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Quadratic Fractional Programming under Asymptotic Analysis

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This paper considers the quadratic fractional programming problem, which minimizes a ratio of two functions; a quadratic (not necessarily convex) function over an a affine function on an unbounded set. As is well-known, if the quadratic function is convex or quasiconvex, then the quadratic fractional function is pseudoconvex, a particular case of the quasiconvex minimization problem. Thus, we develop optimality conditions for the general case by introducing a generalized asymptotic function to deal with quasiconvexity. We established two characterization results for the nonemptiness and compactness for the set of minimizers of any quasiconvex function. In addition, an extension for the Frank-Wolfe theorem from the quadratic to the quadratic fractional problem will be given. Finally, applications to pseudoconvex quadratic fractional programming are also provided.

Keywords: Asymptotic functions, second order asymptotic functions, nonconvex optimization, optimality conditions, quasiconvexity, Frank-Wolfe theorem, quadratic fractional programming.

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1. Introduction

Given a subset K of a finite dimensional space $\mathbb{R}^n$, and functions $h\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$, with $h(x) = \frac{1}{2}x^\top Ax + a^\top x + \alpha$ a quadratic (not necessarily convex) function and an affine function $g(x) = b^\top x + \beta$, with $A \in \mathbb{R}^{n\times n}$ being a symmetric matrix, $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, let us consider the objective function $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) := \frac{h(x)}{g(x)}$ whenever $g(x) \neq 0$, that is, the ratio of a quadratic and an affine function. Thus the quadratic fractional minimization problem (QFMP from now on) is defined by

$$\min_{x\in K} f(x) \;=\; \min_{x\in K} \frac{\frac{1}{2}x^\top Ax + a^\top x + \alpha}{b^\top x + \beta}. \tag{1}$$

This problem is an important issue in continuous optimization due to its applications in several fields of the mathematical sciences, especially for economics purposes, for instance, in maximization of productivity as maximization of return/risk and minimization of cost/time or cost/profit among others (see [7, 29, 30]). Moreover, it is also the generalization of important problems in operation research. In

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fact, if $A = 0$ (the null matrix), then (1) coincide with the linear fractional minimization problem (LFMP from now on) given by

$$\min_{x \in K} \frac{a^\top x + \alpha}{b^\top x + \beta}. \tag{2}$$

On the other hand, if A is not the null matrix, $b = 0$ and $\beta = 1$, then (1) coincide with the well-known quadratic minimization problem (QMP from now on) given by

$$\min_{x \in K} \frac{1}{2} x^\top A x + a^\top x + \alpha. \tag{3}$$

This problem has been studied deeply in the literature due to its applications in almost all fields of the mathematical sciences (see [21]).

In general, those problems are not convex. In fact, even in the case when the quadratic function h is convex, problem (1) is not necessarily convex, i.e., some generalized convexity notions will be needed (see [7, 15]).

The main notion to deal with (1) is the notion of *quasiconvexity*. Recall that,

$$\begin{aligned} f \text{ is convex} \quad &\Longleftrightarrow \quad \text{epi } f \text{ is a convex set.} \\ f \text{ is quasiconvex} \quad &\Longleftrightarrow \quad S_\lambda(f) \text{ is convex, for all } \lambda \in \mathbb{R}, \end{aligned}$$

where epi f is the epigraph of f and $S_\lambda(f)$ the level set of f at height λ.

It is important to point out that many different approaches to provide optimality conditions for the QFMP can be found in the literature (see [7, 25, 30] and references therein). However, in this paper, we study problem (1) from another point of view, using the information of the behavior of the quadratic fractional function at infinity, i.e., no local descriptions will be needed.

To that end, the notions of asymptotic (recession) cone and function will be useful for our further analysis. Recall that, the asymptotic cone and function has been proved to be powerful tools to deal with optimization problems with unbounded data in the convex case (see [4, 27, 28]). But when dealing with nonconvex functions, the usual asymptotic function does not provide adequate information on the level sets of the original function. The issue of finding a correct definition of asymptotic function in the quasiconvex case was dealt with in [1, 12, 26].

All these notions were compared in [14], where the authors conclude that the notion of *q-asymptotic function* is the "best" suited for quasiconvex functions. Several connections of these concepts with others issues of generalized convexity were developed in [18, 20], supporting the conclusions of [14]. Here we present a variant which is even better for algorithms purposes and easier to be computed.

In this paper, we provide optimality conditions and characterization results for the nonemptiness and compactness of the solution set in the quasiconvex minimization problem. Moreover, an extension of the Frank-Wolfe theorem from the quadratic to the fractional quadratic problem will be given for asymptotically linears sets, a family of sets strictly bigger than polyhedrals. Finally, we apply our results in the pseudoconvex quadratic fractional problem, which includes the cases when the quadratic function h is convex or quasiconvex.

The structure of the paper is as follows: In Section 2 we set up notation and preliminaries. We review some standard facts on generalized convexity and asymptotic analysis. In Section 3, we develop new theoretical tools to deal with quasiconvexity, and general optimality conditions will be given. Finally, in Section 4 we established an extension of the Frank-Wolfe theorem for quadratic fractional problems and applications to pseudoconvex quadratic fractional programming.

2. Preliminaries and basic definitions

In this paper, by $\langle \cdot, \cdot \rangle$ we mean the scalar (inner) product and by $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ the norm on $\mathbb{R}^n$. Let $K \subseteq \mathbb{R}^n$, its *closure* is denoted by cl K, its *boundary* by bd K, its *topological interior* by int K and its *relative interior* as ri K. The *orthogonal complement* of K is denoted as $K^{\perp}$ and the *polar* (*positive*) *cone* of K as K^*.

Given any function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the *effective domain* of f is defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We say that f is a *proper function* if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\operatorname{dom} f$ is nonempty (clearly, $f(x) = +\infty$ for every $x \notin \operatorname{dom} f$). By $\operatorname{epi} f := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$ we mean the *epigraph* of f and by $S_{\lambda}(f) := \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ the level set of f at height $\lambda \in \mathbb{R}$. Finally, $\operatorname{argmin}_{\mathbb{R}^n} f$ (resp. $\operatorname{argminloc}_{\mathbb{R}^n} f$) denotes the set of all minimal (resp. local minimal) points of f and $B(x,\delta)$ is the *open ball* with center at $x \in \mathbb{R}^n$ and radius $\delta > 0$.

Recall that a point x_0 is said to be a *local minimizer* of f if there exists $\delta > 0$ such that $f(x_0) \leq f(x)$ for all $x \in B(x_0, \delta)$.

A proper function f with convex domain is said to be:

(i) *semistrictly quasiconvex*, if for every $x, y \in \operatorname{dom} f$ with $f(x) \neq f(y)$,

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \ \forall \ \lambda \in]0,1[;$$

(ii) *quasiconvex*, if for every $x, y \in \operatorname{dom} f$

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \ \forall \ \lambda \in]0,1[.$$

(iii) If f is differentiable, f is said to be *pseudoconvex* if for every $x, y \in \operatorname{dom} f$

$$f(x) > f(y) \Longrightarrow \nabla f(x)^{\top}(y-x) < 0.$$

Every convex function is quasiconvex. If f is lower semicontinuous (lsc from now on), every semistrictly quasiconvex function is quasiconvex. Finally, if f is differentiable, every pseudoconvex function is quasiconvex.

For further results on generalized convexity an applications in economics, variational inequalities and multiobjective optimization we refer to [7, 15].

2.1. First order asymptotic analysis

As explained in [4], the notions of asymptotic cone and function have been employed in optimization theory in order to handle unbounded situations (see [3] for instance), that is, when standard compactness assumptions are absent.

For a nonempty set $K \subseteq \mathbb{R}^n$, its *asymptotic cone* is defined by

$$K^{\infty} := \left\{ u \in \mathbb{R}^n : \ \exists \, t_k \to +\infty, \ \exists \, x_k \in K, \ \frac{x_k}{t_k} \to u \right\}.$$

In case K is a closed convex set, it is known that the notion of asymptotic (recession) cone (see [4, 27]) is equivalent to

$$K^{\infty} = \left\{ u \in \mathbb{R}^n : \ x_0 + \lambda u \in K, \ \forall \, \lambda \geq 0 \right\} \text{ for any } x_0 \in K. \tag{4}$$

The *asymptotic function* $f^{\infty} \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ of a proper function f is the function for which

$$\text{epi } f^{\infty} := (\text{epi } f)^{\infty}.$$

From this we have

$$f^{\infty}(u) = \inf \left\{ \liminf_{k \to +\infty} \frac{f(y_k)}{t_k} : \ y_k \in \text{dom } f, \ t_k \to +\infty, \ \frac{y_k}{t_k} \to u \right\}.$$

Moreover, when f is lsc and convex, then [4, Proposition 2.5.2] implies that

$$f^{\infty}(u) = \sup_{t > 0} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to +\infty} \frac{f(x_0 + tu) - f(x_0)}{t}, \tag{5}$$

where both formulas do not depends of the choice of the point $x_0 \in \text{dom } f$.

A proper function f is said to be *coercive* if $f(x) \to +\infty$ as $\|x\| \to +\infty$. If $f^{\infty}(u) > 0$ for all $u \neq 0$, then f is coercive. In addition, if f is convex and lsc, then f is coercive iff

$$f^{\infty}(u) > 0, \ \forall \, u \neq 0 \iff \operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact.} \tag{6}$$

To obtain a similar characterization for quasiconvex functions, the authors in [12] introduce the so-called *q-asymptotic function*

$$f_q^{\infty}(u) := \sup_{x \in \text{dom } f} \sup_{t > 0} \frac{f(x + tu) - f(x)}{t}. \tag{7}$$

The q-asymptotic function can geometrically be considered as *the maximal rate of steepness of f in the direction u on its effective domain.*

If f is quasiconvex (resp. lsc), then f_q^{∞} is quasiconvex (resp. lsc) by [12, Proposition 3.28]. In general, $f^{\infty} \leq f_q^{\infty}$. If f is convex and lsc, then $f^{\infty} = f_q^{\infty}$ by [14, Proposition 5.4].

Given a lsc and quasiconvex function f with convex domain, by [12, Theorem 4.7] we have

$$f_q^{\infty}(u) > 0, \ \forall \ u \neq 0 \iff \underset{\mathbb{R}^n}{\operatorname{argmin}} \ f \neq \emptyset \text{ and compact.} \tag{8}$$

By its connections in generalized convexity theory and applications in scalar and multiobjective optimization (see [12, 14, 18, 19, 20]), the q-asymptotic function seems to be the most adequate definition to deal with quasiconvex functions.

2.2. Second order asymptotic analysis

The second order asymptotic cone was introduced to obtain more information from the infinity of the set and function in the convex case (see [13, 16]).

Given a nonempty set $K \subseteq \mathbb{R}^n$ and $u \in \mathbb{R}^n$, the *second order asymptotic direction of K at u* is defined by

$$K^{\infty 2}[u] := \left\{ v \in \mathbb{R}^n : \ \exists \ x_k \in K, \ \exists \ s_k, t_k \to +\infty, \ \left(\frac{x_k}{s_k} - t_k u \right) \to v \right\}. \tag{9}$$

$K^{\infty 2}[u]$ is always a closed cone. If $u = 0$ then $K^{\infty 2}[0] = K^{\infty}$.

Take $u \in \mathbb{R}^n$ with $f^{\infty}(u) \in \mathbb{R}$. Then the *directional second order asymptotic function* of f at u, denoted by $f^{\infty 2}(u; \cdot)$, is defined as the function for which

$$\text{epi } f^{\infty 2}(u; \cdot) := (\text{epi } f)^{\infty 2}[(u, f^{\infty}(u))]. \tag{10}$$

From (10) we derive the next formula. Let $u \in \mathbb{R}^n$ be such that $f^{\infty}(u) \in \mathbb{R}$. Then for every $v \in \mathbb{R}^n$ (see [13, Proposition 4.1])

$$f^{\infty 2}(u; v) = \inf \left\{ \liminf_{k \to \infty} \left(\frac{f(x_k)}{s_k} - t_k f^{\infty}(u) \right) : \begin{array}{l} x_k \in \text{dom } f, \ s_k, t_k \to +\infty, \\ \left(\frac{x_k}{s_k} - t_k u \right) \to v \end{array} \right\}.$$

When f is a convex function and $x \in \text{ridom } f$, then for every u such that $f^{\infty}(u) \in \mathbb{R}$ and $v \in (\text{dom } f)^{\infty 2} [u]$ (see [13, Proposition 4.10])

$$f^{\infty 2}(u; v) = \sup_{s>0} \inf_{t>0} \frac{f(x + tu + sv) - tf^{\infty}(u) - f(x)}{s} \tag{11}$$

$$= \lim_{s \to +\infty} \lim_{t \to +\infty} \frac{f(x + tu + sv) - tf^{\infty}(u) - f(x)}{s}. \tag{12}$$

A further analysis for second order asymptotic cones and directional second order functions in the convex and nonconvex case can be found in [13, 16].

If f is quasiconvex, then (11)-(12) do not hold. Thus, motivated by the good properties of (7) in the first order case and (11) in the second order case, the following slight modification of (11) for the quasiconvex case has been introduced in [14].

Given $u \in \mathbb{R}^n$ such that $f^{\infty}(u) \in \mathbb{R}$, then the *directional second order q-asymptotic function* of f at u is defined by

$$f_q^{\infty 2}(u; v) := \sup_{\substack{x \in \mathrm{dom} f \\ s>0}} \inf_{t>0} \frac{f(x+tu+sv) - tf^{\infty}(u) - f(x)}{s}. \tag{13}$$

Applications of directional second order asymptotic functions in scalar and multiobjective optimization may be found in [14, 16, 19].

3. Quasiconvex minimization problems

We introduce a generalized asymptotic function to deal with quasiconvexity. The usual properties, calculus rules and two characterization results for the nonemptiness and compactness of the solution set in the quasiconvex minimization problem will be given. In this section, the function f is always defined by $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$.

3.1. Level-set asymptotic function

We start this subsection with the following example. Consider the continuous quasiconvex function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{|x|}$. It is evident that $\mathrm{argmin}_{\mathbb{R}^n} f = \{0\}$, a nonempty and compact set. But even in this trivial example, the q-asymptotic function is $f_q^{\infty}(u) = +\infty$ for all $u \neq 0$, i.e., cannot be computed using algorithms. In order to solve this situation, we propose a variant of (7).

Definition 3.1. Let f be proper and $\lambda \in \mathbb{R}$ be such that $S_{\lambda}(f) \neq \emptyset$. We define the *level set asymptotic function of f at height λ* by the function

$$f_{\lambda}^{\infty}(u) := \sup_{x \in S_{\lambda}(f)} \sup_{t>0} \frac{f(x+tu) - f(x)}{t}. \tag{14}$$

The level set asymptotic function of f at u can geometrically be considered as the *rate of maximum steepness from the level set of f at height λ in the direction u.*

The basic properties of f_{λ}^{∞} are listed below, but since their proofs are similar to those for the q-asymptotic function (see [12, Section 5]), they are omitted.

Proposition 3.2. *Let f be proper and $\lambda \in \mathbb{R}$ be such that $S_{\lambda}(f) \neq \emptyset$. Then the following assertions hold:*

(a) $f_{\lambda}^{\infty}(0) = 0$ *and* $f_{\lambda}^{\infty}(\alpha u) = \alpha f_{\lambda}^{\infty}(u)$, *for all* $\alpha > 0$ *and all* $u \in \mathbb{R}^n$.

(b) $\mathrm{epi}\ (f_{\lambda}^{\infty}) = \displaystyle\bigcap_{y \in S_{\lambda}(f)} \bigcap_{\varepsilon>0} \varepsilon \left(\mathrm{epi}\ f - (y, f(y))\right).$

(c) *If f is quasiconvex (resp. lsc), then f_{λ}^{∞} is quasiconvex (resp. lsc).*

Take $\lambda \in \mathbb{R}$ with $S_{\lambda}(f) \neq \emptyset$. It is easy to see that (see [14, Proposition 2.9])

$$f^{\infty}(u) \leq f_{\lambda}^{\infty}(u) \leq f_q^{\infty}(u), \ \forall\ u \in \mathbb{R}^n. \tag{15}$$

If f is lsc and convex, then all the previous functions are equals by [14, Proposition 5.4]. If f is quasiconvex, then the first inequality may be strict by [14, Example 2.2]. The last inequality may be also strict as the following example shows.

Example 3.3. Take the lsc quasiconvex function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} |x|, & \text{if } |x| \leq 1, \\ 1 + |x|, & \text{if } |x| > 1. \end{cases}$$

Then $f^{\infty}_{q}(u) = +\infty$ for all $u \neq 0$.

On the other hand, set any $0 \leq \lambda < 1$, thus $S_{\lambda}(f) = [0, \lambda]$, a compact set. Moreover, for any $u > 0$ we have

$$\begin{aligned} f^{\infty}_{\lambda}(u) &= \sup_{0 \leq x \leq \lambda} \sup_{t > 0} \frac{f(x + tu) - x}{t} \\ &= \max \left\{ \sup_{0 \leq x \leq \lambda} \sup_{0 < t \leq \frac{1-x}{u}} \frac{x + tu - x}{t}, \sup_{0 \leq x \leq \lambda} \sup_{t > \frac{1-x}{u}} \frac{1 + x + tu - x}{t} \right\} \\ &= \max\{u, 2u\} < +\infty. \end{aligned}$$

Hence, $0 < f^{\infty}_{\lambda}(u) < f^{\infty}_{q}(u)$ for all $u \neq 0$. $\square$

The q-asymptotic function is $+\infty$ while the level set asymptotic function is finite. This is not a consequence of the discontinuity of f. In fact, take $f\colon \mathbb{R} \to \mathbb{R}$ by the quasiconvex continuous function $f(x) = x$ if $x \leq 0$, and $f(x) = \sqrt{x}$ if $x > 0$. Here $f^{\infty}_{q}(1) = +\infty$ while $f^{\infty}_{\lambda}(1) \leq 2$, a finite number.

Remark 3.4. For asymptotic algorithms (see [2, 3, 4] and references therein), the level set asymptotic function would be easier to be computed by using asymptotic algorithms than f^{∞}_{q}. $\square$

The level set asymptotic function has an obvious monotonicity property on λ, that is, if $\lambda_1 \leq \lambda_2$ with $S_{\lambda_1}(f) \neq \emptyset$, then $f^{\infty}_{\lambda_1} \leq f^{\infty}_{\lambda_2}$. This remark reveals the connection between f^{∞}_{λ} and f^{∞}_{q}, that is

$$f^{\infty}_{q}(u) = \sup_{\lambda > 0} f^{\infty}_{\lambda}(u) = \lim_{\lambda \to +\infty} f^{\infty}_{\lambda}(u), \ \forall\ u \in \mathbb{R}^n. \tag{16}$$

Thus, only one level set could be the key to obtain all the information about the behavior of f, for instance, take a lsc function bounded from above, then $f^{\infty}_{\lambda_0} = f^{\infty}_{q}$ for some $\lambda_0 \in \mathbb{R}$, and all the global links for f^{∞}_{q} with generalized directional derivatives provided in [20, Section 4] are true for $f^{\infty}_{\lambda_0}$.

3.2. Characterization Results

In this subsection, we show the potential of our asymptotic function in the quasiconvex minimization problem. Given a proper function f, we are interested in the study of the global minimization problem given by

$$\min_{x \in \mathbb{R}^n} f(x). \tag{17}$$

Our first main result, which provides a characterization for the nonemptiness and compactness of the solution set, is given below.

Theorem 3.5. *Let f be a proper, lsc and quasiconvex function with convex domain. For any $\lambda \in \mathbb{R}$ with $S_\lambda(f) \neq \emptyset$, we have*

$$\mathrm{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact } \iff f^\infty_\lambda(u) > 0, \ \forall\, u \neq 0. \tag{18}$$

Proof. ($\Rightarrow$) Let $\lambda \in \mathbb{R}$ be such that $S_\lambda(f) \neq \emptyset$. Thus $\lambda \geq \min_{\mathbb{R}^n} f$. Suppose to the contrary that there exists $u \neq 0$ such that $f^\infty_\lambda(u) \leq 0$, that is,

$$f^\infty_\lambda(u) \leq 0 \iff f(x + tu) \leq f(x), \ \forall\, x \in S_\lambda(f), \ \forall\, t > 0.$$

Set $\overline{x} \in \mathrm{argmin}_{\mathbb{R}^n} f$, thus $f(\overline{x} + tu) \leq f(\overline{x}) \leq f(y)$ for all $y \in S_\lambda(f)$ and all $t > 0$, that is, $\overline{x} + tu \in \mathrm{argmin}_{\mathbb{R}^n} f$ for all $t > 0$, a contradiction.

($\Leftarrow$) Take $\lambda > \inf_{\mathbb{R}^n} f$ and suppose to the contrary that the solution set is empty, then there exists a minimizing sequence $\{x_l\}_{l \in \mathbb{N}} \subseteq S_\lambda(f)$ such that $f(x_{l+1}) < f(x_l)$ for all $l \in \mathbb{N}$.

(i) If $\sup_{l \in \mathbb{N}} \|x_l\| < +\infty$, there exists a subsequence $\{x^1_l\}_{l \in \mathbb{N}} \subseteq \{x_l\}_{l \in \mathbb{N}}$ such that $x^1_l \to \overline{x}$. We claim that $\overline{x} \in \mathrm{argmin}_{\mathbb{R}^n} f$. In fact, since x_l is a minimizing sequence and f is lsc, then

$$f(\overline{x}) \leq \liminf_{l \to +\infty} f(x^1_l) = \inf_{x \in \mathbb{R}^n} f(x).$$

(ii) If $\sup_{l \in \mathbb{N}} \|x_l\| = +\infty$, then $\frac{x_l}{\|x_l\|} \to u \in (S_\lambda(f))^\infty, u \neq 0$. Take $y \in S_\lambda(f)$ and $t > 0$, the quasiconvexity of f implies that

$$f\left((1 - \frac{t}{\|x_l\|})y + \frac{t}{\|x_l\|}x_l\right) \leq \max\{f(x_l), f(y)\} = f(y),$$

for l large enough. Since f is lsc, $f(y + tu) \leq f(y)$ for all $y \in S_\lambda(f)$ and all $t > 0$, thus $f^\infty_\lambda(u) \leq 0$. A contradiction. Hence $\mathrm{argmin}_{\mathbb{R}^n} f \neq \emptyset$.

For the boundedness: Suppose to the contrary that there exists a sequence $x_l \in \mathrm{argmin}_{\mathbb{R}^n} f$ such that $\|x_l\| \to +\infty$ with $\frac{x_l}{\|x_l\|} \to u \neq 0$.

Take $\overline{x} \in \mathrm{argmin}_{\mathbb{R}^n} f$ a fixed point and $t > 0$, thus

$$f\left((1 - \frac{t}{\|x_l\|})\overline{x} + \frac{t}{\|x_l\|}x_l\right) \leq \max\{f(x_l), f(\overline{x})\} = f(\overline{x}),$$

by the lsc property of f we have $f(\overline{x} + tu) \leq f(\overline{x}) \leq f(y)$ for all $y \in \mathrm{dom}\, f$ and all $t > 0$, thus $\overline{x} + tu \in \mathrm{argmin}_{\mathbb{R}^n} f$ for all $t > 0$. A contradiction.

The closedness is a consequence of the lsc assumption on f. $\square$

Our characterization is more useful than (8). In fact, for trivial examples the calculus of our asymptotic function remain trivial while f^∞_q is not. A more interesting application will be discussed in the next section.

Example 3.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the noncoercive continuous quasiconvex function given by

$$f(x) = \begin{cases} |x|, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| > 1. \end{cases}$$

Given any $\lambda \in [0, 1[$, then $S_{\lambda}(f) = [-\lambda, \lambda]$ is a compact set while dom f is all $\mathbb{R}$, thus the calculus of f^{∞}_{λ} is easier than f^{∞}_{q}. Take $\lambda = 0$ for simplicity (for any $0 < \lambda < 1$ we just proceed as in Example 3.3), thus $S_0(f) = \{0\}$ and

$$f^{\infty}_{0}(u) = \sup_{t>0} \frac{f(tu)}{t} = \max\left\{ \sup_{0<t\leq\frac{1}{u}} \frac{f(tu)}{t}, \ \sup_{t>\frac{1}{u}} \frac{f(tu)}{t} \right\} = \max\{|u|, u\} = |u|.$$

Hence, $\operatorname{argmin}_{\mathbb{R}} f$ is nonempty and compact. □

If f is coercive, then $f^{\infty}_{\lambda}(u) > 0$ for all $u \neq 0$. The reverse statement is not true as the previous example shows.

Motivated by the good properties of the level set asymptotic function, we introduce a slight modification of (13).

Definition 3.7. Let f be proper and $\lambda \in \mathbb{R}$ be such that $S_{\lambda}(f) \neq \emptyset$. Given $u \in (\text{dom } f)^{\infty}$ with $f^{\infty}(u) \in \mathbb{R}$, we define the *second order level set asymptotic function* of f at u by

$$f^{\infty 2}_{\lambda}(u; v) := \sup_{\substack{x \in S_{\lambda}(f) \\ s>0}} \ \inf_{t>0} \ \frac{f(x + tu + sv) - tf^{\infty}(u) - f(x)}{s}, \ \forall \ v \in \mathbb{R}^n. \tag{19}$$

It is easy to see that $f^{\infty 2}_{\lambda}(0; v) = f^{\infty}_{\lambda}(v)$ for all $v \in \mathbb{R}^n$, and, if $u \in \mathbb{R}$ with $f^{\infty}(u) \in \mathbb{R}$, then

$$f^{\infty 2}(u; v) \leq f^{\infty 2}_{\lambda}(u; v) \leq f^{\infty 2}_{q}(u; v), \ \forall \ v \in \mathbb{R}^n. \tag{20}$$

If f is convex, then $f^{\infty 2}(u; v) = f^{\infty 2}_{\lambda}(u; v) = f^{\infty 2}_{q}(u; v)$ for all $v \in \mathbb{R}^n$.

Another characterization result is given in the next proposition.

Proposition 3.8. *Let f be a proper, lsc and quasiconvex function, and let $\lambda \in \mathbb{R}$ be such that $S_{\lambda}(f) \neq \emptyset$. Thus* $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$ *and compact iff the following assertions hold:*

(a) $f^{\infty}(u) \geq 0$, *for all* $u \in \mathbb{R}^n \setminus \{0\}$;

(b) $[u \in (\text{dom } f)^{\infty}, \ f^{\infty}(u) = 0] \Longrightarrow f^{\infty 2}_{\lambda}(u; u) \geq 0$;

(c) $\widetilde{R}_{\lambda} := \{u \in \mathbb{R}^n : \ f^{\infty}(u) = 0, \ f^{\infty 2}_{\lambda}(u; u) = 0\} = \{0\}$.

Proof. ($\Rightarrow$) By Theorem 3.5 one has $f^{\infty}_{\lambda}(u) > 0$ for all $u \neq 0$. Hence (a) holds. Furthermore, the only vector u satisfying the hypothesis in (b) is $u = 0$ and so $f^{\infty 2}_{\lambda}(u; u) = 0$, while clearly $\widetilde{R}_{\lambda} = 0$ and (c) holds.

($\Leftarrow$) Assume that $f_{\lambda}^{\infty}(u) \leq 0$ for some $u \neq 0$; since $f^{\infty}(u) \leq f_{\lambda}^{\infty}(u) \leq 0$, by (a) we have $f^{\infty}(u) = 0$. Moreover, since $f_{\lambda}^{\infty}(u) \leq 0$, thus $f(x+tu) \leq f(x)$ for all $t > 0$ and all $x \in S_{\lambda}(f)$. It follows that $u \in (\mathrm{dom}\ f)^{\infty}$, and so $f_{\lambda}^{\infty 2}(u;u) \geq 0$ by (b).

Observe that

$$f(x+(t+s)u) - tf^{\infty}(u) - f(x) = f(x+(t+s)u) - f(x) \leq 0,$$

for all $x \in S_{\lambda}(f)$ and all $t, s > 0$, whence $f_{\lambda}^{\infty 2}(u;u) \leq 0$, thus $f_{\lambda}^{\infty 2}(u;u) = 0$, which means that $u \in \widetilde{R}_{\lambda}$ with $u \neq 0$, contradicting (c). $\square$

Our necessary and sufficient conditions are easier to check than other optimality conditions for quasiconvex problems. In fact, the following condition was introduced in the pioner paper of [5], similar versions can be found in [2, 10].

Consider the cone $R_0 := \{v \in \mathbb{R}^n : \ f^{\infty}(v) = 0\}$ and the condition;

$(*)$: If the sequence $x_k \in \mathrm{dom}\ f$, $\|x_k\| \to +\infty$ is such that $\frac{x_k}{\|x_k\|} \to u \in R_0$ and for all $z \in \mathrm{dom}\ f$, there exists $k_z \in \mathbb{N}$ such that $f(x_k) \leq f(z)$ when $k \geq k_z$, then there exist $\overline{x} \in \mathrm{dom}\ f$ and $\overline{k}$ such that

$$\|\overline{x}\| < \|x_{\overline{k}}\| \ \text{ and } \ f(\overline{x}) \leq f(x_k), \ \forall\ k \geq \overline{k}.$$

A characterization result using $(*)$ for quasiconvexity can be found in [10, Theorem 5.7]. It is evident that, Theorem 3.5 and Proposition 3.8 are easier to use than [10, Theorem 5.7] since condition $(*)$ is, in practice, hard to prove.

4. Quadratic fractional programming

As was explained in the introduction, the QFMP is an important issue in continuous optimization due to its applications in economics. In this section, we study the general case (without convexity assumption on the numerator) and the particular case of pseudoconvex quadratic fractional problem, using the tools of the previous section.

As in the introduction, consider $f(x) = \frac{h(x)}{g(x)} = \frac{\frac{1}{2}x^{\top}Ax + a^{\top}x + \alpha}{b^{\top}x + \beta}$. Take any fixed $\gamma > 0$, we choose the feasible set K by $K := \{x \in \mathbb{R}^n : g(x) \geq \gamma\}$ because is closed (and convex). Thus the QFMP is defined by

$$\min_{x \in K} \frac{\frac{1}{2}x^{\top}Ax + a^{\top}x + \alpha}{b^{\top}x + \beta}. \tag{21}$$

Recall that, a nonempty closed set C from $\mathbb{R}^n$ is said to be *asymptotically linear* (see [4, Definition 2.3.1]), if for each $\rho > 0$ and all sequence $\{x_l\}_{l \in \mathbb{N}} \subseteq C$ satisfying $\|x_l\| \to +\infty$, $\frac{x_l}{\|x_l\|} \to u \in C^{\infty}$, there exists $l_0 \in \mathbb{N}$ such that $x_l - \rho u \in C$ for all $l \geq l_0$.

It is easy to see that K is asymptotically linear since every polyhedral set is asymptotically linear, but there are asymptotically linear sets that are not polyhedral (see after [4, Definition 2.3.2]).

4.1. A Frank-Wolfe theorem

In this subsection, we established necessary and sufficient conditions for the existence of minimizers in the quadratic fractional problem without any convexity assumption on the numerator. The next result can be seen as an extension of the Frank-Wolfe theorem from the quadratic to the quadratic fractional case. As far as we know, there is no result in this direction.

Theorem 4.1. *Let f be any quadratic fractional function with K as above. Thus*

(a) *If $\nu := \inf_{x\in K} f(x) > -\infty$, then $u^\top Au \geq 0$ for all $u \in K^\infty$, and*

$$u^\top Au = 0,\ u \in K^\infty \implies \langle \nabla h(x) - \nu b, u\rangle \geq 0,\ \forall\ x \in K. \tag{22}$$

(b) *If $u^\top Au \geq 0$ for all $u \in K^\infty$, and*

$$u^\top Au = 0,\ u \in K^\infty \implies \langle \nabla h(x) - f(x)b, u\rangle \geq 0,\ \forall\ x \in K, \tag{23}$$

then $\operatorname{argmin}_K f \neq \emptyset$.

Proof. Observe that $K^\infty = \{x \in \mathbb{R}^n :\ g(x) \geq \gamma\}^\infty = \{u \in \mathbb{R}^n :\ \langle b, u\rangle \geq 0\}$.

(a): Take $x \in K$ and $u \in K^\infty$, thus $x + tu \in K$ for all $t > 0$ and

$$f(x+tu) = \frac{h(x+tu)}{g(x+tu)} = \frac{h(x) + t\langle \nabla h(x), u\rangle + \frac{1}{2}t^2u^\top Au}{g(x) + t\langle b, u\rangle} \geq \nu,\ \forall\ t > 0,$$

since $g(x+tu) = g(x) + t\langle b, u\rangle \geq \gamma > 0$, thus

$$h(x) + t\langle \nabla h(x), u\rangle + \frac{1}{2}t^2u^\top Au \geq \nu(g(x) + t\langle b, u\rangle),\ \forall\ t > 0, \tag{24}$$

that is, $\dfrac{h(x)}{t^2} + \dfrac{1}{t}\langle \nabla h(x), u\rangle + \dfrac{1}{2}u^\top Au \geq \dfrac{\nu}{t^2}g(x) + \dfrac{\nu}{t}\langle b, u\rangle,\ \forall\ t > 0.$

Taking $t \to +\infty$ we get $u^\top Au \geq 0$. Furthermore, if $u^\top Au = 0$, from (24) we obtain

$$\frac{h(x)}{t} + \langle \nabla h(x), u\rangle \geq \frac{\nu}{t}g(x) + \nu\langle b, u\rangle,\ \forall\ t > 0,$$

and taking $t \to +\infty$, (22) follows.

(b): For every $l \in \mathbb{N}$, setting $B_l := \{x \in K : \|x\| \leq l\}$, we may assume that $B_l \neq \emptyset$ for all l. We consider the problem

$$\min_{x\in B_l} f(x), \tag{25}$$

which always has a solution. For every $l \in \mathbb{N}$, take x_l such that

$$x_l = \min\{\|x\| :\ x \in \operatorname*{argmin}_{B_l} f\}.$$

If $\sup_{l\in\mathbb{N}}\|x_l\| < +\infty$, we proceed analogously to (i) in the proof of Theorem 3.5.

If $\sup_{l\in\mathbb{N}}\|x_l\| = +\infty$, then $\|x_l\| \to +\infty$ and $\frac{x_l}{\|x_l\|} \to u \in K^{\infty},\ u \neq 0$.

Since K is asymptotically linear, given $\rho > 0$ there exists $l_0 \in \mathbb{N}$ such that $x_l - \rho u \in K$ for all $l \geq l_0$. Moreover, since $\frac{x_l}{\|x_l\|} \to u$, we may assume that $\|\frac{x_l}{\|x_l\|} - u\| < 1$ and $\frac{\rho}{\|x_l\|} < 1$ for all $l \geq l_0$. Thus

$$x_l - \rho u = \left(1 - \frac{\rho}{\|x_l\|}\right) x_l + \frac{\rho}{\|x_l\|}\left(x_l - \|x_l\| u\right), \tag{26}$$

and $\|x_l - \rho u\| < \|x_l\|$ for all $l \geq l_0$.

On the other hand, for any $x \in K$ there exists $l_1 \in \mathbb{N}$ such that $f(x_l) \leq f(x)$ for all $l \geq l_1$, thus $h(x_l) = \frac{1}{2}x_l^{\top}Ax_l + a^{\top}x_l + \alpha \leq (b^{\top}x_l + \beta)f(x)$ and

$$\frac{1}{2}\frac{x_l^{\top}}{\|x_l\|}A\frac{x_l}{\|x_l\|} + \frac{a^{\top}}{\|x_l\|}\frac{x_l}{\|x_l\|} + \frac{\alpha}{\|x_l\|^2} \leq \left(b^{\top}\frac{x_l}{\|x_l\|} + \frac{\beta}{\|x_l\|}\right)\frac{f(x)}{\|x_l\|}, \quad \forall\, l \geq l_1.$$

It follows that $u^{\top}Au \leq 0$, whence $u^{\top}Au = 0$ by assumption. Moreover, again by assumption we have $\langle \nabla h(x) - f(x)b, u\rangle \geq 0$ for all $x \in K$.

Take any $l \geq l_2 := \max\{l_1, l_0\}$, thus

$$\begin{aligned}
&\langle \nabla h(x_l) - f(x_l)b, u\rangle \geq 0 \iff -\rho\langle g(x_l)\nabla h(x_l), u\rangle \leq -h(x_l)\rho\langle b, u\rangle \\
&\iff g(x_l)h(x_l) - g(x_l)\rho\langle \nabla h(x_l), u\rangle + g(x_l)\frac{1}{2}\rho^2u^{\top}Au \leq g(x_l)h(x_l) - h(x_l)\rho\langle b, u\rangle \\
&\iff g(x_l)\left(h(x_l) - \rho\langle \nabla h(x_l), u\rangle + \frac{1}{2}\rho^2u^{\top}Au\right) \leq h(x_l)\left(g(x_l) - \rho\langle b, u\rangle\right) \\
&\iff g(x_l)h(x_l - \rho u) \leq h(x_l)g(x_l - \rho u) \iff \frac{h(x_l - \rho u)}{g(x_l - \rho u)} \leq \frac{h(x_l)}{g(x_l)},
\end{aligned}$$

that is, $f(x_l - \rho u) \leq f(x_l)$ with $\|x_l - \rho u\| < \|x_l\|$ for all $l \geq l_2$, a contradiction to the choice of x_l.

Hence $\mathrm{argmin}_K f \neq \emptyset$. □

In general, (22) and (23) are not equivalent. In fact, since $f(x) \geq \nu$ for all $x \in K$ and $\langle b, u\rangle \geq 0$ for $u \in K^{\infty}$, thus $\langle \nabla h(x) - f(x)b, u\rangle \leq \langle \nabla h(x) - \nu b, u\rangle$ for all $x \in K$, i.e.,

$$\langle \nabla h(x) - f(x)b, u\rangle \geq 0 \implies \langle \nabla h(x) - \nu b, u\rangle \geq 0. \tag{27}$$

If $\langle b, u\rangle = 0$ and $\beta = 1$, then $f = h$ is a quadratic (not necessarily convex) function. Thus, (22) and (23) are equivalent, and Theorem 4.1 coincides with the Frank-Wolfe theorem for asymptotically linear sets.

Corollary 4.2. ([11, Theorem 4.1]) *Let K be a nonempty closed convex set from $\mathbb{R}^n$ and h as above. If K is asymptotically linear, then the following assertions are equivalent:*

(a) $\inf_{x\in K} h(x) > -\infty$.

(b) $u^\top Au \geq 0$ *for all* $u \in K^\infty$, *and*

$$u^\top Au = 0,\ u \in K^\infty \implies \langle \nabla h(x), u\rangle \geq 0,\ \forall\, x \in K. \tag{28}$$

(c) $\mathrm{argmin}_K h \neq \emptyset$.

Other extensions for the Frank-Wolfe theorem, in different directions, may be found in [22] and in the recent paper of Martínez-Legaz et al. [24].

4.2. The pseudoconvex case

In this subsection, under a pseudoconvexity assumption on f, we provide a new characterization for the nonemptiness and compactness for the solution set.

Recall that, a quadratic function h as above is convex iff its matrix A is positive semidefinite iff $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$. If h is convex on K, then f is pseudoconvex on K by [7, Theorem 3.2.10]. If h is pseudoconvex on K, then h is quasiconvex on K by [7, Theorem 3.2.9]. Finally, f is quasiconvex on K iff f is pseudoconvex on K [7, Theorem 7.2.1].

In order to apply Theorem 3.5 we extend f to the whole $\mathbb{R}^n$ setting $f(x) := +\infty$ on $\mathbb{R}^n \backslash K$, i.e., f is lsc on $\mathbb{R}^n$. Thus, we refer to f as the *extended quadratic fractional function.*

Theorem 4.3. *Let f be an extended quadratic fractional function and K as above. If* $\mathrm{argmin}_K\, f \neq \emptyset$ *and compact, then for every* $u \in K^\infty$ *with* $u^\top Au \leq 0$ *we have*

$$\langle \nabla h(x) - f(x)b, u\rangle \leq 0,\ \forall\, x \in S_\lambda(f) \implies u = 0. \tag{29}$$

In addition, if f is the extended pseudoconvex quadratic fractional function, then the reverse also holds.

Proof. $(\Rightarrow)$: Since $\mathrm{argmin}_K f \neq \emptyset$, there exists $\overline{x} \in K$ with $\nu = f(\overline{x})$. Take $\lambda \geq f(\overline{x})$. For every $u \in K^\infty$ with $u^\top Au \leq 0$, we suppose to the contrary that $u \neq 0$ with

$$\langle \nabla h(x) - f(x)b, u\rangle \leq 0,\ \forall\, x \in S_\lambda(f). \tag{30}$$

It follows from Theorem 4.1 that $u^\top Au \geq 0$, thus $u^\top Au = 0$. Moreover,

$$\langle \nabla h(x) - f(\overline{x})b, u\rangle \geq 0,\ \forall\, x \in K,$$

in particular, $\langle \nabla h(\overline{x}) - f(\overline{x})b, u\rangle \geq 0$. Since $\overline{x} \in S_\lambda(f)$, using (30) we obtain

$$\langle \nabla h(\overline{x}) - f(\overline{x})b, u\rangle = 0 \iff \langle \nabla h(\overline{x}), u\rangle = f(\overline{x})\langle b, u\rangle. \tag{31}$$

Thus,

$$\begin{aligned} f(\overline{x} + tu) &= \frac{h(\overline{x}) + t\langle \nabla h(\overline{x}), u\rangle}{g(\overline{x}) + t\langle b, u\rangle} = \frac{h(\overline{x}) + tf(\overline{x})\langle b, u\rangle}{g(\overline{x}) + t\langle b, u\rangle} \\ &= \frac{h(\overline{x})g(\overline{x}) + h(\overline{x})t\langle b, u\rangle}{g(\overline{x})(g(\overline{x}) + t\langle b, u\rangle)} = f(\overline{x}),\ \forall\, t > 0, \end{aligned}$$

then $\overline{x} + tu \in \mathrm{argmin}_K f$ for all $t > 0$, a contradiction. Hence $u = 0$.

$(\Leftarrow)$: We apply Theorem 3.5. Observe that $f_\lambda^\infty(u) \leq 0$ iff

$$\sup_{x\in S_\lambda(f)} \sup_{t>0} \left(\frac{\frac{\frac{1}{2}(x+tu)^\top A(x+tu)+a^\top(x+tu)+\alpha}{b^\top(x+tu)+\beta} - \frac{\frac{1}{2}x^\top Ax+a^\top x+\alpha}{b^\top x+\beta}}{t} \right) \leq 0$$

$$\iff \sup_{x\in S_\lambda(f)} \sup_{t>0} \left(\frac{tg(x)x^\top Au + tg(x)\langle a,u\rangle + \frac{1}{2}t^2 g(x)u^\top Au - th(x)\langle b,u\rangle}{t(g(x)+t\langle b,u\rangle)g(x)} \right) \leq 0$$

$$\iff \sup_{x\in S_\lambda(f)} \sup_{t>0} \left(\frac{g(x)(x^\top Au + \langle a,u\rangle + \frac{1}{2}tu^\top Au) - h(x)\langle b,u\rangle}{g(x)(g(x)+t\langle b,u\rangle)} \right) \leq 0. \tag{32}$$

For $x \in K$, $g(x) + t\langle b,u\rangle \geq \gamma > 0$ for all $t > 0$, so (32) is equivalent to

$$\sup_{x\in S_\lambda(f)} \sup_{t>0} \left(x^\top Au + \langle a,u\rangle + \frac{1}{2}tu^\top Au - f(x)\langle b,u\rangle \right) \leq 0. \tag{33}$$

Relation (33) implies that $u^\top Au \leq 0$, since otherwise, taking $t \to +\infty$ leads a contradiction. Thus, for any $u \in K^\infty$, (33) is equivalent to

$$u^\top Au \leq 0 \ \wedge \ (\langle \nabla h(x)u - f(x)b, u\rangle \leq 0, \ \forall\, x \in S_\lambda(f)) \,.$$

Since $\{u \in K^\infty : \ u^\top Au \leq 0, \ \langle \nabla h(x)u - f(x)b, u\rangle \leq 0, \ \forall\, x \in S_\lambda(f)\} = \{0\}$, Theorem 3.5 implies $\operatorname{argmin}_K f \neq \emptyset$ and compact. □

If $A = 0$, then we obtain a characterization for the LFMP.

Corollary 4.4. *Let f be the linear fractional function and K as above. Thus* $\operatorname{argmin}_K f \neq \emptyset$ *and compact iff*

$$\text{for every } u \in K^\infty, \ \langle a - f(x)b, u\rangle \leq 0, \ \forall\, x \in S_\lambda(f) \implies u = 0. \tag{34}$$

If $b = 0$ and $\beta = 1$, then $f = h$. Thus we obtain another result for the QMP.

Corollary 4.5. *Let h be the quadratic function and C a closed convex set from $\mathbb{R}^n$. If* $\operatorname{argmin}_C h \neq \emptyset$ *and compact then*

$$\text{for every } u \in C^\infty, \ \langle \nabla h(x), u\rangle \leq 0, \ \forall\, x \in S_\lambda(h) \implies u = 0. \tag{35}$$

In addition, if h is the quasiconvex quadratic function, then the reverse also holds.

If $\beta \geq \gamma$, another sufficient condition is given in the next proposition. To that end, consider the following sets;

$$B_1 := \{v \in K^\infty : \ v^\top Av > 0\},$$
$$B_2 := \{v \in K^\infty : \ v^\top Av \leq 0, \ \beta\langle a,v\rangle - \alpha\langle b,v\rangle > 0\}.$$

Proposition 4.6. *Let f be the extended pseudoconvex quadratic function as above and K as above. If $K^\infty \subseteq B_1 \cup B_2$, then* $\operatorname{argmin}_K f \neq \emptyset$ *and compact.*

Proof. Since $g(0) = \beta \geq \gamma$, $0 \in K$ and $f(0) = \frac{\alpha}{\beta}$. Hence, for $0 \in S_{\frac{\alpha}{\beta}}(f)$, set $u \in K^{\infty}$ with $u \neq 0$, thus

$$\begin{aligned} f^{\infty}_{\frac{\alpha}{\beta}}(u) &= \sup_{x \in S_{\frac{\alpha}{\beta}}(f)} \sup_{t>0} \left(\frac{g(x)(x^{\top}Au + \langle a, u \rangle + \frac{1}{2}tu^{\top}Au) - h(x)\langle b, u \rangle}{g(x)(g(x) + t\langle b, u \rangle)} \right) \\ &\geq \sup_{t>0} \left(\frac{\beta \frac{1}{2}tu^{\top}Au + \beta\langle a, u \rangle - \alpha\langle b, u \rangle}{\beta(t\langle b, u \rangle + \beta)} \right). \end{aligned} \tag{36}$$

Since $\langle u, b \rangle \geq 0$ and $\beta > 0$, thus $\beta(\langle b, u \rangle + \beta) > 0$ for all $t > 0$. Then

$$f^{\infty}_{\frac{\alpha}{\beta}}(u) > 0 \iff u^{\top}Au > 0 \ \vee \ \left(u^{\top}Au \leq 0 \wedge \beta\langle a, u \rangle - \alpha\langle b, u \rangle > 0\right).$$

Thus, Theorem 3.5 implies $\operatorname{argmin}_K f \neq \emptyset$ and compact. $\square$

Remark 4.7. (i) Our asymptotic approach is simpler than the one given in [6], where the authors use the usual asymptotic directions and functions for approximating nonconvex functions.

(ii) The term $u^{\top}Au$ is exactly the *second order asymptotic function* of the quadratic function $h(x) = \frac{1}{2}x^{\top}Ax + a^{\top}x + \alpha$ introduced in [17, Definition 3.1], and was denoted by $h^{\infty\infty}(u; u)$.

(iii) Our sufficient condition is different and easier than the one given in [17, Theorem 4.2]. $\square$

If $A = 0$, then $f(x) = \frac{a^{\top}x+\alpha}{b^{\top}x+\beta}$. For K as before, thus f is semistrictly quasiconvex on K (see [7, 29]). Since $A = 0$, thus B_2 reduces to

$$\widehat{B_2} := \{v \in \mathbb{R}^n : \ \beta\langle a, v \rangle - \alpha\langle b, v \rangle > 0\}.$$

The next corollary is different and easier than the one given in [19, Example 4.1].

Corollary 4.8. *If $K^{\infty} \subseteq \widehat{B_2}$, then $\operatorname{argmin}_K f_1 \neq \emptyset$ and compact.*

Algorithms and Methods to deal with fractional programming problems are very rich areas in operation research (see [8, 9, 30]).

4.3. Necessary conditions for local minimizers

We finish this section with a necessary condition for the set of local minimizers of a proper function $\varphi \colon \mathbb{R}^n \to \overline{\mathbb{R}}$. Another necessary condition, using the q-asymptotic function, can be found in [18].

Lemma 4.9. *If $\operatorname*{argminloc}_{\mathbb{R}^n} \varphi \neq \emptyset$, then $\varphi^{\infty}_{\lambda}(u) \geq 0$ for all $u \in \mathbb{R}^n$.*

Proof. Take $x_0 \in \operatorname{argminloc}_{\mathbb{R}^n} \varphi$, thus there exists $\delta > 0$ with $\varphi(x_0) \leq \varphi(x)$ for all $x \in B(x_0, \delta)$. Then for all $u \neq 0$, calling $\eta = \delta \left\|u\right\|^{-1}$, we have

$$\varphi(x_0) \leq \varphi(x_0 + \varepsilon u), \ \forall \ \varepsilon < \eta.$$

Finally, for any $\lambda \in \mathbb{R}$ with $\lambda > f(x_0)$, it follows that

$$0 \leq \frac{\varphi(x_0 + \varepsilon u) - \varphi(x_0)}{\varepsilon} \leq \sup_{x \in S_\lambda(\varphi)} \sup_{t>0} \frac{\varphi(x+tu) - \varphi(x)}{t},$$

and $\varphi^\infty_\lambda(u) \geq 0$. $\square$

Take $\varphi\colon \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = x^2 - x^4$. Here φ has a local minimum at $x = 0$, but $\varphi^\infty \equiv -\infty$. Thus Lemma 4.9 does not hold for the usual asymptotic function.

Now, we apply Lemma 4.9 to extend the necessary conditions given in [11, Lemma 5.2] from quadratic to quadratic fractional programming. Notice that there is no convexity assumption on the numerator.

Proposition 4.10. *Let f be any quadratic fractional function, and K, as above. If $\operatorname*{argminloc}_K f \neq \emptyset$, then for every $u \in K^\infty$ one of the following assertions holds:*

(a) $u^\top A u > 0$.
(b) *If $u^\top A u \leq 0$, then there exists $x_u \in S_\lambda(f)$ such that*

$$\langle \nabla h(x_u) - f(x_u) b, u \rangle \geq 0. \tag{37}$$

Proof. Take $x_0 \in \operatorname*{argminloc}_K f$ and $\lambda \geq f(x_0)$, thus

$$\begin{aligned}
f^\infty_\lambda(u) \geq 0 &\iff \sup_{x\in S_\lambda(f)} \left(\sup_{t>0} g(x)(x^\top A u + \frac{1}{2} t u^\top A u + a^\top u) - b^\top u h(x) \right) \geq 0 \\
&\iff \sup_{x\in S_\lambda(f)} \left(g(x)(x^\top A u + a^\top u) - b^\top u h(x) + \sup_{t>0} \frac{1}{2} t u^\top A u \right) \geq 0 \\
&\iff u^\top A u > 0 \ \vee \left(u^\top A u \leq 0, \ \sup_{x\in S_\lambda(f)} g(x)(x^\top A u + a^\top u) - b^\top u h(x) \geq 0 \right),
\end{aligned}$$

and the result follows. $\square$

If $b = 0$ and $\beta = \gamma = 1$, then $K = \mathbb{R}^n$ and $f = h$. Thus the problem (1) reduces to the well-known quadratic minimization problem. In particular, if we consider the constrained minimization problem on the set

$$K := \{x \in \mathbb{R}^n : \ Hx = d, \ g(x) = \frac{1}{2} x^\top C x + c^\top x + \delta \leq 0\},$$

with $C \in \mathbb{R}^{n\times n}$ symmetric, $H \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n, d \in \mathbb{R}^m$ and $\delta \in \mathbb{R}$, Proposition 4.10 coincides with [11, Lemma 5.2 (a)] and the necessary condition of Theorem 3.5 with [11, Lemma 5.2 (b)].

Necessary and sufficient conditions for a point to be a local minimum in quadratic programming, over a polyhedral set, can be found in [23].

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