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Generalized Takagi-Van der Waerden Functions and their Subdifferentials

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We prove that for every countable dense subset D of a separable Hilbert space $\mathcal{H}$, there exists a continuous function with empty subdifferential at every $x \notin D$, meanwhile for every $x \in D$ the subdifferential at x is $\mathcal{H}$.

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1. Introduction and preliminaries

Less known than Weierstrass function, the Takagi-Van der Waerden's function, provides an easy example of a continuous nowhere differentiable function (see [15], [16]). It is defined in the following way; we consider D the set of all dyadic real numbers, we decompose it as an increasing union of the following sets

$$D_n = \left\{ \frac{k}{2^n} : k \in \mathbf{Z} \right\},$$

then we define the Takagi-Van der Waerden function $T\colon \mathbb{R} \to \mathbb{R}$ as

$$T(x) = \sum_{n=1}^{\infty} d(x, D_n)$$

where $d(x, D_n)$ denotes the distance of the point x to the closed set D_n. Gora and Stern, see [11], proved that T has empty Fréchet subdifferential at every

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$x \notin D$, meanwhile its α-Holder subdifferential is $\mathbb{R}$ at every $x \in D$. Integrating a Takagi-Van der Waerden type function, it is possible to obtain a C^1 function with empty proximal subdifferential almost everywhere, see [7]. In this sense we should cite the following result: for every countable dense subset D of $\mathbb{R}$, there exists a Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\partial_P f(x) = (-1, 1)$ for every $x \in D$ and $\partial_P f(x) = \varnothing$ if $x \notin D$, see [2].

The aim of this paper is to prove that for every dense subset D of a separable real Hilbert space $\mathcal{H}$, we may construct a Takagi-Van de Waerden type function, that we will call generalized Takagi-Van der Waerden function, enjoying the same properties, namely: empty subdifferential at every $x \notin D$, and the biggest possible subdifferential, that is the whole $\mathcal{H}$, at every $x \in D$. As far as we know this result is not known even in finite dimensional euclidean spaces. On the other hand, although there are many generalizations of the Takagi-Van der Waerden function, see [1] and the references therein, we think that the generalization that we provide has not appeared in the literature before.

We introduce some nonsmooth definitions and results. As a general reference for these concepts, you may use [6], [8] or [14].

For a lower semicontinuous function $f\colon \mathcal{H} \to (-\infty, +\infty]$, and a point $x \in \operatorname{dom} f$, we define the *Fréchet subdifferential* of f at x, $\partial f(x)$, as the set of $\zeta \in \mathcal{H}$ that satisfy

$$\liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle \zeta, h \rangle}{|h|} \geq 0.$$

Let us remember that, in the finite dimensional case, for every $v \in \mathcal{H}$, we define the *one-sided subderivative* of f at x in the direction v, $df(x)(v)$, as follows:

$$df(x)(v) = \liminf_{w \to v, t \to 0^+} \frac{f(x+tw) - f(x)}{t}.$$

We will use the following property of the subdifferential, for a proof see [8] or [14] for instance.

Proposition 1.1. *If $\mathcal{H}$ is a finite dimensional Hilbert space, then $\zeta \in \partial f(x)$ if and only if*

$$\langle \zeta, v \rangle \leq df(x)(v) \quad \text{for every } v \in \mathcal{H}.$$

Smaller than the Fréchet subdifferential is the *proximal subdifferential*, $\partial_P f(x)$, which consists of all the $\zeta \in \mathcal{H}$ that satisfy the so call proximal inequality that means that there exist positive constants σ and δ, such that

$$f(x+h) \geq f(x) + \langle x, h \rangle - \sigma |h|^2$$

for every h with $|h| < \delta$. See [6] for an exhaustive study of this subdifferential. If we replace $|h|^2$ by $|h|^\alpha$, $\alpha > 1$, in the proximal inequality, we have the

α-Holder subdifferential $\partial_\alpha f(x)$. It is clear that, for every $1 < \alpha_1 < \alpha_2$,

$$\partial_{\alpha_2} f(x) \subset \partial_{\alpha_1} f(x) \subset \partial f(x).$$

From the Fréchet subdifferential, we may define the *approximate subdifferential*, $\partial_a f(x)$, see [13], as well as the *limiting subdifferential*, $\partial_L f(x)$, see [6].

From now on, $\mathcal{H}$ will be a separable real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$, norm $|\cdot|$, and open unit ball B. Let $D \subset B$ be countable and dense. Let $\{\alpha_n\} \in l^1$ be a decreasing sequence of positive numbers. We consider $\mathcal{D} = \{D_n\}_n$ and increasing sequence of discrete subsets of D, satisfying

$$D = \bigcup_n D_n \quad \text{and} \quad B \subset \bigcup_{x \in D_n} B(x, \alpha_n)$$

for every n. Such decompositions always exist (see Proposition 1.5 below).

Given D and a decomposition $\mathcal{D} = \{D_n\}_n$, we define the Takagi-Van der Waerden function $T_{\mathcal{D}} \colon B \to \mathbb{R}$ associated to the decomposition $\mathcal{D}$ of D, as

$$T_{\mathcal{D}}(x) = \sum_{k=1}^{\infty} g_k(x) = \lim_n G_n(x)$$

where $g_k(x) = d(x, D_k)$ is the distance of x to D_k, and $G_n = g_1 + \cdots + g_n$. If no confusion arises we will write T instead of $T_{\mathcal{D}}$. $T_{\mathcal{D}}$ is clearly continuous since $\{\alpha_n\} \in l^1$, and a posteriori as a consequence of Theorem 1.3, it is not locally Lipschitz. The aim of this paper is to prove that for every $x \in D$

$$\partial T_{\mathcal{D}}(x) = \mathcal{H}$$

and that for every $x \notin D$, under some restrictions on $\mathcal{D}$,

$$\partial T_{\mathcal{D}}(x) = \varnothing.$$

This is a very extreme situation, since it is known that for any function f, $\partial f(x) \neq \varnothing$ for every x in a dense subset of the domain of f, see [6] for instance. On the other hand $\# \partial f(x) \leq 1$ for a residual set and, in the finite dimensional case, $\# \partial f(x) \leq 1$ a.e. (see [10] or [3])

In the one dimensional case, it is easy to observe that for every decomposition $\mathcal{D}$ of D, the associated Takagi-Van der Waerden function satisfies the following

Theorem 1.2. *If $x \in D$, then $\partial T_{\mathcal{D}}(x) = \mathbb{R}$, otherwise $\partial T_{\mathcal{D}}(x) = \varnothing$*

Proof. If $x \in D$, there exists n_0 such that $x \in D_{n_0}$. For every $\zeta \in \mathbb{R}$ let n be such that $n > 2n_0 + |\zeta|$. If $2|h| < d\left(x, D_n \setminus \{x\}\right)$ then $g_k(x+h) = |h|$ for every k, $n_0 \le k \le n$. Thus

$$
\begin{aligned}
T_{\mathcal{D}}(x+h) - T_{\mathcal{D}}(x) - \langle \zeta, h\rangle \ge{}& \\
&\ge \sum_{k=1}^{n_0-1} \big(g_k(x+h) - g_k(x)\big) + \sum_{k=n_0}^{n} g_k(x+h) - \langle \zeta, h\rangle \ge \\
&\ge -(n_0-1)|h| + (n-n_0+1)|h| - |\zeta||h| \ge |h|.
\end{aligned}
$$

This implies that $\zeta \in \partial_\alpha T_{\mathcal{D}}(x) \subset \partial T_{\mathcal{D}}(x)$ for every $\alpha > 1$. As a matter of fact, we have proved that the function $T_{\mathcal{D}}(z) - \langle \zeta, z\rangle$ has a local minimum at x.

On the other hand, let $x \notin D$. We may assume, by means of a translation, that $x = 0$. We consider the sequence $\{x_n\}$, where $|x_n| = g_n(0)$ (if there are two possibilities, we choose one). For every n, we define $I_n = \{k \le n : x_k > 0\}$ and $J_n = \{k \le n : x_k < 0\}$.

If $\lim_n \# I_n = \lim_n \# J_n = +\infty$, we may consider a subsequence $\{x_{n_j}\} \subset (0, +\infty)$ such that $x_{n_j+1} < 0$. We have

$$
\begin{aligned}
dT_{\mathcal{D}}(0)(1) = \liminf_{t\to 0^+} \frac{T_{\mathcal{D}}(t) - T_{\mathcal{D}}(0)}{t} &\le \liminf_j \frac{T_{\mathcal{D}}(x_{n_j}) - T_{\mathcal{D}}(0)}{x_{n_j}} \le \\
&\le \liminf_j \frac{G_{n_j}(x_{n_j}) - G_{n_j}(0)}{x_{n_j}} \le \liminf_j \sum_{k=1}^{n_j} \frac{|x_{n_j} - x_k| - |x_k|}{|x_{n_j}|} \le \\
&\le \liminf_j \big(\# J_{n_j} - \# I_{n_j}\big)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
dT_{\mathcal{D}}(0)(-1) = \liminf_{t\to 0^+} \frac{T_{\mathcal{D}}(-t) - T_{\mathcal{D}}(0)}{t} &\le \liminf_j \frac{T_{\mathcal{D}}(x_{n_j+1}) - T_{\mathcal{D}}(0)}{|x_{n_j+1}|} \le \\
&\le \liminf_j \big(\# I_{n_j+1} - \# J_{n_j+1}\big).
\end{aligned}
$$

If $\zeta \in \partial T_{\mathcal{D}}(0)$ then, from Proposition 1.1, we have $\zeta \le dT_{\mathcal{D}}(0)(1)$ and $\zeta(-1) \le dT_{\mathcal{D}}(0)(-1)$. Hence

$$
\begin{aligned}
0 &\le dT_{\mathcal{D}}(0)(1) + dT_{\mathcal{D}}(0)(-1) \le \\
&\le \liminf_j \big(\# J_{n_j} - \# I_{n_j}\big) + \liminf_j \big(\# I_{n_j+1} - \# J_{n_j+1}\big) \le \\
&\le \liminf_j \big(\# J_{n_j} - \# I_{n_j} + \# I_{n_j+1} - \# J_{n_j+1}\big) = -1,
\end{aligned}
$$

since $\# I_{n_j+1} = \# I_{n_j}$ and $\# J_{n_j+1} = \# J_{n_j} + 1$, which is a contradiction.

If one of the limits is finite, then we have

$$dT_{\mathcal{D}}(0)(1) \le \liminf_n \frac{T_{\mathcal{D}}(x_n) - T_{\mathcal{D}}(0)}{x_n} \le \liminf_n \frac{G_n(x_n) - G_n(0)}{x_n} \le$$
$$\le \liminf_n \sum_{k=1}^{n} \frac{|x_n - x_k| - |x_k|}{x_n} = -\infty$$

provided that $\lim_n \#J_n < +\infty$, and analogously

$$dT_{\mathcal{D}}(0)(-1) = -\infty$$

if $\lim_n \#I_n < +\infty$. Both situations imply that $\partial T_{\mathcal{D}}(0) = \varnothing$. □

Incidentally, we remember that Garg proved in a different context that the Weierstrass function has subdifferential $\mathbb{R}$ at a countable dense subset of $[0, 1]$, and empty subdifferential otherwise, (see [9]). This example, as well as the example provided by Gora and Stern in [11], which is a particular case of Theorem 1.2, are extremal for one dimensional functions, since it is easy to observe that the subdifferential has cardinal bigger than one for a countable subset of $\mathbb{R}$ at most.

Let us observe that the proof of $\partial T(x) = \mathbb{R}$ holds for arbitrary separable Hilbert spaces provided that the sets D_n are discrete. As a matter of fact, under this conditions, it holds for Banach spaces, in particular if $\mathcal{H} = \mathbb{R}^N$ and we define the functions g_k with different norms. Hence we have:

Theorem 1.3. *For every $\zeta \in \mathcal{H}$, the function $T_{\mathcal{D}}(z) - \langle \zeta, z\rangle$ attains a minimum at every $x \in D$. In particular, we have*

$$\partial_\alpha T_{\mathcal{D}}(x) = \mathcal{H}.$$

The existence of a dense subset where the subdifferential is the whole $\mathcal{H}$, give us the following corollary:

Corollary 1.4. *The functions $T_{\mathcal{D}}$ defined as above satisfy for every $x \in \mathcal{H}$*

$$\partial_a T_{\mathcal{D}}(x) = \partial_L T_{\mathcal{D}}(x) = \mathcal{H}.$$

The existence of functions enjoying this property was known for the one dimensional case, see [17] example 8.13 (2), for instance.

This result may also be compared with the example presented in [5], where the authors provide a Lipschitz function on a separable Banach space with approximate subdifferential $\partial_a F(x) = B^*$ for every x. The existence of this kind of functions was established before by Baire Category techniques, see [4].

As we pointed out above, Theorem 1.3 does not depend on the decomposition of the set D. However, in order to prove the other statement of the main Theorem, we require some more restrictions. That is the aim of the following proposition.

Proposition 1.5. *Let $\mathcal{H}$ be a separable Hilbert space, $D \subset B$ countable and dense. Then there exist a sequence of positive real numbers, $\{\alpha_n\}$, that satisfies $\alpha_{n+1} < \frac{1}{4}\alpha_n$, and a decomposition $\mathcal{D} = \{D_n\}$ of D that satisfies:*

(1) $D = \cup_n D_n$ *and* $D_n \subset D_{n+1}$ *for every* n.

(2) $B \subset \cup_{x \in D_n} B(x, \alpha_n)$.

(3) $|x - y| \geq \frac{\alpha_n}{4}$ *for every* $x, y \in D_n$, $x \neq y$.

Proof. Let $D = \{p_n\}_n$. We define α_n and D_n inductively as follows. Let $\alpha_1 = 2$ and $D_1 = \{p_1\}$. Assuming α_n and D_n have been selected with the additional condition $p_1, \ldots, p_n \in D_n$, we choose

$$0 < \alpha_{n+1} < \min\left\{\frac{\alpha_n}{4},\, d(p_{n_0}; D_n)\right\}$$

where $n_0 = \min\{m \in \mathbb{N} :\ p_m \notin D_n\}$. Let us observe that $d\,(p_{n_0}; D_n) > 0$ by (3). As

$$A = \left\{x \in B :\ d(x; D_n \cup \{p_{n_0}\}) > \frac{3\alpha_{n+1}}{4}\right\}$$

is an open set, and D is a dense subset of B, then

$$A = \bigcup_{p \in A \cap D} B(p; r_p)$$

with $0 < r_p < \frac{\alpha_{n+1}}{5}$. Thus

$$A \subset \bigcup_{p \in A \cap D} B\left(p; \frac{\alpha_{n+1}}{5}\right)$$

and by the Vitali's Covering Lemma (see [12]) there exists a countable set $D' \subset D \cap A$ such that the $\left\{B\left(p; \frac{\alpha_{n+1}}{5}\right) :\ p \in D'\right\}$ is a collection of disjoint balls and

$$A \subset \bigcup_{p \in D'} B(p; \alpha_{n+1}).$$

Finally let $D_{n+1} = D_n \cup \{p_{n_0}\} \cup D'$. □

Remark 1.6. Let us observe that the sets D_n are discrete, and that for finite dimensional Hilbert spaces they are necessarily finite.

Definition 1.7. We say that a decomposition that satisfies property (3) of Proposition 1.5 is a *balanced decomposition.*

Of course a discrete decomposition may fail to be balanced. It is easy to find a no balanced decomposition of D, in the finite dimensional case, such that every D_n is finite, and consequently discrete.

2. Main Theorem

We have seen, Theorem 1.3 that $\partial T_{\mathcal{D}}(x) = \mathcal{H}$ for every $x \in D$. In this section we will prove that $\partial T_{\mathcal{D}}(x) = \varnothing$ for every $x \notin D$.

Let $x_0 \notin D$. By means of a translation, we may assume that $x_0 = 0$. We will assume that the decomposition $\mathcal{D} = \{D_n\}$ is a balanced decomposition, and that the sequence $\{\alpha_n\}$ that defines $\mathcal{D}$ satisfies the following inequalities

$$\alpha_{n+1} \leq \frac{1}{n^6}\alpha_n \tag{1}$$

for every n.

We begin with a lemma that give us more information about the distribution of the points of D_n.

Lemma 2.1. *For every $w \in \mathcal{H}$ with $|w| = 1$ there exists $z_{n+1} \in D_{n+1}$, satisfying $(2n^2 - 1)\alpha_{n+1} \leq |z_{n+1}| \leq (1 + 2n^2)\alpha_{n+1}$ and*

$$\left| \frac{z_{n+1}}{|z_{n+1}|} - w \right| < \frac{1}{n^2}.$$

Proof. Let $a_n = 2n^2\alpha_{n+1}$. For every $z \neq 0$

$$\left\langle \frac{z}{|z|}, w \right\rangle = \frac{1}{2a_n} \frac{|z|^2 + a_n^2 - |z - a_n w|^2}{|z|}$$

As for every r, $0 < r < a_n$, the function $\varphi(t) = t^{-1}(t^2 + a_n^2 - r^2)$ has a minimum absolute for $t > 0$ in $t = \sqrt{a_n^2 - r^2}$ then

$$\frac{t^2 + a_n^2 - r^2}{t} \geq 2\,\frac{a_n^2 - r^2}{\sqrt{a_n^2 - r^2}} = 2\,\sqrt{a_n^2 - r^2}.$$

In consequence, for every $z \in B(a_n w, \alpha_{n+1})$

$$\left\langle \frac{z}{|z|}, w \right\rangle \geq \frac{1}{a_n}\sqrt{a_n^2 - |z - a_n w|^2} > \sqrt{1 - \frac{1}{4n^4}} \geq 1 - \frac{1}{4n^4}. \tag{2}$$

As $B(a_n w, \alpha_{n+1}) \cap D_{n+1} \neq \varnothing$ there exists $z_{n+1} \in D_{n+1}$ satisfying (2), and therefore

$$\left| w - \frac{z_{n+1}}{|z_{n+1}|} \right|^2 = 2 - 2\left\langle w, \frac{z_{n+1}}{|z_{n+1}|} \right\rangle < \frac{1}{2n^4}.$$

Moreover

$$(2n^2 - 1)\alpha_{n+1} = a_n - \alpha_{n+1} \leq |z_{n+1}| \leq \alpha_{n+1} + a_n = (1 + 2n^2)\alpha_{n+1}. \qquad \square$$

In order to study the subdifferential of $T = T_{\mathcal{D}}$, we require to estimate incremental quotients of the form

$$\frac{g_n(z) - g_n(0)}{|z|}.$$

That is the aim of the following two lemmas.

Lemma 2.2. *Let $\{x_k\}_{k=1}^n$ be a finite sequence of elements of $\mathcal{H} \setminus \{0\}$. If, for every $k \in \{1, \dots, n\}$, we denote $v_k = \frac{x_k}{|x_k|}$ and $s_n = v_1 + \cdots + v_n$, then for every $z \neq 0$ and $k = 1, \dots, n$,*

$$\frac{|x_k - z| - |x_k|}{|z|} \leq -\left\langle v_k, \frac{z}{|z|} \right\rangle + \frac{|z|}{2|x_k|}$$

and consequently

$$\sum_{k=1}^n \frac{|x_k - z| - |x_k|}{|z|} \leq -\left\langle s_n, \frac{z}{|z|} \right\rangle + \frac{1}{2} \sum_{k=1}^n \frac{|z|}{|x_k|}.$$

Proof. We have that

$$\frac{|x_k - z| - |x_k|}{|z|} = \frac{|x_k - z|^2 - |x_k|^2}{|z|(|z - x_k| + |x_k|)} = \frac{|z| - 2\langle x_k, \frac{z}{|z|}\rangle}{|z - x_k| + |x_k|}.$$

If $|z|^2 - 2\langle x_k, z\rangle \geq 0$, then

$$|x_k - z|^2 = |x_k|^2 + |z|^2 - 2\langle x_k, z\rangle \geq |x_k|^2$$

and consequently

$$\frac{|x_k - z| - |x_k|}{|z|} \leq \frac{|z| - 2\langle x_k, \frac{z}{|z|}\rangle}{2|x_k|} = -\left\langle v_k, \frac{z}{|z|} \right\rangle + \frac{|z|}{2|x_k|}.$$

Otherwise, if $|z|^2 - 2\langle x_k, z\rangle < 0$, then $|x_k - z|^2 < |x_k|^2$ and

$$\frac{|x_k| - |x_k - z|}{|z|} = \frac{-|z| + 2\langle x_k, \frac{z}{|z|}\rangle}{|z - x_k| + |x_k|} \geq \frac{-|z| + 2\langle x_k, \frac{z}{|z|}\rangle}{2|x_k|}$$

hence
$$\frac{|x_k - z| - |x_k|}{|z|} \leq -\left\langle v_k, \frac{z}{|z|} \right\rangle + \frac{|z|}{2|x_k|}.$$ □

Lemma 2.3. *For every n there exists a vector u_n such that*

$$\limsup_{x \to 0} \frac{g_n(x) - g_n(0) - \langle u_n, x\rangle}{|x|} \leq 0. \tag{3}$$

Proof. There exists a sequence $\{z_k\}_k \subset D_n$ and a vector z such that $\{|z_k|\}_k$ converges to $g_n(0)$ and $\{z_k\}_k$ converges weakly to z. As, for every k,

$$g_n^2(x) \leq |x - z_k|^2 = |x|^2 + |z_k|^2 - 2\langle x, z_k\rangle$$

then $g_n^2(x) \leq |x|^2 + g_n^2(0) - 2\langle x, z\rangle$. Therefore, if $u_n = -\frac{z}{g_n(0)}$, for $x \neq 0$,

$$\begin{aligned}\frac{g_n(x) - g_n(0) - \langle u_n, x\rangle}{|x|} &= \frac{1}{|x|}\frac{g_n^2(x) - g_n^2(0)}{g_n(x) + g_n(0)} + \langle \frac{z}{g_n(0)}, \frac{x}{|x|}\rangle \leq \\ &\leq \frac{|x|}{g_n(x) + g_n(0)} + \langle z, \frac{x}{|x|}\rangle \left(\frac{1}{g_n(0)} - \frac{2}{g_n(x) + g_n(0)}\right)\end{aligned}$$

and then

$$\begin{aligned}&\limsup_{x\to 0} \frac{g_n(x) - g_n(0) - \langle u_n, x\rangle}{|x|} \leq \\ &\quad \leq \lim_{x\to 0}\left[\frac{|x|}{g_n(x) + g_n(0)} + \langle z, \frac{x}{|x|}\rangle \left(\frac{1}{g_n(0)} - \frac{2}{g_n(x) + g_n(0)}\right)\right] = 0. \qquad \square\end{aligned}$$

We cannot guarantee the existence of $x_k \in D_k$ such that $g_k(0) = |x_k|$ unless $\dim\mathcal{H} < +\infty$, however, the fact that $\mathcal{D}$ is a balanced decomposition, implies the following

Remark 2.4. If $g_n(0) < \frac{\alpha_n}{8}$ then there exists a $x_n \in D_n$ such that $g_n(0) = |x_n|$. Moreover x_n is unique.

We will denote $\mathcal{N} = \{n : |g_n(0)| \geq n^{-2}\alpha_n\}$. We will distinguish two situations according to the finiteness of $\mathcal{N}$. First we assume that $\mathcal{N}$ is finite.

Lemma 2.5. *Let us assume that $\mathcal{N} = \varnothing$. If for every n, $x_n \in D_n$ is such that $|x_n| = g_n(0)$ and $s_n = \sum_{k=1}^n \frac{x_k}{|x_k|}$ then, for $n \geq 5$,*

$$\frac{T(x_n) - T(0)}{|x_n|} \leq -\left\langle s_n, \frac{x_n}{|x_n|}\right\rangle + \frac{1}{n}. \tag{4}$$

Proof. Let $n \geq 5$. Let m, $1 < m \leq n$, be such that $g_k(0) = g_n(0)$ for $m \leq k \leq n$ and $g_{m-1}(0) > g_m(0)$. Note that m can be equal to n. As $x_n \in D_n$ and, for $m \leq k \leq n$, $|x_n| = g_n(0) = g_k(0) \leq \frac{\alpha_n}{8}$ then $x_n = x_k \in D_k$ and therefore $g_k(x_n) = 0$. On the other hand

$$|x_n - x_{m-1}| = |x_m - x_{m-1}| \geq \frac{\alpha_m}{4}$$

and consequently

$$|x_{m-1}| \geq |x_n - x_{m-1}| - |x_n| \geq \frac{\alpha_m}{4} - \frac{\alpha_n}{n^2} \geq \frac{\alpha_m}{5}.$$

Hence, by Lemma 2.2

$$\begin{aligned}\frac{T(x_n)-T(0)}{|x_n|} &\le \frac{G_n(x_n)-G_n(0)}{|x_n|} = \frac{G_{m-1}(x_n)-G_{m-1}(0)}{|x_n|} + \sum_{k=m}^{n} \frac{-g_k(0)}{|x_n|} \le \\ &\le -\Big\langle s_{m-1}, \frac{x_n}{|x_n|}\Big\rangle + \frac{m-1}{2}\frac{|x_n|}{|x_{m-1}|} - (n-m+1) = \\ &= -\Big\langle s_n, \frac{x_n}{|x_n|}\Big\rangle + \frac{m-1}{2}\frac{|x_n|}{|x_{m-1}|} \le -\Big\langle s_n, \frac{x_n}{|x_n|}\Big\rangle + \frac{5(m-1)\alpha_n}{2\alpha_m} \le \\ &\le -\Big\langle s_n, \frac{x_n}{|x_n|}\Big\rangle + \frac{5}{2(n-1)^5} \le -\Big\langle s_n, \frac{x_n}{|x_n|}\Big\rangle + \frac{1}{n}. \qquad \square\end{aligned}$$

From this lemma we may deduce the result when $\mathcal{N}$ is finite.

Proposition 2.6. *If there exists n_0 such that $g_n(0) < \alpha_n n^{-2}$ for all $n \ge n_0$ then $\partial T(0) = \varnothing$.*

Proof. First let assume that $n_0 = 1$. Let $\zeta \in \mathcal{H}$. By Lemma 2.5, for $n \ge 5$

$$\frac{T(x_n)-T(0)-\langle \zeta, x_n\rangle}{|x_n|} \le -\Big\langle s_n+\zeta, \frac{x_n}{|x_n|}\Big\rangle + \frac{1}{n}.$$

In consequence if $\limsup \Big\langle s_n+\zeta, \frac{x_n}{|x_n|}\Big\rangle > 0$ then $\zeta \notin \partial T(0)$. On the other hand, if $\limsup \Big\langle s_n+\zeta, \frac{x_n}{|x_n|}\Big\rangle \le 0$ there exists n_1 such that

$$\Big\langle s_n+\zeta, \frac{x_n}{|x_n|}\Big\rangle \le \frac{1}{4}$$

for every $n \ge n_1$. This implies that

$$|s_{n-1}+\zeta|^2 = |s_n+\zeta|^2 + 1 - 2\Big\langle s_n+\zeta, \frac{x_n}{|x_n|}\Big\rangle \ge |s_n+\zeta|^2 + \frac{1}{2}$$

and therefore that $\{|s_n+\zeta|\}_{n\ge n_1}$ is a decreasing sequence. If $\ell = \lim |s_n+\zeta| = \inf\{|s_n+\zeta| :\ n \ge n_1\}$ then $\ell^2 \ge \ell^2 + \frac{1}{2}$ that is a contradiction.

If $n_0 > 1$, let $T_1 = \sum_{n\ge n_0} g_n$. Then $T = T_1 + G_{n_0-1}$. Let $\zeta \in \mathcal{H}$. By Lemma 2.3 we have

$$\limsup_{x\to 0} \frac{G_{n_0-1}(x) - G_{n_0-1}(0) - \langle w_{n_0-1}, x\rangle}{|x|} \le 0 \tag{5}$$

where $w_{n_0-1} = u_1 + \cdots + u_{n_0-1}$. Arguing as in the first part of the proof it can be shown that there exists a sequence $\{z_n\}_n$ that converges to 0 such that

$$\ell_1 = \lim_n \frac{T_1(z_n) - T_1(0) - \langle \zeta + s_{n_0-1}, z_n\rangle}{|z_n|} < 0.$$

We have, by (5), that

$$\liminf_{x\to 0} \frac{T(x)-T(0)-\langle \zeta, x\rangle}{|x|} \leq \liminf_{n} \frac{T(z_n)-T(0)-\langle \zeta, z_n\rangle}{|z_n|} \leq$$
$$\leq \liminf_{n} \frac{G_{n_0-1}(z_n)-G_{n_0-1}(0)+\langle s_{n_0-1}, z_n\rangle}{|z_n|} + \ell_1 \leq \ell_1 < 0.$$

Therefore $\zeta \notin \partial T(0)$. □

Once established the result when $\mathcal{N}$ is finite, we consider the alternative case. If $\mathcal{N}$ is infinite, we may write $\mathcal{N}\cup\{1\}$ as an increasing sequence of positive integers, $\{n_k\}_{k\geq 0}$ where $n_0 = 1$. Our aim is to construct a suitable sequence $\{s_n\}$ that will play a similar role than that one of Lemma 2.5. We will define it by blocks.

Given $\varepsilon > 0$, and N, we construct a finite sequence $\{x_k\}_{k=1}^N$ as follows: if there exists some $x_k \in D_k$ such that $g_k(0) = |x_k|$, we choose it (more precisely one of them). Otherwise we choose $x_k \in D_k$ satisfying $|x_k| - g_k(0) \leq \frac{\varepsilon}{N}$. It is clear that we may construct this sequence satisfying $|x_1| \geq \cdots \geq |x_N|$. We define $\tilde{s}_n = \sum_{k=1}^n \frac{x_k}{|x_k|}$ for every $n \leq N$. Moreover, if we denote $m_n = \#\{k:\ 1 \leq k \leq n,\ g_k(0) \neq |x_k|\}$, by Lemma 2.2 we deduce

Remark 2.7.

$$\frac{G_n(z)-G_n(0)}{|z|} \leq -\Big\langle \tilde{s}_n, \frac{z}{|z|}\Big\rangle + \frac{1}{2}\sum_{k=1}^{n} \frac{|z|}{|x_k|} + \frac{m_n}{N}\,\frac{\varepsilon}{|z|} \tag{6}$$

for every $z \neq 0$, and every $n \leq N$.

We proceed to define the sequence $\{s_n\}$. If $j \geq 1$, for $n_{2(j-1)} \leq n < n_{2j}$, let s_n be $\tilde{s}_n$ as above, with $N = n_{2j}$ and $\varepsilon = \frac{g_{n_{2j}}(0)}{n_{2j}}$. The following Lemma summarizes the properties of this sequence.

Lemma 2.8. *If $\mathcal{N}$ is infinite, then there exists a sequence $\{s_n\}_n \in \mathcal{H}$ such that, for every n:*

(1) $\quad s_{n+1} = s_n + \dfrac{x_{n+1}}{|x_{n+1}|}$ *if* $|x_{n+1}| = g_{n+1}(0) < \dfrac{\alpha_{n+1}}{(n+1)^2}$,

(2) $\quad \dfrac{G_n(z)-G_n(0)}{|z|} \leq -\Big\langle s_n, \dfrac{z}{|z|}\Big\rangle + \dfrac{n}{2}\dfrac{|z|}{|g_n(0)|} + \dfrac{\alpha_{n+1}}{|z|},\ z \neq 0,$

(3) $\quad \dfrac{G_{n-1}(x_n)-G_{n-1}(0)}{|x_n|} \leq -\Big\langle s_n, \dfrac{x_n}{|x_n|}\Big\rangle + \dfrac{n}{2}\dfrac{|x_n|}{g_{n-1}(0)} + \dfrac{1}{n}$ *if* $g_n(0) < \dfrac{\alpha_n}{n^2}$,

(4) *if* $|s_n - s_{n-1}| \neq 1$ *then* $|s_{n+1} - s_n| = 1$.

Proof. First, let us note that if $m \neq n_{2k}$ for all $k = 0, 1, \dots,$ then there exists $j \geq 1$ such that $n_{2j-2} \leq m-1 < m < n_{2j}$ and then

$$|s_m - s_{m-1}| = |\tilde{s}_m - \tilde{s}_{m-1}| = \left| \frac{x_m}{|x_m|} \right| = 1,$$

since m and $m-1$ belong to the same block. In particular, if $|s_m - s_{m-1}| \neq 1$ then $m = n_{2k}$ for some $k \geq 0$. This proves (4) since n_{2k} and $n_{2k}+1$ always belong to the same block.

On the other hand, if $|x_{n+1}| = g_{n+1}(0) < \dfrac{\alpha_{n+1}}{(n+1)^2}$ then $n+1 \notin \mathcal{N}$, and therefore $n+1 \neq n_k$ for all k, this implies that n and $n+1$ belong to the same block, hence $s_{n+1} = s_n + \frac{x_{n+1}}{|x_{n+1}|}$ that proves (1).

Moreover if $j \geq 1$ is such that $n_{2j-2} \leq n-1 < n < n_{2j}$ then, by equation (6) and our choice of s_n,

$$\frac{G_{n-1}(x_n) - G_{n-1}(0)}{|x_n|} \leq -\left\langle s_{n-1}, \frac{x_n}{|x_n|} \right\rangle + \frac{n}{2} \frac{|x_n|}{g_{n-1}(0)} + \frac{g_{n_{2j}}(0)}{n_{2j}} \frac{1}{|x_n|}.$$

and as $|x_n| = g_n(0) \geq g_{n_{2j}}(0)$ it follows (3).

Finally, from equation (6) it is immediate that $\{s_n\}_n$ verifies (2) since if n belongs to the block $[n_{2(j-1)}, n_{2j})$, then $n+1 \leq n_{2j} = N$ and consequently $\varepsilon < \alpha_{n_{2j}} \leq \alpha_{n+1}$. $\square$

Let us remember the condition on the sequence $\{\alpha_n\}$, that is $\alpha_{n+1} \leq \frac{1}{n^6}\alpha_n$. We will use it in the next lemma.

Lemma 2.9. *Let us assume that $\mathcal{N}$ is infinite. If $g_n(0) \geq \frac{\alpha_n}{n^2}$, and s_n is as in Lemma 2.8, then for every w with $|w| = 1$ there exists $z_{n+1} \in D_{n+1}$, satisfying*

(1) $(2n^2 - 1)\alpha_{n+1} \leq |z_{n+1}| \leq (1 + 2n^2)\alpha_{n+1}$,

(2) $\left| \dfrac{z_{n+1}}{|z_{n+1}|} - w \right| < \dfrac{1}{n^2}$, *and*

(3) $\dfrac{T(z_{n+1}) - T(0)}{|z_{n+1}|} \leq \dfrac{G_n(z_{n+1}) - G_n(0)}{|z_{n+1}|} \leq -\left\langle s_n, \dfrac{z_{n+1}}{|z_{n+1}|} \right\rangle + \dfrac{3}{n}$.

Proof. Let z_{n+1} be as in Lemma 2.1. It remains to prove (3). By Lemma 2.8,

$$\begin{aligned}
\frac{T(z_{n+1}) - T(0)}{|z_{n+1}|} &\leq \frac{G_n(z_{n+1}) - G_n(0)}{|z_{n+1}|} \leq -\left\langle s_n, \frac{z_{n+1}}{|z_{n+1}|} \right\rangle + n\frac{|z_{n+1}|}{2g_n(0)} + \frac{1}{n} \leq \\
&\leq -\left\langle s_n, \frac{z_{n+1}}{|z_{n+1}|} \right\rangle + \frac{n^3(2n^2+1)\alpha_{n+1}}{2\alpha_n} + \frac{1}{n} \leq \\
&\leq -\left\langle s_n, \frac{z_{n+1}}{|z_{n+1}|} \right\rangle + \frac{n^3(2n^2+1)}{2n^6} + \frac{1}{n} \leq -\left\langle s_n, \frac{z_{n+1}}{|z_{n+1}|} \right\rangle + \frac{3}{n}.
\end{aligned}$$

$\square$

We deduce that, when $g_n(0) \geq \frac{\alpha_n}{n^2}$, we have the following estimation.

Corollary 2.10. *Let us assume that $\mathcal{N}$ is infinite. If $g_n(0) \geq \frac{\alpha_n}{n^2}$, and s_n is as in Lemma 2.8, then for every w with $|w| = 1$ there exists $z_{n+1} \in D_{n+1}$ such that*

$$\frac{T(z_{n+1}) - T(0)}{|z_{n+1}|} \leq -\langle s_n, w\rangle + \frac{4}{n}.$$

Proof. It is enough to observe that

$$\left|\langle s_n, w\rangle - \left\langle s_n, \frac{z_{n+1}}{|z_{n+1}|}\right\rangle\right| \leq |s_n| \left|w - \frac{z_{n+1}}{|z_{n+1}|}\right| < n\frac{1}{n^2} = \frac{1}{n}. \qquad \square$$

Alternatively, if $g_n(0) < \frac{\alpha_n}{n^2}$ we have:

Lemma 2.11. *Let us assume that $\mathcal{N}$ is infinite. If $g_n(0) = |x_n| < \frac{\alpha_n}{n^2}$, $n \geq 4$, and s_n is as in Lemma 2.8, then $g_n(0) < g_{n-1}(0)$ and*

$$\frac{T(x_n) - T(0)}{|x_n|} \leq \frac{G_n(x_n) - G_n(0)}{|x_n|} \leq \left\langle s_n, \frac{x_n}{|x_n|}\right\rangle + \frac{4}{n}.$$

Proof. For all $x \in D_{n-1}$

$$|x| \geq |x_n - x| - |x_n| \geq \frac{\alpha_n}{4} - \frac{\alpha_n}{n^2} \geq \frac{\alpha_n}{5}$$

and therefore $g_{n-1}(0) \geq \frac{\alpha_n}{5}$. Thus, by Lemma 2.8,

$$\begin{aligned}\frac{G_n(x_n) - G_n(0)}{|x_n|} &\leq -1 - \left\langle s_{n-1}, \frac{x_n}{|x_n|}\right\rangle + \frac{n}{2}\frac{|x_n|}{g_{n-1}(0)} + \frac{1}{n} \leq \\ &\leq -\left\langle s_n, \frac{x_n}{|x_n|}\right\rangle + \frac{5n\alpha_n}{2n^2\alpha_n} + \frac{1}{n}.\end{aligned}$$

Hence

$$\frac{T(x_n) - T(0)}{|x_n|} \leq \frac{G_n(x_n) - G_n(0)}{|x_n|} \leq -\left\langle s_n, \frac{x_n}{|x_n|}\right\rangle + \frac{4}{n}. \qquad \square$$

From Corollary 2.10 and Lemma 2.11, we deduce the result when $\mathcal{N}$ is infinite.

Proposition 2.12. *If $\mathcal{N}$ is infinite then $\partial T(0) = \varnothing$.*

Proof. Let $\{s_n\}_n$ be as in Lemma 2.8 and let $\zeta \in \mathcal{H}$. For every $n \in \mathcal{N}$ there exists w_n with $|w_n| = 1$ such that $|s_n + \zeta| = \langle z_n + \zeta, w_n\rangle$. By Corollary 2.10 there exists $z_{n+1} \in D_{n+1}$ such that

$$\frac{T(z_{n+1}) - T(0) - \langle \zeta, z_{n+1}\rangle}{|z_{n+1}|} \leq -\langle s_n + \zeta, w_n\rangle + \frac{4}{n} = -|s_n + \zeta| + \frac{4}{n}.$$

If $\limsup_{n\in\mathcal{N}} |s_n + \zeta| > 0$ then $\zeta \notin \partial T(0)$. If $\lim_{n\in\mathcal{N}} s_n = -\zeta$ then there exists $n_0 \in \mathcal{N}$ such that $|s_n - s_m| < \frac{1}{2}$ for all $n, m \in \mathcal{N}$, $n, m \geq n_0$. In particular, by Lemma 2.8(4), there exist infinite indices $n > n_0$ such that $n+1 \notin \mathcal{N}$ and $n \in \mathcal{N}$. For these indices $s_{n+1} - s_n = \frac{x_{n+1}}{|x_{n+1}|}$ and, by Lemma 2.11,

$$\begin{aligned}\frac{T(x_{n+1}) - T(0) - \langle \zeta, x_{n+1}\rangle}{|x_{n+1}|} &\leq -\Big\langle s_{n+1} + \zeta, \frac{x_{n+1}}{|x_{n+1}|}\Big\rangle + \frac{4}{n+1} = \\ &= -\Big\langle s_n + \zeta, \frac{x_{n+1}}{|x_{n+1}|}\Big\rangle - 1 + \frac{4}{n+1} \xrightarrow[n\to\infty]{} -1\,,\end{aligned}$$

and therefore $\zeta \notin \partial T(0)$. □

Finally, we deduce the Theorem from Propositions 2.6 and 2.12.

Theorem 2.13. *For every $x \notin D$ we have $\partial T(x) = \varnothing$.*

Joining both Theorem 1.3 and Theorem 2.13, we obtain the result we are looking for.

Theorem 2.14. *If D is a countable dense subset of the unit ball of a separable Hilbert space, $\mathcal{H}$, then there exists a decomposition of D, such that the associated Takagi-Van der Waerden function T satisfies:*

$$\partial T(x) = \mathcal{H} \quad \textit{for every} \quad x \in D$$

and $\partial T(x) = \varnothing$ otherwise.

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