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On Singular Sets of H-Monotone Maps*

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The purpose of our paper is to extend a result of Alberti and Ambrosio about singularity sets of monotone multivalued maps to the sub-Riemmanian setting of Heisenberg groups. We prove that the k-th horizontal singular set of a H-monotone multivalued map of the Heisenberg group $\mathbb{H}^n$, with values in $\mathbb{R}^{2n}$, is $(2n+1-k, \mathbb{H})$-rectifiable.

Keywords: Heisenberg group, H-monotonicity, singular set, convex functions.

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1. Introduction

Alberti and Ambrosio [1] studied various properties of monotone set valued maps defined on Euclidean spaces $\mathbb{R}^n$. A set valued map T from $\mathbb{R}^n$ to $\mathbb{R}^n$, denoted by $T\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, is said to be *monotone* if for any pair of points x_1 and $x_2 \in \mathbb{R}^n$ and for every $y_1 \in T(x_1)$ and $y_2 \in T(x_2)$, it holds that $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean scalar product. We say that T is *maximal* if it is maximal with respect to the inclusion in the class of monotone maps.

It is clear that this concept is a generalization of monotonicity of functions of the real line $\mathbb{R}$ to higher dimensional spaces. It has several applications, for instance in the theory of optimal mass transportation. To illustrate this, let us recall that if $u\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a convex function, then the subdifferential $x \rightrightarrows \partial u(x) = \{y \in \mathbb{R}^n \,|\, u(z) \geq u(x) + \langle y, x - z\rangle\,, \forall\, y \in \mathbb{R}^n\}$ is a set valued maximal monotone map. Optimal transportation maps with respect to the quadratic cost between two absolut continuous probability measures are given by gradients of convex functions ∇u and are well defined at the points where ∂u is single valued. Therefore, it is important to understand the size of the singularity sets of ∂u i.e. the sets where ∂u is single valued.

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More generally, in [1] the following question has been studied: can we classify points $x \in \mathbb{R}^n$ according to the size of $T(x)$ in case of a monotone mapping $T\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$? In order to study this question, we define the *k-th singular set* of T:

$$\Sigma^k(T) = \{x \in \mathbb{R}^n \mid \dim T(x) \geq k\}.$$

In their paper, Alberti and Ambrosio proved that, for each $k = 1, ..., n$, the set $\Sigma^k(T)$ is at most $(n-k)$-rectifiable. The idea of the proof uses Minty's theorem according to which the resolvent map $(T+I)^{-1}$ is a single valued Lipschitz function.

The goal of our paper is to prove a sub-Riemannian counterpart of the result of Alberti and Ambrosio. We focus our attention to the setting of the Heisenberg group $\mathbb{H}^n$, that is the simplest Carnot group of step 2 with one dimensional center $\mathbb{T} = \{(x, y, t) \in \mathbb{R}^{2n+1} \mid (x, y) = (0, 0)\}$. Let us recall that points $p \in \mathbb{H}^n$ have coordinates $p = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. (Later on we will use also the notation $p = (p_1, ..., p_{2n}, p_{2n+1}) \in \mathbb{H}^n$.) The group operation in $\mathbb{H}^n$ is given by

$$p \cdot p' = (x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2\, \langle J(x, y)\, ,\, (x', y')\rangle)\,,$$

where J is the standard $(2n \times 2n)$-symplectic matrix. The identity element of $\mathbb{H}^n$, denoted by e, is the null vector of $\mathbb{R}^{2n+1}$ and the inverse of p is $p^{-1} = (x, y, t)^{-1} = (-x, -y, -t)$.

Let us also recall the so-called *Korányi norm*, defined for $p = (x, y, t) \in \mathbb{H}^n$ as

$$\|p\| = \sqrt[4]{\|(x, y)\|^4_{\mathbb{R}^{2n}} + |t|^2},$$

and the associated left invariant metric by $d(p, q) = \|p^{-1}q\|$. We denote by $B(p, r)$ the closed ball with center $p \in \mathbb{H}^n$ and radius $r > 0$ with respect to this metric and by $B_{\mathbb{R}^{2n}}(z, r)$ the Euclidean ball in $\mathbb{R}^{2n}$ with center $z \in \mathbb{R}^{2n}$ and radius $r > 0$. We note that while the Korányi metric generates the Euclidean topology, it is not bi-Lipschitz equivalent to the Euclidean metric. For example the Hausdorff dimension of $\mathbb{H}^n$ with respect to the Korányi metric is equal to $2n + 2$ that is strictly greater than its topological dimension making $(\mathbb{H}^n, d)$ a singular metric space of fractal type. This observation holds true for the whole class of metrics that are bi-Lipschitz equivalent to d, including all left invariant and homogeneous metrics on $\mathbb{H}^n$. For the intrinsic geometry of $\mathbb{H}^n$ the concept of *horizontal plane* plays an important role. This is defined at a point $p \in \mathbb{H}^n$ by left translation, as the set $p \cdot H_0$, where H_0 is identified with $\mathbb{R}^{2n}$. We note that, for every p and $p' \in \mathbb{H}^n$, $p \in H_{p'}$ if and only if $p' \in H_p$.

Calogero and Pini [8] introduced the notion of horizontal monotonicity, or *H-monotonicity* as a generalization of the notion of monotonicity to the setting of the Heisenberg group as follows. A set valued map $T\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$ is H-monotone if $\langle \xi_1(p) - \xi_1(p'), v - v'\rangle \geq 0$, for every p and $p' \in \mathbb{H}^n$ with $p' \in H_p$ and for every $v \in T(p)$ and $v' \in T(p')$, where $\langle \cdot, \cdot\rangle$ stands for the standard Euclidean scalar product of $\mathbb{R}^{2n}$. In the above, we denote by $\xi_1(x, y, t) = (x, y) \in \mathbb{R}^{2n}$ the projection of $\mathbb{R}^{2n}$. In the same way as in Euclidean case, one can consider maximal H-monotone maps. From the definition it is easy to see that $T\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$ is maximal H-monotone, then $T(p)$ is a closed and convex set in $\mathbb{R}^{2n}$, for every $p \in \mathbb{H}^n$.

Let us note, however, two important differences between this concept and its Euclidean counterpart. The first such difference is the fact that the target space is $\mathbb{R}^{2n}$ that is one topological dimension less than $\mathbb{H}^n$ and therefore there is no single valued Lipschitz resolvent as in the case of Minty's theorem. The second difference is, that monotonicity in this new setting provides information only for pairs of points p, p' for which $p' \in H_p$. This is a major restriction in our assumptions, which roughly speaking means that monotonicity holds only along horizontal directions. Nevertheless, in a recent work Balogh, Calogero and Pini [5] proved that if the effective domain of T, $\mathrm{dom}(T) = \{p \in \mathbb{H}^n \mid T(p) \neq \emptyset\}$, is the whole $\mathbb{H}^n$, then T is upper semicontinuous at every point $p \in \mathbb{H}^n$.

The purpose of the present paper is to study the Hausdorff dimension of the set of points $p \in \Sigma^k(T)$ such that $T(p)$ is contained, in some sense, in a horizontal subgroups of $\mathbb{H}^n$. This fact allows us to use the geometric measure theoretical machinery related to the decomposition in homogeneous subgroups of $\mathbb{H}^n$ developed in the recent years by Franchi, Mattila, Serapioni and Serra Cassano [19, 14, 15].

To be more precise, let us denote $\Sigma^k_H(T)$ the *horizontal singularity set* of dimension k defined as follows. In order to introduce this concept let us note first that $T(p) \subset \mathbb{R}^{2n}$ is a convex set. Let k be the dimension of this set. Then there is a unique affine subspace of dimension k of $\mathbb{R}^{2n}$ containing it. Let us denote by V_p the (unique) vector space generating that affine space. In this way we can define

$$\Sigma^k_H(T) \;=\; \left\{p \in \mathbb{H}^n \;\middle|\; V_p \in G^H(2n,k)\right\},$$

where $G^H(2n,k)$ is the Grassmannian of horizontal k-dimensional subspaces of $\mathbb{R}^{2n}$ (see [6]) defined as

$$G^H(2n,k) \;=\; \{V \in G(2n,k) \mid \langle v, Ju\rangle = 0,\, \forall v, u \in V\}. \tag{1}$$

The statement of our main result is as follows:

Theorem 1.1. *Let $T\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$ be a maximal H-monotone set valued map with $\mathrm{dom}(T) = \mathbb{H}^n$. If $1 \le k \le n$, then $\Sigma^k_H(T)$ is $(d, \mathbb{H})$-rectifiable, for $d = 2n+1-k$.*

In the above statement we use the appropriate sub-Riemannian version of rectifiability that has been recently developed in [16, 14, 15, 13] [3], [4], [10]. Recall that for $n+1 \le d \le 2n$ we say that a subset $S \subset \mathbb{H}^n$ is a d-dimensional $\mathbb{H}$-*regular surface* if it is locally the zero-level set of a function $f\colon \mathcal{U} \subset \mathbb{H}^n \longrightarrow \mathbb{R}^{2n+1-d}$, for $\mathcal{U}$ open, such that f is continuous and its Pansu differential [20] $\nabla_H f$ exists and it is continuous and surjective (see [12]).

We say that $E \subseteq \mathbb{H}^n$ is $(d, \mathbb{H})$-*rectifiable* if there exists a sequence of d-dimensional $\mathbb{H}$-regular surfaces $(S_i)_{i\in\mathbb{N}}$ such that $\mathcal{S}^{d+1}\left(E \setminus \cup_{i\in\mathbb{N}} S_i\right) = 0$, where with $\mathcal{S}^{d+1}$ we denote the $(d+1)$-dimensional spherical Hausdorff measure with respect to the Korányi metric.

Note that our result implies that the Hausdorff dimension with respect to the Korányi metric of $\dim_H \Sigma_H^k(T)$ is at most $2n+2-k$.

In the proof of our theorem we use in an essential way the main result of [5] about the upper semicontinuity of T and the condition of $(d, \mathbb{H})$ rectifiabiltiy due to Mattila, Serapioni and Serra Cassano [19] in terms of *intrinsic cones*.

The paper is organized as follows: in Section 2 we recall notations and preliminary results about homogeneous subgroups and intrinsic cones in $\mathbb{H}^n$. Section 3 is devoted to the proof of Theorem 1.1. Finally, in Section 4, we conclude with applications and final remarks.

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2. Complementary subgroups and intrinsic cones in $\mathbb{H}^n$

In Euclidean spaces, one can decompose $\mathbb{R}^n$ in the Cartesian product of two subspaces. Now, following [4], we aim to introduce in $\mathbb{H}^n$ an analogous decompositon.

We consider the class of homogeneous subgroups of the Heisenberg group: we say that a subgroup of $\mathbb{H}^n$ is *homogeneous* if it is invariant under the intrinsic dilations $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$, for $\lambda \in \mathbb{R}^+$. A homogeneous subgroup $\mathbb{G}$ can be endowed with a norm, that is the restriction to $\mathbb{G}$ of the Korányi norm.

One can classify homogeneous subgroups as follows: they are *horizontal*, if they are contained in $\{(x, y, t) \in \mathbb{H}^n \mid t = 0\}$, or *vertical*, if they contain the subgroup $\mathbb{T}$. It is important to notice that a horizontal subgroup $\mathbb{V}$ is an element of $G^H(2n, k)$ and therefore its dimension k will satisfy with $1 \leq k \leq n$. In this case the algebraic and the Hausdorff dimensions coincide. On the other hand, a vertical subgroup $\mathbb{W}$ can have any dimension $1 \leq d \leq 2n+1$, and its Hausdorff dimension is $d+1$.

We are now ready to introduce a notion of decomposition in homogeneous subgroups of $\mathbb{H}^n$. We say that $\mathbb{H}^n$ is a *semidirect product* of homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$, and we write $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, if $\mathbb{W}$ is a vertical normal subgroup and $\mathbb{V}$ a horizontal subgroup. Moreover, we assume that $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups, i.e. $\mathbb{W} \cap \mathbb{V} = \{e\}$. This implies that for any $p \in \mathbb{H}^n$ there is a unique $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{V}} \in \mathbb{V}$ such that $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}}$. This unique decomposition generates vertical and horizontal projections $p \to p_{\mathbb{W}}$ and $p \to p_{\mathbb{V}}$.

Following [19], for $n+1 \leq d \leq 2n$ a vertical normal subgroup $\mathbb{W}$ of algebraic dimension d belongs to the *intrinsic Grassmannian of the d-subgroups*, denoted by $\mathcal{G}(\mathbb{H}^n, d)$. One can easily show that $\mathcal{G}(\mathbb{H}^n, d)$ endowed with the natural metric $d(\mathbb{W}_1, \mathbb{W}_2) = \sup_{\|p\|=1} \left\| p_{\mathbb{W}_2}^{-1} \cdot p_{\mathbb{W}_1} \right\|$ is a compact metric space (see [21]).

We recall from [14] the notion of *intrinsic closed cone* with base $\mathbb{W}$, axis $\mathbb{V}$, vertex $q \in \mathbb{H}^n$ and opening $\alpha > 0$, defined as

$$C_{\mathbb{W},\mathbb{V}}(q,\alpha) := q \cdot C_{\mathbb{W},\mathbb{V}}(e,\alpha),$$

where

$$C_{\mathbb{W},\mathbb{V}}(e,\alpha) := \{p \in \mathbb{H}^n \mid \|p_{\mathbb{W}}\| \le \alpha \|p_{\mathbb{V}}\|\}.$$

Notice the obvious inclusion property: $C_{\mathbb{W},\mathbb{V}}(q,\alpha) \subset C_{\mathbb{W},\mathbb{V}}(q,\beta)$, when $\alpha < \beta$, and also the homogeneity property: for $\lambda \in \mathbb{R}^+$, $\delta_\lambda\left(C_{\mathbb{W},\mathbb{V}}(e,\alpha)\right) = C_{\mathbb{W},\mathbb{V}}(e,\alpha)$.

We can now recall a slightly modified result of Mattila, Serapioni and Serra Cassano about $(d,\mathbb{H})$-rectifiability of a set (Corollary 3.16 in [19]). (The Euclidean version of it (see e.g. [18]) is sometimes called as the *butterfly lemma.*) For a detailed proof of this statement we refer to [21].

Proposition 2.1. *Let $n+1 \le d \le 2n$ and let $E \subset \mathbb{H}^n$ be a Borel set with the property that for all $p \in E$ there exist $r_p > 0$, $\alpha_p > 0$ and $\mathbb{W}_p \in \mathcal{G}(\mathbb{H}^n, d)$ such that*

$$E \cap B(p, r_p) \cap C_{\mathbb{W}_p,\mathbb{V}_p}(p,\alpha_p) = \{p\}.$$

Then E is $(d,\mathbb{H})$-rectifiable.

The main idea of the proof of Proposition 2.1 uses the compactness of $\mathcal{G}(\mathbb{H}^n, d)$ and can be made along the lines of [2], Theorem 2.61.

We conclude this section with the following observation: For a given homogeneous vertical subgoup $\mathbb{W}$ there could be infinitely many choices of complementary horizontal subgroups $\mathbb{V}$ such that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$. To avoid this ambiguity, in the rest of the paper, for given $\mathbb{W}$ we shall always consider the unique horizontal subgroup $\mathbb{V}$ with the property that $\mathbb{V}$ is the orthogonal complement of $\mathbb{W} \cup \mathbb{R}^{2n} \times \{0\}$. In the same way, for a given homogeneous horizontal subgroup $\mathbb{V}$ we consider its unique "orthogonal" vertical complement $\mathbb{W}$.

3. Proof of the main result

We are now ready to give the proof of Theorem 1.1. We start with some preparations. Let us first introduce a notation. For $k \in \mathbb{N}$, if $v_1, ..., v_k$ are k linearly independent unit vectors of $\mathbb{R}^{2n}$ and $r > 0$, we denote by $\Delta(v_1, ..., v_k, r)$ the *k-simplex*,

$$\Delta(v_1, ..., v_k, r) := \left\{ \sum_{i=1}^{k} \lambda_i v_i \in \mathbb{R}^{2n} \,\middle|\, \sum_{i=1}^{k} |\lambda_i| \le r \,,\, \lambda_i \in \mathbb{R} \right\}.$$

Definition 3.1. Let $T\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$ be a maximal H-monotone set valued map and let $p \in \mathbb{H}^n$ be fixed. If $\dim(T(p)) = k$, then there exist $\eta \in \mathbb{R}^{2n}$, $r_1 > 0$ and $v_1, ..., v_k \in \mathbb{R}^{2n}$, linearly independent unit vectors, such that

$$\eta + \Delta(v_1, ..., v_k, r_1) \subset T(p). \tag{2}$$

We define the k-dimensional vector space $V_p = \mathrm{span}\{v_1, ..., v_k\} \subseteq \mathbb{R}^{2n}$.

We recall that $T(p) \subset \mathbb{R}^{2n}$ is convex, for every $p \in \mathbb{H}^n$, therefore a linear subset of the type of V_p exists. Definition 3.1 is correct in the sense, that V_p does not depend on the choices of η, v_i and r_1 involved in the definition. Indeed, since $T(p)$ is a k-dimensional convex subset of $\mathbb{R}^{2n}$ there exists a unique affine subspace of $\mathbb{R}^{2n}$ containing it. The translation to the origin of this affine subspace will be V_p.

Recall that in [5] it is proved that if the effective domain of T, $\operatorname{dom}(T) = \{p \in \mathbb{H}^n \,|\, T(p) \neq \emptyset\}$, is the whole $\mathbb{H}^n$, then T is upper semicontinuous at every point $p \in \mathbb{H}^n$ i.e. if $p_k \longrightarrow p$, as $k \to \infty$, and $v_k \in T(p_k)$ for every $k \in \mathbb{N}$, then there exists a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ such that $v_{n_k} \longrightarrow v \in T(p)$, as $k \to \infty$.

Before starting the proof of our main theorem let us observe that the singularity set $\Sigma^k_H(T)$ can be split in a countable union as

$$\Sigma^k_H(T) = \bigcup_{h\in\mathbb{N}} \Sigma^{k,h}_H(T). \tag{3}$$

where

$$\Sigma^{k,h}_H(T) := \left\{ p \in \Sigma^k_H(T) \;\middle|\; \exists\, \eta + \Delta\left(v_1, ..., v_k, \frac{1}{h}\right) \subset T(p) \right\}. \tag{4}$$

One can imagine the sets $\Sigma^{k,h}_H(T)$ as being the set of points p in the singularity set such that $T(p)$ contains a convex set with a definite size $1/h$. Let us check why this is true. First, by definition, $\Sigma^{k,h}_H(T) \subset \Sigma^k_H(T)$, for every $h \in \mathbb{N}$.

For the opposite inclusion, let $p \in \Sigma^k_H(T)$. We want to show that there exists $h \in \mathbb{N}$ such that $p \in \Sigma^{k,h}_H(T)$. We know that $\dim T(p) = k$ and that $T(p)$ is a convex set. This implies that there exists a k-dimensional ball B contained in $T(p)$. Let $r > 0$ be the radius of B and η its center. There exists $\tilde{h} \in \mathbb{N}$ such that $\frac{1}{\tilde{h}} \leq r$.

Let $v_1, ..., v_k$ be linearly independent unit vectors such that $V_p = \operatorname{span}\{v_1, ..., v_k\}$. It follows that $\eta_p + \Delta\left(v_1, ..., v_k, \frac{1}{\tilde{h}}\right) \subset B \subset T(p)$. Therefore $p \in \Sigma^{k,\tilde{h}}_H(T)$.

Since the union in (3) is countable, we can reduce our investigation to $\Sigma^{k,h}_H(T)$, for every $h \in \mathbb{N}$. Let us choose an arbitrary $h \in \mathbb{N}$. We aim to prove that $\Sigma^{k,h}_H(T)$ is $(2n + 1 - k, \mathbb{H})$-rectifiable. In order to do that we need the following:

Lemma 3.2. *Let $p_0 \in \Sigma^{k,h}_H(T)$ and $\{p_m\}_{m\in\mathbb{N}} \subset \Sigma^{k,h}_H(T)$ be such that $p_m \longrightarrow p_0$ as $m \to \infty$. Then there exist a subsequence of $\{p_m\}_{m\in\mathbb{N}}$, denoted in the same way, $\eta_0 \in \mathbb{R}^{2n}$ and $v^1_0, ..., v^k_0$ linearly independent unit vectors such that:*

(i) $\operatorname{span}\{v^1_0, ..., v^k_0\} \in G^H(2n, k)$*:*

(ii) $\eta_0 + \Delta\left(v^1_0, ..., v^k_0, \frac{1}{h}\right) \subset T(p_0)$*;*

(iii) *for each $w_0 \in \eta_0 + \Delta\left(v^1_0, ..., v^k_0, \frac{1}{h}\right)$, there exists $w_m \in T(p_m)$, for every $m \in \mathbb{N}$, such that $w_m \longrightarrow w_0$, as $m \to \infty$.*

Proof. Let us select $\{v^1_m, ..., v^k_m\}$ an orthonormal basis of V_{p_m}, for every $m \in \mathbb{N}$. Fix $i \in \{1, ..., k\}$. It holds that $\|v^i_m\|_{\mathbb{R}^{2n}} = 1$, for every $m \in \mathbb{N}$. Therefore, up to a subsequence, there exists $v^i_0 \in \mathbb{R}^{2n}$ such that $v^i_m \longrightarrow v^i_0$, as $m \to \infty$.

We prove that $v_0^1, ..., v_0^k$ are linearly independent unit vectors. Let i and $j \in \{1, ..., k\}$. We know that $\langle v_m^i, v_m^j \rangle = \delta_{ij}$, for every i and $j \in \{1, ..., k\}$. If we let $m \to \infty$, we get that $\left\langle v_0^i, v_0^j \right\rangle = \delta_{ij}$. A second step is to prove that $\mathrm{span}\{v_0^1, ..., v_0^k\} \in G^H(2n, k)$. Since $\mathrm{span}\{v_m^1, ..., v_m^k\} \in G^H(2n, k)$, it holds that $\langle v_m^i, Jv_m^j \rangle = 0$, for every $m \in \mathbb{N}$ and for every $i \neq j$. Again, if $m \to \infty$, we obtain $\left\langle v_0^i, Jv_0^j \right\rangle = 0$, for every $i \neq j$. This in turn implies that $\mathrm{span}\{v_0^1, ..., v_0^k\} \in G^H(2n, k)$ as required.

Now, since $p_m \in \Sigma_H^{h,k}(T)$, there exists $\eta_m \in T(p_m)$ such that

$$\eta_m + \Delta\left(v_m^1, ..., v_m^k, \frac{1}{h}\right) \subset T(p_m).$$

By hypothesis, we know that T is a maximal H-monotone set valued map. Moreover, $p_m \longrightarrow p_0$, as $m \to \infty$. Therefore, by upper semicontinuity, it follows that there exists $\eta_0 \in T(p_0)$ such that, possibly restricting to a subsequence, $\eta_m \longrightarrow \eta_0$, as $m \to \infty$.

It remains to show that

$$\eta_0 + \Delta\left(v_0^1, ..., v_0^k, \frac{1}{h}\right) \subset T(p_0).$$

We argue by contradiction. Let $\eta_0 + \sum_{i=1}^k \tilde{\lambda}_i v_0^i \notin T(p_0)$, for some $\tilde{\lambda}_1, ..., \tilde{\lambda}_k \in \mathbb{R}$ such that $\sum_{i=1}^k |\tilde{\lambda}_i| < \frac{1}{h}$. By definition of $\Sigma_H^{h,k}(T)$, it holds that

$$w_m := \eta_m + \sum_{i=1}^k \tilde{\lambda}_i v_m^i \in T(p_m).$$

Again by upper semicontinuity of the set valued map T, it holds that there exists $w_0 \in T(p_0)$ such that $w_m \longrightarrow w_0$, as $m \to \infty$.

On the other hand, one has

$$w_m = \eta_m + \sum_{i=1}^k \tilde{\lambda}_i v_m^i \longrightarrow \eta_0 + \sum_{i=1}^k \tilde{\lambda}_i v_0^i,$$

as $m \to \infty$. Hence, by uniqueness of the limit, we can conclude that

$$w_0 = \eta_0 + \sum_{i=1}^k \tilde{\lambda}_i v_0^i \in T(p_0),$$

which provides the desired contradiction. □

We can now turn to the

Proof of Theorem 1.1. Let us fix an arbitrary $h \in \mathbb{N}$. We will to prove that $\Sigma_H^{k,h}(T)$ is $(2n + 1 - k, \mathbb{H})$-rectifiable.

Our strategy is to apply Proposition 2.1: we need to show that, for every $p \in \Sigma_H^{k,h}(T)$, there exist a radius $r_p > 0$, an opening $\alpha_p > 0$ and a vertical homogeneous subgroup $\mathbb{W}_p \in \mathcal{G}(\mathbb{H}^n, 2n+1-k)$ such that

$$\Sigma_H^{k,h}(T) \cap B(p, r_p) \cap C_{\mathbb{W}_p, \mathbb{V}_p}(p, \alpha_p) \;=\; \{p\}\,,$$

where $\mathbb{V}_p$ is the unique orthogonal horizontal subgroup of $\mathbb{H}^n$ of dimension k such that $\mathbb{H}^n = \mathbb{W}_p \cdot \mathbb{V}_p$ is a semidirect product.

Without loss of generality, we can assume that $0 \in \Sigma_H^{k,h}(T)$ and we restrict our investigation to $p = 0$.

For $0 \in \Sigma_H^{k,h}(T)$, let $V_0 = \mathbb{V}_0 \in G^H(2n, k)$ its associated horizontal subspace. There exist a point $\eta_0 \in T(0)$ and a k-simplex $\Delta\left(v_0^1, ..., v_0^k, \frac{1}{h}\right)$ such that $\eta_0 + \Delta\left(v_0^1, ..., v_0^k, \frac{1}{h}\right) \subset T(0)$. This fact implies that there is a k-dimensional disk $D_0 \subset \mathbb{V}_0$, centered at zero, such that $\eta_0 + D_0 \subset T(0)$.

Let $\mathbb{W}_0$ be the vertical subgroup of $\mathbb{H}^n$ of linear dimension $2n+1-k$, orthogonal to $\mathbb{V}_0$ such that $\mathbb{H}^n = \mathbb{W}_0 \cdot \mathbb{V}_0$ is a semidirect product.

For simplicity of calculations, we can assume that $\mathbb{V}_0 = \{(v_1, ..., v_k, 0, ..., 0) \in \mathbb{H}^n \,|\, v_i \in \mathbb{R},\, i = 1, ..., k\}$ and $\mathbb{W}_0 = \{(0, ..., 0, w_{k+1}, ..., w_{2n}, t) \in \mathbb{H}^n \,|\, w_j \in \mathbb{R}, t \in \mathbb{R},\, j = k+1, ..., 2n\}$.

The proof of the theorem is a direct consequence of the following claim:

Claim 3.3. *There exist $r_0 > 0$ and $\beta_0 > 0$ such that*

$$\Sigma_H^{k,h}(T) \cap B(0, r) \cap C_{\mathbb{W}_0, \mathbb{V}_0}(0, \beta) \;=\; \{0\},$$

for every $0 < r < r_0$ and $0 < \beta < \beta_0$.

The proof of the claim is by contradiction. Let us assume that for every $\beta > 0$ and for every $r > 0$ there exists $p \neq 0$ such that

$$p \;\in\; \Sigma_H^{k,h}(T) \cap B(0, r) \cap C_{\mathbb{W}_0, \mathbb{V}_0}(0, \beta).$$

Let us choose $r = \frac{1}{m}$ and $\beta = \frac{1}{m}$, for $m \in \mathbb{N}$. Then, for every $m \in \mathbb{N}$, there exists $p^{(m)} \neq 0$ such that

$$p^{(m)} \;\in\; \Sigma_H^{k,h}(T) \cap B\left(0, \frac{1}{m}\right) \cap C_{\mathbb{W}_0, \mathbb{V}_0}\left(0, \frac{1}{m}\right).$$

It is clear that $\lim_{m \to \infty} p^{(m)} \;=\; 0$. Moreover, since $p^{(m)} \in \Sigma_H^{k,h}(T)$, there exists $\mathbb{V}_{p^{(m)}} \in G^H(2n, k)$, here denoted by $\mathbb{V}_m$, defined again in (1).

Now, let us choose arbitrarily a vector $w_0 \in \eta_0 + D_0$. Let $v_0 \in D_0$ be such that $w_0 = \eta_0 + v_0$. Thanks to Lemma 3.2, we can select a subsequence of $\left\{p^{(m)}\right\}_{m \in \mathbb{N}}$, still denoted in the same way, such that we can find $w_m \in T\left(p^{(m)}\right)$, for every $m \in \mathbb{N}$, in order to have $w_m \longrightarrow w_0$, as $m \to \infty$.

By definition of intrinsic cone, we have that

$$C_{\mathbb{W}_0,\mathbb{V}_0}\Big(0,\frac{1}{m}\Big) = \Big\{p = (p_1,...,p_{2n},p_{2n+1}) \in \mathbb{H}^n \;\Big|\; (p_{k+1}^2+...+p_{2n}^2)^2 + \Big(p_{2n+1} - 2\sum_{i=1}^{k} p_i \cdot p_{n+i}\Big)^2 \le \frac{1}{m^4}(p_1^2+...+p_k^2)^2\Big\}. \tag{5}$$

Therefore, since $p^{(m)} \in C_{\mathbb{W}_0,\mathbb{V}_0}\left(0,\frac{1}{m}\right)$, one can deduce that

$$\left(\left(p_{k+1}^{(m)}\right)^2 + ... + \left(p_{2n}^{(m)}\right)^2\right)^2 \le \frac{1}{m^4}\left(\left(p_1^{(m)}\right)^2 + ... + \left(p_k^{(m)}\right)^2\right)^2.$$

This implies that there exists $\alpha = (\alpha_1,...,\alpha_k,0,...,0)$, with $\|\alpha\|_{\mathbb{R}^{2n}} = 1$ such that

$$\lim_{m\to\infty} \frac{\xi_1\left(p^{(m)}\right)}{\|\xi_1\left(p^{(m)}\right)\|_{\mathbb{R}^{2n}}} = \alpha.$$

More specifically, we have for $i = 1,...,k$ and $j = k+1,...,2n$:

$$\lim_{m\to\infty} \frac{p_i^{(m)}}{\|\xi_1\left(p^{(m)}\right)\|_{\mathbb{R}^{2n}}} = \alpha_i, \quad \text{and} \quad \lim_{m\to\infty} \frac{p_j^{(m)}}{\|\xi_1\left(p^{(m)}\right)\|_{\mathbb{R}^{2n}}} = 0. \tag{6}$$

Since $\|\alpha\|_{\mathbb{R}^{2n}} = 1$, there exists an index i such that $\alpha_i \neq 0$. Without loss of generality, we can assume that $\alpha_1 \neq 0$. Therefore, possibly restricting to a subsequence, $p^{(m)} \neq 0$, for every $m \in \mathbb{N}$.

Consider now $\tilde{p}^{(m)} \in H_{p^{(m)}} \cap H_0$, for every $m \in \mathbb{N}$. By definition of horizontal plane at the point $p^{(m)} = \left(p_1^{(m)},...,p_{2n}^{(m)},p_{2n+1}^{(m)}\right) \in \mathbb{H}^n$, there exists $(u_1,...,u_{2n},0) \in \mathbb{H}^n$ such that

$$\begin{aligned}\tilde{p}^{(m)} &= \left(\tilde{p}_1^{(m)},...,\tilde{p}_{2n}^{(m)},\tilde{p}_{2n+1}^{(m)}\right)\\ &= \Big(p_1^{(m)}+u_1,...,p_{2n}^{(m)}+u_{2n},p_{2n+1}^{(m)} + 2\sum_{i=1}^{n} p_i^{(m)}u_{n+i} - 2\sum_{j=1}^{n} p_{n+j}^{(m)}u_j\Big).\end{aligned}$$

Since we asked for $\tilde{p}^{(m)} \in H_0$, in order to satisfy condition

$$p_{2n+1}^{(m)} + 2\sum_{i=1}^{n} p_i^{(m)}u_{n+i} - 2\sum_{j=1}^{n} p_{n+j}^{(m)}u_j = 0,$$

we choose

$$u_i = \begin{cases} -\frac{p_i^{(m)}}{2}, & \text{if } i \neq n+1,\ i = 1,...,2n,\\ -\frac{p_{2n+1}^{(m)}}{2p_1^{(m)}} - \frac{p_{n+1}^{(m)}}{2}, & \text{if } i = n+1, \end{cases} \tag{7}$$

Consider now $\tilde{w}^{(m)} \in T\left(\tilde{p}^{(m)}\right)$. Since $\tilde{p}^{(m)} \longrightarrow 0$, as $m \to \infty$, by upper semicontinuity of T, possibly restricting to a subsequence, there exists $\tilde{w}_0 \in T(0)$, such that $\tilde{w}^{(m)} \longrightarrow \tilde{w}_0$, as $m \to \infty$. By convexity of $T(0)$, there exists $\tilde{v}_0 \in \mathbb{V}_0$ such that $\tilde{w}_0 = \eta_0 + \tilde{v}_0$.

Now, we use the definition of H-monotonicity of the set valued map T, valued at the points $\tilde{p}^{(m)}$ and 0. It holds that

$$\left\langle \xi_1\left(\tilde{p}^{(m)}\right)\, ,\, \tilde{w}^{(m)} - w_0 \right\rangle \,\geq\, 0.$$

Writing explicity in coordinates, we have

$$\left\langle \left(\frac{p_1^{(m)}}{2}, ..., \frac{p_n^{(m)}}{2}, -\frac{p_{2n+1}^{(m)}}{2p_1^{(m)}} + \frac{p_{n+1}^{(m)}}{2}, \frac{p_{n+2}^{(m)}}{2}, ..., \frac{p_{2n}^{(m)}}{2}, \right), \, \tilde{w}^{(m)} - w_0 \right\rangle \,\geq\, 0.$$

Divide by $\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}$, one has

$$\frac{1}{2\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}} \left\langle \left(p_1^{(m)}, ..., p_n^{(m)}, -\frac{p_{2n+1}^{(m)}}{p_1^{(m)}} + p_{n+1}^{(m)}, p_{n+2}^{(m)}, ..., p_{2n}^{(m)} \right), \, \tilde{w}^{(m)} - w_0 \right\rangle \geq 0.$$

The task now is to compute the limit for $m \longrightarrow \infty$. Taking (6) into consideration it only remains to show that

$$\lim_{m\to\infty} \frac{p_{2n+1}^{(m)}}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}} p_1^{(m)}} \,=\, 0.$$

By definition of intrinsic cone (see (5)), it holds that

$$\left| p_{2n+1}^{(m)} - 2\sum_{i=1}^{k} p_i^{(m)} \cdot p_{n+i}^{(m)} \right| \,\leq\, \frac{1}{m^2} \left(\left(p_1^{(m)}\right)^2 + ... + \left(p_k^{(m)}\right)^2 \right).$$

Therefore, we can estimate

$$\begin{aligned}
\frac{|p_{2n+1}^{(m)}|}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}} |p_1^{(m)}|} \,&\leq\, \frac{1}{m^2} \frac{\left(\left(p_1^{(m)}\right)^2 + ... + \left(p_k^{(m)}\right)^2 \right)}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}} |p_1^{(m)}|} + \frac{2\sum_{i=1}^{k} |p_i^{(m)}| \cdot |p_{n+i}^{(m)}|}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}} |p_1^{(m)}|} \\
= \frac{1}{m^2} \frac{\left\|\xi_1\left(p^{(m)}\right)\right\|^2_{\mathbb{R}^{2n}}}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}} |p_1^{(m)}|} &+ \frac{2\sum_{i=1}^{k} |p_i^{(m)}| \cdot |p_{n+i}^{(m)}|}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}} |p_1^{(m)}|} \frac{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}} \\
= \frac{1}{m^2} \frac{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}}{|p_1^{(m)}|} &+ 2\frac{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}}{|p_1^{(m)}|} \sum_{i=1}^{k} \frac{|p_i^{(m)}|}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}} \frac{|p_{n+i}^{(m)}|}{\left\|\xi_1\left(p^{(m)}\right)\right\|_{\mathbb{R}^{2n}}} \longrightarrow 0,
\end{aligned}$$

as $m \to \infty$. The final limit follows from (6) as $m \to \infty$.

Thus, we can conclude that

$$\left\langle \left(\frac{\alpha_1}{2}, ..., \frac{\alpha_k}{2}, 0, ..., 0 \right), \tilde{w}_0 - w_0 \right\rangle \,\geq\, 0. \tag{8}$$

Consider now H-monotonicity at the points $\tilde{p}^{(m)}$ and $p^{(m)}$. It holds that

$$\left\langle \xi_1\left(\tilde{p}^{(m)}\right) - \xi_1\left(p^{(m)}\right), \tilde{w}^{(m)} - w_m \right\rangle \ge 0.$$

Repeating the previous argument, we obtain

$$\left\langle \left(-\frac{\alpha_1}{2}, ..., -\frac{\alpha_k}{2}, 0, ..., 0\right), \tilde{w}_0 - w_0 \right\rangle \ge 0. \tag{9}$$

Combining together inequalities (8) and (9), we have

$$\left\langle \left(\frac{\alpha_1}{2}, ..., \frac{\alpha_k}{2}, 0, ..., 0\right), \tilde{w}_0 - w_0 \right\rangle = \\ = \left\langle \left(\frac{\alpha_1}{2}, ..., \frac{\alpha_k}{2}, 0, ..., 0\right), (\eta_0 + \tilde{v}_0) - (\eta_0 + v_0) \right\rangle = 0.$$

This equality provides a contradiction. Let us see why. We recall that $\alpha \in \mathbb{V}_0$, $\tilde{v}_0 \in \mathbb{V}_0$ is fixed and that we chose v_0 arbitrarily in the k-dimensional disk $D_0 \subset \mathbb{V}_0$. Therefore, the equality

$$\left\langle \left(\frac{\alpha_1}{2}, ..., \frac{\alpha_k}{2}, 0, ..., 0\right), \tilde{v}_0 \right\rangle = \left\langle \left(\frac{\alpha_1}{2}, ..., \frac{\alpha_k}{2}, 0, ..., 0\right), v_0 \right\rangle,$$

is verified if and only if α is the zero vector, which is a contradiction. The claim is finally proved.

The proof of the Theorem is now also complete. Thanks to the claim, we now know that there exist $r_0 > 0$ and $\beta_0 > 0$ such that

$$\Sigma_H^{k,h}(T) \cap B(0,r) \cap C_{\mathbb{W}_0,\mathbb{V}_0}(0,\beta) = \{0\}$$

for every $0 < r < r_0$ and for every $0 < \beta < \beta_0$. Now, we can apply Proposition 2.1, which guarantees that $\Sigma_H^{k,h}(T)$ is $(2n+1-k)$-rectifiable. $\square$

4. Final remarks

In this final section, we would like to make some remarks about our main result. First of all, we recall the definition of *H-convex function* on $\mathbb{H}^n$. Consider a *H-convex set* $\Omega \subseteq \mathbb{H}^n$, i.e. a set with the following property: for every $p \in \Omega$ and for every $p' \in H_p \cap \Omega$, $p \cdot \delta_\lambda\left(p^{-1} \cdot p'\right) \in \Omega$, for each $\lambda \in [0,1]$. We say that a function $u \colon \Omega \longrightarrow \mathbb{R}$ is *H-convex* if

$$u\left(p \cdot \delta_\lambda\left(p^{-1} \cdot p'\right)\right) \le u(p) + \lambda\left(u(p') - u(p)\right),$$

for every $p' \in \Omega \cap H_p$ and $\lambda \in [0,1]$. This concept was first introduced in [11] and also in [17]. It is interesting to notice that if $u \colon \mathbb{H}^n \longrightarrow \mathbb{R}$ is convex in the Euclidean sense, then it is also a H-convex function. The converse is in general not true.

We need to introduce also the notion of *horizontal subdifferential*. Let $u \colon \mathbb{H}^n \longrightarrow \mathbb{R}$ be a function. We say that $v \in \mathbb{R}^{2n}$ is a *horizontal subdifferential* of u at $p_0 \in \mathbb{H}^n$ if

$$u(p) \ge u(p_0) + \langle v, \xi_1(p) - \xi_1(p_0) \rangle,$$

for every $p \in H_{p_0}$. We denote by $\partial_H u(p_0)$ the set of all the horizontal subdifferentials of u at the point $p_0 \in \mathbb{H}^n$. As a characterization, in [7], if $u\colon \mathbb{H}^n \longrightarrow \mathbb{R}$, the authors show that $\partial_H u(p) \neq \emptyset$, for every $p \in \mathbb{H}^n$, if and only if u is a H-convex function.

Now, let us fix $u\colon \mathbb{H}^n \longrightarrow \mathbb{R}$ to be a H-convex function. We can define the set valued map $\partial_H u\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$, $p \mapsto \partial_H u(p)$. One can prove [9] that this set valued map is maximal H-monotone. Therefore, Theorem 1.1 applies.

As a step further, it would be interesting to study a different singular set, the so-called *k-th singular set*, for $k = 0, 1, ..., 2n$, defined for a set valued map $T\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$ as

$$\Sigma^k(T) \;=\; \{p \in \mathbb{H}^n \mid \dim\left(T(p)\right) = k\}\,.$$

In general, $\Sigma^k_H(T) \neq \Sigma^k(T)$. Let us give an example which shows this fact. Consider the Euclidean convex function (which is also H-convex) $u\colon \mathbb{H}^2 \longrightarrow \mathbb{R}^4$ given by $u(x_1, x_2, y_1, y_2, t) = (x_1^2 + y_1^2)^{\frac{1}{2}}$ and compute its horizontal subdifferential at a point $p = (x_1, x_2, y_1, y_2, t) \in \mathbb{H}^2$:

$$\partial_H u(p) = \begin{cases} B_1, & \text{if } p = (0, x_2, 0, y_2, t),\\ \frac{1}{\left(x_1^2+y_1^2\right)^{\frac{1}{2}}}\left(x_1, 0, y_1, 0\right), & \text{if } p \neq (0, x_2, 0, y_2, t), \end{cases}$$

where $B_1 = \{(x_1, 0, y_1, 0) \in \mathbb{R}^4 \,|\, x_1^2 + y_1^2 \leq 1\}$. Since $\dim B_1 = 2$, it holds that $\Sigma^2(\partial_H u) = \{(0, x_2, 0, y_2, t) \in \mathbb{H}^2 \,|\, x_2, y_2, t \in \mathbb{R}\}$, which has Hausdorff dimension 4. On the other hand, we notice that there is no horizontal subgroup of $\mathbb{H}^2$ which contains B_1. Therefore, $\Sigma^2_H(\partial_H u) = \emptyset$.

This last example leads us to a final conjecture, which is the analogous result of Theorem 1.1 for k-th singular sets of a H-monotone set valued map.

Conjecture 4.1. *Let $T\colon \mathbb{H}^n \rightrightarrows \mathbb{R}^{2n}$ be a maximal H-monotone operator with* $\mathrm{dom}(T) = \mathbb{H}^n$*. Then, for every $0 \leq k \leq 2n$, the Hausdorff dimension of $\Sigma^k(T)$ is smaller than or equal to $2n + 2 - k$, i.e.*

$$\dim_H\left(\Sigma^k(T)\right) \;\leq\; 2n + 2 - k. \tag{10}$$

Even the simplest case $n = 1$ and $k = 2$ is not clear. In this case, the conjecture reads as follows: if $T\colon \mathbb{H}^1 \rightrightarrows \mathbb{R}$ is a maximal H-monotone set valued map, then $\Sigma^k(T)$ has Hausdorff dimension at most $4 - k$, for $k = 0, 1, 2$. Notice that for $k = 0$ the estimate is trivially true. For $k = 1$ we observe that $\Sigma^1(T) = \Sigma^1_H(T)$ and our theorem applies. However in the case $n = 1$ and $k = 2$ we were not able to show that $\dim_H(\Sigma^2(T)) \leq 2$.

Furthermore, notice that the dimension estimate introduced in the Conjecture is sharp. Indeed, consider the following Euclidean convex function

$$f(p) \;=\; f(x, y, t) \;=\; \left(x^2 + y^2\right)^{\frac{1}{2}}.$$

Then f is a H-convex function as well. By simple computations, one can check that the horizontal subdifferential of f (that is a H-monotone set valued map) is given by

$$\partial_H f(p) = \begin{cases} \overline{B(0,1)}, & \text{if } p = (0,0,t), \\ \frac{1}{(x^2+y^2)^{\frac{1}{2}}}\,(x,y)\,, & \text{if } p \neq (0,0,t), \end{cases}$$

for every $t \in \mathbb{R}$. This implies that $\Sigma^2(\partial_H f) = \{(0,0,t) \in \mathbb{H}^n \,|\, t \in \mathbb{R}\}$. Hence, $\dim_H \Sigma^2(\partial_H f) = 2$.

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