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Zero Free Inner Function On Every Single Slice

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We present a completely new construction of a holomorphic function whose real part has given radial limits a.e. on every slice. As an application, we obtain a holomorphic function which is inner on every slice.

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1. Introduction

Assume that Ω is a bounded, strictly convex and circular domain with the boundary of class C^2.

In the papers [1, 2, 3, 6, 7], the authors give examples of various inner functions on the domain with a holomorphic support function (see [7]). In this paper we present a new construction of inner function.

In contrast to the functions from the papers mentioned above, our function is zero free and, even more, has given radial limits on every single slice. We construct a holomorphic function with a given real part on the boundary using homogeneous polynomials. Our polynomials enable a very good control of the boundary values, so we can guarantee radial limits a.e. on every single slice.

2. Results

Definition 2.1. For a given function g on a circular set S we consider a *slice function* described by the formula

$$g_z(\lambda) := g(\lambda z) \quad \text{for } z \in S, \quad \lambda \in \mathbb{U} := \{\mu \in \mathbb{C} : |\mu| < 1\}.$$

Furthermore, we need the following *mean values* of the function g:

$$M(g) := \int_0^1 g(e^{2\pi it})dt, \quad M_z(g) := M(g_z).$$

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It is known that any continuous function g on $\mathbb{T} := \partial\mathbb{U}$ can be extended to a harmonic function $P[g]$ on $\mathbb{U}$ as a solution of the Dirichlet problem by the following formulas:

$$P_r(e^{i\theta}) := \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}, \quad P[g](re^{i\theta}) := \int_0^{2\pi} g(e^{i\theta})P_r(e^{i(\theta-\psi)})d\psi.$$

We need the following fact about bounded holomorphic functions:

Theorem 2.2. [3, Theorem 3.2] *There exists a natural number $N_1 = N_1(\partial\Omega)$ such that if $\varepsilon \in (0,1)$, T is a compact subset of Ω, h is a continuous, strictly positive function on $\partial\Omega$, then there exist holomorphic functions $f_1, ..., f_{N_1} \in A(\Omega)$ such that:*

(1) $\|f_j\|_T \leq \varepsilon$,
(2) $|f_j| < h$ *on* $\partial\Omega$ *for* $j = 1, ..., N_1$,
(3) $\frac{h}{2} < \max_{j=1,...,N_1} |f_j|$ *on* $\partial\Omega$.

Since Ω is bounded, convex and circular, we can assume that $f_1, ..., f_{N_1}$ produced by the above theorem are polynomials.

We combine the above result and a useful property of homogeneous polynomials with bounded growth (see [4, 5]):

Theorem 2.3. [5, Theorem 2.5] *There exists $N_2 \in \mathbb{N}$ such that we can choose $M_0 = M_0(\partial\Omega) \in \mathbb{N}$ with the following property: for each natural number $N \geq M_0$ and given $m_1, ..., m_{N_2}$ with $N \leq m_1 \leq m_2 \leq ... \leq m_{N_2} \leq 2N$ there exist homogeneous polynomials p_{m_i} with m_i degree such that:*

(1) $|p_{m_i}| \leq 1$ *on* $\partial\Omega$,
(2) $\max_{k=1,...,N_2} |p_{m_k}| \geq 1/2$ *on* $\partial\Omega$.

Let us consider a Hilbert space of continuous functions on interval $[0,1]$ with the scalar product $\langle f, g\rangle := \int_0^1 f(t)\overline{g(t)}dt$ and the natural norm $\|f\| := \sqrt{\langle f, f\rangle}$.

We prove the following Lemma on periodical functions:

Lemma 2.4. *Assume that $n \in \mathbb{N}$ and g is a function with period $1/n$ and g is orthogonal to* 1 *on* $[0,1]$. *If h is a continuous function on $[0,1]$ then $|\langle hg, 1\rangle| \leq \|g\|\,\omega\left(1/n, h\right)$ where $\omega(\Delta, h) := \sup\{|h(x)-h(y)| : \|x-y\| \leq \Delta\}$.*

Proof. Let us observe that:

$$0 = \langle g, 1\rangle = \int_0^1 g(t)\,dt = \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} g(t)\,dt = \sum_{k=0}^{n-1}\int_0^{\frac{1}{n}} g(t)\,dt = n\int_0^{\frac{1}{n}} g(t)\,dt.$$

In particular, $\int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n}} g(t)\,dt = 0$. Now we can estimate:

$$
\begin{aligned}
|\langle hg,1\rangle| &= \left|\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} h(t)g(t)dt - \sum_{k=0}^{n-1} h(k/n)\int_{\frac{k}{n}}^{\frac{k+1}{n}} g(t)dt\right| \\
&\le \sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(h(t)-h(k/n)\right)g(t)dt\right| \le \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} |h(t)-h(k/n)|\,\|g\|\,dt \\
&\le \sum_{k=0}^{n-1}\|g\|\int_{\frac{k}{n}}^{\frac{k+1}{n}} \omega\left(1/n,h\right)dt = \|g\|\,\omega\left(1/n,h\right). \qquad \square
\end{aligned}
$$

Lemma 2.5. *The inequality* $\sqrt{1+x}\le 1+\frac{1}{2}x-\frac{1}{16}x^2$ *holds for* $x\in(-1,1)$.

Proof. It is enough to estimate:

$$
\begin{aligned}
\sqrt{1+x} &= \sum_{n=0}^{\infty}\begin{pmatrix} 1/2 \\ n\end{pmatrix}x^n = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\ldots \\
&\le 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\le 1+\frac{x}{2}-\frac{x^2}{16}-\frac{x^2}{16}(1-x) \\
&\le 1+\frac{x}{2}-\frac{x^2}{16}. \qquad \square
\end{aligned}
$$

Lemma 2.6. *There exists a constant* $\alpha\in(0,1)$ *such that for a given* $\varepsilon>0$, T *a compact subset of* Ω *and* h *a positive continuous function on* $\partial\Omega$*, we can choose a polynomial* p *such that:*

(1) $|p|<\varepsilon$ *on* T,

(2) $|p|<h$ *on* $\partial\Omega$,

(3) $M_z(\sqrt{h-\Re p})\le \alpha M_z(\sqrt{h})$ *for* $z\in\partial\Omega$.

Proof. Let us choose N_1, N_2 from Theorems 2.2, 2.3 respectively. Now we can define:

$$
\alpha = 1-\frac{1}{2}\cdot\frac{1}{64}\cdot\frac{1}{8N_1^2N_2^2}.
$$

Due to Theorem 2.2, there exist polynomials $f_1,\ldots,f_{N_1}$ such that:

- $N_1N_2\,|f_j|<h$ on $\partial\Omega$ for $j=1,\ldots,N_1$,
- $\frac{h}{2}<\max_{j=1,\ldots,N_1}N_1N_2\,|f_j|$ on $\partial\Omega$.

Now due to Theorem 2.3, for a given $n\in\mathbb{N}$ sufficiently large we can choose homogeneous polynomials p_{nm} of nm degree such that:

- $|p_{nm}|\le 1$ on $\partial\Omega$,
- $\max_{m=N+1,\ldots,N+N_2}|p_{nm}|\ge\frac{1}{2}$ on $\partial\Omega$ for $N\ge N_2$,

We define:

$$J := \{0, 1, ..., N_1 - 1\},$$

$$I(n, j) := \{n(N_1N_2 + jN_2), n(N_1N_2 + jN_2 + 1), ..., n(N_1N_2 + jN_2 + N_2 - 1)\},$$

$$Q_n := \sum_{j\in J} \sum_{m\in I(n,j)} f_j p_m.$$

On $\partial\Omega$ we can estimate: $|Q_n| \le \sum_{j\in J} \sum_{m\in I(n,j)} |f_j p_m| \le \sum_{j\in J} |f_j| N_2 < h.$

There exists $r \in (0, 1)$ such that $T \subset r\Omega$, so we can estimate on T:

$$|Q_n| \le \sum_{j\in J} \sum_{m\in I(n,j)} |f_j p_m| \le \sum_{j\in J} \sum_{m\in I(n,j)} \|f_j\|_{\overline{\Omega}} r^m \le \sum_{j\in J} N_2 \|f_j\|_{\overline{\Omega}} r^n < \varepsilon$$

for n large enough. Now using Lemma 2.5 we can estimate:

$$\begin{aligned} M_z(\sqrt{h - \Re Q_n}) &\le M_z\left(\sqrt{h}\sqrt{1 - \frac{\Re Q_n}{h}}\right) \\ &\le M_z\left(\sqrt{h}\left(1 - \frac{\Re Q_n}{2h} - \frac{1}{16}\left(\frac{\Re Q_n}{h}\right)^2\right)\right) \\ &\le M_z\left(\sqrt{h}\right) - \frac{1}{4}M_z\left(\frac{Q_n + \overline{Q_n}}{\sqrt{h}}\right) - \frac{1}{64}M_z\left(\frac{\left(Q_n + \overline{Q_n}\right)^2}{h^{3/2}}\right). \end{aligned}$$

Let us observe that for given $z \in \partial\Omega$, $j \in J$, $m \in I(n, j)$ the function $t \to p_m(e^{2\pi it}z)$ has ${}^1\!/\!{}_n$ period and is orthogonal to 1 on $[0, 1]$ so we can use Lemma 2.4 and conclude the following inequalities for $z \in \partial\Omega$:

- $\left|M_z\left(\frac{Q_n+\overline{Q_n}}{2\sqrt{h}}\right)\right| \le N_1N_2 \max_{j\in J} \omega\left({}^1\!/\!{}_n, f_j h^{-1/2}\right),$

- $\left|M_z\left(\frac{Q_n^2}{h^{3/2}}\right)\right| \le N_1^2N_2^2 \max_{j,k\in J} \omega\left({}^1\!/\!{}_n, f_j f_k h^{-3/2}\right),$

- $\left|M_z\left(\frac{\overline{Q_n^2}}{h^{3/2}}\right)\right| \le N_1^2N_2^2 \max_{j,k\in J} \omega\left({}^1\!/\!{}_n, f_j f_k h^{-3/2}\right),$

- $\left|M_z\left(\frac{Q_n\overline{Q_n} - \sum_{j\in J}\sum_{m\in I(n,j)} |f_j p_m|^2}{h^{3/2}}\right)\right| \le N_1^2N_2^2 \max_{j,k\in J} \omega\left({}^1\!/\!{}_n, f_j \overline{f_k} h^{-3/2}\right)$

  (if $m \in I(n, j_1) \cap I(n, j_2)$ then $j_1 = j_2$!).

We obtain the following estimation for $z \in \partial\Omega$:

$$M_z\left(\sum_{j\in J}\sum_{m\in I(n,j)}\frac{|f_j p_m|^2}{h^{3/2}}\right)=\sum_{j\in J}M_z\left(\frac{|f_j|^2}{h^{3/2}}\sum_{m\in I(n,j)}|p_m|^2\right)$$
$$\geq \sum_{j\in J}M_z\left(\frac{|f_j|^2}{4h^{3/2}}\right)=M_z\left(\sum_{j\in J}\frac{|f_j|^2}{4h^{3/2}}\right)\geq M_z\left(\frac{h^2}{16N_1^2N_2^2h^{3/2}}\right)$$
$$\geq \frac{1}{16N_1^2N_2^2}M_z\left(\sqrt{h}\right)$$

In particular, for n large enough we have the following inequality:

$$M_z(\sqrt{h-\Re Q_n})\leq M_z\left(\sqrt{h}\right)-\frac{1}{4}M_z\left(\frac{Q_n+\overline{Q_n}}{\sqrt{h}}\right)-\frac{1}{64}M_z\left(\frac{\left(Q_n+\overline{Q_n}\right)^2}{h^{3/2}}\right)$$
$$\leq M_z\left(\sqrt{h}\right)+\frac{N_1N_2}{2}\max_{j\in J}\omega\left({}^1\!/_n,\, f_jh^{-1/2}\right)+\frac{2N_1^2N_2^2}{64}\max_{j,k\in J}\omega\left({}^1\!/_n,\, f_jf_kh^{-3/2}\right)$$
$$+\frac{2N_1^2N_2^2}{64}\max_{j,k\in J}\omega\left({}^1\!/_n,\, f_j\overline{f_k}h^{-3/2}\right)-\frac{1}{64}\frac{2}{16N_1^2N_2^2}M_z\left(\sqrt{h}\right)$$
$$\leq\left(1-\frac{1}{2}*\frac{1}{64}*\frac{1}{8N_1^2N_2^2}\right)M_z\left(\sqrt{h}\right)=\alpha M_z\left(\sqrt{h}\right).$$

So we can define $p=Q_n$ for n large enough, which finishes the proof. □

Using the above lemma several times, we prove the following fact:

Theorem 2.7. *There exists a constant $\alpha\in(0,1)$ such that for a given sequence of positive numbers $\varepsilon_n>0$, compact subsets $T_n\subset\Omega$ and h a positive continuous function on $\partial\Omega$, we can choose polynomials $p_0,p_1,\dots$ such that:*

(1) $|p_n|\leq\varepsilon_n$ *on* T_n,

(2) $|p_n|<h-\Re\sum_{k=0}^{n-1}p_k$ *on* $\partial\Omega$,

(3) $M_z\left(\sqrt{h-\Re\sum_{k=0}^{n}p_k}\right)\leq\alpha^nM_z(\sqrt{h})$ *for* $z\in\partial\Omega$.

Proof. Let $\alpha\in(0,1)$ be a constant from Lemma 2.6. We choose $p_0=0$. Obviously, we have the properties (1)–(3) fulfilled. Now suppose that we have just defined $p_0,\dots,p_n$ and now due to Lemma 2.6 there exists p_{n+1} such that

- $|p_{n+1}|<\varepsilon_{n+1}$ on T_{n+1},
- $|p_{n+1}|<h-\sum_{k=0}^{n}\Re p_k$ on $\partial\Omega$,
- $M_z\left(\sqrt{(h-\sum_{k=0}^{n}\Re p_k)-\Re p_{n+1}}\right)\leq\alpha M_z\left(\sqrt{h-\sum_{k=0}^{n}\Re p_k}\right)$ for $z\in\partial\Omega$.

Properties (1)–(2) are fulfilled. Property (3) follows from:

$$M_z\left(\sqrt{h-\Re\sum_{k=0}^{n+1}p_k}\right)\leq\alpha M_z\left(\sqrt{h-\sum_{k=0}^{n}\Re p_k}\right)\leq\alpha^{n+1}M_z(\sqrt{h}),$$ □

As a direct consequence we have the following

Theorem 2.8. *There exists a constant $C > 0$ such that for a given $s \in \mathbb{N}$, h a continuous positive function on $\partial\Omega$, $\varepsilon > 0$ and K a compact subset of Ω, we can choose a non constant holomorphic function f with:*

(1) $|f| < \varepsilon$ *on* K,

(2) $f_z \in H^{1/2}$ *and* $M_z\left(\sqrt{|f^*|}\right) \le CM_z\left(\sqrt{h}\right)$ *for* $z \in \partial\Omega$,

(3) $\Re f_z \le P[h_z]$ *on* $\mathbb{U}$ *for* $z \in \partial\Omega$,

(4) $\Re f_z^* = h_z$ *a.e. on* $\mathbb{T}$ *for* $z \in \partial\Omega$.

Proof. Let $\alpha \in (0,1)$ be a constant from Theorem 2.7. We can define:

$$C := \sum_{k=0}^{\infty} \alpha^{k-1} = \frac{1}{\alpha(1-\alpha)}.$$

Now, due to Theorem 2.7 there exists a sequence of polynomials p_m such that:

(1) $|p_n| \le 2^{-n}$ on $(1-2^{-n})\,\overline{\Omega}$,

(2) $|p_n| \le 2^{-n-1}\varepsilon$ on K,

(3) $|p_n| < h - \Re\sum_{k=0}^{n-1} p_k$ on $\partial\Omega$,

(4) $M_z\left(\sqrt{h - \Re\sum_{k=0}^{n} p_k}\right) \le \alpha^n M_z(\sqrt{h})$ for $z \in \partial\Omega$.

Let us define $f = \sum_m p_m$. The properties (1)–(2) imply that f is a holomorphic non constant function on Ω with $|f| < \varepsilon$ on K. Since

$$\begin{aligned} M_z\left(\sqrt{\left|\sum_{k=n}^{m} p_k\right|}\right) &\le M_z\left(\sqrt{\sum_{k=n}^{m} |p_k|}\right) \le \sum_{k=n}^{m} M_z\left(\sqrt{|p_k|}\right) \\ &\le \sum_{k=n}^{m} M_z\left(\sqrt{h - \Re\sum_{j=0}^{k-1} p_j}\right) \le M_z\left(\sqrt{h}\right)\sum_{k=n}^{m}\alpha^{k-1} \end{aligned}$$

for $z \in \partial\Omega$ we conclude that $\left(\sum_{k=0}^{m} p_k\right)_z$ is a Cauchy sequence in $H^{1/2}$, so that $f_z \in H^{1/2}$. Moreover:

$$\begin{aligned} M_z\left(\sqrt{|f|}\right) &\le M_z\left(\sqrt{\left|\sum_{k=0}^{\infty} p_k\right|}\right) \le \sum_{k=0}^{\infty} M_z\left(\sqrt{|p_k|}\right) \\ &\le M_z\left(\sqrt{h}\right)\sum_{k=0}^{\infty}\alpha^{k-1} \le CM_z\left(\sqrt{h}\right). \end{aligned}$$

It is obvious that there exist radial limits f_z^* a.e. on $\mathbb{T}$ for all $z \in \partial\Omega$. Moreover, property (3) gives us that $\Re f_z \le P[h_z]$ on $\mathbb{U}$ for $z \in \partial\Omega$. We can estimate for $z \in \partial\Omega$:

$$
\begin{aligned}
M_z\left(\sqrt{h-\Re f^*}\right) &\le M_z\left(\sqrt{h-\Re\sum_{k=0}^{m}p_k+\left(\Re f^*-\Re\sum_{k=0}^{m}p_k\right)}\right)\\
&\le M_z\left(\sqrt{h-\Re\sum_{k=0}^{m}p_k}\right)+M_z\left(\sqrt{\Re f^*-\Re\sum_{k=0}^{m}p_k}\right)\\
&\le \alpha^m M_z\left(\sqrt{h}\right)+M_z\left(\sqrt{\left|f^*-\sum_{k=0}^{m}p_k\right|}\right)\underset{m\to\infty}{\longrightarrow}0
\end{aligned}
$$

and the proof is complete. □

Definition 2.9. We say that a positive LSC function h has the $L^{1/2}$ *property* iff there exists a sequence $\{h_k\}$ of continuous functions such that $0 < h_1 \le h_2 \le \dots \le \lim_{m\to\infty} h_m = h$ and $\sum_m M_z\left(\sqrt{h_{m+1}-h_m}\right) < \infty$ for all $z \in \partial\Omega$.

Corollary 2.10. *Assume that h is a positive LSC function on $\partial\Omega$ with the $L^{1/2}$ property. Then there exists non constant holomorphic function f with $f_z \in H^{1/2}$, $\Re f_z \le P[h_z]$ on $\mathbb{U}$ and $\Re f_z^* = h_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$.*

Proof. There exists a sequence $\{h_n\}$ of continuous functions such that: $0 < h_1 < h_2 < \dots < \lim_{m\to\infty} h_m = h$ and $\sum_m M_z\left(\sqrt{h_{m+1}-h_m}\right) < \infty$ for all $z \in \partial\Omega$. If $h_i \le h_j$, it is enough to define $\tilde{h}_i = h_i - \varepsilon 2^{-i}$ for $\varepsilon > 0$ such that $\varepsilon < h_1$.

Let $C > 0$ be a constant from Theorem 2.8. We construct a Cauchy sequence of holomorphic functions f_n such that:

(1) $(h_n = \Re f_n^*)_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$,
(2) $|f_m - f_n| \le 2^{-n}$ on $(1-2^{-n})\,\overline{\Omega}$, for $m > n$,
(3) $(\Re f_n)_z \le P\left[(h_n)_z\right]$ on $\mathbb{U}$ for $z \in \partial\Omega$,
(4) $M_z\left(\sqrt{|f_n^*|}\right) \le C M_z\left(\sqrt{h_1} + \sum_{m<n}\sqrt{h_{m+1}-h_m}\right)$ for $z \in \partial\Omega$.

The holomorphic function f_1 exists due to Theorem 2.8. Now suppose that $f_1, \dots, f_n$ are chosen in the way that (1)-(3) are fulfilled. Due to Theorem 2.8 there exists a holomorphic function g_{n+1} such that:

- $\left(h_{n+1} - h_n = \Re g_{n+1}^*\right)_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$,
- $|g_{n+1}| \le 2^{-n-1}$ on $(1-2^{-n-1})\,\overline{\Omega}$,
- $(\Re g_{n+1})_z \le P\left[(h_{n+1}-h_n)_z\right]$ on $\mathbb{U}$ for $z \in \partial\Omega$,
- $M_z\left(\sqrt{\left|g_{n+1}^*\right|}\right) \le C M_z\left(\sqrt{h_{n+1}-h_n}\right)$ for all $z \in \partial\Omega$.

We can define $f_{n+1} := f_n + g_{n+1}$ which fulfills properties (1)–(4), and the function $f = \lim_{m\to} f_m$ on Ω.

In particular we have $M_z\left(\sqrt{|f^*|}\right) \leq CM_z\left(\sqrt{h_1} + \sum_m \sqrt{h_{m+1} - h_m}\right)$, and therefore $f_z \in H^{1/2}$. Moreover $\Re f_z \leq P[h_z]$ on $\mathbb{U}$ for $z \in \partial\Omega$ and $\Re f_z^* = h_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$. $\square$

Example 2.11. There exists a holomorphic function $f \in H^\infty(\Omega)$ such that $0 < |f| < 1$ on Ω and $|f_z^*| = 1$ a.e. on $\mathbb{T}$ for all z.

Proof. Let us choose $h \equiv 1$. Now due to Corollary 2.10 there exists $g \in H^{1/2}$ such that $\Re g_z^* = 1$ a.e. on $\mathbb{T}$ for all $z \in \partial\Omega$, so $f = \exp(g)$ fulfills the required property. $\square$

Example 2.12. There exists a holomorphic function g such that $|g_z^*| = 1$ a.e. on $\mathbb{T}$ but $g_z \notin H^p$ for $p > 0$ and $z \in \partial\Omega$.

Proof. Let f be a function from Example 2.11. We prove that $g = \frac{1}{f}$ has required properties. Obviously $|g_z^*| = 1$ exists a.e. on $\mathbb{T}$ for $z \in \partial\Omega$. Let us choose $p \in (0, \infty)$. If $g_z \in H^p$ then $M_z(|g|^p) = 1$ but $|f(0)| < 1$, so $1 < |g(0)|^p \leq M_z(|g|^p) = 1$ which is impossible. If $p = \infty$ and $g_z \in H^\infty$ then $M_z(|g|) = 1$ which is also impossible. $\square$

Example 2.13. There exists a bounded holomorphic zero free function $f \in H^\infty(\Omega)$ such that if $p > 0$, g is a holomorphic function with $|(f * g)_z^*| = 1$ a.e on $\mathbb{T}$ for all $z \in \partial\Omega$, then there exists $z \in \partial\Omega$ such that $g_z \notin H^p$.

Proof.

$$h(w) := \begin{cases} 2 & \text{for} \quad w \notin \mathbb{T}z_0 \\ 1 & \text{for} \quad w \in \mathbb{T}z_0 \end{cases}.$$

There exists a continuous functions h_n such that

$$0 < h_1 \leq h_2 \leq ... \leq \lim_{m\to\infty} h_m = h,$$

$h_n(\mathbb{T}z_0) = 1$ and $h_n(w) = 2$ iff $\operatorname{dist}(w, \mathbb{T}z_0) > \frac{1}{n}$. We can easily observe that h has $L^{1/2}$ property since for a slice $\mathbb{T}z$ there exists $n_z \in \mathbb{N}$ such that $h_n(\mathbb{T}z) = h(\mathbb{T}z)$ for $n > n_z$. Due to Corollary 2.10 there exists $f \in H^\infty(\Omega)$ such that $f_z^* = h_z$ a.e. on $\mathbb{T}$ for all $z \in \partial\Omega$. Let now g be a holomorphic function such that $g_z \in H^p$ for $z \in \partial\Omega$. Suppose that $|(f * g)_z^*| = 1$ a.e. on $\mathbb{T}$ for all $z \in \partial\Omega$.

We can observe that $|g_z^*| = 1/h_z$ a.e. on $\mathbb{T}$ for all $z \in \partial\Omega$.

If $p < \infty$ then let us define the following continuous function on $\partial\Omega$:

$$\phi_r(z) := \int_0^1 \left|g(re^{2\pi it}z)\right|^p dt.$$

Due to $\phi_r \nearrow \phi_1$ we can conclude that ϕ_1 is LSC. We can observe:

$$\phi_1(z) = \int_0^1 \left|g^*(e^{2\pi it}z)\right|^p dt = 1/h^p(z).$$

Since $1/h^p$ is not LSC we have a contradiction.

If $p = \infty$, then $g_z \in H^1$. It is therefore enough to repeat the same arguments as for $p = 1$. □

Theorem 2.14. *Assume that h is a positive LSC function on $\partial\Omega$, $h_z \in L^\infty$. Then there exists non constant holomorphic function f on Ω with $\Re f_z \le P[h_z]$ on $\mathbb{U}$ and $\Re f_z^* = h_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$.*

Proof. Without loss of generality we can assume that $h > 1$. There exists a sequence $\{h_n\}$ of continuous functions such that $1 = h_0 < h_1 < h_2 < ... < \lim_{m\to\infty} h_m = h$.

We construct a Cauchy sequence $\{f_n\}$ of holomorphic polynomials and positive real numbers $\{r_n\} \subset (0,1)$ such that:

(1) $M_z\left(\sqrt{h_n - \Re f_n}\right) \le 2^{-n}$ for $z \in \partial\Omega$,

(2) $|f_n(z) - f_n(w)| < 2^{-n}$ for $z, w \in [r_n, 1]\overline{\Omega}$,

(3) $|f_{n+1} - f_n| \le 2^{-n}$ on $r_n\overline{\Omega}$,

(4) $\Re\left(f_n\right)_z \le P\left[(h_n)_z\right]$ on $\overline{\mathbb{U}}$ for $z \in \partial\Omega$,

(5) $\frac{1}{2} \le r_n < r_{n+1} < ... < \lim_{n\to\infty} r_n = 1$.

Let $f_0 \equiv 0$ and $r_0 = 0.5$. Suppose that we have constructed polynomials $f_0, .., f_n$ and numbers $r_0 < r_1 < ... < r_n$. Due to Theorem 2.7 there exists a polynomial p such that:

- $|p| \le 2^{-n}$ on $r_n\overline{\Omega}$,
- $\Re p < h_{n+1} - \Re f_n$ on $\partial\Omega$,
- $M_z\left(\sqrt{h_{n+1} - \Re\left(p + f_n\right)}\right) \le 2^{-n-1}$ for $z \in \partial\Omega$.

Now let us define $f_{n+1} := f_n + p$. The properties (1),(3) are evident. Moreover, we can choose $r_{n+1} > \min\left\{r_n, \frac{1}{n+1}\right\}$ such that property (2) is fulfilled. The inequality $\Re f_{n+1} < h_{n+1}$ implies the property (4), so $f_0, ..., f_{n+1}, r_0, ..., r_{n+1}$ fulfills (1)–(5).

Let $f := \lim_{n\to\infty} f_n$. Since h_z is bounded $\exp(f_z)$ has radial limits a.e. on $\mathbb{T}$. This implies that $\Re f_z$ has radial limits a.e. on $\mathbb{T}$. Property (4) implies that $\Re f_z^* \le P[h_z]$ on $\mathbb{U}$ for $z \in \partial\Omega$. We choose $z \in \partial\Omega$ and define:

$$D_{m,n} := \left\{t \in [0,1] : h\left(e^{2\pi it}z\right) - \Re f_m\left(e^{2\pi it}z\right) < 2^{-n}\right\}.$$

Since h_z is bounded we observe:

$$\lim_{m\to\infty} M_z\left(\sqrt{h - \Re f_m}\right) \le \lim_{m\to\infty} M_z\left(\sqrt{h - h_m}\right) + M_z\left(\sqrt{h_m - \Re f_m}\right) = 0,$$

and the one-dimensional Lebesgue measure of $D := \bigcap_{n\in\mathbb{N}} \bigcup_{m>n} D_{m,n}$ is equal to one. If now $t \in D$ and $n \in \mathbb{N}$ then there exists a natural number $m = m(n) \in \mathbb{N}$ such that $t \in D_{m,n}$ and $m > n$. In particular, we can estimate:

$$\begin{aligned}\Re f\left(e^{2\pi it}r_m z\right) &= \Re\left(f_m + \sum_{k\geq m}\left(f_{k+1}-f_k\right)\right)\left(e^{2\pi it}r_m z\right)\\ &\underset{(3)}{\geq} \Re f_m\left(e^{2\pi it}r_m z\right) - \sum_{k\geq m}2^{-k} \geq \Re f_m\left(e^{2\pi it}r_m z\right) - 2^{-m+1}\\ &\underset{(2)}{\geq} \Re f_m\left(e^{2\pi it}z\right) - 2^{-m} - 2^{-m+1}\\ &\geq h\left(e^{2\pi it}z\right) - 2^{-n} - 2^{-m} - 2^{-m+1} \geq h\left(e^{2\pi it}z\right) - 3*2^{-n}.\end{aligned}$$

This implies that $\Re f_z^* = h_z$ a.e. on $\mathbb{T}$. □

Corollary 2.15. *Assume that h is a positive LSC function on $\partial\Omega$, $h_z \in L^\infty$ for $z \in \partial\Omega$. Then there exists non constant holomorphic function f on Ω with $|\exp(f_z)| \leq P[h_z]$ on $\mathbb{U}$ and $|\exp(f_z^*)| = h_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$. Moreover $\exp(-f_z) \notin H^p$ for $z \in \partial\Omega$, $0 < p < \infty$.*

Proof. Assume for a moment that $h > 1$. Since $\ln h > 0$, so due to Theorem 2.14 we can conclude that there exists holomorphic function f on Ω such that $|\exp(f_z)| \leq \exp P[\ln(h_z)] \leq P[h_z]$ on $\mathbb{U}$ and $|\exp(f_z^*)| = h_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$. If we observe the construction of f closely, we may assume the equality $f(0) = 0$.

If there exists $z \in \partial\Omega$ and $0 < p < \infty$ such that $\exp(-f_z) \in H^p$ then

$$1 = |\exp(-pf(0))| \leq M_z(|\exp(-pf^*)|) = M_z\left(\frac{1}{h^p}\right) < M_z(1) = 1,$$

which is impossible.

If h is only positive then there exists $c > 0$ such that $\tilde{h} := e^c h > 1$ on $\partial\Omega$, so we can construct $\tilde{f}$ as above and it is enough to define $f := \tilde{f} - c$. □

Finally we illustrate how strange the function constructed above can be.

Example 2.16. There exists a zero free bounded holomorphic function f such that, if we define

$$A_z := \left\{t \in [0,1] : \left|f_z^*\left(e^{2\pi it}\right)\right| = 1\right\} \quad \text{and} \quad B_z := \left\{t \in [0,1] : \left|f_z^*\left(e^{2\pi it}\right)\right| = 2\right\},$$

then both sets A_z, B_z have positive measures and $[0,1]\backslash(A_z \cup B_z)$ has zero measure for all $z \in \partial\Omega$.

Proof. There exists $r > 0$ such that $K(0, \sqrt{d}r) \subset \Omega \subset \mathbb{C}^d$. We define the following compact subsets of $\partial\Omega$:

$$K_i := \{z \in \partial\Omega : r \leq \Re z_i = z_i\}.$$

We observe that $\partial\Omega = \mathbb{T}\bigcup_{i=1}^d K_i$. There exists a Smith–Volterra–Cantor subset $T \subset \mathbb{T}$ which is nowhere dense but has a positive Lebesgue measure μ. In fact we can assume that $0 < d\mu(T) < 1$.

We now define a compact subset of $\partial\Omega$:

$$K := \left\{ e^{2\pi i t} z : z \in \bigcup_{i=1}^{d} K_i,\ t \in T \right\}.$$

In particular, $0 < \mu(T) \le \mu(K \cap \mathbb{T}z) \le d\mu(T) < 1$ for $z \in \partial\Omega$. We define the LSC function

$$h(w) = \begin{cases} 2 & \text{for} \quad w \in \partial\Omega \setminus K, \\ 1 & \text{for} \quad w \in K. \end{cases}$$

Due to Theorem 2.14 (or Corollary 2.15) there exists a zero free holomorphic function f such that $|f| \le 2$ on Ω and $|f_z^*| = h_z$ a.e. on $\mathbb{T}$ for $z \in \partial\Omega$. Since $0 < \mu\,(K \cap \mathbb{T}z) < 1$ for $z \in \partial\Omega$ the function f has the required property. □

References

[1] A. B. Aleksandrov: The existence of inner functions in the ball, Sb. Math. 46(2) (1983) 143–159.

[2] A. B. Aleksandrov: Inner functions on compact spaces, Functional Anal. Appl. 18 (1984) 87–98.

[3] P. Kot: A holomorphic function with given almost all boundary values on a domain with holomorphic support function, J. Convex Analysis 14(4) (2007) 693–704.

[4] P. Kot: Homogeneous polynomials on strictly convex domains, Proc. Amer. Math. Soc. 135 (2007) 3895–3903.

[5] P. Kot: Bounded holomorphic functions with given maximal modulus on all circles, Proc. Amer. Math. Soc. 137 (2009) 179–187.

[6] E. Löw: A construction of inner functions on the unit ball in $\mathbb{C}^p$, Invent. Math. 67 (1982) 223–229.

[7] E. Löw: Inner functions and boundary values in $H^\infty(\Omega)$ and $A(\Omega)$ in smoothly bounded pseudoconvex domains, Math. Z. 185 (1984) 191–210.

[8] J. Ryll, P. Wojtaszczyk: On homogeneous polynomials on a complex ball, Trans. Amer. Math. Soc. 276 (1983) 107–116.

[9] P. Wojtaszczyk: On values of homogeneous polynomials in discrete sets of points, Studia Math. 84 (1986) 97–104.