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Inequalities for Orlicz Mixed Quermassintegrals*

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Our main aim is to generalize the mixed quermassintegrals $W_i(K,L)$ of convex bodies to Orlicz spaces. In the framework of Orlicz-Brunn-Minkowski theory, we introduce new affine geometric quantities $W_{\varphi,i}(M,K,L)$, for $0 \le i < n$, by calculating the first Orlicz variation of the mixed quermassintegrals, and we call these quantities the Orlicz mixed quermassintegrals of convex bodies M, K and L. Fundamental results concerning the mixed quermassintegrals as well as related Minkoswki and Brunn-Minkowski inequalities are then extended to the Orlicz setting. Diverse inequalities for certain new L_p-mixed quermassintegrals $W_{p,i}(M,K,L)$ are also derived. In particular, we establish a new general log Minkowski type inequality connected with the conjectured log-Brunn-Minkowski inequality. Finally, we introduce the concept of mixed projection quermassintegrals for which we prove an Orlicz-Minkowski type inequality.

Keywords: L_p-addition, Orlicz addition, mixed quermassintegrals, mixed p-quermassintegrals, Orlicz mixed quermassintegrals, Orlicz projection quermassintegrals.

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1. Introduction

One of the most important operations in geometry is vector addition. Given two sets K and L in $\mathbb{R}^n$, their Minkowski addition/sum is defined by

$$K + L = \{x + y : x \in K, y \in L\}.$$

This addtion plays an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to L_p-Brunn-Minkowski theory. If K and L are compact convex sets in $\mathbb{R}^n$ containing the origin and $1 \le p \le \infty$, their L_p addition $K +_p L$ is defined by Firey ([6] or [7]) through the equality

$$h(K +_p L, x)^p = h(K,x)^p + h(L,x)^p, \ x \in \mathbb{R}^n. \tag{1.1}$$

When $p = \infty$, (1.1) is interpreted as $h(K +_\infty L, x) = \max\{h(K,x), h(L,x)\}$, as is customary. Here $h(K,\cdot)$ denotes the support function of the set K, that is, then

$$h(K,x) = \sup\{x \cdot y : y \in K\},$$

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for all $x \in \mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. L_p addition and inequalities are the fundamental and core content in the L_p Brunn-Minkowski theory. For recent important results and more information on this theory, we refer to [12], [13], [14], [15], [20], [22], [23], [24], [25], [26], [27], [30], [31], [35], [36], [37] and the references therein. In recent years, a new extension of L_p-Brunn-Minkowski theory to Orlicz-Brunn-Minkowski theory has been initiated by Lutwak, Yang, and Zhang [28] and [29]. Following this, Gardner, Hug and Weil [9] introduced the Orlicz addition for the first time, constructed a general framework for the Orlicz-Brunn-Minkowski theory and made clear the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies has also been considered from different ways and the L_p-Brunn-Minkowski inequality has been extended to the Orlicz-Brunn-Minkowski inequality (see [38]). The Orlicz centroid inequality for star bodies has been studied in [43]. Advances in the theory can be found in the references [11], [17], [18], [32], [39], [40], [41], [42] and [44]. The Orlicz addition $K +_{\varphi,\varepsilon} L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin is defined (see [9]) so that

$$\varphi\left(\frac{h(K,x)}{h(K +_{\varphi,\varepsilon} L, x)}, \frac{h(L,x)}{h(K +_{\varphi,\varepsilon} L, x)}\right) = 1, \tag{1.2}$$

for all $x \in \mathbb{R}^n$ with $h(K,x) + h(L,x) > 0$, and $h(K +_{\varphi,\varepsilon} L, x) = 0$ for all $x \in \mathbb{R}^n$ with $h(K,x) = h(L,x) = 0$. Here $\varphi \in \Phi_2$, the set of convex functions $\varphi\colon [0,\infty)^2 \to [0,\infty)$ that are increasing in each variable and satisfy $\varphi(0,0) = 0$, $\varphi(1,0) = \varphi(0,1) = 1$. The particular case of interest corresponds to using (1.2) with $\varphi(x_1,x_2) = \varphi_1(x_1) + \varepsilon\varphi_2(x_2)$ for $\varepsilon > 0$ and some $\varphi_1, \varphi_2 \in \Phi$. Here Φ denotes the set of convex functions $\varphi_i\colon [0,\infty) \to (0,\infty)$ that are increasing and satisfy $\varphi_i(1) = 1$ and $\varphi_i(0) = 0$, where $i = 1,2$. Orlicz addition reduces to L_p addition, $1 \le p < \infty$, when $\varphi(x_1,x_2) = x_1^p + x_2^p$, or L_∞ addition, when $\varphi(x_1,x_2) = \max\{x_1,x_2\}$. The Orlicz mixed volume of K and L is defined by

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)dS(K,u),$$

where $S(K,u)$ is the mixed surface area measure of K and $\varphi \in \Phi$. Here K is a convex body containing the origin in its interior and L is a compact convex set containing the origin. Let $\mathcal{K}^n$ be the class of nonempty compact convex subsets of $\mathbb{R}^n$, let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}^n_{oo}$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. A set $K \in \mathcal{K}^n$ is called a convex body if its interior is nonempty. The Orlicz-Minkowksi inequality was obtained in [9]: for any $\varphi \in \Phi$, $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$

$$V_\varphi(K,L) \ge V(K) \cdot \varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \tag{1.3}$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

Aleksandrov [1] and Fenchel and Jessen [5] (see also Busemann [4] and Schneider [33]) have shown that for $K \in \mathcal{K}^n$, and $i = 0, 1, \ldots, n-1$, there exists a regular

Borel measure $S_i(K,\cdot)$ on S^{n-1}, such that each mixed quermassintegral $W_i(K,L)$ has the following representation:

$$W_i(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)dS_i(K,u), \tag{1.4}$$

for all $L\in\mathcal{K}^n$. When $K=L$, the mixed quermassintegral $W_i(K,L)$ becomes the classical quermassintegrals $W_i(K)$. The fundamental inequality for the mixed quermassintegrals (see [19]) says: for any $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$ and $0\le i<n$

$$W_i(K,L)^{n-i}\ge W_i(K)^{n-i-1}W_i(L), \tag{1.5}$$

with equality if and only if K and L are dilates or $L=\{o\}$.

Following the basic spirit of Aleksandrov's [1], Fenchel's and Jensen's [5] introduction of mixed quermassintegrals, and Lutwak's [21] introduction of p-mixed quermassintegrals, we study in the present paper the first order Orlicz variation of the mixed quermassintegrals. In Section 3, we prove that the first order Orlicz variation of the mixed quermassintegrals can be expressed as:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon\to 0^+}W_i(K',K+_{\varphi,\varepsilon}L)=\frac{1}{(\varphi_1)'_l(1)}\cdot W_{\varphi_2,i}(K',K,L),$$

where $K',K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$ and $\varphi_1,\varphi_2\in\Phi$, and $(\varphi_1)'_l(1)$ denotes the value of the left derivative of convex function φ_1 at point 1. This first order variational equation then provides the new geometric quantity $W_{\varphi_2,i}(K',K,L)$, called the i-th Orlicz mixed quermassintegral of K', K and L and defined by

$$W_{\varphi_2,i}(K',K,L):=(\varphi_1)'_l(1)\cdot\left.\frac{d}{d\varepsilon}\right|_{\varepsilon\to 0^+}W_i(K',K+_{\varphi,\varepsilon}L).$$

We also prove the integral representation

$$W_{\varphi,i}(K',K,L)=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)\,dS_i(K',u), \tag{1.6}$$

where $\varphi\in\Phi$. In (1.6), when $K'=K$, the Orlicz mixed quermassintegrals $W_{\varphi,i}(K',K,L)$ become the Orlicz quermassintegrals $W_{\varphi,i}(K,L)$, see [39]. When $K=L$, the $W_{\varphi,i}(K',K,L)$ become the mixed quermassintegrals $W_i(K',K)$. When $\varphi(t)=t^p$ and $p=1$, the $W_{\varphi,i}(K',K,L)$ become the mixed quermassintegrals $W_i(K',L)$. When $\varphi(t)=t^p$, $p\ge 1$ and $K'=K$, the $W_{\varphi,i}(K',K,L)$ become the Lutwak's p-mixed quermassintegrals $W_{p,i}(K,L)$. When $\varphi(t)=t^p$ and $p\ge 1$, the $W_{\varphi,i}(K',K,L)$ become a new mixed quermassintegrals in L_p-space, denoted by $W_{p,i}(K',K,L)$ and called L_p-mixed quermassintegrals of K',K and L. We establish a Minkowski type inequality for the L_p-mixed quermassintegrals $W_{p,i}(K',K,L)$: for any $K',K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$, $p\ge 1$ and $0\le i<n$,

$$W_{p,i}(K',K,L)^{n-i}\ge W_i(K')^{n-i-1}\cdot W_i(K)^{1-p}\cdot W_i(L)^p,$$

with equality if and only if K', K and L are dilates of each other, or $L=\{o\}$.

In Section 4 we establish the following Orlicz-Minkowksi inequality for the Orlicz mixed quermassintegrals: for any $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $\varphi \in \Phi$ and $0 \le i < n$,

$$W_{\varphi,i}(K', K, L) \ge W_i(K', K) \cdot \varphi\left(\frac{\overline{W}_i(K', L)}{W_i(K', K)}\right), \tag{1.7}$$

and under the strict convexity of φ, the equality holds if and only if K', K and L are dilates of each other, or $L = \{o\}$. Let us put, for short,

$$\overline{W}_i(K, L) := W_i(K)^{(n-i-1)/(n-i)} W_i(L)^{1/(n-i)},$$

for any $K \in \mathcal{K}^n_{00}$, $L \in \mathcal{K}^n_0$ and $0 \le i < n$. When $\varphi(t) = t^p, p \ge 1$ and $K' = K$, (1.7) reduces to the following Lutwak's L_p-Minkowski inequality for mixed p-quermassintegrals established in [21]: for any $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \ge 1$ and $0 \le i < n$

$$W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p, \tag{1.8}$$

with equality if and only if K and L are dilates or $L = \{o\}$. When $K' = K$, (1.7) becomes the Orlicz-Minkowksi inequality, see [39], for the Orlicz quermassintegrals: for any $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $\varphi \in \Phi$ and $0 \le i < n$

$$W_{\varphi,i}(K, L) \ge W_i(K) \cdot \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right), \tag{1.9}$$

and under the strict convexity of φ, the equality holds if and only if K and L are dilates, or $L = \{o\}$. When $K = L$, (1.7) becomes the Minkowksi inequality (1.5) for the mixed quermassintegrals. Obviously, putting $i = 0$ and $K' = K$ in (1.7), (1.7) becomes (1.3).

In Section 5, we establish the following Orlicz-Brunn-Minkowksi inequality for the mixed quermassintegrals: for any $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $\varphi \in \Phi_2$, and $1 \le i < n$

$$1 \ge \varphi\left(\frac{\overline{W}_i(K', K)}{W_i(K', K +_{\varphi,\varepsilon} L)}, \frac{\overline{W}_i(K', L)}{W_i(K', K +_{\varphi,\varepsilon} L)}\right), \tag{1.10}$$

and under the strict convexity of φ, the equality holds if and only if K', K and L are dilates of each other, or $L = \{o\}$. When $K' = K +_{\varphi,\varepsilon} L$, (1.10) becomes the Orlicz-Brunn-Minkowski inequality for the quermassintegrals, see [39]:

$$1 \ge \varphi\left(\left(\frac{W_i(K)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}, \left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right), \tag{1.11}$$

and under the strict convexity of φ, the equality holds if and only if K and L are dilates or $L = \{o\}$. When $\varphi(x_1, x_2) = x_1^p + x_2^p$, $p \ge 1$ and $K' = K +_{\varphi,\varepsilon} L$, (1.10) reduces to Lutwak's L_p-Brunn-Minkowski inequality for quermassintegrals (see [21]): for any $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \ge 1$ and $0 \le i < n$

$$W_i(K +_p L)^{p/(n-i)} \ge W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{1.12}$$

with equality if and only if K', K and L are dilates of each other, or $L=\{o\}$. When $\varphi(x_1,x_2)=x_1^p+x_2^p$, $p\geq 1$, $i=0$ and $K'=K+_{\varphi,\varepsilon}L$, (1.10) reduces to Firey's L_p-Brunn-Minkowski inequality for L_p-addition (see [7]):

$$V(K+_p L)^{p/n}\geq V(K)^{p/n}+V(L)^{p/n},$$

with equality if and only if K and L are dilates or $L=\{o\}$. A special case of (1.10) has been recently established by Gardner, Hug and Weil [9] with the inequality

$$1\geq\varphi\left(\left(\frac{V(K)}{V(K+_{\varphi,\varepsilon}L)}\right)^{1/n},\left(\frac{V(L)}{V(K+_{\varphi,\varepsilon}L)}\right)^{1/n}\right), \tag{1.13}$$

which, under the strict convexity of φ, is an equality if and only if K and L are dilates or $L=\{o\}$.

In 2012, Böröczky, Lutwak, Yang, and Zhang [2] conjectured the log-Minkowski inequality

$$\int_{S^{n-1}}\log\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS(K,u)\geq V(K)\log\left(\frac{V(L)}{V(K)}\right) \tag{1.14}$$

for origin-symmetric convex bodies K and L in $\mathbb{R}^n$. This was proved by them only when $n=2$. Gardner, Hug and Weil [9] proved a new version for convex bodies, but not for origin-symmetric convex bodies: for any $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$ such that $L\subset\mathrm{int}K$

$$\int_{S^{n-1}}\log\left(1-\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS(K,u)\leq V(K)\log\left(1-\frac{V(L)^{1/n}}{V(K)^{1/n}}\right)^n, \tag{1.15}$$

with equality if and only if K and L are dilates or $L=\{o\}$.

In Section 6, we provide a new log-Minkowski-type inequality in proving that, for any $K',K\in\mathcal{K}^n_{oo}$ and for any $L\in\mathcal{K}^n_o$ such that $L\subset\mathrm{int}K$ and $1\leq i<n$,

$$\int_{S^{n-1}}\log\left(1-\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K',u)\leq$$

$$\leq W_i(K',K)\log\left(1-\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right)^n, \tag{1.16}$$

with equality if and only if K', K and L are dilates of each other, or $L=\{o\}$. For $K'=K$ and $i=0$, (1.16) becomes (1.15). We also state the following conjecture-extension of the log Minkowski inequality:

$$\frac{1}{n}\int_{S^{n-1}}\log\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K',u)\geq$$

$$\geq W_i(K',K)\log\left(\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right). \tag{1.17}$$

When $K' = K$ and $i = 0$, (1.17) becomes the log-Minkowski inequality (1.14). Combining (1.16) and (1.17) yields the following classical Brunn-Minkowski inequality for quermassintegrals:

$$W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)},$$

with equality if and only if K and L are dilates or $L = \{o\}$.

In 2010, the Orlicz projection body $\mathbf{\Pi}_\varphi K$ of $K \in \mathcal{K}_{oo}^n$ has been introduced by Lutwak, Yang and Zhang [28] through its support function

$$h(\mathbf{\Pi}_\varphi K, u) = \inf\left\{\lambda > 0 : \frac{1}{nV(K)} \int_{S^{n-1}} \varphi\left(\frac{|u \cdot \upsilon|}{\lambda h(K,\upsilon)}\right) h(K,\upsilon) dS(K,\upsilon) \leq 1\right\},$$

for $\varphi \in \Phi$ and $u \in S^{n-1}$.

In Section 7, an Orlicz mixed projection body $\mathbf{\Pi}_{\varphi,i}(K', K)$ of K' and K is defined by its support function

$$h(\mathbf{\Pi}_{\varphi,i}(K', K), u) =$$

$$= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{|u \cdot \upsilon|}{\lambda h(K,\upsilon)}\right) d\bar{W}_{n,i}(K', K, \upsilon) \leq 1\right\}, \qquad (1.18)$$

where $K', K \in \mathcal{K}_{oo}^n$, $\varphi \in \Phi$, $u \in S^{n-1}$ and $0 \leq i < n$, and $d\bar{W}_{n,i}(K', K, \upsilon)$ is as in (4.1). Replacing $|u \cdot \upsilon|$ by $h(L, u)$ in (1.18) defines the quantity

$$\widehat{W}_{\varphi,i}(K', K, L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right) d\bar{W}_{n,i}(K', K, u) \leq 1\right\} \quad (1.19)$$

that we call the i-th Orlicz mixed projection quermassintegral. We prove the following Orlicz-Minkowski type inequality for the Orlicz mixed projection quermassintegrals: for any $\varphi \in \Phi$ and $K', K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $1 \leq i < n$

$$\widehat{W}_{\varphi,i}(K', K, L) \geq \frac{\overline{W}_i(K', L)}{W_i(K', K)}, \qquad (1.20)$$

and under the strict convexity of φ and the condition $W_i(L) > 0$, the equality holds if and only if K', K and L are dilates of each other.

2. Notation and preliminaries

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the closed unit ball centered at the origin. The surface of B is S^{n-1}. For a compact set K in $\mathbb{R}^n$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. If K is a nonempty closed (not necessarily bounded) convex set, then

$$h(K, x) = \sup\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the support function of K, where $x \cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. A support function is positively homogeneous of degree 1, that is,

$$h(K, rx) = rh(K, x),$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$. Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $d(K, L) = |h(K, \cdot) - h(L, \cdot)|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Let Φ_n, $n \in \mathbb{N}$, denote the set of convex functions φ: $[0, \infty)^n \to [0, \infty)$ that are strictly increasing in each variable and satisfy $\varphi(0) = 0$ and $\varphi(e_j) = 1 > 0$, $j = 1, \ldots, n$. When $n = 1$, we shall write Φ instead of Φ_1. The left derivative and right derivative of a real-valued function f are denoted by $(f)'_l$ and $(f)'_r$, respectively.

2.1. Mixed quermassintegrals

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [3])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.1), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, the mixed volume $V(K_1, \ldots, K_n)$ is written as $W_i(K)$ and called quermassintegral (or i-th quermassintegral) of K. If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, $K_n = L$. The mixed volumes $V(K_1, \ldots, K_n)$ are written as $W_i(K, L)$ and called mixed quermassintegrals of K and L. Aleksandrov [1] and Fenchel and Jessen [5] (see also Busemann [4] and Schneider [33]) have shown that for $K \in \mathcal{K}^n_{oo}$, and $i = 0, 1, \ldots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on S^{n-1}, such that the mixed quermassintegral $W_i(K, L)$ has the following representation:

$$W_i(K, L) = \frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon L) - W_i(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u) \, dS_i(K, u). \tag{2.2}$$

Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$ (see e.g. [8], p.353),

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) \, dS(K_1, \ldots, K_{n-1}, u). \tag{2.3}$$

In fact, the measure $S(K_1, \ldots, K_{n-1}, \cdot)$ can be defined by requiring that (2.3) holds for all $K \in \mathcal{K}^n$. If $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is written as $S_i(K, L, \cdot)$. When $L = B$, $S_i(K, L, \cdot)$ is written as $S_i(K, \cdot)$ and called the i-th mixed surface area measure. A fundamental inequality for mixed quermassintegrals says: For any $K, L \in \mathcal{K}^n$ and $0 \le i < n-1$,

$$W_i(K, L)^{n-i} \ge W_i(K)^{n-i-1} W_i(L), \tag{2.4}$$

with equality if and only if K and L are dilates (see [4] and [19]).

2.2. Mixed p-quermassintegrals

Mixed quermassintegrals are, of course, the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The mixed quermassintegrals $W_{p,0}(K, L), W_{p,1}(K, L), \ldots, W_{p,n-1}(K, L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition are defined by (see e.g. [21])

$$W_{p,i}(K, L) = \frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}, \tag{2.5}$$

for all $K \in \mathcal{K}^n_{oo}, L \in \mathcal{K}^n_o$, and real $p \ge 1$. The mixed p-quermassintegral $W_{p,i}(K, L)$ has the following integral representation:

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \, dS_{p,i}(K, u), \tag{2.6}$$

where $S_{p,i}(K, \cdot)$ denotes the Borel measure on S^{n-1}. The measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to a certain Borel measure $S_i(K, \cdot)$ on S^{n-1} such that the Radon-Nikodym derivative is given by

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{2.7}$$

The measure $S_{n-1}(K, \cdot)$ is independent of the body K, and is just the ordinary Lebesgue measure, S, on S^{n-1}. $S_i(B, \cdot)$ denotes the i-th surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B, \cdot) = S$ for all i. The surface area measure $S_0(K, \cdot)$ just is $S(K, \cdot)$. When $i = 0$, $S_{p,i}(K, \cdot)$ is written as $S_p(K, \cdot)$ (see [25], [26]). A fundamental inequality for mixed p-quermassintegrals says that, for any $K, L \in \mathcal{K}^n_{oo}, p > 1$ and $0 \le i < n$,

$$W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p, \tag{2.8}$$

with equality if and only if K and L are dilates. L_p-Brunn-Minkowski inequality for quermassintegrals have been established by Lutwak [21]. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $p \ge 1$ and $0 \le i < n$, then

$$W_i(K +_p L)^{p/(n-i)} \ge W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{2.9}$$

with equality if and only if K and L are dilates or $L = \{o\}$. Obviously, putting $i = 0$ in (2.6), the mixed p-quermassintegrals $W_{p,i}(K, L)$ become the well-known L_p-mixed volume $V_p(K, L)$, defined by (see e.g. [25])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u). \tag{2.10}$$

2.3. The Orlicz mixed volume

For $\varphi \in \Phi$, $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_o^n$, the Orlicz mixed volume of K and L, denoted by $V_\varphi(K, L)$, is defined in [9] by

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)\, dS(K,u). \tag{2.11}$$

The authors of [9] obtained the Orlicz-Minkowksi inequality

$$V_\varphi(K,L) \geq V(K)\cdot \varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right), \tag{2.12}$$

for all $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

The Orlicz quermassintegrals of K and L, denoted $W_{\varphi,i}(K,L)$, were defined by

$$W_{\varphi,i}(K,L) =: \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)\, dS_i(K,u), \tag{2.13}$$

see [39], for all $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $0 \leq i < n$. Obviously, when $\varphi(t) = t^p$ and $p \geq 1$, the Orlicz quermassintegrals reduces to the p-mixed quermassintegrals $W_{p,i}(K,L)$ defined in (2.6). When $i = 0$, (2.13) reduces to (2.11).

2.4. Orlicz addition

Given $m \geq 2, \varphi \in \Phi_m$, $K_j \in \mathcal{K}_0^n$ and $j = 1, \ldots, m$, the Orlicz addition of $K_1, \ldots, K_m$, denoted $+_\varphi(K_1, \ldots, K_m)$, is given by its support function, see [9]:

$$h(+_\varphi(K_1,\ldots,K_m),x) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h(K_1,x)}{\lambda},\ldots,\frac{h(K_m,x)}{\lambda}\right) \leq 1\right\} \tag{2.14}$$

for all $x \in \mathbb{R}^n$. Equivalently, the Orlicz addition $+_\varphi(K_1, \ldots, K_m)$ can be defined implicitly (and uniquely) by

$$\varphi\left(\frac{h(K_1,x)}{h(+_\varphi(K_1,\ldots,K_m),x)},\ldots,\frac{h(K_m,x)}{h(+_\varphi(K_1,\ldots,K_m),x)}\right) = 1, \tag{2.15}$$

for all $x \in \mathbb{R}^n$. An important special case is obtained when

$$\varphi(x_1,\ldots,x_m) = \sum_{j=1}^m \varphi_j(x_j),$$

for some fixed $\varphi_j \in \Phi$ such that $\varphi_1(1) = \cdots = \varphi_m(1) = 1$. We then write $+_\varphi(K_1, \ldots, K_m) = K_1 +_\varphi \cdots +_\varphi K_m$. This means that $K_1 +_\varphi \cdots +_\varphi K_m$ is defined either by

$$h(K_1 +_\varphi \cdots +_\varphi K_m, u) = \sup\left\{\lambda > 0 : \sum_{j=1}^m \varphi_j\left(\frac{h(K_j,x)}{\lambda}\right) \leq 1\right\}, \tag{2.16}$$

for all $x \in \mathbb{R}^n$, or by the corresponding special case of (2.15).

For real $p \geq 1$, $K, L \in \mathcal{K}^n_{oo}$ and $\alpha, \beta \geq 0$ (not both zero), the Firey linear combination $\alpha \cdot K +_p \beta \cdot L \in \mathcal{K}^n_o$ can be defined by (see [6] and [7])

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

Obviously, Firey and Minkowski scalar multiplications are related by $\alpha \cdot K = \alpha^{1/p} K$. The Orlicz linear combination, denoted by $+_\varphi(K, L, \alpha, \beta)$ for $K, L \in \mathcal{K}^n_o$, $\varphi_1, \varphi_2 \in \Phi$ and $\alpha, \beta \geq 0$, is defined (see [9]) by

$$\alpha\varphi_1\left(\frac{h(K,x)}{h(+_\varphi(K,L,\alpha,\beta),x)}\right) + \beta\varphi_2\left(\frac{h(L,x)}{h(+_\varphi(K,L,\alpha,\beta),x)}\right) = 1 \tag{2.17}$$

if $\alpha h(K,x) + \beta h(L,x) > 0$, and by $h(+_\varphi(K,L,\alpha,\beta),x) = 0$ for the case that $\alpha h(K,x) + \beta h(L,x) = 0$. It is easy to verify that when $\varphi_1(t) = \varphi_2(t) = t^p$, $p \geq 1$, the Orlicz linear combination $+_\varphi(K,L,\alpha,\beta)$ equals the Firey combination $\alpha \cdot K +_p \beta \cdot L$. Henceforth we shall write $K +_{\varphi,\varepsilon} L$ instead of $+_\varphi(K,L,1,\varepsilon)$, for $\varepsilon \geq 0$, $\alpha = 1$, and $\beta = \varepsilon$, and assume throughout that this is defined by (2.17).

3. Orlicz mixed quermassintegrals

In the following, for short we set $K_\varepsilon := K +_{\varphi,\varepsilon} L$, where $\varepsilon > 0$ and $K, L \in \mathcal{K}^n_o$.

Lemma 3.1. ([9]) *If $\varphi \in \Phi_m$, then the Orlicz addition $+_\varphi\colon (\mathcal{K}^n_0)^m \to \mathcal{K}^n_0$ is continuous, $GL(n)$ covariant, monotonic, projection covariant and has the identity property.*

Lemma 3.2. ([9]) *If $K, L \in \mathcal{K}^n_o$, then*

$$K_\varepsilon \to K, \tag{3.1}$$

in the Hausdorff metric as $\varepsilon \to 0^+$.

Lemma 3.3. *Let $\varphi \in \Phi_2$, and $\varphi_1, \varphi_2 \in \Phi$. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $1 \leq i < n$, then*

$$(\varphi_1)'_l(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon\to 0^+} W_i(K', K +_{\varphi,\varepsilon} L) = \frac{1}{n}\int_{S^{n-1}} \varphi_2\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)\, dS_i(K',u). \tag{3.2}$$

Proof. From (2.2), (2.17) and in view of L'Hospital's rule, we have

$$\lim_{\varepsilon\to 0^+} \frac{W_i(K',K_\varepsilon) - W_i(K',K)}{\varepsilon} =$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon n}\int_{S^{n-1}} \left(\frac{h(K,u)}{\varphi_1^{-1}\left(1 - \varepsilon\varphi_2\left(\dfrac{h(L,u)}{h(K_\varepsilon,u)}\right)\right)} - h(K,u)\right) dS_i(K',u)$$

$$= \frac{1}{n}\lim_{\varepsilon\to 0^+} \int_{S^{n-1}} \frac{A}{B}\, dS_i(K'.u),$$

where

$$A = h(K,u) \cdot \frac{d\varphi_1^{-1}(y)}{dy} \cdot \left(\varphi_2 \left(\frac{h(L,u)}{h(K_\varepsilon,u)} \right) - \varepsilon \cdot \frac{d\varphi_2(z)}{dz} \cdot \frac{h(L,u)}{h(K_\varepsilon,u)^2} \frac{dh(K_\varepsilon,u)}{d\varepsilon} \right),$$

$$B = \left(\varphi_1^{-1} \left(1 - \varepsilon \varphi_2 \left(\frac{h(L,u)}{h(K_\varepsilon,u)} \right) \right) \right)^2, \ \ y = 1 - \varepsilon \varphi_2 \left(\frac{h(L,u)}{h(K_\varepsilon,u)} \right) \text{ and } z = \frac{h(L,u)}{h(K_\varepsilon,u)}.$$

Since φ_1, φ_2 and φ_1^{-1} are continuous functions, and noting that $y \to 1^-$ as $\varepsilon \to 0^+$, and

$$\frac{d\varphi_1^{-1}(y)}{dy} = \lim_{y \to 1^-} \frac{\varphi_1^{-1}(y) - \varphi_1^{-1}(1)}{y-1} = \frac{1}{(\varphi_1)_l'(1)},$$

we obtain

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon \to 0^+} W_i(K', K_\varepsilon) = \frac{1}{n(\varphi_1)_l'(1)} \cdot \int_{S^{n-1}} \varphi_2 \left(\frac{h(L,u)}{h(K,u)} \right) h(K,u) dS_i(K',u). \qquad \square$$

Denoting by $W_{\varphi,i}(K', K, L)$, for any $\varphi \in \Phi$ and $1 \leq i < n$, the integral on the right-hand side of (3.2) with φ_2 replaced by φ, we see that either side of the equation (3.2) is equal to $W_{\varphi_2,i}(K', K, L)$ and therefore this new geometric quantity $W_{\varphi,i}(K', K, L)$ (Orlicz mixed quermassintegrals) has been born. It will play the same role in Orlicz-Brunn-Minkowski theory as $W_{p,i}(K, L)$ in L_p-Brunn-Minkowski theory.

Definition 3.4. (Orlicz mixed quermassintegrals) For $\varphi \in \Phi$, the *Orlicz mixed quermassintegral*, $W_{\varphi,i}(K', K, L)$, for $0 \leq i < n$, is defined by

$$W_{\varphi,i}(K', K, L) =: \frac{1}{n} \int_{S^{n-1}} \varphi \left(\frac{h(L,u)}{h(K,u)} \right) h(K,u) dS_i(K',u), \tag{3.3}$$

for all $K', K, L \in \mathcal{K}_o^n$.

When $i = 0$, $W_{\varphi,i}(K', K, L)$ is denoted by $V_\varphi(K', K, L)$ and called the *Orlicz mixed volume* of the convex bodies K', K and L. When $K' = K$, the Orlicz mixed volume of the convex bodies $V_\varphi(K', K, L)$ becomes the Orlicz mixed volumes $V_\varphi(K, L)$. When $\varphi_1(t) = \varphi_2(t) = t^p$ and $p \geq 1$, $V_\varphi(K', K, L)$ becomes the L_p-mixed volume $V_p(K', K, L)$ of the convex bodies K', K and L.

Lemma 3.5. *If $\varphi_1, \varphi_2 \in \Phi$ and $K, L \in \mathcal{K}_o^n$, and $0 \leq i < n$, then*

$$W_{\varphi_2,i}(K', K, L) = (\varphi_1)_l'(1) \cdot \lim_{\varepsilon \to 0^+} \frac{W_i(K', K_\varepsilon) - W_i(K', K)}{\varepsilon}. \tag{3.4}$$

From Definition 3.4 or Lemma 3.5, we easily find that the Orlicz mixed quermassintegral $W_{\varphi,i}(K', K, L)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 3.6. *If $K', K, L \in \mathcal{K}_o^n$, $0 \leq i < n$ and $\varphi \in \Phi$, then for $A \in SL(n)$*

$$W_{\varphi,i}(AK', AK, AL) = W_{\varphi,i}(K', K, L). \tag{3.5}$$

4. Orlicz-Minkowski inequality for the Orlicz mixed quermassintegrals

Definition 4.1. (i-th mixed normalized cone measure) If $K', K \in \mathcal{K}_o^n$, the *$i$-th mixed normalized cone measure*, denoted by $\bar{W}_{n,i}(K', K, \upsilon)$, is defined by

$$d\bar{W}_{n,i}(K', K, \upsilon) = \frac{h(K, \upsilon)}{nW_i(K', K)} \, dS_i(K', \upsilon). \tag{4.1}$$

For $K' = K$, $\bar{W}_{n,i}(K', K, \upsilon)$ becomes the i-th normalized cone measure $\bar{W}_{n,i}(K, \upsilon)$, and, see [39],

$$d\bar{W}_{n,i}(K, \upsilon) = \frac{h(K, \upsilon)}{nW_i(K)} dS_i(K, \upsilon). \tag{4.2}$$

For $K' = K$ and $i = 0$, $\bar{W}_{n,i}(K', K, \upsilon)$ becomes to the normalized cone measure $\bar{V}_n(K, \upsilon)$, by (see [2] and [9])

$$d\bar{V}_n(K, \upsilon) = \frac{h(K, \upsilon)}{nV(K)} dS(K, \upsilon).$$

In the following, we start with two auxiliary results (Lemmas 4.2 and 4.3), which will be the base of our further study. The Orlicz-Minkowski inequality for the Orlicz mixed quermassintegrals is established in Theorem 4.4.

Lemma 4.2. [16] (Jensen's inequality) *Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(g(x)) d\mu(x) \geq \phi\left(\int_X g(x) d\mu(x)\right). \tag{4.3}$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$.

Lemma 4.3. *Let $0 < a \leq \infty$ be an extended real number, and let $I = [0, a)$ be a possibly infinite interval. Suppose that $\varphi : I \to [0, \infty)$ is convex with $\varphi(0) = 0$. If $K', K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_o^n$ are such that $L \subset \mathrm{int}(aK)$, then*

$$\frac{1}{nW_i(K', K)} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K', u) \geq \varphi\left(\frac{\overline{W}_i(K', L)}{W_i(K', K)}\right). \tag{4.4}$$

If φ is strictly convex, equality holds if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. Since $L \subset \mathrm{int}(aK)$, we have $0 \leq \frac{h(L,u)}{h(K,u)} < a$ for all $u \in S^{n-1}$, and

$$\int_{S^{n-1}} d\bar{W}_{n,i}(K', K, \upsilon) = 1, \tag{4.5}$$

so the equation (4.5) defines a Borel probability measure $\bar{W}_{n,i}(K', K, \upsilon)$ on S^{n-1}. Hence, from (4.1), Jensen's inequality, Minkowski's inequality (2.4) for the mixed quermassintegrals and the fact that φ is increasing, we obtain

$$\frac{1}{nW_i(K',K)}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K',u) =$$
$$= \int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{h(K,u)}\right)d\bar{W}_{n,i}(K',K,u) \geq \varphi\left(\frac{W_i(K',L)}{W_i(K',K)}\right)$$
$$\geq \varphi\left(\frac{W_i(K')^{(n-i-1)/(n-i)}\cdot W_i(L)^{1/(n-i)}}{W_i(K',K)}\right) = \varphi\left(\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right).$$

In the following, we discuss the equal condition of (4.4). If ϕ is strictly convex, suppose that K', K and L are dilations of each other, namely: there exist $\lambda, \mu > 0$ such that $L = \lambda K$ and $K' = \mu L$. Hence

$$\frac{1}{nW_i(K',K)}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K',u) =$$
$$= \frac{1}{nW_i(\lambda\mu K,K)}\int_{S^{n-1}}\varphi\left(\frac{h(\lambda K,u)}{h(K,u)}\right)h(K,u)dS_i(\lambda\mu K,u)$$
$$= \varphi(\lambda) = \varphi\left(\frac{\overline{W}_i(K',\lambda K)}{W_i(K',K)}\right) = \varphi\left(\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right). \tag{4.6}$$

This implies that equality in (4.4) holds. On the other hand, suppose that equality holds in (4.4). Then the Jensen inequality and the Minkowski inequality must satisfy the equality sign. When φ is strictly convex, from the equality condition of Jensen's inequality (4.3), it follows that $h(L,u)/h(K,u)$ must be a constant for $S_i(K',\cdot)$-almost all $u \in S^{n-1}$, namely: $h(L,u)$ and $h(K,u)$ are proportional, this yields that L and K must be dilates. When φ is strictly convex, from the equality condition of Minkowski inequality (2.4), it follows that K' and L must be dilates. Combinations of these equality conditions with the case of (4.6) yield that if ϕ is strictly convex, the equality in (4.4) holds if and only if K', K and L are dilations of each other. □

Theorem 4.4. (Orlicz-Minkowski inequality for Orlicz mixed quermassintegrals) *Let $\varphi \in \Phi$. If $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $0 \leq i < n$, then*

$$W_{\varphi,i}(K',K,L) \geq W_i(K',K)\cdot\varphi\left(\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right). \tag{4.7}$$

If φ is strictly convex, equality holds if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. This follows immediately from (3.3) and Lemma 4.3 with $a = \infty$. □

Corollary 4.5. (L_p-Minkowski inequality for L_p-mixed quermassintegrals) *If $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \geq 1$ and $0 \leq i < n$, then*

$$W_{p,i}(K',K,L)^{n-i} \geq W_i(K')^{n-i-1}\cdot W_i(K)^{1-p}\cdot W_i(L)^p, \tag{4.8}$$

with equality if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. This follows from (2.4) and (4.7) with $\varphi(t) = t^p$ and $p \geq 1$. □

For $K' = K$, (4.8) becomes Lutwak's L_p-Minkowski inequality (1.8) for p-mixed quermassintegals. For $K = L$, (4.8) becomes the Minkowski inequality (1.5) for mixed quermassintegals.

Corollary 4.6. *If $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \geq 1$ and $0 \leq i < n$, then*

$$W_{p,i}(K', K, L) \geq W_i(K', K)^{1-p} \cdot \overline{W}_i(K', L)^p, \tag{4.9}$$

with equality if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. This follows immediately from (4.7) with $\varphi(t) = t^p$ and $p \geq 1$. □

For $K' = K$, (4.9) becomes Lutwak's L_p-Minkowski inequality (1.8). For $p = 1$, (4.9) becomes the Minkowski inequality (1.5).

5. Orlicz-Brunn-Minkowski inequality for mixed quermassintegrals

Lemma 5.1. [44] *Let $K, L \in \mathcal{K}^n_o$, $\varepsilon > 0$ and $\varphi \in \Phi$.*
(1) *If K and L are dilates, then K and $K +_{\varphi,\varepsilon} L$ are dilates.*
(2) *If K and $K +_{\varphi,\varepsilon} L$ are dilates, then K and L are dilates.*

Theorem 5.2. *Let $\varphi \in \Phi_2$. If $K, K' \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, and $1 \leq i < n$, then for $\varepsilon > 0$*

$$1 \geq \varphi\left(\frac{\overline{W}_i(K', K)}{W_i(K', K_\varepsilon)}, \frac{\overline{W}_i(K', L)}{W_i(K', K_\varepsilon)}\right). \tag{5.1}$$

If φ is strictly convex, equality holds if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. From (1.2), (2.2), (2.15), (3.3) and Theorem 4.4, we obtain for $\varphi_1, \varphi_2 \in \Phi$

$$\begin{aligned}
W_i(K', K_\varepsilon) &= \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(K,u)}{h(K_\varepsilon,u)}, \frac{h(L,u)}{h(K_\varepsilon,u)}\right) h(K_\varepsilon,u)\, dS_i(K',u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\varphi_1\left(\frac{h(K,u)}{h(K_\varepsilon,u)}\right) + \varepsilon\varphi_2\left(\frac{h(L,u)}{h(K_\varepsilon,u)}\right)\right) h(K_\varepsilon,u)\, dS_i(K',u) \\
&= W_{\varphi_1,i}(K', K_\varepsilon, K) + \varepsilon W_{\varphi_2,i}(K', K_\varepsilon, L) \\
&\geq W_i(K', K_\varepsilon)\left(\varphi_1\left(\frac{\overline{W}_i(K', K)}{W_i(K', K_\varepsilon)}\right) + \varepsilon\varphi_2\left(\frac{\overline{W}_i(K', L)}{W_i(K', K_\varepsilon)}\right)\right) \\
&= W_i(K', K_\varepsilon) \cdot \varphi\left(\frac{\overline{W}_i(K', K)}{W_i(K', K_\varepsilon)}, \frac{\overline{W}_i(K', L)}{W_i(K', K_\varepsilon)}\right).
\end{aligned} \tag{5.2}$$

Obviously, (5.2) easily yields (5.1). If equality holds in (5.1), then it also holds in (5.2), with K, L and φ replaced by K_ε, K and φ_1 (and by K_ε, L and φ_2), respectively. So if φ is strictly convex, then φ_1 and φ_2 are also, so both K and L are multiples of K_ε, these combining Lemma 5.1, it follows that If φ is strictly convex, equality holds if and only if K', K and L are dilates of each other, or $L = \{o\}$. □

Corollary 5.3. *If $K, K' \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \geq 1$ and $1 \leq i < n$, then*

$$W_i(K', K +_p L)^p \geq W_i(K')^{p(n-i-1)/(n-i)} \left(W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}\right), \qquad (5.3)$$

with equality if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. This follows immediately from Theorem 5.2 with $\varphi(x_1, x_2) = x_1^p + x_2^p$, $p \geq 1$ and $\varepsilon = 1$. □

Putting $K' = K +_p L$ in (5.3), (5.3) becomes Lutwak's L_p Brunn-Minkowski inequality (1.12) stated in the introduction.

Theorem 5.4. *Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals implies Orlicz Minkowski inequality for the mixed quermassintegrals.*

Proof. From (1.5), (3.4) and by using the Orlicz-Brunn-Minkowski inequality (5.1), we obtain

$$\begin{aligned}
&\frac{1}{(\varphi_1)'_l(1)} \cdot W_{\varphi_2,i}(K', K, L) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} W_i(K', K_\varepsilon) \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\big(W_i(K', K_\varepsilon) - W_i(K', K)\big) \geq \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\big(W_i(K', K_\varepsilon) - W_i(K', K)\big) \\
&= \lim_{\varepsilon\to 0^+} \frac{1 - \dfrac{W_i(K', K)}{W_i(K', K_\varepsilon)}}{1 - \varphi_1\left(\dfrac{W_i(K', K)}{W_i(K', K_\varepsilon)}\right)} \cdot \frac{1 - \varphi_1\left(\dfrac{W_i(K', K)}{W_i(K', K_\varepsilon)}\right)}{\varepsilon} \cdot W_i(K', K_\varepsilon) \\
&= \lim_{t\to 1^-} \frac{1-t}{1-\varphi_1(t)} \cdot \lim_{\varepsilon\to 0^+} \frac{1 - \varphi_1\left(\dfrac{W_i(K', K)}{W_i(K', K_\varepsilon)}\right)}{\varepsilon} \cdot \lim_{\varepsilon\to 0^+} W_i(K', K_\varepsilon) \\
&\geq \frac{1}{(\varphi_1)'_l(1)} \cdot \lim_{\varepsilon\to 0^+} \varphi_2\left(\frac{W_i(K', L)}{W_i(K', K_\varepsilon)}\right) \cdot W_i(K', K) \\
&\geq \frac{1}{(\varphi_1)'_l(1)} \cdot \varphi_2\left(\frac{\overline{W}_i(K', L)}{W_i(K', K)}\right) \cdot W_i(K', K). \qquad (5.4)
\end{aligned}$$

Replacing φ_2 by φ in (5.4) yields the Orlicz-Minkowski inequality (4.7).

From the equality conditions of the Orlicz-Brunn-Minkowski inequality (5.1), it follows that if ϕ is strictly convex, the equality in (5.4) holds if and only if K', K and L are dilates of each other, or $L = \{o\}$, and the proof is complete. □

From the proof of Theorem 5.2, we can see that the Orlicz Minkowski inequality for the Orlicz mixed quermassintegrals implies also the Orlicz Brunn-Minkowski inequality for the mixed quermassintegrals, and combining this with Theorem 5.4, we find:

Theorem 5.5. *The Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals is equivalent to the Orlicz Minkowski inequality for the mixed quermassintegrals. Namely, given* $\varphi_2 \in \Phi$ *and* $\varphi \in \Phi_2$, $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ *and* $1 \leq i < n$, *then*

$$W_{\varphi_2,i}(K',K,L) \geq W_i(K',K) \cdot \varphi_2\left(\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right)$$
$$\iff \quad 1 \geq \varphi\left(\frac{\overline{W}_i(K',K)}{W_i(K',K_\varepsilon)}, \frac{\overline{W}_i(K',L)}{W_i(K',K_\varepsilon)}\right). \tag{5.5}$$

If φ *is strictly convex, equality holds if and only if* K', K *and* L *are dilates of each other, or* $L = \{o\}$.

Corollary 5.6. *The Orlicz-Brunn-Minkowski inequality for the Orlicz quermassintegrals is equivalent to Orlicz-Minkowski inequality for the Orlicz mixed quermassintegrals. Namely: Let* $\varphi_2 \in \Phi$ *and* $\varphi \in \Phi_2$. *If* $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ *and* $0 \leq i < n$, *then*

$$W_{\varphi_2,i}(K,L) \geq W_i(K)\varphi_2\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right)$$
$$\iff \quad 1 \geq \varphi\left(\frac{W_i(K)^{1/(n-i)}}{W_i(K_\varepsilon)^{1/(n-i)}}, \frac{W_i(L)^{1/(n-i)}}{W_i(K_\varepsilon)^{1/(n-i)}}\right). \tag{5.6}$$

If φ *is strictly convex, equality holds if and only if* K *and* L *are dilates or* $L = \{o\}$.

Proof. The result follows immediately from Theorem 5.5 with $K' = K$. □

6. The log-Minkowski type inequality

Assume that $K, L \in \mathcal{K}^n_{oo}$, then the log Minkowski combination, $(1-\lambda)\cdot K +_o \lambda \cdot L$, is defined by

$$(1-\lambda)\cdot K +_o \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h(K,u)^{1-\lambda} h(L,u)^{\lambda}\},$$

for all real $\lambda \in [0,1]$. Böröczky, Lutwak, Yang and Zhang [2] conjecture that for origin-symmetric convex bodies K and L in $\mathbb{R}^n$ and $0 \leq \lambda \leq 1$,

$$V((1-\lambda)\cdot K +_o \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{6.1}$$

They proved (6.1) only when $n = 2$ and K, L must be origin-symmetric convex bodies.They also showed that (6.1), for all n, is equivalent to the following log-Minkowski inequality

$$\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K,u)}\right) d\bar{V}_n(K,v) \geq \frac{1}{n}\log\left(\frac{V(L)}{V(K)}\right), \tag{6.2}$$

where $\bar{V_n}(K,\cdot)$ is the normalized cone measure for K. In fact, replacing K and L by $K+L$ and K, respectively, (6.2) becomes the following result:

$$\int_{S^{n-1}} \log\left(\frac{h(K,u)}{h(K+L,u)}\right) d\bar{V_n}(K+L,u) \geq \log\left(\left(\frac{V(K)}{V(K+L)}\right)\right)^{1/n}. \tag{6.3}$$

Gardner, Hug and Weil [9] gave a new version of (6.3) for the nonempty compact convex subsets K and L, not origin-symmetric convex bodies as follows: If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$, then

$$\int_{S^{n-1}} \log\left(\frac{h(K,u)}{h(K+L,u)}\right) d\bar{V_n}(K+L,u) \leq \log\left(\frac{V(K+L)^{1/n} - V(L)^{1/n}}{V(K+L)^{1/n}}\right), \tag{6.4}$$

with equality if and only if K and L are dilates or $L=\{o\}$. They also showed that combining (6.3) and (6.4) yields the classical Brunn-Minkowski inequality.

$$V(K+L)^{1/n} - V(L)^{1/n} \geq V(K)^{1/n}.$$

In particular, if (6.2) holds (as it does, for origin-symmetric convex bodies when $n=2$), then (6.2) and (6.4) together furnish the classical Brunn-Minkowski inequality.

In the following, we give some new general versions of (6.4).

Theorem 6.1. *If $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ are such that $L \subset \mathrm{int} K$ and $1 \leq i < n$, then*

$$\log\left(\frac{W_i(K',K) - \overline{W}_i(K',L)}{W_i(K',K)}\right) \geq \int_{S^{n-1}} \log\left(\frac{h(K,u)-h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K',K,u), \tag{6.5}$$

with equality if and only if K', K and L are dilates of each other, or $L=\{o\}$.

Proof. Let $\varphi(t) = -\log(1-t)$, and notice that $\varphi(0)=0$ and φ is strictly increasing and strictly convex on $[0,1)$ with $\varphi(t) \to \infty$ as $t \to 1^-$. Hence inequality (6.5) is a direct consequence of Lemma 4.3 with $a=1$. □

For $K'=K$, (6.5) becomes the following result.

Corollary 6.2. *If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ are such that $L \subset \mathrm{int} K$ and $1 \leq i < n$, then*

$$\log\left(\frac{W_i(K)^{1/(n-i)} - W_i(L)^{1/(n-i)}}{W_i(K)}\right) \geq \int_{S^{n-1}} \log\left(\frac{h(K,u)-h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K,u), \tag{6.6}$$

with equality if and only if K and L are dilates or $L=\{o\}$. Here, $d\bar{W}_{n,i}(K,u)$ is as in (4.2).

Theorem 6.3. *If $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ are such that $L \subset \mathrm{int} K$ and $1 \leq i < n$, then*

$$\log\left(\frac{W_i(K',K+L) - \overline{W}_i(K',K)}{W_i(K',K+L)}\right) \geq \int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K+L,u)}\right) d\bar{W}_{n,i}(K',K+L,u), \tag{6.7}$$

with equality if and only if K', K and L are dilates of each other, or $L=\{o\}$.

Proof. If $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_o^n$, then $K+L \in \mathcal{K}_{oo}^n$. Noting that $K, L \subset \mathrm{int}(K+L)$ and using Theorem 6.1 with replacing K and L by $K+L$ and K, respectively, (6.6) easily follows. $\square$

When $K' = K+L$, (6.6) becomes the following result.

Corollary 6.4. *If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $1 \le i < n$, then*

$$\log\left(\frac{W_i(K+L)^{1/(n-i)} - W_i(K)^{1/(n-i)}}{W_i(K+L)^{1/(n-i)}}\right) \ge \int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K+L,u)}\right) d\bar{W}_{n,i}(K+L,u), \tag{6.8}$$

with equality if and only if K and L are dilates or $L = \{o\}$.

Putting $i = 0$ in (6.8), (6.8) reduces to (6.4).

Theorem 6.5 *If $K', K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $1 \le i < n$, then*

$$\log \int_{S^{n-1}} \exp\left(\frac{h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K',K,u) \ge \frac{\overline{W}_i(K',L)}{W_i(K',K)}, \tag{6.9}$$

with equality if and only if K', K and L are dilates of each other, or $L = \{o\}$.

Proof. The result follows from Lemma 4.3 with $\varphi(t) = e^t - 1$ and $a = \infty$. $\square$

Remark 6.6. Similarly, L_p-Minkowski inequality (4.8) for L_p-mixed quermass-integrals $W_{p,i}(K',K,L)$ can be written as

$$\left(\int_{S^{n-1}} \left(\frac{h(L,u)}{h(K,u)}\right)^p d\bar{W}_{n,i}(K',K,u)\right)^{1/p} \ge \frac{\overline{W}_i(K',L)}{W_i(K',K)}, \tag{6.10}$$

with equality if and only if K', K and L are dilates of each other, or $L = \{o\}$. When $p = 1$, (6.10) becomes a new form of the Minkowski inequality (2.4) written as

$$\int_{S^{n-1}} \left(\frac{h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K',K,u) \ge \frac{\overline{W}_i(K',L)}{W_i(K',K)},$$

with equality if and only if K' and L are dilates or $L = \{o\}$. Namely,

$$W_i(K',L)^{n-i} \ge W_i(K')^{n-i-1} W_i(L),$$

with equality if and only if K' and L are dilates, or $L = \{o\}$.

We also point out a conjecture which is an extension of the log Minkowski inequality as follows.

Conjecture 6.7. *If $K', K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $1 \le i < n$, then*

$$\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K,u)}\right) dS_i(K',K,u) \ge \log\left(\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right). \tag{6.11}$$

When $K' = K$ and $i = 0$, (6.11) becomes the log-Minkowski inequality (6.2). Noting that $L \subset \mathrm{int}(K + L)$ and from (6.11) with K replaced by $K + L$, we have

$$\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K+L,u)}\right) dS_i(K', K+L, u) \geq \log\left(\frac{\overline{W}_i(K',L)}{W_i(K',K+L)}\right). \quad (6.12)$$

It's very interesting to see that combining (6.12) and (6.7) together split the following Brunn-Minkowski type inequality for the mixed quermassintegrals:

$$W_i(K', K+L) \geq \overline{W}_i(K', K) + \overline{W}_i(K', L), \quad (6.13)$$

with equality if and only if K', K and L are dilates of each other, or $L = \{o\}$. Putting $K' = K + L$ in (6.13), (6.13) becomes the classical Brunn-Minkowski inequality for quermassintegrals

$$W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)},$$

with equality if and only if K and L are dilates or $L = \{o\}$.

7. Orlicz-Minkowski inequality for Orlicz mixed projection quermassintegrals

In 2010, Lutwak, Yang and Zhang [28] defined the Orlicz projection body $\mathbf{\Pi}_\varphi K$ of $K \in \mathcal{K}^n_{oo}, u \in S^{n-1}$ through its support function

$$h(\mathbf{\Pi}_\varphi K, u) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{|u \cdot v|}{\lambda h(K,v)}\right) d\bar{V}_n(K,v) \leq 1\right\}, \quad (7.1)$$

for all $u \in S^{n-1}$, where $\bar{V}_n(K, \cdot)$ is the normalized cone measure for K. Here, we define the Orlicz mixed projection body.

Definition 7.1. Let $K', K \in \mathcal{K}^n_{oo}$, $\varphi \in \Phi$ and $0 \leq i < n$. The *Orlicz mixed projection body* $\mathbf{\Pi}_{\varphi,i}(K', K)$ is defined by

$$h(\mathbf{\Pi}_{\varphi,i}(K',K),u) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{|u \cdot v|}{\lambda h(K,v)}\right) d\bar{W}_{n,i}(K',K,v) \leq 1\right\}, \quad (7.2)$$

for $u \in S^{n-1}$.

Obviously, when $K' = K$ and $i = 0$, (7.2) becomes (7.1). In (7.2), for $L \in \mathcal{K}^n_o$, replacing $|u \cdot v|$ by $h(L, u)$ yields

$$\widehat{W}_{\varphi,i}(K',K,L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right) d\bar{W}_{n,i}(K',K,u) \leq 1\right\}, \quad (7.3)$$

that we call the *Orlicz mixed projection quermassintegral.* When $K' = K$, (7.3) becomes the quantity defined in [9]:

$$\widehat{W}_{\varphi,i}(K,L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right) d\bar{W}_{n,i}(K,u) \leq 1\right\}.$$

We prove the following Orlicz-Minkowski type inequality for the Orlicz mixed projection quermassintegrals $\widehat{W}_{\varphi,i}(K',K,L)$.

Theorem 7.2. *If $K',K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$, $\varphi\in\Phi$ and $1\le i<n$, then*

$$\widehat{W}_{\varphi,i}(K',K,L)\ge\frac{\overline{W}_i(K',L)}{W_i(K',K)}. \tag{7.4}$$

If φ is strictly convex and $W_i(L)>0$, equality holds if and only if K', K and L are dilates of each other.

Proof. Replacing K by λK, $\lambda>0$ in (4.4) with $a=\infty$, we have

$$\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right)d\bar{W}_{n,i}(K',K,u)\ge\varphi\left(\frac{1}{\lambda}\cdot\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right). \tag{7.5}$$

Suppose that

$$\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right)d\bar{W}_{n,i}(K',K,u)=1. \tag{7.6}$$

From (7.3), we have

$$\widehat{W}_{\varphi,i}(K',K,L)=\lambda. \tag{7.7}$$

On the other hand, from (7.5) and (7.6), we have

$$\varphi\left(\frac{1}{\lambda}\cdot\frac{\overline{W}_i(K',L)}{W_i(K',K)}\right)\le 1.$$

Noting that φ is increasing and $\varphi(1)=1$, we obtain

$$\frac{\overline{W}_i(K',L)}{W_i(K',K)}\le\lambda. \tag{7.8}$$

From (7.7) and (7.8) we see that (7.4) follows.

Now, we discuss the equality condition for (7.4). Suppose that equality holds, φ is strictly convex and $W_i(L)>0$. From (7.3), the exist $\mu=\widehat{W}_{\varphi,i}(K',K,L)>0$ satisfying

$$\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{\mu h(K,u)}\right)d\bar{W}_{n,i}(K',K,v)=1.$$

Hence $\quad\mu=\dfrac{\overline{W}_i(K',L)}{W_i(K',K)}\quad$ and $\quad\varphi\left(\dfrac{1}{\mu}\cdot\dfrac{\overline{W}_i(K',L)}{W_i(K',K)}\right)=1.$

Therefore the equality in (7.5) holds for $\lambda=\mu$. From the equality condition of (4.4), it follows that K', K and L are dilates of each other. $\square$

When $K' = K$, (7.4) becomes the following result.

Corollary 7.3. *If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $\varphi \in \Phi$ and $1 \le i < n$, then*

$$\widehat{W}_{\varphi,i}(K, L) \ge \left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}.$$

If φ is strictly convex and $W_i(L) > 0$, equality holds if and only if K', K and L are dilates of each other.

Remark 7.4. When $\varphi(t) = t^p$ and $p \ge 1$ in (7.3), it easily follows that, if $K', K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \ge 1$ and $0 \le i < n$, then

$$\widehat{W}_{\varphi,i}(K', K, L) = \left(\frac{W_{p,i}(K', K, L)}{W_i(K', K)}\right)^{1/p}. \tag{7.10}$$

Putting $\varphi(t) = t^p$ and $p \ge 1$ in (7.4), from (7.10), and in view of $W_i(L) > 0$, (7.4) reduces to the following result:

$$W_{p,i}(K', K, L) \ge W_i(K', K)^{1-p} \cdot \overline{W}_i(K', L)^p, \tag{7.11}$$

with equality if and only if K', K and L are dilates of each other. Putting $K' = K$ in (7.11), (7.11) reduces to Lutwak's L_p-Minkowski inequality (1.8) stated in the introduction.

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