where $M(A,B)$ is called a *conflict set*), Siersma makes the following remark:

"*In several sources the smoothness of the conflict set is mentioned or supposed in the case of convex sets A and B. ...Although the smoothness in the general convex case without differentiability conditions seems to be 'folklore', we could not find a complete proof in the literature.*"

Then a geometric proof (attributed to Goddijn) for the smoothness $M(A,B)$ in dimension two is given in [15].

In this article, we will first provide a short proof different from that in [15] to establish that the midset $M(A,B)$ is always a $(k-1)$-dimensional C^1-submanifold of $\mathbb{R}^k$ for any two disjoint $A, B \in \Omega_k$. The key observation needed for this is that for $A \in \Omega_k$, the square d_A^2 of the distance function is a C^1-map with gradient $\nabla(d_A^2)(x) = 2(x - P_A(x))$. This well-known fact can be found in several places; see for instance the Example in page 286 of Moreau [12], or Lemma 5, Equation (44), and Theorem 16(l) of Colombo and Thibault [5]. This fact can also be deduced from Lemma 2.21, Lemma 2.22, and Theorem 2.23 of Birbrair and Denkowski [2] since the *medial axis* of a closed convex set is empty.

The same technique allows us to deduce also that the ε-boundary $S_\varepsilon(A)$ is a $(k-1)$-dimensional C^1-submanifold of $\mathbb{R}^k$ for every $A \in \Omega_k \setminus \{\mathbb{R}^k\}$ and every $\varepsilon > 0$. From a geometrical view point, the C^1-smoothness of $S_\varepsilon(A)$ and $M(A,B)$ is perhaps a little surprising since the boundaries of the convex sets under consideration need not be smooth and can have sharp corners. In a related context, we also refer the reader to the recent article [14] (and some of the references therein), where the authors study generalized conics as midsets.

Next we will make use of the C^1-smoothness of the midset in the study of the distance of a point in $\mathbb{R}^k$ to a finite union $A = \bigcup_{j=1}^n A_j$ of pairwise disjoint members $A_1, \ldots, A_n$ of Ω_k. That is, we will study Voronoi type decomposition of $\mathbb{R}^k$ using a collection of pairwise disjoint closed convex sets. Voronoi decomposition and the corresponding Delaunay triangulation (see [1]), mainly for a finite collection of points and sometimes for a special collection of sets such as closed intervals, are important concepts in Computer Science (especially in $\mathbb{R}^2$ and $\mathbb{R}^3$). Our treatment is much more general by considering closed convex sets which are not necessarily bounded. Letting $X_p = \{x \in \mathbb{R}^k : |\{a \in A : \|x-a\| = dist(x,A)\}| = p\}$, we will prove among other things that $X_1 \cup X_2$ is a connected dense open subset of $\mathbb{R}^k$ and that $\overline{X_2} = \bigcup_{p=2}^n X_p$.

2. $S_\varepsilon(A)$ and $M(A,B)$ are $(k-1)$-dimensional C^1-submanifolds of $\mathbb{R}^k$

Notations are as stated in the beginning of the article: Ω_k is the collection of all nonempty closed convex subsets of $\mathbb{R}^k$, and $P_A\colon \mathbb{R}^k \to A$, $d_A\colon \mathbb{R}^k \to \mathbb{R}$ respectively are the nearest point mapping and distance function for $A \in \Omega_k$. In this section, first we will show that the ε-boundary $S_\varepsilon(A)$ of any $A \in \Omega_k \setminus \{\mathbb{R}^k\}$ and the midset $M(A,B)$ of any two disjoint members $A, B \in \Omega_k$ are $(k-1)$-dimensional C^1-submanifolds of $\mathbb{R}^k$. Then we will show that the submanifolds $M(A,C)$ and $M(B,C)$ intersect transversely in $\mathbb{R}^k$ for pairwise disjoint members $A, B, C \in \Omega_k$.

For a differentiable function $f\colon \mathbb{R}^k \to \mathbb{R}$, let $\nabla f(x)$ denote the gradient vector at $x \in \mathbb{R}^k$. As mentioned earlier, Lemma 2.1 can be found in, for instance, [2, 5, 12].

Lemma 2.1. *Let $k \in \mathbb{N}$, and $A \in \Omega_k$. Then d_A^2 is continuously differentiable and $\nabla(d_A^2)(x) = 2(x - P_A(x))$ for every $x \in \mathbb{R}^k$.*

Theorem 2.2. *Let $k \geq 2$ be an integer. Then,*

(1) *$S_\varepsilon(A)$ is a $(k-1)$-dimensional C^1-submanifold of $\mathbb{R}^k$ for every $A \in \Omega_k \setminus \{\mathbb{R}^k\}$ and $\varepsilon > 0$.*

(2) *$M(A,B)$ is a $(k-1)$-dimensional C^1-submanifold of $\mathbb{R}^k$ for any two disjoint $A, B \in \Omega_k$.*

Proof. Let $g, h\colon \mathbb{R}^k \to \mathbb{R}$ be $g = d_A^2$ and $h = d_A^2 - d_B^2$. Note that g, h are continuously differentiable, $S_\varepsilon(A) = g^{-1}(\varepsilon^2)$, and $M(A, B) = h^{-1}(0)$. If $x \in S_\varepsilon(A)$, then $x \neq P_A(x)$, and hence $\nabla g(x) = 2(x - P_A(x)) \neq 0$ by Lemma 2.1. Similarly, if $x \in M(A, B)$, then $P_A(x) \neq P_B(x)$ because $A \cap B = \emptyset$ by assumption, and hence $\nabla h(x) = 2(P_B(x) - P_A(x)) \neq 0$ by Lemma 2.1. Therefore, ε^2, 0 are regular values of g, h respectively. Consequently $g^{-1}(\varepsilon^2)$ and $h^{-1}(0)$ are $(k-1)$-dimensional C^1-submanifolds of $\mathbb{R}^k$ (see Theorem 3.2 of [9]). $\square$

We remark that if we put $A_\varepsilon = \{x \in \mathbb{R}^k : d_A(x) \leq \varepsilon\}$, then it can be seen (using Lemma 3.3 for instance) that $M(A, B) = M(A_\varepsilon, B_\varepsilon)$ for all sufficiently small $\varepsilon > 0$, and therefore part (2) of Theorem 2.2 can also be deduced from part (1) and the result of van Manen (Theorem 1.1 of [16]) mentioned earlier.

Before proceeding to the next result, we mention two simple examples in $\mathbb{R}^2$ to point out that the sets $S_\varepsilon(A)$ and $M(A, B)$ are not C^2-smooth in general. Let $A \subset \mathbb{R}^2$ be the line segment joining $(-1, 0)$ and $(0, 0)$, and $\varepsilon = 1$. Then $S_\varepsilon(A) \cap \{(x, y) \in \mathbb{R}^2 : x \geq -1 \text{ and } y \geq 0\}$ is the graph of the function $f\colon [-1, 1] \to \mathbb{R}$ given by $f(x) = 1$ for $-1 \leq x \leq 0$ and $f(x) = \sqrt{1 - x^2}$ for $0 \leq x \leq 1$. Since $f''(x) = 0$ for $-1 < x < 0$ and $f''(x) = -(1 - x^2)^{-3/2}$ for $0 < x < 1$, we have that $f''(0^-) = 0 \neq -1 = f''(0^+)$, and thus f is not C^2-smooth. Next, if $A = \{(0, 0)\} \subset \mathbb{R}^2$ and $B = \{(x, 1) \in \mathbb{R}^2 : x \geq 0\}$, then $M(A, B) \cap \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$ is the graph of the function $g\colon [-1, 1] \to \mathbb{R}$ given by $g(x) = 1/2$ for $-1 \leq x \leq 0$ and $g(x) = (1 - x^2)/2$ for $0 \leq x \leq 1$. Since $g''(0^-) = 0 \neq -1 = g''(0^+)$, the function g is not C^2-smooth.

However, it may be noted that by an old result of Whitney [19], every C^1-manifold admits a compatible C^∞-smooth manifold structure.

The following result about the 2-fold intersection of midsets will be used in the next section while studying the structure of the set of points equidistant from three or more closed convex sets.

Theorem 2.3. *Let $k \geq 2$ and $A, B, C \in \Omega_k$ be pairwise disjoint. Then,*

(1) *$M(A, C)$ intersects $M(B, C)$ transversally. This means for each $x \in M(A, C) \cap M(B, C)$, the algebraic sum of the tangent spaces to $M(A, C)$ and $M(B, C)$ at x is $\mathbb{R}^k$.*

(2) *$M(A, C) \cap M(B, C)$ is either empty or is a $(k-2)$-dimensional C^1-submanifold of $\mathbb{R}^k$.*

Proof. (1) If $M(A, C) \cap M(B, C) = \emptyset$, then the transversality condition is vacuously satisfied. So assume $M(A, C) \cap M(B, C) \neq \emptyset$, and consider $x \in M(A, C) \cap M(B, C)$. After a translation of the sets A, B, C, we may suppose without loss of generality that $x = 0 \in \mathbb{R}^k$. Let T_1 and T_2 respectively be the tangent spaces to $M(A, C)$ and $M(B, C)$ at $0 \in \mathbb{R}^k$. Then $dim(T_1) = k - 1 = dim(T_2)$ by Theorem 2.2. Therefore, if the transversality condition fails at 0, then we must have $T_1 = T_2$. Assume $T_1 = T_2$, and we will derive a contradiction. Let $f, g\colon \mathbb{R}^k \to \mathbb{R}$ be $f = d_A^2 - d_C^2$ and $g = d_B^2 - d_C^2$. By Lemma 2.1, $\nabla f(0) = 2(P_C(0) - P_A(0))$ and $\nabla g(0) = 2(P_C(0) - P_B(0))$. Since these gradient vectors are perpendicular to the $(k-1)$-dimensional space $T_1 = T_2$, there must exist some $t \in \mathbb{R}$ such that $\nabla f(0) = t\nabla g(0)$. That is, $P_C(0) - P_A(0) = t(P_C(0) - P_B(0))$, or $(1-t)P_C(0) + tP_B(0) = P_A(0)$. This means that the three vectors $P_A(0)$, $P_B(0)$, and $P_C(0)$ are collinear. But that is impossible since $P_A(0), P_B(0), P_C(0)$ are three distinct non-zero vectors in $\mathbb{R}^k$ with $\|P_A(0)\| = \|P_B(0)\| = \|P_C(0)\|$.

(2) This is an immediate consequence of part (1) by a standard result in Differential Topology (for instance, see Theorem 3.3 of [9] and Section 5 of Chapter 1 in [8]) since $M(A, C)$ and $M(B, C)$ are $(k-1)$-dimensional C^1-submanifolds of $\mathbb{R}^k$. It must be remarked that $M(A, C) \cap M(B, C)$ need not be connected. $\square$

3. Voronoi type decomposition of $\mathbb{R}^k$ using finitely many pairwise disjoint closed convex sets

Our aim in this section is to analyze the nature of the sets X_p's defined below in connection with the distance of a point to finitely many closed convex sets. The importance of this more general type of Voronoi decomposition is already mentioned in the last paragraph of the Introduction. We stress the fact that the closed convex sets that we consider need not be bounded. The reader may kindly note that the notations that we fix here will be directly used in the sequel without further comments.

Recall that Ω_k is the collection of all nonempty closed convex subsets of $\mathbb{R}^k$. Let $k, n \geq 2$ be integers, $A_1, \ldots, A_n \in \Omega_k$ be pairwise disjoint members, and $A = \bigcup_{j=1}^n A_j$. For $x \in \mathbb{R}^k$, let $P_A(x)$ be the subset of A defined as

$$\begin{aligned} P_A(x) &= \{a \in A : \|x - a\| = dist(x, A)\} \\ &= \{P_{A_j}(x) : \|x - P_{A_j}(x)\| = dist(x, A)\}, \end{aligned}$$

and note that $1 \leq |P_A(x)| \leq n$ for every $x \in \mathbb{R}^k$. For $1 \leq p \leq n$, let

$$X_p = \{x \in \mathbb{R}^k : |P_A(x)| = p\}.$$

In particular, X_1 is the collection of all $x \in \mathbb{R}^k$ for which there is a unique nearest point in $A = \bigcup_{j=1}^n A_j$. Clearly, $\mathbb{R}^k = X_1 \cup \cdots \cup X_n$ is a partition of $\mathbb{R}^k$. Moreover, $\bigcup_{p=2}^n X_p$ is the *conflict set* (for example, as in [3]). For $J \subset \{1, \ldots, n\}$, let

$$U_J = \{x \in \mathbb{R}^k : d_{A_j}(x) < d_{A_m}(x) \text{ for every } j \in J \text{ and every } m \in \{1, \ldots, n\}\backslash J\},$$

which is open in $\mathbb{R}^k$. For $|J| = p$ fixed, the set U_J generalizes the concept of the p-th order Voronoi diagram [1]. As a special case, let

$$U_j = U_{\{j\}} = \{x \in \mathbb{R}^k : d_{A_j}(x) < d_{A_m}(x) \text{ for every } m \in \{1, \ldots, n\} \setminus \{j\}\}$$

for $1 \leq j \leq n$, which is related to the notion of a *territory* in [3]. Also, for $a, b \in \mathbb{R}^k$, let $[a, b]$ be the line segment joining a and b.

First we note that the multi-valued map P_A is upper semi-continuous; this is related to Theorem 2.17 in [2].

Theorem 3.1. *If $V \subset \mathbb{R}^k$ is open, then $W := \{x \in \mathbb{R}^k : P_A(x) \subset V\}$ is open in $\mathbb{R}^k$.*

Proof. Consider $x \in W$. We need to find $\delta > 0$ with $B(x, \delta) \subset W$. Let $r = dist(x, A)$ and $J = \{1 \leq j \leq n : d_{A_j}(x) = r\}$. Let $a_j = P_{A_j}(x) \in A_j$ for $j \in J$. Then $P_A(x) = \{a_j : j \in J\} \subset V$. Choose $\delta > 0$ such that $\bigcup_{j \in J} B(a_j, \delta) \subset V$ and $d_{A_m}(x) > r + 2\delta$ for every $m \in \{1, \ldots, n\} \setminus J$. For any $y \in B(x, \delta)$, we observe that $dist(y, A) \leq \|y - x\| + dist(x, A) < \delta + r$ and $d_{A_m}(y) > d_{A_m}(x) - \delta > r + 2\delta - \delta = r + \delta$ for every $m \in \{1, \ldots, n\} \setminus J$. Hence $P_A(y) \subset \bigcup_{j \in J} B(a_j, \delta) \subset V$ for every $y \in B(x, \delta)$. Thus $B(x, \delta) \subset W$. $\square$

However, P_A is not lower semi-continuous in general: if $k = n = 2$, $A_1 = \{(1, 0)\}$, and $A_2 = \{(-1, 0)\}$, then $P_A(x, y) = A_1 \cup A_2$ if $x = 0$, $P_A(x, y) = A_1$ if $x > 0$ and $P_A(x, y) = A_2$ if $x < 0$.

Theorem 3.2.

(1) *$U_J \subset \bigcup_{p \leq |J|} X_p$ for $J \subset \{1, \ldots, n\}$ and $X_p \subset \bigcup_{|J|=p} U_J$ for $1 \leq p \leq n$.*

(2) *For each $q \in \{1, \ldots, n\}$, $\bigcup_{|J| \leq q} U_J = \bigcup_{p=1}^{q} X_p$, and hence $\bigcup_{p=1}^{q} X_p$ is open in $\mathbb{R}^k$.*

(3) *X_1 is open and dense in $\mathbb{R}^k$.*

(4) *X_p is a nowhere dense G_δ subset of $\mathbb{R}^k$ for $2 \leq p \leq n$.*

(5) *$\overline{X_p} \subset \bigcup_{j=p}^{n} X_j$ and X_p is an open subset of $\overline{X_p}$ for $1 \leq p \leq n$.*

Proof. (1) If $x \in U_J$, then $x \notin \bigcup_{p > |J|} X_p$ by the definition of U_J, and hence $x \in \mathbb{R}^k \setminus (\bigcup_{p > |J|} X_p) = \bigcup_{p \leq |J|} X_p$. Next, if $x \in X_p$, then for $J = \{1 \leq j \leq n : P_A(x) \cap A_j \neq \emptyset\}$, we see $|J| = p$ and $x \in U_J$.

(2) This follows from (1).

(3) X_1 is open by (2). The denseness of X_1 is implicitly proved in [20]. The argument is as follows: note that $\mathbb{R}^k \setminus \bigcup_{1 \leq i < j \leq n} M(A_i, A_j) \subset X_1$, and use the fact (proved in [20]) that each $M(A_i, A_j)$ is a nowhere dense closed subset of $\mathbb{R}^k$.

(4) X_p is nowhere dense by (3) because $X_p \subset \mathbb{R}^k \setminus X_1$. Since $X_p = (\bigcup_{j=1}^{p} X_j) \setminus (\bigcup_{j=1}^{p-1} X_j)$, it follows by (2) that X_p is the intersection of an open subset of $\mathbb{R}^k$ with a closed subset of $\mathbb{R}^k$. And every closed subset of $\mathbb{R}^k$ is a G_δ set. Thus X_p is a G_δ set.

(5) If $p = 1$, the result follows from (3). So assume $p \geq 2$. Let $V = \bigcup_{j=1}^{p} X_j$

and $W = \bigcup_{j=1}^{p-1} X_j$, which are open in $\mathbb{R}^k$ by (2). Since $X_p \cap W = \emptyset$, we see $\overline{X_p} \subset \mathbb{R}^k \setminus W = \bigcup_{j=p}^{n} X_j$. Hence $\overline{X_p} \cap V = X_p \cap V = X_p$, which shows that X_p is an open subset of $\overline{X_p}$. □

The following elementary observations will be useful for our further investigations.

Lemma 3.3. *Let $0 \in \mathbb{R}^k \setminus A$, $j \in \{1, \dots, n\}$, and let $a_j = P_{A_j}(0) \in A_j$. Then,*

(1) *$P_{A_j}(\lambda a_j) = a_j$ for every $\lambda \in (-\infty, 1)$.*

(2) *If $d_{A_j}(0) = dist(0, A) =: r$, then $d_{A_j}(\delta a_j) = (1-\delta)r < d_{A_m}(\delta a_j)$ for every $m \in \{1, \dots, n\} \setminus \{j\}$ and every $\delta \in (0,1)$.*

Proof. A proof of (1) can be found in [13]. Part (2) is also possibly known, but we provide an elementary argument for the sake of completeness. Fix $\delta \in (0,1)$ and let $y = \delta a_j$. By (i), we have $d_{A_j}(y) = (1-\delta)r$. If possible, let there be $m \in \{1, \dots, n\} \setminus \{j\}$ and $b \in A_m$ with $\|b - y\| \le (1-\delta)r$. Consider the three points $0, b, y$. If they are not collinear, then by the strict convexity of the Euclidean norm of $\mathbb{R}^k$, we get $\|b\| < \|y\| + \|b - y\| \le \delta r + (1-\delta)r = r$, a contradiction since $\|b\| \ge d_{A_m}(0) \ge dist(0, A) = r$. Next suppose $0, b, y$ are collinear. If $0 \in [b, y]$, then $r \le \|b\| < \|b - y\| \le (1-\delta)r$, which is impossible. If $y \in [0, b]$, then $a_j \in [y, b]$ and $a_j \ne b$ since $A_j \cap A_m = \emptyset$ and $d_{A_j}(0) \le d_{A_m}(0)$, and thus we get $(1-\delta)r = \|a_j - y\| < \|b - y\|$, a contradiction to the choice of b. If $b \in [0, y]$, then also $\|b\| \le \|y\| = \delta r < r$, a contradiction. □

Theorem 3.4. (1) *Each U_j is path connected. Moreover, U_j's are precisely the connected components of X_1. Thus X_1 has exactly n connected components.*

(2) *$X_1 \cup X_2$ is a connected dense open subset of $\mathbb{R}^k$.*

Proof. (1) Clearly, U_j's are pairwise disjoint open sets in $\mathbb{R}^k$ with $A_j \subset U_j$ and $\bigcup_{j=1}^{n} U_j = X_1$. Therefore, it remains to show U_j's are path connected. Fix j and consider $x, y \in U_j$. Let $a = P_{A_j}(x) \in A_j$ and $b = P_{A_j}(y) \in A_j$. By Lemma 3.3, we have that $P_{A_j}(u) = a$ for every $u \in [x, a]$, $P_{A_j}(v) = b$ for every $v \in [y, b]$ and $[x, a] \cup [y, b] \subset U_j$. Moreover, $[a, b] \subset A_j \subset U_j$ by the convexity of A_j. Thus $[x, a] \cup [a, b] \cup [b, y]$ is a path in U_j from x to y.

(2) We may suppose $n \ge 3$. By Theorem 3.2, $X_1 \cup X_2$ is open and dense in $\mathbb{R}^k$. Let $Y = \bigcup_{1 \le j < m < s \le n} (M(A_j, A_s) \cap M(A_m, A_s))$, which is a nowhere dense closed subset of $\mathbb{R}^k$ having dimension at most $k-2$ by Theorem 2.3(2), and therefore $\mathbb{R}^k \setminus Y$ is connected by Theorem 1.8.13 of [6]. Since $\mathbb{R}^k \setminus Y \subset X_1 \cup X_2 \subset \mathbb{R}^k = \overline{\mathbb{R}^k \setminus Y}$, we obtain that $X_1 \cup X_2$ is connected. The reader may note here the standard fact that any nonempty connected open subset of $\mathbb{R}^k$ is polygonally connected (i.e., any two points can be joined by a polygonal path), and in particular path connected. □

Lemma 3.5. *Let $k \ge 2$ and $n \ge 3$. If $3 \le p \le n$, $x \in X_p$, and $\varepsilon > 0$, then there is $2 \le q < p$ such that $B(x, \varepsilon) \cap X_q \ne \emptyset$.*

Proof. Since $p \geq 3$, we have that $x \in \mathbb{R}^k \setminus A$. After a translation, assume $x = 0$ and let $r = dist(0, A)$. By Theorem 3.2(1), there is $J \subset \{1, \ldots, n\}$ such that $|J| = p$ and $0 \in U_J$. After a relabeling, assume without loss of generality (for notational simplicity) that $J = \{1, \ldots, p\}$. Since U_J is open, there is $\delta > 0$ such that $2\delta < \min\{1, \varepsilon\}$ and $B(0, 2\delta) \subset U_J$. We claim that $B(0, 2\delta) \cap X_q \neq \emptyset$ for some q with $2 \leq q < p$. If possible, let the claim be false, and we will derive a contradiction.

Since the claim is assumed to be false, we must have $B(0, 2\delta) \subset X_1 \cup X_p$ because $B(0, 2\delta) \subset U_J \subset \bigcup_{j=1}^{p} X_j$ by Theorem 3.2(1). For $1 \leq j \leq p$, let $V_j = B(0, 2\delta) \cap U_j$. Note that V_j's are pairwise disjoint open subsets of $\mathbb{R}^k$, and moreover V_j's are nonempty since $\delta P_{A_j}(0) \in V_j$ by Lemma 3.3(2). Also, $B(0, 2\delta) \setminus X_p = B(0, 2\delta) \cap X_1 = \bigcup_{j=1}^{p} V_j$ because $B(0, 2\delta) \subset U_J \cap (X_1 \cup X_p)$. Since $p \geq 3$, it now follows that $B(0, 2\delta) \setminus X_p$ is not connected. On the other hand, $X_p \cap U_J \subset \bigcup_{1 \leq j < m < s \leq p} (M(A_j, A_s) \cap M(A_m, A_s))$, and the latter set is a closed subset of $\mathbb{R}^k$ having dimension at most $k - 2$ by Theorem 2.3(2), and therefore $B(0, 2\delta) \setminus X_p$ must be connected by Theorem 1.8.13 of [6]. This is the required contradiction. $\square$

Theorem 3.6. *Let $k, n \geq 2$. Then,*

(1) $\overline{X_2} = \bigcup_{p=2}^{n} X_p$.

(2) *X_2 is unbounded, and in particular $X_2 \neq \emptyset$.*

Proof. (1) If $n = 2$, then $X_2 = \mathbb{R}^k \setminus X_1$ is closed in $\mathbb{R}^k$ by the openness of X_1, and hence (1) is trivially true. So assume $n \geq 3$. We have $\overline{X_2} \subset \bigcup_{p=2}^{n} X_p$ by Theorem 3.2(5), and the reverse inclusion follows by a repeated application of Lemma 3.5.

(2) If X_2 is bounded, then $\bigcup_{p=2}^{n} X_p$ is also bounded by part (1). Recall from Theorem 3.4(1) that the open sets $U_1, \ldots, U_n$ are the connected components of X_1. Since $\mathbb{R}^k \setminus X_1 = \bigcup_{p=2}^{n} X_p$ is bounded, exactly one U_j must be unbounded. Without loss of generality, suppose U_1 is the unique unbounded component of X_1. Let $m \in \{2, \ldots, n\}$, $a_m \in A_m$, and $a_1 = P_{A_1}(a_m) \in A_1$. For every $t \in (1, \infty)$, we observe the following: the element $y_t := a_1 + t(a_m - a_1)$ satisfies $d_{A_m}(y_t) \leq \|y_t - a_m\| < \|y_t - a_1\|$ and moreover $P_{A_1}(y_t) = a_1$ by Lemma 3.3(1) so that $\|y_t - a_1\| = d_{A_1}(y_t)$. Thus $d_{A_m}(y_t) < d_{A_1}(y_t)$ and therefore $y_t \notin U_1$ for every $t \in (1, \infty)$. Since $\lim_{t \to \infty} \|y_t\| = \infty$, we get a contradiction to the the earlier assumption that the unique unbounded component of X_1 is U_1. $\square$

As X_1 is dense in $\mathbb{R}^k$, we have $\overline{X_1} = \mathbb{R}^k = \bigcup_{p=1}^{n} X_p$. Theorem 3.6(1) says that $\overline{X_2} = \mathbb{R}^k \setminus X_1 = \bigcup_{p=2}^{n} X_p$. Therefore, we may wonder whether $\overline{X_3}$ is equal to $\bigcup_{p=3}^{n} X_p$. That this is false is illustrated by the following two simple examples in $\mathbb{R}^2$. (i) Let $n = 4$ and $A_1, \ldots, A_4$ be the singletons $\{(1, 1)\}$, $\{(-1, 1)\}$, $\{(-1, -1)\}$, and $\{(1, -1)\}$. Then $X_4 = \{(0, 0)\}$ but $X_3 = \emptyset$. (ii) Let $n = 5$,

$A_1, \ldots, A_4$ be the four singletons mentioned above and let $A_5 = \{(99, 0)\}$. Then the center of the unique circle passing through the points $(1, 1)$, $(1, -1)$ and $(99, 0)$ is the only element of X_3, and in particular X_3 is closed. And $(0, 0) \in X_4 \setminus X_3 = X_4 \setminus \overline{X_3}$.

We conclude with a question. It would be interesting to obtain a deeper understanding of the structure of the sets X_p, and one of the relevant questions is the following: is it true that each X_p has only finitely many connected components?

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