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A Necessary and Sufficient Condition on the Stability of the Infimum of Convex Functions

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Let us say that a convex function $f\colon C\to[-\infty,\infty]$ on a convex set $C\subseteq\mathbb{R}$ is infimum-stable if, for any sequence (f_n) of convex functions $f_n\colon C\to[-\infty,\infty]$ converging to f pointwise, one has $\inf_C f_n\to\inf_C f$. A simple necessary and sufficient condition for a convex function to be infimum-stable is given. The same condition remains necessary and sufficient if one uses Moore-Smith nets (f_ν) in place of sequences (f_n). This note is motivated by certain applications to stability of measures of risk/inequality in finance/economics.

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1. Summary and discussion

In this paper, the general notion of the convexity of functions is assumed, following [6]. Namely, let C be any convex subset of $\mathbb{R}$, and then take any function $f\colon C\to[-\infty,\infty]$. The function f is called convex if its epigraph

$$\operatorname{epi} f := \{(x,\tau)\in C\times\mathbb{R}\colon \tau\ge f(x)\}$$

is a convex set. The effective domain of f is

$$\operatorname{dom} f := \{x\in C\colon f(x)<\infty\}.$$

The convexity of f can also be expressed by the usual inequality

$$f\big((1-\lambda)x+\lambda y\big)\le(1-\lambda)f(x)+\lambda f(y) \tag{1}$$

for all $x\in C$, $y\in C$, and $\lambda\in(0,1)$ – but with the exception of the case when $\{f(x),f(y)\}=\{\infty,-\infty\}$, that is, when $f(x)$ and $f(y)$ are infinite values of opposite signs; cf. e.g. [6, Theorem 4.1]. Equivalently, one may require (1) only for all x and y in $\operatorname{dom} f$ $\big($and still for all $\lambda\in(0,1)\big)$.

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Generally, whenever possible, let us allow variables in expressions to take infinite values. At that, the corresponding values of the expressions are defined "by continuity", as usual; e.g., the value of the expression $\sqrt{\rho-\sigma-1}$ at $(\rho,\sigma)=(\infty,-\infty)$ should be understood as $\lim\limits_{\substack{\rho\to\infty,\\ \sigma\to-\infty}} \sqrt{\rho-\sigma-1}=\infty$.

One can now give

Definition 1.1. We say that the function f is *(sequentially) infimum-stable* if, for any sequence (f_n) of convex functions $f_n\colon C\to[-\infty,\infty]$ converging to f pointwise, one has

$$\inf_C f_n \to \inf_C f.$$

As usual, assume the conventions $\inf\emptyset:=\infty$ and $\sup\emptyset:=-\infty$. Also as usual, let the symbol card denote the cardinality of the given set.

If $\operatorname{card} C\le 1$ then, obviously, f is necessarily infimum-stable. To avoid this triviality, let us further assume that $\operatorname{card} C>1$, so that C is an interval with endpoints

$$c_{\min}:=\inf C \quad\text{and}\quad c_{\max}:=\sup C,$$

with

$$-\infty\le c_{\min}<c_{\max}\le\infty. \tag{2}$$

Let us also say that f is *monotonic* if f is nondecreasing or nonincreasing.

We shall provide a simple necessary and sufficient condition for a convex function on C to be infimum-stable. The form of this condition depends on which (if any) of the endpoints $c_{\min}$ and $c_{\max}$ of the interval C are infinite, so that there are four possible cases. The main case, when $c_{\min}=-\infty$ and $c_{\max}=\infty$, is described by

Theorem 1.2. *Suppose that $C=\mathbb{R}$ and the function f is convex. Then the following two conditions are equivalent:*

(I) *f is infimum-stable;*

(II) *either (i)* $\operatorname{card}\operatorname{dom} f>1$ *and at that f is not monotonic or (ii)* $\inf\limits_{\mathbb{R}} f=-\infty$.

The proofs are given in Section 2.

The remaining three cases concerning the finiteness of $c_{\min}$ and/or $c_{\max}$ are presented in the following three corollaries.

Corollary 1.3. *Suppose that $-\infty<c_{\min}<c_{\max}<\infty$ and the function f is convex. Then the following two conditions are equivalent to each other:*

(I) *f is infimum-stable;*

(II) *either (i)* $\operatorname{card}\operatorname{dom} f>1$ *or (ii)* $\inf\limits_C f=-\infty$.

Corollary 1.4. *Suppose that $-\infty=c_{\min}<c_{\max}<\infty$ and the function f is convex. Then the following two conditions are equivalent to each other:*

(I) *f is infimum-stable;*

(II) *either (i)* $\operatorname{card}\operatorname{dom} f>1$ *and f is not nondecreasing or (ii)* $\inf\limits_C f=-\infty$.

Corollary 1.5. *Suppose that* $-\infty < c_{\min} < c_{\max} = \infty$ *and the function* f *is convex. Then the following two conditions are equivalent to each other:*

(I) f *is infimum-stable;*

(II) *either (i)* $\operatorname{card}\operatorname{dom} f > 1$ *and* f *is not nonincreasing or (ii)* $\inf\limits_C f = -\infty$.

When, in addition to the conditions in Corollary 1.3 that the set C be bounded and the function f on C be convex, it is known that $f > -\infty$ on C, then the nececcary and sufficient condition for f to be infimum-stable can be simplified:

Corollary 1.6. *Suppose that* $-\infty < c_{\min} < c_{\max} < \infty$*, the function* f *is convex, and* $f > -\infty$ *on* C*. Then* f *is infimum-stable if and only if* $\operatorname{card}\operatorname{dom} f > 1$.

The above results are significantly easier to prove, if one additionally assumes that the functions f and f_n take only real values. Indeed, if f is not monotonic on C, then $f(v) < f(u) \wedge f(w)$ for some u, v, and w in C such that $u < v < w$ and hence the convexity of f yields $\inf\limits_C f = \inf\limits_{C\cap[u,w]} f$. So, for all large enough n one has $f_n(v) < f_n(u) \wedge f_n(w)$ and hence $\inf\limits_C f_n = \inf\limits_{C\cap[u,w]} f_n$. By [6, Theorem 10.6], the real-valued f_n's are uniformly bounded and equi-continuous on $C \cap [u, w]$. So, by the Arzelà–Ascoli theorem, $f_n \to f$ uniformly on $C \cap [u, w]$. Thus,

$$\inf_C f_n = \inf_{C\cap[u,w]} f_n \longrightarrow \inf_{C\cap[u,w]} f = \inf_C f.$$

Alternatively, if $C = \mathbb{R}$ and again f and f_n take only real values, then the convexity condition implies that these functions are continuous and attain a minimum value, which yields another kind of short proof [8].

However, it is oftentimes useful to allow f and f_n to take the value ∞. In particular, one may restate any constrained minimization problem, of minimizing a convex function f over a convex set $C \subset \mathbb{R}$, as an unconstrained one, of minimizing the corresponding convex extension $\tilde f$ of f over the entire set $\mathbb{R}$, where

$$\tilde f := \begin{cases} f & \text{on } C, \\ \infty & \text{on } \mathbb{R} \setminus C. \end{cases} \tag{3}$$

In fact, such extensions will be used in Section 2, in the proofs of Corollaries 1.3, 1.4, and 1.5.

Also, the value ∞ of convex functions arises naturally, for instance, in applications in probability, where the moment generating function of a random variable X (defined by the formula $M_X(t) := \mathsf{E}\, e^{tX}$ for $t \in \mathbb{R}$) may assume the value ∞; clearly, this function and even its logarithm are convex. In fact, the present note was motivated by certain applications to measures of risk/inequality in finance/economics; see the proofs of Propositions 2.3 and 2.4 in [5].

As for allowing f and f_n to take the value $-\infty$ also, this provides a further generalization, without significantly complicating the proofs.

Another application of Theorem 1.2 is to the Legendre-Fenchel transform of f, namely $f^*\colon \mathbb{R} \to [-\infty, \infty]$, $f^*(t) := \sup\limits_{x\in\mathbb{R}}[tx - f(x)]$.

This application is quite straightforward:

Corollary 1.7. *Suppose that* $C = \mathbb{R}$ *and the function* f *is convex and not monotonic, with* $\operatorname{card}\operatorname{dom} f > 1$. *Then the function* f^* *is continuous at* 0.

A series of papers (see e.g. [2]) concerns a perturbed function $f\colon X \times U \to \mathbb{R}$, where U is the set of values of a perturbation parameter. Among other conditions, $f(x,u)$ is assumed there to be at least continuous, jointly in $x \in X$ and $u \in U$, with the conclusion that the value function $u \mapsto \inf\limits_{x\in X} f(x,u)$ is continuous.

One should also mention [4, Theorem 1.2] concerning, essentially, the case when f and f_n are *real-valued* convex functions defined on a *compact* convex subset of $\mathbb{R}^d$.

Using [7, Theorem 7.17] and [1, Theorem 3.7] concerning the epi-convergence, one can deduce the implication (II) $\Longrightarrow$ (I) of our Theorem 1.2 – in the case when the function f is lower-semicontinuous.

Instead of sequences (f_n), one can, more generally, deal here with (Moore–Smith) nets $(f_\nu)_{\nu\in\mathrm{N}}$ of convex functions f_ν, where N is an arbitrary directed set; see e.g. [3, Chapter 2] concerning the relevant terms of general topology. Equivalently, one can define an ostensibly stronger notion of infimum-stability in terms of a topology, say $\pi_{\mathcal{C}(C)}$, on the set $\mathcal{C}(C)$ of all convex functions $f\colon C \to [-\infty,\infty]$ on a convex set $C \subseteq \mathbb{R}$. Namely, $\pi_{\mathcal{C}(C)}$ should be the topology induced on $\mathcal{C}(C)$ by the Tychonoff product topology π_C on the set $[-\infty,\infty]^C$ of all functions $f\colon C \to [-\infty,\infty]$ on C, so that a subbase of the topology $\pi_{\mathcal{C}(C)}$ consists of all sets of the form $F_{x,I} := \{f \in \mathcal{C}(C)\colon f(x) \in I\}$, where x is any point in C and I is any interval of one of the following three forms: (a,b), $[-\infty,b)$, or $(a,\infty]$, for arbitrary real a and b.

Definition 1.8. A function $f \in \mathcal{C}(C)$ is *topologically infimum-stable* if the mapping $\mathcal{C}(C) \ni g \mapsto \inf g$ is continuous at "point" f with respect to the topology $\pi_{\mathcal{C}(C)}$.

Closely following the lines of the proofs of Theorem 1.2 and Corollaries 1.3, 1.4, 1.5, and 1.6, stated above, and substituting terms such as "net" and "subnet" for "sequence" and "subsequence", one sees that all these statements hold with nets $(f_\nu)_{\nu\in\mathrm{N}}$ in place of sequences (f_n); at that, the term "eventually" should be understood in a standard way, as e.g. in [3, page 65]: a property P_ν holds eventually if there is some $\mu \in \mathrm{N}$ such that P_ν holds for each $\nu \in \mathrm{N}$ satisfying the condition $\nu \geq \mu$. Thus, one obtains

Theorem 1.9. *For any convex set* $C \subseteq \mathbb{R}$, *a function* $f \in \mathcal{C}(C)$ *is topologically infimum-stable if and only if it is sequentially infimum-stable.*

2. Proofs

Proof of Theorem 1.2. Let us first consider the implication

(II) $\Longrightarrow$ (I). Assume that the function $f\colon \mathbb{R} \to [-\infty,\infty]$ is convex and satisfies the condition (II), and take any sequence (f_n) of convex functions $f_n\colon \mathbb{R} \to$

$[-\infty,\infty]$ converging to f pointwise. Let

$$m := \inf_{\mathbb{R}} f \quad \text{and} \quad m_n := \inf_{\mathbb{R}} f_n. \tag{4}$$

We need to show that $m_n \to m$. If this is not true, then there is a sequence (n_k) of natural numbers such that $n_k \to \infty$ and m_{n_k} converges to a point in $[-\infty,\infty]\setminus\{m\}$ as $k\to\infty$. So, without loss of generality (w.l.o.g.) there is a limit

$$m_\infty := \lim_n m_n. \tag{5}$$

Take any $c \in (m,\infty)$. Then, by the definition of m, there is some $x_c \in \mathbb{R}$ such that $f(x_c) < c$. By the pointwise convergence of f_n to f, it follows that eventually (that is, for all large enough n) $f_n(x_c) < c$ and hence $m_\infty \le \limsup_n f_n(x_c) \le c$. Thus (in fact, without any conditions on f), $m_\infty \le m$. So, w.l.o.g.

$$m_\infty < m. \tag{6}$$

In particular, it follows that $m > -\infty$. On the other hand, the condition $\operatorname{card}\operatorname{dom} f > 1$ shows that $m < \infty$. Thus,

$$m \in \mathbb{R}. \tag{7}$$

So, by vertical shifting, w.l.o.g. one may assume that

$$m = 1; \tag{8}$$

we shall actually use this latter assumption only where convenient.

Thus, alternative (i) of condition (II) takes place: $\operatorname{card}\operatorname{dom} f > 1$ and f is not monotonic. So, there are real numbers x_1 and x_2 such that $x_1 < x_2$ and $f(x_1) < f(x_2)$, which in particular implies $f(x_1) < \infty$. Hence, by (1), for any $x \in (x_2,\infty)$ with $f(x) < \infty$ one has

$$f(x_2) \le \frac{x-x_2}{x-x_1} f(x_1) + \frac{x_2-x_1}{x-x_1} f(x)$$

or, equivalently,

$$f(x) \ge f(x_2) + \frac{x-x_2}{x_2-x_1}\big(f(x_2) - f(x_1)\big), \tag{9}$$

which shows that $f(x)\to\infty$ as $x\to\infty$. Similarly, $f(x)\to\infty$ as $x\to-\infty$. So,

$$f(x) \underset{|x|\to\infty}{\longrightarrow} \infty. \tag{10}$$

Therefore and in view of (7), introducing now

$$L := \{x\in\mathbb{R}\colon f(x) \le m+1\}, \quad u := \inf L, \quad \text{and} \quad v := \sup L,$$

one sees that $v < \infty$ and $u > -\infty$. Also, by the convexity of f and the condition $\operatorname{card}\operatorname{dom} f > 1$, L is a nonempty interval with the endpoints u and v, and

$$m = \inf_{[u,v]} f.$$

Moreover, $u < v$. Indeed, if $u = v$, then the nonempty interval L must be the singleton set $\{u\}$. So, $f > m + 1$ on $\mathbb{R} \setminus \{u\}$ and hence $f(u) = m$. Therefore (cf. (9)), for each $x \in (u, \infty)$ and all $\varepsilon \in (0, x - u)$ one has

$$f(x) \geq f(u) + \frac{x-u}{\varepsilon}\big(f(u+\varepsilon) - f(u)\big) \geq f(u) + \frac{x-u}{\varepsilon} \underset{\varepsilon\downarrow 0}{\longrightarrow} \infty,$$

so that $f = \infty$ on (u, ∞); similarly, $f = \infty$ on $(-\infty, u)$, which contradicts the condition $\operatorname{card}\operatorname{dom} f > 1$. Thus,

$$-\infty < u < v < \infty \quad \text{and} \quad f(x) > m + 1 \text{ for all real } x \notin [u, v].$$

So, eventually $f_n(u-1) \wedge f_n(v+1) > m+1$, whereas $f(w) < m+1$ for some $w \in L \subseteq [u, v]$ and hence eventually $f_n(w) < m + 1$. Now the convexity of f_n implies (cf. (9)) that uniformly over all $x \in [v + 1, \infty)$ with $f_n(x) < \infty$ eventually one has

$$f_n(x) \geq f_n(v+1) + \frac{x-v-1}{v+1-w}\big(f_n(v+1) - f_n(w)\big) \geq f_n(v+1) > m+1,$$

and so, $\inf\limits_{[v+1,\infty)} f_n \geq m + 1$. Similarly, eventually $\inf\limits_{(\infty,u-1]} f_n \geq m+1$, whence

$$m_n = \inf_{[u-1,v+1]} f_n. \tag{11}$$

Introduce next

$$x_{\min} := \inf \operatorname{dom} f \quad \text{and} \quad x_{\max} := \sup \operatorname{dom} f; \tag{12}$$

by the condition $\operatorname{card}\operatorname{dom} f > 1$, $\operatorname{dom} f$ is a nonempty interval with endpoints $x_{\min}$ and $x_{\max}$, so that

$$-\infty \leq x_{\min} < x_{\max} \leq \infty. \tag{13}$$

Thus, in view of (7), the convex function f is real-valued on the interval $(x_{\min}, x_{\max})$ and therefore has the left and right derivatives, f'_- and f'_+, on $(x_{\min}, x_{\max})$. Now one can also introduce

$$x_+ := x_{\max} \wedge \inf E_+ \quad \text{and} \quad x_- := x_{\min} \vee \sup E_-, \quad \text{where}$$

$$E_+ := \{x \in (x_{\min}, x_{\max})\colon f'_+(x) > 0\} \ \text{ and } \ E_- := \{x \in (x_{\min}, x_{\max})\colon f'_-(x) < 0\}.$$

Note that for any $x \in E_-$ and $y \in E_+$ one has $x \leq y$ – because for any x and y such that $x_{\min} < y < x < x_{\max}$ one has $f'_+(y) \leq f'_-(x)$, by the convexity of f. So,

$$x_{\min} \leq x_- \leq x_+ \leq x_{\max}. \tag{14}$$

Comparing this with the strict inequality in (13), one sees that $x_+ > x_{\min}$ or $x_- < x_{\max}$. These latter two cases are quite similar to each other, and in fact they can be deduced from each other by the horizontal flip $\mathbb{R} \times [-\infty, \infty] \ni (x, \tau) \mapsto (-x, \tau)$ applied to the graphs of the functions f_n and f. So, w.l.o.g.

$$x_{\min} < x_+ \le x_{\max}. \tag{15}$$

The right derivative, f'_+, of the convex function f is nondecreasing on $(x_{\min}, x_{\max})$. Therefore, the set E_+ is a (possibly empty) interval with endpoints x_+ and $x_{\max}$. So, $f'_+ \le 0$ and hence f is nonincreasing on the interval $(x_{\min}, x_+)$, and the latter interval is nonempty, in view of (15). Thus, there exists the left limit of the function f at the point x_+:

$$p := f\big((x_+)-\big) \in [m, \infty). \tag{16}$$

Observe that

$$f > p \text{ in a right neighborhood of } x_+. \tag{17}$$

Indeed, if $x_+ = x_{\max}$, then $f = \infty > p$ on the entire interval $(x_+, \infty) = (x_{\max}, \infty)$. So, in view of (15), one may now assume that $x_+ < x_{\max}$, so that x_+ is in the interval $(x_{\min}, x_{\max})$, which latter is the interior of the set $\operatorname{dom} f$. Therefore and in view of (7), the convex function f is continuous at x_+, with $f(x_+) = f\big((x_+)-\big) = p$. Because f is strictly increasing on the interval $(x_+, x_{\max})$ and continuous at x_+, f is strictly increasing on the interval $[x_+, x_{\max})$ as well, which implies that $f > f(x_+) = p$ on $(x_+, x_{\max})$. Thus, (17) is verified.

Take now any $\delta \in (0, \infty)$. Then take any

$$x_{++} \in (x_+, x_+ + \delta) \tag{18}$$

such that

$$f(x_{++}) > p; \tag{19}$$

such a point x_{++} exists by (17). Next, take any

$$\varepsilon \in \big(0, \delta \wedge 1 \wedge (x_+ - x_{\min}) \wedge \sqrt{f(x_{++}) - p}\big), \tag{20}$$

which is possible in view of (15) and (19). Then, successively take

$$y \in \big(\max\big[\tfrac{1}{2}(x_+ + x_{\min}), x_+ - \varepsilon\big],\ x_+ - \tfrac{\varepsilon}{2}\big), \tag{21}$$

$$z \in (y + \tfrac{\varepsilon}{2},\ x_+) \text{ such that } f(z) < p + \varepsilon^2, \tag{22}$$

$$w \in \big(x_{\min},\ y - \tfrac{1}{2}(x_+ - x_{\min})\big); \tag{23}$$

here, (i) the choice of y is possible by the choice of ε; (ii) the choice of w is possible by the choice of y; and (iii) the choice of z is possible by the choice of y, the condition $\varepsilon \ne 0$, and the definition of p in (16). The conditions on ε, y, z, w listed in (20), (21), (22), (23) imply

$$\begin{gathered} x_{\min} < w < y < z < x_+, \quad y > x_+ - \varepsilon, \quad z - y > \varepsilon/2, \\ y - w > \tfrac{1}{2}\,(x_+ - x_{\min}), \quad \text{and} \quad f(x_{++}) > p + \varepsilon^2. \end{gathered} \tag{24}$$

Since f is nonincreasing on $(x_{\min}, x_+)$, one has $f(y) \geq p$. This and the inequalities $f(z) < p + \varepsilon^2$ in (22) and $f(x_{++}) > p + \varepsilon^2$ in (24) show that eventually

$$f_n(y) \geq p - \varepsilon^2 \quad \text{and} \quad f_n(z) < p + \varepsilon^2 < f_n(x_{++}). \tag{25}$$

Also, the values $f(y)$ and $f(z)$ are real, in view of (7) and because $\{y, z\} \subset \operatorname{dom} f$. So, eventually the values $f_n(y)$ and $f_n(z)$ are real. Now, by the convexity of f_n, (25), (24), and (16), uniformly over all $x \in [u-1, y]$ one eventually has

$$\begin{aligned} f_n(x) &\geq \frac{z-x}{z-y} f_n(y) - \frac{y-x}{z-y} f_n(z) \geq p - \frac{z+y-2x}{z-y} \varepsilon^2 \geq p - 2(z+y-2x)\varepsilon \\ &\geq p - 2\big(2x_+ - 2(u-1)\big)\varepsilon \geq m - 4\big(x_+ - (u-1)\big)\varepsilon, \end{aligned} \tag{26}$$

so that eventually

$$\inf_{[u-1,y]} f_n \geq m - c_1\varepsilon \geq m - c_1\delta, \tag{27}$$

where $c_1 := \max\big[0, 4\big(x_+ - (u-1)\big)\big]$; the last inequality in (27) holds because of (20). Somewhat similarly, uniformly over all $x \in [y, x_{++}]$ one eventually has

$$f_n(x) \geq \frac{x-w}{y-w} f_n(y) - \frac{x-y}{y-w} f_n(w) \geq p - \delta - \frac{4\big(f(w)+1\big)}{x_+ - x_{\min}} \delta; \tag{28}$$

here we used the relations $\frac{x-w}{y-w} \geq 1$, $p \geq m = 1 > \varepsilon > 0$, $f_n(y) \geq p - \varepsilon^2 \geq p - \varepsilon = 0 \vee (p-\varepsilon)$, $\varepsilon < \delta$, $0 \leq x - y \leq \varepsilon + \delta < 2\delta$, $y - w > \frac{1}{2}(x_+ - x_{\min})$, and $0 \leq f(w) - m = f(w) - 1 < f_n(w) < f(w) + 1$, which hold at least eventually. So, eventually

$$\inf_{[y,x_{++}]} f_n \geq m - c_2\delta, \tag{29}$$

where $c_2 := 1 + 4\big(f(w)+1\big)/(x_+ - x_{\min})$.

Further, uniformly over all $x \in (x_{++}, \infty)$ with $f_n(x) < \infty$ one eventually has

$$f_n(x) \geq f_n(x_{++}) + \frac{x - x_{++}}{x_{++} - z}\big(f_n(x_{++}) - f_n(z)\big) \geq f_n(x_{++}) > p + \varepsilon^2 \geq m; \tag{30}$$

the second inequality here holds by (25). So, eventually

$$\inf_{(x_{++},\infty)} f_n \geq m. \tag{31}$$

Combining this with (11), (27), and (29), one sees that, for any real $\delta > 0$, eventually

$$m_n \geq m - c\delta, \tag{32}$$

where $c := c_1 \vee c_2$.

Thus, one obtains a contradiction with (5)–(6), so that the implication (II) $\Longrightarrow$ (I) is verified.

It remains to consider the reverse implication, (I) $\Longrightarrow$ (II). Equivalently, let us verify the implication $\neg$(II) $\Longrightarrow$ $\neg$(I), where the symbol $\neg$ denotes the negation, as usual. Thus, let us assume that the condition (II) in Theorem 1.2 fails to hold. We have then to show that f is not infimum-stable.

If $\operatorname{card}\operatorname{dom} f = 0$, then $f = \infty$ on $\mathbb{R}$ and hence $m = \infty$. On the other hand, defining convex functions f_n by the formula $f_n(x) := x + n$ for all natural n and all $x \in \mathbb{R}$, one sees that $f_n \to f$ pointwise, whereas $m_n = -\infty \not\to \infty = m$. So, f is not infimum-stable in the case $\operatorname{card}\operatorname{dom} f = 0$.

Next, if $\operatorname{card}\operatorname{dom} f = 1$ and $m > -\infty$, then $\operatorname{dom} f$ is the singleton set $\{x_0\}$ for some $x_0 \in \mathbb{R}$, and $m = f(x_0) \in \mathbb{R}$. So, in view of possible vertical and horizontal shifting, let us make w.l.o.g. the simplifying assumptions that $x_0 = 0$ and $f(x_0) = 1$. Then $m = f(0) = 1$ and $f = \infty$ on $\mathbb{R} \setminus \{0\}$. Let us now define convex functions f_n by the formula $f_n(x) := |1+nx|$ for all natural n and all $x \in \mathbb{R}$. Then $f_n \to f$ pointwise, whereas $m_n = 0 \not\to 1 = m$. So, f is not infimum-stable in the case $\operatorname{card}\operatorname{dom} f = 1$ as well.

By the assumption $\neg$(II), it remains to consider the case when $\operatorname{card}\operatorname{dom} f > 1$, f is monotonic, and $m > -\infty$. Then w.l.o.g. f is nondecreasing (say), whence $f\big((-\infty)+\big) \le f(x) < \infty$ for any x in the set $\operatorname{dom} f$, which is nonempty by the assumption $\operatorname{card}\operatorname{dom} f > 1$, and so, $f\big((-\infty)+\big) < \infty$. Therefore, defining convex functions f_n by the formula $f_n(x) := f(x) + x/n$ for all natural n and all $x \in \mathbb{R}$, one has $m_n \le f_n\big((-\infty)+\big) = f\big((-\infty)+\big) - \infty = -\infty$ for all n, which implies, in view of the assumption $m > -\infty$, that $m_n = -\infty \not\to m$, whereas $f_n \to f$ pointwise. It follows that f is not infimum-stable in this remaining case as well.

Thus, the proof of the implication (I) $\Longrightarrow$ (II) is complete, and so is the entire proof of Theorem 1.2. $\square$

Proof of Corollary 1.3. (II) $\Longrightarrow$ (I): Assume that the function $f\colon C \to [-\infty, \infty]$ is convex and satisfies the condition (II) of Corollary 1.3, and take any sequence (f_n) of convex functions $f_n\colon C \to [-\infty, \infty]$ converging to f pointwise. Extend the function f, defined on C, to the function $\tilde f$ as in (3), and similarly extend f_n to $\tilde f_n$, for each n. Then, on the entire real line $\mathbb{R}$, the functions $\tilde f$ and $\tilde f_n$ are convex, and $\tilde f_n \to \tilde f$ pointwise. Moreover, the condition (II) of Corollary 1.3 implies that the condition (II) of Theorem 1.2 is satisfied with the extended function $\tilde f$ in place of f there; indeed, if $-\infty < c_{\min} < c_{\max} < \infty$ and $\operatorname{card}\operatorname{dom} f > 1$, then $\tilde f$ is not monotonic. So, by Theorem 1.2, $\inf\limits_C f_n = \inf\limits_{\mathbb{R}} \tilde f_n \longrightarrow \inf\limits_{\mathbb{R}} \tilde f = \inf\limits_C f$. Thus, the implication (II) $\Longrightarrow$ (I) in Corollary 1.3 is verified.

(I) $\Longrightarrow$ (II): Equivalently, let us verify the implication $\neg$(II) $\Longrightarrow$ $\neg$(I). Thus, let us assume that the condition (II) in Corollary 1.3 fails to hold. We have then to show that f is not infimum-stable.

If $\operatorname{card}\operatorname{dom} f = 0$, then $f = \infty$ on C. By horizontal shifting and the condition (2), w.l.o.g. the point 0 is in the interior of the set C. Defining now convex functions f_n by the formula $f_n(x) := n|1 + nx|$ for all natural n and all $x \in C$, one sees that $f_n \to f$ pointwise, whereas eventually one has $-\frac{1}{n} \in C$ and hence

$\inf\limits_C f_n = 0 \not\to \infty = \inf\limits_C f$. So, f is not infimum-stable in the case $\operatorname{card}\operatorname{dom} f = 0$.

It remains to consider the case when $\operatorname{card}\operatorname{dom} f = 1$ and $\inf\limits_C f > -\infty$. Then $\operatorname{dom} f$ is the singleton set $\{x_0\}$ for some $x_0 \in C$, and $\inf\limits_C f = f(x_0) \in \mathbb{R}$. By (2), either $x_0 > c_{\min}$ or $x_0 < c_{\max}$. These two cases are quite similar to each other. So, assume w.l.o.g. that $x_0 > c_{\min}$. Moreover, by vertical and/or horizontal shifting, w.l.o.g. $x_0 = 0$ and $f(x_0) = 1$. So, $f(0) = 1$, $f = \infty$ on $C \setminus \{0\}$, and $\inf C = c_{\min} < 0 \in C$. Let us now define convex functions f_n by the formula $f_n(x) := |1 + nx|$ for all natural n and all $x \in C$. Then $f_n \to f$ pointwise, whereas eventually $\inf\limits_C f_n = 0 \not\to 1 = \inf\limits_C f$. So, f is not infimum-stable whenever the condition (II) in Corollary 1.3 fails to hold. □

Proof of Corollary 1.4. The proof of the implication (II) $\implies$ (I) of this corollary repeats the corresponding part of the proof of Corollary 1.3 almost literally. The main difference is that here — instead of the conditions $-\infty < c_{\min} < c_{\max} < \infty$ and $\operatorname{card}\operatorname{dom} f > 1$ — one should use the conditions $-\infty = c_{\min} < c_{\max} < \infty$ and $\operatorname{card}\operatorname{dom} f > 1$ to find that the function $\tilde f$ is not nonincreasing and then conclude that $\tilde f$ is not monotonic (given that f is not nondecreasing and hence $\tilde f$ is not nondecreasing).

As for the reverse implication, (I) $\implies$ (II), in Corollary 1.4, its proof is almost literally the same as the corresponding part of the proof of Theorem 1.2. The differences are few and small: “Theorem 1.2”; “monotonic”; $\mathbb{R}$ as the domain of f and f_n; m; and m_n have to be replaced, respectively, by “Corollary 1.4”; “nondecreasing”; C; $\inf\limits_C f$; and $\inf\limits_C f_n$. □

Proof of Corollary 1.5. This proof is quite similar to that of Corollary 1.4. Alternatively, Corollary 1.5 can be obtained immediately from Corollary 1.4 by horizontal flipping. □

Proof of Corollary 1.6. Suppose that indeed $-\infty < c_{\min} < c_{\max} < \infty$, the function f is convex, and $f > -\infty$ on C. In view of Corollary 1.3, it is enough to show that then $\inf\limits_C f > -\infty$. This conclusion is obvious if $\operatorname{card}\operatorname{dom} f = 0$. If $\operatorname{card}\operatorname{dom} f = 1$, then the condition $f > -\infty$ on C implies that there is a point $x_0 \in C$ such that $f(x_0) \in \mathbb{R}$ and $f = \infty$ on $C \setminus \{x_0\}$, so that again the conclusion $\inf\limits_C f > -\infty$ follows.

Finally, if $\operatorname{card}\operatorname{dom} f > 1$, then there is a point $x_0 \in C$ such that f is finite in a neighborhood of x_0. So, by the convexity of f, there is a finite right derivative, $f'_+(x_0)$, of f at x_0. So, $f(x) \geq f(x_0) + f'_+(x_0)(x - x_0)$ for all $x \in C$ and therefore

$$\inf_C f \geq \inf_{x\in C}[f(x_0) + f'_+(x_0)(x - x_0)] > -\infty,$$

since the set C is bounded. Thus, in all cases the conclusion $\inf\limits_C f > -\infty$ holds. □

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