Journal of Convex Analysis
Volume 26 (2019), No. 1, 105–116

Vertices, Edges and Facets of the Unit Ball

F. J. Garcia-Pacheco
Department of Mathematics, University of Cadiz, 11519 Puerto Real, Spain
garcia.pacheco@uca.es

Received: June 28, 2017
Revised manuscript received: December 13, 2017
Accepted: December 17, 2017

It has been recently proved that every real Banach space can be endowed with an equivalent norm in such a way that the new unit sphere contains a convex subset with non-empty interior relative to the unit sphere. In fact, under good conditions like separability or being weakly compactly generated, this renorming can be accomplished to have a dense amount of convex sets in the unit sphere with non-empty relative interior. Therefore, not all equivalent norms on a Banach space show some degree of strict convexity. In the opposite direction, for a long time it was unknown whether there exists a non-strictly convex real Banach space of dimension strictly greater than 2 with a dense amount of extreme points in the unit sphere. This question has recently solved in three dimensions. The idea behind this solution is to construct a 3-dimensional unit ball whose boundary is made of extreme points except for two non-trivial segments (which are opposite to each other). This unit ball is a deformation of an ellipsoid. In this manuscript we follow this line of research and prove that every Banach space with dimension strictly greater than 2 admitting a strictly convex equivalent renorming admits a non-strictly convex equivalent norm whose unit ball verifies that all of its proper faces are segments.

Keywords: Vertices, edges, facets, faces, infinite dimensional Banach space, strictly convex, renorming.

2010 Mathematics Subject Classification: 46B20

1. Introduction

1.1. Preliminaries and background

Every real Banach space can be endowed with an equivalent norm in such a way that the new unit sphere contains a convex subset with non-empty interior relative to the unit sphere. Indeed, it suffices to consider the equivalent norm on X whose unit ball is given by

$$\mathsf{B}_X \cap (x^*)^{-1}\left(\left[\frac{-1}{2}, \frac{1}{2}\right]\right)$$

where $x^* \in \mathsf{NA}(X)$, that is, x^* is a norm-attaining functional on X, and $\|x^*\| = 1$. This renorming does not work on complex spaces because, as shown in [8], the unit sphere of complex spaces cannot have a convex subset with non-empty relative interior. Refinements of this simple renorming can be found in [1, 2, 5, 10]. In fact, under good conditions like separability or being weakly compactly generated,

ISSN 0944-6532 / $ 2.50

this renorming can be improved to have a dense amount of convex sets in the unit sphere with non-empty relative interior (see [6, 7, 11]). These equivalent norms, overall the last ones, are very far from being strictly convex and basically show no signs of strict convexity. Therefore, not all equivalent norms on a Banach space show some degree of strict convexity. In the opposite direction, for a long time it was unknown whether there exists a non-strictly convex real Banach space of dimension strictly greater than 2 with a dense amount of extreme points in the unit sphere. Notice that if a 2-dimensional real Banach space has a dense amount of extreme points in its unit sphere, then it must be strictly convex. The previous question has been recently solved in three dimensions (see [9]). The idea behind this solution is to construct a 3-dimensional unit ball whose boundary is made of extreme points except for two non-trivial segments (which are opposite to each other). This unit ball is a deformation of an ellipsoid. In this manuscript we follow this line of research and prove that every real Banach space with dimension strictly greater than 2 admitting a strictly convex equivalent renorming admits a non-strictly convex equivalent norm whose unit ball verifies that all of its proper faces are segments (see Theorem 3.4).

1.2. A brief introduction to smoothness

All Banach spaces considered throughout this paper will be over the reals. Recall that a point x in the unit sphere S_X of a Banach space X is said to be a *smooth point* of B_X provided that there is only one functional in S_{X^*} attaining its norm at x. This unique functional is usually denoted by $\mathsf{J}_X(x)$. The set of smooth points of the (closed) unit ball B_X of X is usually denoted as $\mathrm{smo}(\mathsf{B}_X)$. This way X is said to be *smooth* provided that $\mathsf{S}_X = \mathrm{smo}(\mathsf{B}_X)$.

In case $x \in \mathsf{S}_X$ is not a smooth point then $\mathsf{J}_X(x)$ is defined as $x^{-1}(1) \cap \mathsf{B}_{X^*}$, that is, the set $\{x^* \in \mathsf{B}_{X^*} : x^*(x) = 1\}$.

If X is a smooth Banach space, then the dual map of X is defined as the map $\mathsf{J}_X : X \to X^*$ such that $\|\mathsf{J}_X(x)\| = \|x\|$ and $\mathsf{J}_X(x)(x) = \|x\|^2$ for all $x \in X$. It is well known that the dual map is $\|\cdot\|$-w^* continuous and verifies that $\mathsf{J}_X(\lambda x) = \lambda \mathsf{J}_X(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

1.3. A brief introduction to faces

The following definition is very well known amid the Banach Space Geometers.

Definition 1.1. Let X be a Banach space and consider a non-empty convex subset C of B_X. Then:

(1) C is said to be a *face* of B_X provided that C verifies the extremal condition with respect to B_X, that is, if $x, y \in \mathsf{B}_X$ and $t \in (0,1)$ are so that $tx + (1-t)y \in C$, then $x, y \in C$.

(2) C is said to be an *exposed face* of B_X provided that there exists $f \in \mathsf{S}_{X^*}$ such that $C = \mathrm{supp}(f, \mathsf{B}_X) := f^{-1}(1) \cap \mathsf{B}_X$.

It is immediate that every exposed face is a *proper face*, and every proper face must be contained in the unit sphere. Also notice that $\mathsf{J}_X(x) = \operatorname{supp}(x, \mathsf{B}_{X^*})$ for every $x \in \mathsf{S}_X$, that is, $\mathsf{J}_X(x)$ is an exposed face of B_{X^*}. We will refer to these exposed faces as w^*-exposed faces. It is also immediate that a Banach space is *reflexive* if and only if all the exposed faces of its dual unit ball are w^*-exposed faces.

A face, an exposed face, or a w^*-exposed face which is a singleton is called an *extreme point*, an *exposed point*, or a *w^*-exposed point*, respectively.

Definition 1.2. Let C be an exposed face of the unit ball of a Banach space X. Then C is said to be

(1) a *vertex* of B_X if C is a singleton,

(2) an *edge* of B_X if C is a non-trivial segment, and

(3) a *facet* of B_X if C has non-empty interior relative to the unit sphere S_X.

It is immediate that vertices and exposed points are the same thing. According to [8, Theorem 2.1]), no proper face of the unit ball of a complex normed space can have non-empty interior relative to the unit sphere. This is why in this manuscript we will only consider real spaces.

2. Duality of vertices and facets

2.1. Topological properties of facets

In this subsection we unveil the topological properties of the facets, reaching the conclusion that the interior of a facet relative to the unit sphere is dense in the facet. Some of the technical lemmas employed in this subsection will also be used in the further subsections.

Lemma 2.1. *Let X be a Banach space. Let $C \subset \mathsf{S}_X$ be a convex subset and let $f \in \mathsf{S}_{X^*}$ so that $C \subseteq \operatorname{supp}(f, \mathsf{B}_X)$. Then the interior of C relative to S_X coincides with the interior of C relative to $f^{-1}(1)$.*

Proof. (a) $\operatorname{int}_{\mathsf{S}_X}(C) \subseteq \operatorname{int}_{f^{-1}(1)}(C)$. Indeed, suppose that c belongs to the interior of C relative to S_X. Let $\varepsilon > 0$ such that $\frac{c+u}{\|c+u\|} \in C$ for all $u \in \mathsf{B}_X(0,\varepsilon)$. We will show that $\mathsf{B}_X(c,\varepsilon) \cap f^{-1}(1) \subseteq C$. Let then $u \in \mathsf{B}_X(0,\varepsilon)$ such that $c+u \in f^{-1}(1)$. We have that

$$\frac{1}{\|c+u\|} = f\left(\frac{c+u}{\|c+u\|}\right) = 1.$$

Thus, $c+u \in C$.

(b) $\operatorname{int}_{\mathsf{S}_X}(C) \supseteq \operatorname{int}_{f^{-1}(1)}(C)$. Conversely, suppose that c belongs to the interior of C relative to $f^{-1}(1)$. So, let $\varepsilon > 0$ such that $f(c+u) > 0$ and $\frac{c+u}{f(c+u)} \in C$ for all $u \in \mathsf{B}_X(0,\varepsilon)$. We will show that $\mathsf{B}_X(c,\varepsilon) \cap \mathsf{S}_X \subseteq C$. Let then $u \in \mathsf{B}_X(0,\varepsilon)$ such that $\|c+u\| = 1$. We have that

$$\frac{1}{f(c+u)} = \left\|\frac{c+u}{f(c+u)}\right\| = 1.$$

Thus, $c+u \in C$. $\square$

Lemma 2.2. *Let X be a Banach space and C a convex subset of S_X. If $c \in \mathrm{int}_{\mathsf{S}_X}(C)$ and $d \in \mathsf{S}_X \setminus \{c\}$ is so that $[c,d] \subseteq \mathsf{S}_X$, then there exists $e \in \mathrm{int}_{\mathsf{S}_X}(C) \setminus \{c,d\}$ such that $c \in (e,d)$, and thus $[e,d] \subseteq \mathsf{S}_X$.*

Proof. There exists $r > 0$ so that $\mathsf{B}_X(c,r) \cap \mathsf{S}_X \subseteq \mathrm{int}_{\mathsf{S}_X}(C)$. By applying the Hahn-Banach Separation Theorem, we can find $f \in \mathsf{S}_{X^*}$ such that $C \subseteq \mathrm{supp}(f, \mathsf{B}_X)$. Let $t < 0$ sufficiently close to 0 in such a way that $e := td + (1-t)c \in \mathsf{B}_X(c,r)$. It is clear that $c \in (e,d)$ so it only remains to show that $e \in \mathrm{int}_{\mathsf{S}_X}(C)$. In accordance with Lemma 2.1, we may suppose without any loss of generality that $\mathsf{B}_X(c,r) \cap f^{-1}(1) \subseteq \mathrm{int}_{\mathsf{S}_X}(C)$. Now it is obvious that $f(e) = 1$ which implies that $e \in \mathrm{int}_{\mathsf{S}_X}(C)$. □

Theorem 2.3. *Let X be a Banach space and C a convex subset of S_X.*

(1) *If $\mathrm{int}_{\mathsf{S}_X}(C) \neq \varnothing$, then $\mathrm{cl}\,(\mathrm{int}_{\mathsf{S}_X}(C)) = C$.*

(2) *If Y is a closed vector subspace of X, then $\mathrm{int}_{\mathsf{S}_X}(C) \cap Y \subseteq \mathrm{int}_{\mathsf{S}_Y}(C \cap Y)$.*

Proof. (1) Fix any $c \in \mathrm{int}_{\mathsf{S}_X}(C)$. By applying the Hahn-Banach Separation Theorem, we can find $f \in \mathsf{S}_{X^*}$ such that $C \subseteq \mathrm{supp}(f, \mathsf{B}_X)$. By Lemma 2.1 we have that C has non-empty interior relative to $f^{-1}(1)$. In fact, c belongs to such interior. Since translations are homeomorphisms, we deduce that $C - c$ has non-empty interior relative to $\ker(f)$. Since $\ker(f)$ is a vector space containing the convex set $C - c$, we deduce that the interior of $C - c$ relative to $\ker(f)$ is dense in $C - c$. Undoing the translation we conclude that the interior of C relative to $f^{-1}(1)$, which equals the interior of C relative to S_X, is dense in C.

(2) Let $c \in \mathrm{int}_{\mathsf{S}_X}(C) \cap Y$. If $\varepsilon > 0$ is so that $\mathsf{B}_X(c,\varepsilon) \cap \mathsf{S}_X \subseteq C$, then

$$\mathsf{B}_Y(c,\varepsilon) \cap \mathsf{S}_Y = \mathsf{B}_X(c,\varepsilon) \cap \mathsf{S}_X \cap Y \subseteq C \cap Y.$$ □

It is not difficult to search for counterexamples that show that the inclusion in the second item of the previous theorem is not an equality in general.

Corollary 2.4. *If X is a Banach space and C a facet of B_X, then $\mathrm{cl}\,(\mathrm{int}_{\mathsf{S}_X}(C)) = C$ and $C \cap Y$ is a facet of B_Y for every closed vector subspace Y of X intersecting $\mathrm{int}_{\mathsf{S}_X}(C)$.*

Proof. If we bear in mind Theorem 2.3, then it only remains to show that $C \cap Y$ is a facet of B_Y. Indeed, it is immediate to check that $C \cap Y$ is a face of B_Y. Finally, in virtue of Theorem 2.8 and Theorem 2.3(2), we conclude that $C \cap Y$ is a facet of B_Y. □

Recall that a closed subspace M of a Banach space is said to be an *isometric reflection subspace* of X provided that M is complemented in X and $\|m+n\| = \|m-n\|$ for all $m \in M$ and all $n \in N$, where N is the complement of M. Note that in this situation

$$\|m\| \leq \frac{\|m+n\|}{2} + \frac{\|m-n\|}{2} = \|m+n\|$$

therefore if M and N are in isometric reflection, then both are 1-complemented in X.

A vector $x \in \mathsf{S}_X$ is called an *isometric reflection vector* provided that $\mathbb{R}x$ is an *isometric reflection subspace*. The isometric reflection complement of $\mathbb{R}x$ is the kernel of a functional $x^* \in \mathsf{S}_{X^*}$ which verifies that $x^*(x) = 1$ and is known as the *isometric reflection functional* associated to x. As we mentioned in the previous paragraph, if $x \in \mathsf{S}_X$ is an isometric reflection vector with isometric reflection functional x^*, then the projection $p\colon X \to \ker(x^*)$ given by $y \mapsto p(y) := y - x^*(y)x$ has norm 1.

Lemma 2.5. *Let X be a Banach space. If $x \in \mathsf{S}_X$, $x^* \in \mathsf{S}_{X^*}$, $x^*(x) = 1$, and $z \in \ker(x^*)$, then*

$$\left\| x - \frac{x+z}{\|x+z\|} \right\| \leq 2\|z\|.$$

Proof.

$$\begin{aligned}\left\| x - \frac{x+z}{\|x+z\|} \right\| &= \frac{\| \|x+z\|x - x - z\|}{\|x+z\|} \leq \frac{|\|x+z\| - 1| + \|z\|}{x^*(x+z)} \\ &\leq \frac{|\|x+z\| - \|x\|| + \|z\|}{x^*(x+z)} \leq \frac{\|z\| + \|z\|}{1} = 2\|z\|.\end{aligned}$$

□

Theorem 2.6. *Let X be a Banach space. Let $X \in \mathsf{S}_X$ and $x^* \in \mathsf{S}_{X^*}$ with $x^*(x) = 1$ and consider the projection $p\colon X \to \ker(x^*)$ given by $y \mapsto p(y) := y - x^*(y)x$.*

(1) $\operatorname{supp}(x^*, \mathsf{B}_X) \subseteq \mathsf{B}_X(x, \|p\|) \cap \mathsf{S}_X$.

(2) *If $0 < \varepsilon \leq \|p\|$ and $\mathsf{B}_X(x, \varepsilon) \cap \mathsf{S}_X \subseteq \operatorname{supp}(x^*, \mathsf{B}_X)$, then $\|x + z\| = \|x - z\|$ for all $z \in \ker(x^*)$ with $\|z\| \leq \frac{\varepsilon}{2}$. Even more, $\|x + z\| = 1$ for all $z \in \ker(x^*)$ with $\|z\| \leq \frac{\varepsilon}{2}$.*

Proof. (1) Let $y \in \operatorname{supp}(x^*, \mathsf{B}_X)$. By hypothesis, $\|y - x\| = \|y - x^*(y)x\| = \|p(y)\| \leq \|p\|$.

(2) Let $z \in \ker(x^*)$ with $\|z\| \leq \frac{\varepsilon}{2}$. If we take into account Lemma 2.5, then we have that $\frac{x+z}{\|x+z\|} \in \mathsf{B}_X(x, \varepsilon) \cap \mathsf{S}_X \subseteq \operatorname{supp}(x^*, \mathsf{B}_X)$, thus $x^*\left(\frac{x+z}{\|x+z\|}\right) = 1$ which means that $\|x + z\| = 1$. This shows the result. □

2.2. Geometric properties of facets

We remind the reader that a convex component of a non-convex set is a maximal convex subset.

Lemma 2.7. *Let X be a Banach space and C a convex subset of S_X.*

(1) *If C is a convex component of S_X, then C is an exposed face of B_X.*

(2) *If C is a face of B_X and is maximal among the proper faces of B_X, then C is a convex component of S_X and hence an exposed face of B_X.*

Proof. (1) This is a direct application of the Hahn-Banach Separation Theorem. Indeed, such theorem allows the existence of $f \in \mathsf{S}_{X^*}$ with $C \subseteq \operatorname{supp}(f, \mathsf{B}_X)$. Since $\operatorname{supp}(f, \mathsf{B}_X)$ is a convex subset of S_X, the maximality of C forces that $C = \operatorname{supp}(f, \mathsf{B}_X)$.

(2) It is essentially the same proof as right above. □

Theorem 2.8. *Let X be a Banach space and C a convex subset of S_X. The following conditions are equivalent:*
(1) *C is a facet of B_X.*
(2) *C is a face of B_X and $\mathrm{int}_{\mathsf{S}_X}(C) \neq \varnothing$.*
(3) *C is a convex component of S_X and $\mathrm{int}_{\mathsf{S}_X}(C) \neq \varnothing$.*

Proof. Observe that (1) $\Rightarrow$ (2) is a direct consequence of the definition of facet, and (3) $\Rightarrow$ (1) is immediate from Lemma 2.7(1). Therefore, we will only need to show (2) $\Rightarrow$ (3). Indeed, let D be a convex subset of S_X containing C. Fix $c \in \mathrm{int}_{\mathsf{S}_X}(C)$. We will show that $D \subseteq C$. So, let any $d \in D$. We have that the segment $[d, c] \subseteq D \subseteq \mathsf{S}_X$. In accordance to Lemma 2.2 there exists $e \in C \setminus \{c\}$ such that $c \in (e, d)$. By hypothesis, C is a face of B_X therefore $d \in C$. □

Lemma 2.9. *Let X be a Banach space and C a convex subset of S_X with the properties $\mathrm{int}_{\mathsf{S}_X}(C) \neq \varnothing$, and $f, g \in \mathsf{S}_{X^*}$.*
(1) *If there exists $x \in \mathrm{int}_{\mathsf{S}_X}(C)$ with $f(x) = 1$, then $f(C) = \{1\}$.*
(2) *If f is constant on C, then $f(C)$ is either $\{1\}$ or $\{-1\}$.*
(3) *If $f = g$ on C, then $f = g$.*

Proof. (1) Let $c \in C \setminus \{x\}$. Since $[c, x] \subseteq C \subseteq \mathsf{S}_X$, by taking into account Lemma 2.2, we deduce that there exists $e \in C \setminus \{x, c\}$ such that $x \in (e, c)$. Since $\mathrm{supp}(f, \mathsf{B}_X)$ is a face of B_X, we conclude that $c \in \mathrm{supp}(f, \mathsf{B}_X)$.

(2) By the Hahn-Banach-Separation Theorem we can find $g \in \mathsf{S}_{X^*}$ with $C \subseteq \mathrm{supp}(f, \mathsf{B}_X)$. Notice that C has non-empty interior relative to $f^{-1}(1)$ in virtue of Lemma 2.1. In fact, that lemma assures that the previously mentioned interior coincides with the interior of C relative to S_X. So, let $x \in \mathrm{int}_{\mathsf{S}_X}(C)$. Then $C - x$ is a neighborhood of 0 in $\ker(g)$. As a consequence, $f|_{\ker(g)}$ vanishes in a neighborhood of 0 in $\ker(g)$, therefore $f|_{\ker(g)}$ vanishes in $\ker(g)$, which implies that $\ker(g) \subseteq \ker(f)$ concluding that $f = \pm g$ since both functionals have norm 1.

(3) Assume to the contrary that $f - g \neq 0$. Then $\frac{f-g}{\|f-g\|}$ is constant on C, so by (2) of this lemma we have that $\frac{f-g}{\|f-g\|}(C)$ is either $\{1\}$ or $\{-1\}$, which contradicts the hypothesis that $f = g$ on C. □

Theorem 2.10. *Let X be a Banach space. If C is a convex subset of S_X, then* $\mathrm{int}_{\mathsf{S}_X}(C) \subseteq \mathrm{smo}(\mathsf{B}_X)$.

Proof. Fix any $c \in \mathrm{int}_{\mathsf{S}_X}(C)$ and suppose that there are $f, g \in \mathsf{S}_{X^*}$ with $f(x) = g(x) = 1$. By Lemma 2.9(1) we have that $f(C) = g(C) = \{1\}$, which means that $f = g$ on C. Finally, in virtue of (3) of Lemma 2.9 we conclude that $f = g$. □

Corollary 2.11. *Let X be a Banach space. If $x \in \mathsf{S}_X$, then*

$$\mathrm{int}_{\mathsf{S}_{X^*}}\left(x^{-1}(1) \cap \mathsf{B}_{X^*}\right) \subseteq \left\{g \in x^{-1}(1) \cap \mathsf{B}_{X^*} : \{x\} = g^{-1}(1) \cap \mathsf{B}_X\right\}.$$

Proof. Let $g \in \operatorname{int}_{\mathsf{S}_{X^*}}(x^{-1}(1) \cap \mathsf{B}_{X^*})$. If we take into consideration Theorem 2.10, we conclude that g is a smooth point of B_{X^*}, which in particular means that g cannot attain its norm at more than one point. Since g already attains its norm at x, we conclude that $\{x\} = g^{-1}(1) \cap \mathsf{B}_X$. $\square$

The final theorem in this subsection shows that facets in dual unit balls must be w^*-exposed faces.

Theorem 2.12. *Let X be a Banach space. If $x^{**} \in \mathsf{S}_{X^{**}}$ is so that $\operatorname{supp}(x^{**}, \mathsf{B}_{X^*})$ is a facet of B_{X^*}, then $x^{**} \in \mathsf{S}_X$.*

Proof. By the famous Bishop-Phelps Theorem we can find

$$x^* \in \operatorname{int}_{\mathsf{S}_{X^*}}(\operatorname{supp}(x^{**}, \mathsf{B}_{X^*})) \cap \mathsf{NA}(X),$$

where $\mathsf{NA}(X)$ denotes the set of norm-attaining functionals on X. So then let $x \in \mathsf{S}_X$ such that $x^*(x) = 1$. By applying Theorem 2.10 we have that x^* is a smooth point of B_{X^*}. As a consequence, $x^{**} = x$. $\square$

2.3. The duality between vertices and facets

As we mentioned in the introduction, $\mathsf{J}_X(x) = \operatorname{supp}(x, \mathsf{B}_{X^*})$ is a w^*-exposed face of B_{X^*} for every $x \in \mathsf{S}_X$. By definition, if x is a smooth point, then $\mathsf{J}_X(x)$ is a w^*-exposed point of B_{X^*} and, in particular, a vertex of B_{X^*}.

Theorem 2.13. *Let X be a Banach space and $f \in \mathsf{S}_{X^*}$. If $\operatorname{supp}(f, \mathsf{B}_X)$ is a facet of B_X, then f is a vertex of B_{X^*}.*

Proof. Let $x \in \operatorname{int}_{\mathsf{S}_X}(\operatorname{supp}(f, \mathsf{B}_X))$. According to Theorem 2.10 we have that x is a smooth point, so by the observation made right above we finally conclude that $f = \mathsf{J}_X(x)$ is a vertex of B_{X^*}. $\square$

The final result is an approach to the converse of Theorem 2.13, which clearly does not hold in general. We recall the reader that a w-strongly exposed point is an exposed point x with supporting functional x^* in such a way that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in B_X with $(x^*(x_n))_{n\in\mathbb{N}}$ converging to 1, then $(x_n)_{n\in\mathbb{N}}$ converges to x.

Theorem 2.14. *Let X be a Banach space. If $x \in \mathsf{S}_X$ is a w-strongly exposed point of B_{X^*}, then $\operatorname{supp}(x, \mathsf{B}_{X^*})$ is a maximal face of B_{X^*}.*

Proof. Let $g \in \mathsf{S}_{X^*}$ be the functional that w-strongly exposes x on B_X. Let $f \in \mathsf{S}_{X^*}$ be such that $[f, g] \subseteq \mathsf{S}_{X^*}$. Fix any arbitrary $t \in (0, 1)$. Since $tf + (1-t)g \in \mathsf{S}_{X^*}$, there must exist a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_X$ such that $(tf + (1-t)g)(y_n))_{n\in\mathbb{N}}$ approaches 1. By passing to subsequences if necessary, we may assume that both $(f(y_n))_{n\in\mathbb{N}}$ and $(g(y_n))_{n\in\mathbb{N}}$ are convergent. Then they must converge to 1. Since x is a w-strongly exposed point of B_X by g, we deduce that $(y_n)_{n\in\mathbb{N}}$ is w-convergent to x. This implies that $f(x) = 1$. $\square$

3. Edges

3.1. Unit balls whose only exposed faces are vertices and edges

In this subsection we will strongly rely on the geometric bodies known as *drops*. However, we first need the following technical lemma.

Lemma 3.1. *Let X be a Banach space.*

(1) *If a non-trivial segment $[x, y] \subset \mathsf{S}_X$ is a face of B_X, then $[x, y]$ has to be a maximal segment.*

(2) *If a proper face C is contained in a segment, then C is either an extreme point or the whole segment.*

Proof. (1) Suppose that $[x, y] \subseteq [a, b] \subseteq \mathsf{B}_X$. Since $x \neq y$ there must exist $t \in (0, 1)$ such that $\frac{x+y}{2} = ta + (1-t)b$. Since $[x, y]$ verifies the extremal condition we deduce that $a, b \in [x, y]$.

(2) Assume that C is not an extreme point. Then the segment cannot be trivial and C must contain an interior point of the segment. Since C verifies the extremal condition we deduce that C is the whole segment. □

The following example shows that in the previous lemma the hypothesis that $[x, y]$ is non-trivial is absolutely necessary.

Example 3.2. Consider any non-strictly convex reflexive Banach space. There exists a proper maximal face of the unit ball which is not a singleton. This proper face is an exposed face in virtue of Lemma 2.7 and thus it is closed. Therefore this proper face is w-compact, and by the famous Krein-Miman Theorem this proper face has extreme points, which are also extreme points of the unit ball. These extreme points are not maximal segments of the unit ball.

Lemma 3.3. *Let X be a Banach space. The following conditions are equivalent:*

(1) *All the proper faces of the unit ball are segments.*

(2) *All the exposed faces of the unit ball are segments.*

Proof. (1) $\Rightarrow$ (2): Trivial since every exposed face is a proper face.

(2) $\Rightarrow$ (1): Let C be a proper face of B_X. By the Hahn-Banach Separation Theorem we know that C is contained in an exposed face. By hypothesis, this exposed face is a segment. Now Lemma 3.1(2) applies. □

Observe that 1-dimensional Banach spaces are always strictly convex and in a 2-dimensional Banach space every norm is either strictly convex or all of its exposed faces are segments.

Theorem 3.4. *Let X be a Banach space with dimension strictly greater than 2 which admits an equivalent strictly convex renorming. Then X can be equivalently renormed so that X is not strictly convex but all of its exposed faces are segments.*

Proof. We begin by assuming that X is already endowed with an equivalent strictly convex renorming. Now fix arbitrary elements $x \in \mathsf{S}_X$ and $x^* \in \mathsf{S}_{X^*}$ such that $x^*(x) = 1$. Consider the absolutely convex hull of $\mathsf{B}_{\ker(x^*)} \cup \{\pm x\}$, which is given by $\mathcal{B} := \{tx + sb : b \in \mathsf{S}_{\ker(x^*)} \text{ and } |t| + |s| \leq 1\}$.

(a) $\mathcal{B}$ is the unit ball of an equivalent norm on X. Indeed, $\mathcal{B}$ is by construction absolutely convex. It is clear that $\mathcal{B}$ is bounded and closed. In fact, $\mathcal{B} \subseteq \mathsf{B}_X$. So it only remains to show that $\mathcal{B}$ has non-empty interior. We will prove that $\mathsf{B}_X\left(0, \frac{1}{3}\right) \subseteq \mathcal{B}$. Let $y \in \mathsf{B}_X\left(0, \frac{1}{3}\right)$. We can write $y = x^*(y)x + (y - x^*(y)x)$. Then

$$|x^*(y)| + \|y - x^*(y)x\| \leq |x^*(y)| + \|y\| + |x^*(y)| \leq 1.$$

This shows that $y \in \mathcal{B}$.

(b) $\mathrm{bd}(\mathcal{B}) = \{t(\pm x) + sb : b \in \mathsf{S}_{\ker(x^*)}, t, s \geq 0 \text{ and } t + s = 1\} = \{[x,b], [-x,b] : b \in \mathsf{S}_{\ker(x^*)}\}$. Indeed, fix $b \in \mathsf{S}_{\ker(x^*)}$ and $b^* \in \mathsf{S}_{\ker(x^*)^*}$ with $b^*(b) = 1$. We can extend b^* to the whole of X by setting $b^*(x) = 1$. Notice then that with this extension we have that $\sup b^*(\mathcal{B}) = 1$ and $[x,b] \subseteq \mathrm{supp}(b^*, \mathcal{B})$. This shows that $\{[x,b], [-x,b] : b \in \mathsf{S}_{\ker(x^*)}\} \subseteq \mathrm{bd}(\mathcal{B})$. Conversely, if $tx + sb \in \mathrm{bd}(\mathcal{B})$ with $t, s \geq 0$, $t + s \leq 1$ and $b \in \mathsf{S}_{\ker(x^*)}$, then the Hahn-Banach Separation Theorem allows the existence of $y^* \in X^*$ such that $\sup y^*(\mathcal{B}) = y^*(tx + sb) = 1$. This forces the situation that $t + s = 1$.

(c) $\mathcal{B}$ is not strictly convex. Indeed, (b) tells us that the segments of the form $[x,b]$ and $[-x,b]$, with $b \in \mathsf{S}_{\ker(x^*)}$, are contained in the boundary of $\mathcal{B}$.

(d) The set of exposed points of $\mathcal{B}$ is $\mathsf{S}_{\ker(x^*)} \cup \{\pm x\}$. Indeed, both x and $-x$ are exposed on $\mathcal{B}$ by x^* and $-x^*$, respectively. If $b \in \mathsf{S}_{\ker(x^*)}$, then b is exposed on $\mathsf{B}_{\ker(x^*)}$ by an element $b^* \in \mathsf{S}_{\ker(x^*)^*}$ because $\ker(x^*)$ is strictly convex, so it suffices to extend b^* to the whole of X by making $b^*(x) = 0$.

(e) The only exposed faces of $\mathcal{B}$ which are not exposed points are the segments of the form $[x,b]$ and $[-x,b]$ with $b \in \mathsf{S}_{\ker(x^*)}$. Indeed, consider, in virtue of the Hahn-Banach Separation Theorem, an element $y^* \in X^*$ with $\sup y^*(\mathcal{B}) = 1$ and $[x,b] \subseteq \mathrm{supp}(y^*, \mathcal{B})$. If there are $t, s \geq 0$ with $t + s = 1$ and $c \in \mathsf{S}_{\ker(x^*)}$ such that $y^*(tx + sc) = 1$, then we necessarily have that either $y^*(x) = 1$ or $y^*(c) = 1$. In the second case, since $\ker(x^*)$ is strictly convex, we necessarily obtain that c is equal to b. As a consequence, $[x,b] = \mathrm{supp}(y^*, \mathcal{B})$. Conversely, let $\mathrm{supp}(y^*, \mathcal{B})$ be an exposed face of $\mathcal{B}$ with $y^* \in X^*$ with $\sup y^*(\mathcal{B}) = 1$ and assume that $\mathrm{supp}(y^*, \mathcal{B})$ is not an exposed point. Then we may assume without any loss that there are $t, s > 0$ with $t + s = 1$ and $b \in \mathsf{S}_{\ker(x^*)}$ such that $y^*(tx + sb) = 1$. This implies that $y^*(x) = y^*(b) = 1$ and thus $\mathrm{supp}(y^*, \mathcal{B}) \subseteq [x,b]$. Finally, Lemma 3.1(2) allows us to deduce that $\mathrm{supp}(y^*, \mathcal{B}) = [x,b]$. $\square$

As an immediate consequence of Theorem 3.4 and the well known fact that reflexive spaces and separable spaces both admit an equivalent strictly convex renorming (see [3]), we obtain the following corollary which concludes this subsection.

Corollary 3.5. *Reflexive Banach spaces and separable Banach spaces with dimension strictly greater than* 2 *can be equivalently renormed to be non-strictly convex but in such a way that all of its exposed faces are segments.*

3.2. The localized boundary

In [4, Proposition 1] it was shown that the smoothness is, in some sense, both topologically and linearly locally invariant.

Proposition 3.6. (De Kock and García-Pacheco, [4]) *Let X be a Banach space. For a given point $x \in \mathsf{S}_X$ the following assertions are equivalent:*

(1) $x \in \operatorname{smo}(\mathsf{B}_X)$.

(2) *For every $d \in X$ it is satisfied that x is a smooth point of the unit ball of* $\operatorname{span}\{x, d\}$.

(3) *For every dense subset D of X it is satisfied that x is a smooth point of the unit ball of* $\operatorname{span}\{x, d\}$ *for all $d \in D$.*

(4) *There exists a dense subset D of X with* $\operatorname{card}(D) = \operatorname{den}(X)$ *and satisfying that x is a smooth point of the unit ball of* $\operatorname{span}\{x, d\}$ *for all $d \in D$.*

We point out to the reader that $\operatorname{den}(X)$ stands for the density character of X. Proposition 3.6 serves as motivation for defining the localized boundary of a w^*-exposed face.

Definition 3.7. (De Kock and García-Pacheco, [4]) Let X be a Banach space with $\dim(X) > 1$ and consider $x \in \mathsf{S}_X$. We define the localized boundary of $\mathsf{J}_X(x)$ as the set $\operatorname{lbd}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x))$ of all $x^* \in \mathsf{J}_X(x)$ for which there exists a dense subset D of $X \setminus \mathbb{R}x$ with $\operatorname{card}(D) = \operatorname{den}(X)$ and satisfying that $x^*|_{Y_d} \in \operatorname{bd}_{\mathsf{S}_{Y_d^*}}(\mathsf{J}_{Y_d}(x))$ for all $d \in D$, where $Y_d := \operatorname{span}\{x, d\}$.

The *localized boundary* was introduced in [4] to study the $\|\cdot\|$-w^* continuity at non-smooth points.

In this final subsection we will prove that the localized boundary of a w^*-exposed face is contained in the topological boundary of such face and we will find the localized boundary of a w^*-exposed face which is an edge.

Lemma 3.8. *Let X be a Banach space of dimension at least 2. Let $x \in \mathsf{S}_X$ and $x^* \in \mathsf{J}_X(x)$. Consider any $d \in X \setminus \mathbb{R}x$ and denote $Y_d :=$* $\operatorname{span}\{x, d\}$. *The following conditions are equivalent:*

(1) $x^*|_{Y_d} \in \operatorname{bd}_{\mathsf{S}_{Y_d^*}}(\mathsf{J}_{Y_d}(x))$.

(2) *Either* $x^*(d) = \max\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$ *or* $x^*(d) = \min\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$.

Proof. (1)⇒(2): Note that $\mathsf{J}_{Y_d}(x) = [x^*|_{Y_d}, z^*|_{Y_d}]$ for some $z^* \in \mathsf{J}_X(x)$. We will distinguish between three cases (a), (b) and (c):

(a) $x^*(d) < z^*(d)$. Then $y^*(d) \in (x^*(d), z^*(d))$ for all $y^* \in \mathsf{J}_X(x)$, so $x^*(d) = \min\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$.

(b) $x^*(d) > z^*(d)$. Then by using a similar reasoning as above we deduce that $x^*(d) = \max\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$.

(c) $x^*(d) = z^*(d)$. In this case, $\mathsf{J}_{Y_d}(x) = \{x^*|_{Y_d}\}$ so that
$x^*(d) = \min\{y^*(d) : y^* \in \mathsf{J}_X(x)\} = \max\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$.

(2)⇒(1): Assume to the contrary that there are $u^*, v^* \in \mathsf{J}_X(x)$ with $\mathsf{J}_{Y_d}(x) = [u^*|_{Y_d}, v^*|_{Y_d}]$ and $x^*|_{Y_d} \notin \{u^*|_{Y_d}, v^*|_{Y_d}\}$. Note that in this situation $u^*|_{Y_d} \neq v^*|_{Y_d}$, so we may assume without any loss that $u^*(d) < v^*(d)$. There exists $t \in (0,1)$ in such a way that $x^*|_{Y_d} = tu^*|_{Y_d} + (1-t)v^*|_{Y_d}$. In particular, $x^*(d) = tu^*(d) + (1-t)v^*(d)$ and $u^*(d) < x^*(d) < v^*(d)$. This contradicts our hypothesis. □

Lemma 3.9. *Let X be a Banach space of dimension at least 2. Let $x \in \mathsf{S}_X$ and $x^* \in \mathsf{J}_X(x)$. If $x^* \in \operatorname{int}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x))$, then*

$$\min\{y^*(d) : y^* \in \mathsf{J}_X(x)\} < x^*(d) < \max\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$$

for all $d \in X \setminus \mathbb{R}x$.

Proof. Suppose to the contrary that $d \in \mathsf{S}_X \setminus \{\pm x\}$ is so that

$$x^*(d) = \max\{y^*(d) : y^* \in \mathsf{J}_X(x)\}$$

(the same reasoning can be applied to the min case). By bearing in mind Lemma 2.2, for all $y^* \in \mathsf{J}_X(x) \setminus \{x^*\}$ there exists $z^* \in \operatorname{int}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x)) \setminus \{x^*, y^*\}$ such that $x^* \in (z^*, y^*)$, which, by assumption, implies that $y^*(d) = x^*(d) = z^*(d)$. In other words, d is constant on $\mathsf{J}_X(x)$. According to Lemma 2.9(2), $x^*(d) \in \{-1, 1\}$. Since $x^* \in \operatorname{smo}(\mathsf{B}_{X^*})$ (see Theorem 2.10), we reach the contradiction that either $d = x$ or $d = -x$. □

As a direct consequence of Lemma 3.8 and Lemma 3.9 we obtain the following result. The details of the proof are left to the reader.

Theorem 3.10. *If X is a Banach space with $\dim(X) > 1$, then*

$$\operatorname{lbd}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x)) \subseteq \operatorname{bd}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x)).$$

We close our presentation by finding the localized boundary of a w^*-exposed face, which is an edge.

Theorem 3.11. *Let X be a Banach space. Let $x \in \mathsf{S}_X$ such that $\mathsf{J}_X(x) = [x^*, y^*]$ where $x^* \neq y^* \in \mathsf{S}_{X^*}$. Then $\operatorname{lbd}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x)) = \{x^*, y^*\}$.*

Proof. In the first place, observe that the Hahn-Banach Theorem allows us to deduce that for all $d \in X \setminus \mathbb{R}x$, $\mathsf{J}_{Y_d}(x) = [x^*|_{Y_d}, y^*|_{Y_d}]$, where $Y_d := \operatorname{span}\{x, d\}$. As a consequence, $\operatorname{lbd}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x)) \supseteq \{x^*, y^*\}$. Let now $z^* \in (x^*, y^*)$ and consider any dense subset $D \subseteq X \setminus \mathbb{R}x$. Notice that $D \cap (X \setminus \ker(x^* - y^*)) \neq \varnothing$, so if we choose $d \in D \cap (X \setminus \ker(x^* - y^*))$, then $z^* \in (x^*|_{Y_d}, y^*|_{Y_d})$. Therefore, $\operatorname{lbd}_{\mathsf{S}_{X^*}}(\mathsf{J}_X(x)) = \{x^*, y^*\}$. □

Acknowledgements. The author would like to express his deepest gratitude towards María Inmaculada Díaz Bellido for her valuable support and neverending encouragement.

References

[1] A. Aizpuru, M. D. Acosta, R. M. Aron, F. J. García-Pacheco: Functionals that do not attain their norm, Bull. Belg. Math. Soc. Simon Stevin 14(3) (2007) 407–418.

[2] M. D. Acosta, M. R. Galán: New characterizations of the reflexivity in terms of the set of norm-attaining functionals, Canad. Math. Bull. 41 (1998) 279–289.

[3] M. M. Day: Normed Linear Spaces, Springer, New York et al. (1973).

[4] M. de Kock, F. J. García-Pacheco: A reverse theorem on the $\|\cdot\|$-w^* continuity of the dual map, J. Function Spaces (2015) 4 pp., Art. ID 864173.

[5] F. J. García-Pacheco: Banach spaces with an infinite number of smooth faces in their unit ball, J. Convex Analysis 15(2) (2008) 215–218.

[6] F. J. García-Pacheco: Nowhere density of the set of non-norm-attaining functionals, in: Operator Algebras, Operator Theory and Applications, Oper. Theory Adv. Appl. 195, Birkhäuser, Basel (2010) 167–172.

[7] F. J. García-Pacheco: The universal renorming, Israel J. Math. 202(1) (2014) 405–422.

[8] F. J. García-Pacheco, A. Miralles: Real renormings on complex Banach spaces, Chin. Ann. Math. Ser. B 29(3) (2008) 239–246.

[9] F. J. García-Pacheco, B. Zheng: Geometric properties on non-complete spaces, Quaest. Math. 34(4) (2011) 489–511.

[10] M. Jiménez-Sevilla, J. P. Moreno: A note on norm-attaining functionals, Proc. Amer. Math. Soc. 126 (1998) 1989–1997.

[11] J. P. Moreno: Geometry of Banach spaces with (α, ε)-property or (β, ε)-property, Rocky Mountain J. Math. 27(1) (1997) 241–256.